Randomness in Quantum Mechanics

—

or: "eth" in quantum theory

"I leave to several futures (not to all) my garden of forking paths"–
J. L. Borges

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Credits and Contents

Numerous useful discussions with: Ph. Blanchard, Bohmians (Dürr, Goldstein), colleagues at ETH (Graf, Hepp), P. Pickl (LMU) and Chr. Schilling; challenge and stimulation coming from experiments by the Haroche-Raimond group in Paris and from papers by Maassen and Kümmerer and subsequent papers by Bauer, Bernard et al.
This is the result of joint work primarily with my former PhD student Baptiste Schubnel; some of our efforts have involved collaboration with M. Ballesteros, J. Faupin and M. Fraas.

Contents

1. Introduction – Some basic questions and claims
2. Direct (von Neumann) measurements
3. Indirect (Kraus) measurements
4. Conclusions – discussion

Thanks to Pierre-Francois Rodriguez for the Borges quote, ...

Quantum Theory somewhat comparable to construction of *Babylonian tower*, with similar consequences …!

Claims:

(1) Without *information loss* and *entanglement* retrieval of information from quantum systems by measurements/ observations would be *impossible*; no info. paradoxes!

(2) Operator algebras (including type III_1 factors!) have been invented to be used in QT, rather than to be ignored!

1. FROM THE CLASSICAL HARMONICS OF NEWTON, LEIBNIZ, BACH, JACOBI ⋯ TO HEISENBERG's QU. HARMONICS

Classical mechanics =
Hamiltonian dynamics on
class. phase space, Γ, w.
Γ = sympl. mf. equipped
w. ω, $\{\cdot\, ,\, \cdot\}_P$.

Pure states = points in Γ;
mixed states = prob. meas. on Γ

"Observables": $\mathcal{A}_0 = C^\infty(\Gamma)$.

<u>Ex.</u> $\Gamma = T^*M$, $M = \mathbb{R}^f$, pt. part.

If point particles el. charged, int. "obs." is a component, x, of el. dipole moment.

System w. $\Gamma = R^{2f}$ integrable iff $\exists\, f$ integrals of motion

$$\underline{A} = (A_1, \cdots, A_f)$$

in involution. $\Rightarrow$ Traj. of system lie on f-D inv. tori, $T^f = \{\underline{A} = \text{const.} = \underline{A}_*\}$

Observable, x, rest. to T^f is periodic fu. of angles

$$\underline{\varphi} = \{\varphi_1, \cdots, \varphi_f\}.$$

$$\Rightarrow \quad x = \sum_{\underline{n} \in \mathbb{Z}^f} \hat{x}_{\underline{n}} \, e^{i \underline{n} \cdot \underline{\varphi}} \, .$$

Eqs. of motion $\Rightarrow$ $\underline{\varphi}(t) = \underline{\omega}_* \cdot t$,

where $\underline{\omega}_* = \left(\partial H / \partial \underline{A} \right) (\underline{A}_*)$,

H: Hamilton fu., $x(0) \in C(T^f(\underline{A}_*))$

$\Rightarrow$ $x(t)$ quasi-per. fu. of t

with frequencies $\omega_{\underline{n}} = \underline{n} \cdot \underline{\omega}_*$

For system of charged particles, e.m. waves of frequencies $\omega_{\underline{n}}$, $\underline{n} \in \mathbb{Z}^f$, emitted. They form <u>abelian group</u>: $\omega_{\underline{n}} + \omega_{\underline{m}} = \omega_{\underline{n}+\underline{m}}$.

Experimentally, this is wrong.

Ritz-Rydberg ... principle:

Frequencies of light em. by system form groupoid, $\mathcal{G}$:

$\omega = \omega_{\underline{\nu}\underline{\mu}}$, w. $\omega_{\underline{\nu}\underline{\mu}} = \omega_{\underline{\nu}\underline{\lambda}} + \omega_{\underline{\lambda}\underline{\mu}}$,

for $(\underline{\nu}\underline{\mu}), (\underline{\nu}\underline{\lambda}), (\underline{\lambda}\underline{\mu})$ in $\mathcal{G}$.

Bohr-Sommerfeld:

$$\omega_{\underline{\nu}\underline{\mu}} = \omega_{\underline{\nu}} - \omega_{\underline{\mu}},$$

$\underline{\nu}, \underline{\mu}$: allowed quantum #

$$\hbar\omega_{\underline{\nu}} = H(\underline{A}_{\underline{\nu}}), \ \underline{A}_{\underline{\nu}} = h\underline{\nu},$$

$\underline{\nu} \in \mathbb{Z}^f$. $\longrightarrow$ Hydrogen atom, harmonic oscillator.

Heisenberg (1925): In Qu. Th.,
"observables", x, only dep.
on data of allowed proc.,
→ pairs of allowed quantum
states → pairs $(\underline{\nu}\,\underline{\mu}) \in \mathcal{G}$, i.e.,

$x \sim \hat{x}_{\underline{n}} \mapsto$ scheme $(x_{\underline{\nu}\underline{\mu}})$, with
$\overline{x}_{\underline{\mu}\underline{\nu}} = x_{\underline{\nu}\underline{\mu}}$ ("reality cond.",
emission → absorption).

$$\hat{x}_{\underline{n}} \longrightarrow x_{\underline{\nu}\underline{\mu}}$$

$$\widehat{(x^1 \cdot x^2)}_{\underline{n}} = \sum_{\underline{m} \in \mathbb{Z}^f} \hat{x}^1_{\underline{n}-\underline{m}} \cdot \hat{x}^2_{\underline{m}}$$

$$\downarrow$$

$$(x^1 * x^2)_{\underline{\nu}\underline{\mu}} = \sum_{\underline{\lambda}} x^1_{\underline{\nu}\underline{\lambda}} \cdot x^2_{\underline{\lambda}\underline{\mu}}$$

convol. prod. $\longrightarrow$ matrix prod.

By R.-R., B.-S. & Maxwell th.

$$x_{\underline{\nu}\underline{\mu}}(t) = x_{\underline{\nu}\underline{\mu}}\, e^{i\omega_{\underline{\nu}\underline{\mu}} \cdot t} \tag{1}$$
$$= \left(e^{i(Ht)/\hbar} * x * e^{-i(Ht)/\hbar}\right)_{\underline{\nu}\underline{\mu}}$$

w: $H = \mathrm{diag}\,(\hbar\omega_{\underline{\nu}})$

In order to be able to "forget" B.-S. (e.g., to quantize non-integr. systems), find alg. generated by $x, \dot{x}, \ldots$

Thomas-Kuhn-Heisenberg sum rules: $p := m\dot{x}$, where x is a component of dipole moment, m = mass of particle (electron). Then

$$[x,p]_{\underline{\nu}\,\underline{\nu}} = [x, m\dot{x}]_{\underline{\nu}\,\underline{\nu}} \quad (2)$$

$$\overset{(1)}{=} 2im \sum_{\underline{\mu}} |x_{\underline{\nu}\,\underline{\mu}}|^2 \, \omega_{\underline{\nu}\,\underline{\mu}}$$

Bohr's correspondence pr.:

For $|\underline{\nu}-\underline{\mu}| \ll |\underline{\nu}|, |\underline{\mu}|, (|\underline{\nu}|, |\underline{\mu}| \gg 1)$

$x_{\underline{\nu}\,\underline{\mu}} \approx \hat{x}_{\underline{\nu}-\underline{\mu}}$, w. $\hat{x}_{\underline{\nu}-\underline{\mu}}$ calculated for traj. in $T^f(\underline{A}_{\frac{\underline{\nu}+\underline{\mu}}{2}})$

For hydrogen atom or harmonic osc., $|\hat{x}_{\underline{\nu}-\underline{\mu}}|^2$ calc. by using virial thm. and

$\omega_{\underline{\nu}\,\underline{\mu}}$ from Bohr-Sommerfeld.

$$\Longrightarrow \quad [x,p]_{\underline{\nu}\,\underline{\nu}} \overset{(2)}{\approx} i\hbar$$
$$= i\hbar \quad (\text{Heis., '25})$$

Born-Jordan '25:

Matrix interpretation;

for $H = p^2/2m + \cdots + v(x, x^2, \cdots)$,

$$[H, [x,p]] = 0$$

$$\Longrightarrow \quad [x,p]_{\underline{\nu}\,\underline{\mu}} = 0, \quad \omega_{\underline{\nu}\,\underline{\mu}} \neq 0$$

$$[x,p] = i\hbar\, \mathbb{1}$$

Dirac, '25: trsf. theory,

$$\{f,g\}_P \longrightarrow \frac{i}{\hbar}[f,g]$$

B-H-J, '25: "quantum geometry"

Miracle: Ritz-Rydberg, B.-S.,
corresp. pr. only approx. valid
($\alpha = e^2/\hbar c \simeq 1/137 \ll 1$); yet,
Heisenberg's matrix mech.
consistent new mech.!

$$(\mathcal{A}_0, \cdot) \longrightarrow (\mathcal{A}_\hbar, *)$$

$\hbar$: deformation parameter

MM ~ WM (Schrödinger '26)

~ Feynman path int., ~'43

Action in Darboux coos.

$$S = \frac{1}{2}\int_\gamma \left\{ \sum_j p_j\, dq^j - H(p,q)\, dt \right\}$$

To deform $\mathcal{A}_0$ to $\mathcal{A}_\hbar$, set

$H = 0 \Longrightarrow q^j(t) = p_j(t) = \text{const.}$

D: surface in Γ, $\partial D =: \gamma$ (loop in Γ), $D_0 := \{(\tau^1, \tau^2) \,|\, |\tau| \leq 1\}$.

$$S = \frac{1}{2}\int_\gamma \Big\{\sum_j p_j \, dq^j\Big\} \underset{\substack{\uparrow \\ \text{Stokes}}}{=} \frac{1}{2}\int_D \omega$$

$$\sim \int_{D_0} d\tau^\alpha \wedge d\tau^\beta \Big\{ \frac{\partial X^A(\tau)}{\partial \tau^\alpha} \eta_{A,\beta}(\tau) - \frac{1}{2}\Omega^{AB}(X(\tau))\, \eta_{A,\alpha}(\tau)\, \eta_{B,\beta}(\tau) \Big\}$$

$S(X,\eta)$ inv. under repar. of D_0 and symplectomor. of Γ. $*$ product of two fus. f, g on Γ given by

TFT: $X(\infty) \equiv X \in \Gamma$.

$$(f \underset{\hbar}{*} g)(X) = \text{"}\int \mathcal{D}X \int \mathcal{D}\eta \; e^{\frac{i}{\hbar} S(X,\eta)} \times f(X(0))\, g(X(1))\text{"}$$

$$= \left(f \cdot g + \frac{i\hbar}{2}\{f, g\} + \cdots\right)(X),$$

$0, 1, \infty \in \partial D_0$.

∞-dim. symmetries of S require gauge fixing (B-V formalism): Problem solved in sense of formal power series in $\hbar$ by Kontsevich; Cattaneo-Felder.

$H_2(\Gamma) \leftrightarrow$ "quant." of $\hbar$.

Shall apply this to C-S!

Concrete examples:

1) $\Gamma = T^*M$ ess. Schrödinger '26

$\Gamma = S^2$ B-H-J '25

$\Gamma = T^*\mathbb{R}^3 \times S^2$ Pauli '25, '27

+ Pauli's exclusion pr.
(Pauli '25, Heisenberg '26)

$\rightarrow$ th. of NR spinning e^-.

2) $\Gamma =$ (compact) Kähler mf.

$\Gamma =$ coadj. orbit

Borel-Weil-Bott, geom. qu.

3) $\Gamma =$ phase space over G^{S^1}

from 3D TFT (C-S & BF theories)

4 deformation parameters

(Planck, 1900)

$\hbar$: as described.

k_B : cont. th. $\rightarrow$ atomism

thermodyn. $\rightarrow$ stat. mech.

c^{-1}: Galilei $\rightarrow$ Lorentz-Poincaré ...

l_P : th. without gravity $\rightarrow$ th. incl. gravity

In nature,

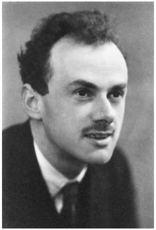
$\hbar, k_B, c^{-1}, l_P \neq 0, = 1$

"quantum gravity"

Need a new "Heisenberg"!

1. Introduction – Some basic questions and claims

Much confusion and disorientation surround the Foundations of Quantum Mechanics – so much thereof that most mathematicians do not want to touch this subject. There are many prejudices that are wrong or, to say the least, inaccurate and confusing. To mention but one example: We tend to teach to our students that the time-evolution of states of a system is described, in quantum mechanics, by the Schrödinger equation, and that the Schrödinger picture and the Heisenberg picture are equivalent. Well, nothing could be farther from the truth when considering systems accessible to observations! – Etc. Not having to make a career, anymore, I consider it to be my duty to attempt to alleviate some of this confusion – I believe I have made a little progress.



Introduction – ctd., some fundamental problems

In our courses, we tend to describe quantum-mechanical systems, S, as pairs of a Hilbert space, $\mathcal{H}_S$, and of a propagator, $(U(t,s))_{t,s\in\mathbb{R}}$, describing time evolution. Unfortunately, these data hardly encode any interesting (invariant) information about S that would enable one to draw conclusions about its physical properties, and they give the erroneous impression that quantum theory might be a deterministic theory.

→Fundamental questions and problems:

- What do we have to add to the usual formalism of quantum theory to arrive at a mathematical structure which – through interpretation – can be given physical meaning, *without the intervention of "observers" (at places where they obviously do not play any role)*?
- What is the origin of the *intrinsic randomness* of quantum theory, given the deterministic character of Schrödinger equation? Does it differ from classical randomness?

Introduction - ctd.

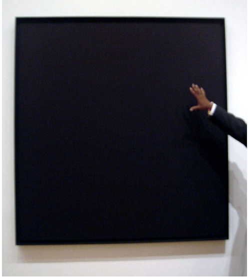
- What is the meaning of *states, "observables"* and *events* ($\nearrow$ R. Haag) in quantum mechanics? Do we understand the *time evolution* of states of quantum systems, and what does it have to do with solutions of the Schrödinger equation?
- What do we mean by an *isolated system* in quantum mechanics, and why is this an important notion*? How can one prepare a system in a prescribed state?
  *Because *only for isolated systems* a general description of the *Heisenberg time evolution of "observables"* is available!

Goal of Lecture

Sketch a theory of *events* and of direct and indirect *observations/ measurements* in quantum mechanics based on two new ideas:

- *Loss of access to information, & entanglement with "lost" (inaccessible) degrees of freedom.*
- *Specification of list of "instruments" serving to observe events.*

Metaphor for the "mysterious holistic aspects" of Quantum Mechanics



QM is QM-as-QM and everything else is everything else

"The one thing to say about art is that it is one thing. Art is art-as-art and everything else is everything else." (Ad Reinhardt)

2. Direct (projective, or von Neumann) Measurements

In classical Hamiltonian mechanics, observable physical quantities of an isolated system, S, are described by real, continuous functions on the phase space, Γ, of S. Their time evolution is governed by the usual Hamiltonian equations of motion formulated in terms of Poisson brackets. *Heisenberg's 1925 paper* on quantum-theoretical *"Umdeutung"* contains revolutionary ideas, further elaborated upon by *Dirac*, of how to replace the basic concepts of Hamiltonian mechanics by new ones leading to a *quantum-mechanical description* of physical systems:

- Physical quantities of a system S are represented by *"symmetric matrices"*, $\widehat{F}$, (s.a. linear operators acting on a Hilbert space, $\mathcal{H}_S$)
- The Poisson bracket, $\{F, G\}$, of two phys. quantities, F and G, in a classical description of S is to be replaced, in QM, by

  $$i\hbar^{-1}[\widehat{F}, \widehat{G}],$$

  where $\widehat{F}$ and $\widehat{G}$ are the s.a. operators representing the physical quantities corresponding (in classical mechanics) to F and G.

Definition of quantum-mechanical systems

- The Heisenberg time evolution of an operator $\widehat{F}$ representing a physical quantity of an isolated systems S is governed by

$$\frac{d}{dt}\widehat{F}(t) = i[\widehat{H}, \widehat{F}(t)],$$

where $t \in \mathbb{R}$ is time, and $\widehat{H}(=\widehat{H}(t))$ is the Hamilton operator of S.

Definition of isolated physical systems in quantum mechanics

Physical quantities of an isolated physical system S are represented by *selfadjoint operators* acting on a Hilbert space $\mathcal{H}_S$; all states of S are *density matrices* on $\mathcal{H}_S$. (Generalization: $\rightarrow C^*$-algebras!)

(I) We define $\mathcal{E}_{[t,s]}$ to be the von Neumann algebra generated by all physical quantities of an isolated physical system S potentially measurable/observable in the time interval $[t, s]$, with $t < s$. In simple cases,

$$\mathcal{E}_{[t,s]} := \{\text{linear combinations of } \prod_i \widehat{X}_i(t_i) | t < t_i < s, \forall i\}, \qquad (1)$$

Definition of qm systems – ctd.

where $\widehat{X}(t)$ is the s.a. operator representing a physical quantity $\widehat{X}$ potentially measurable/obs. at time t, in the Heisenberg picture.

It is natural to require that $\quad \mathcal{E}_{[t,s]} \subseteq \mathcal{E}_{[t',s']} \quad$ whenever $t' \leq t$ and $s' \geq s$. We set

$$\mathcal{E}_{\geq t} := \overline{\bigvee_{s:s>t} \mathcal{E}_{[t,s]}}, \quad \mathcal{Z}_{\geq t} = \text{center of } \mathcal{E}_{\geq t}, \quad \mathcal{E} := \overline{\bigvee_{t<s} \mathcal{E}_{[t,s]}}, \tag{2}$$

where $\overline{(\cdot)}$ indicates closure in the weak topology on $B(\mathcal{H}_S)$. An *event* potentially detectable in $[t,s]$ is an *orthogonal projection* $\Pi \in \mathcal{E}_{[t,s]}$.

(II) An *"instrument"* serving to detect events in S (e.g., reading the value of a physical quantity of S) is an abstract *abelian* C^*-algebra $\mathcal{I}_S$. In quantum mechanics, in order to characterize the information on an isolated system S that can potentially be retrieved, one must specify the list,

$$\mathcal{O}_S := \{\mathcal{I}_S^i\}_{i \in \mathcal{L}_S},$$

of all instruments of S.

Definition of qm systems – ctd.

Remarks.

1. For simplicity, we assume that there are only finitely many instruments, and that all instruments are finite-dimensional. They are then generated by finite families of mutually orthogonal projections: $\mathcal{I}_S = \{\Pi_\alpha\}_{\alpha \in I_S}$, where I_S is a finite set of indices.
2. The notion of an "instrument" is not intrinsic. It depends on the "observer", in the sense that the nature and amount of information available on a physical system S depends on an "observer's" abilities (e.g., on the experimental equipment he/she can use) to retrieve information about S – which may change with time.
3. Very often, the number of instruments available to an "observer" to detect events in S tends to be very small. There are many interesting examples of mesoscopic systems for which the set of instruments consists of the spectral projections of a single self-adjoint operator, $\widehat{X}$, corresponding, e.g., to clicks of some detectors; (example given below).

Definition of qm systems – ctd.

(III) For every time t, there is a (usually reducible) representation, π_t, of all instruments $\{\mathcal{I}_S^i\}_{i \in \mathcal{L}_S}$ as families of commuting projections contained in $\mathcal{E}_{\geq t}$:

$$\pi_t : \mathcal{I}_S = \{\Pi_\alpha\}_{\alpha \in I_S} \mapsto \{\Pi_\alpha(t) \in \mathcal{E}_{\geq t}\}_{\alpha \in I_S} \tag{3}$$

The set of ops. in $\mathcal{E}_{\geq t}$ representing instruments of S is denoted by $\mathcal{O}_S(t)$.

Key idea: *Loss of access to information*

$$\boxed{B(\mathcal{H}) \supseteq \mathcal{E} \supseteq \mathcal{E}_{\geq t} \supset \mathcal{E}_{\geq s} \supseteq \mathcal{O}_S(s), \qquad \text{for any } s > t} \tag{4}$$

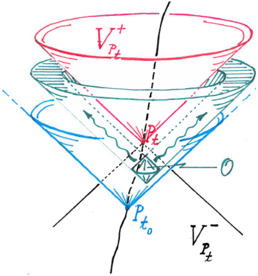
$$\neq \leftarrow I.L.!$$

$(I.L. =$ "Information Loss"!)

Examples of (generally *non-autonomous*) systems exhibiting "Information Loss": "Small systems" (e.g., an n-level atom) temporarily interacting with a quantized wave medium releasing signals that become inaccessible to further observation after some time; (type-I_∞ situation!)

Information Loss and Huyghens Principle

More fundamentally, "Information Loss" always arises in QFT's with massless particles, such as QED, as shown by Buchholz and Roberts; (type-III_1 situation): (4) ($\Rightarrow$ Information Loss and Entanglement with inaccessible degrees of freedom) is then a consequence of *Huyghens' Principle* for the electromagnetic field:



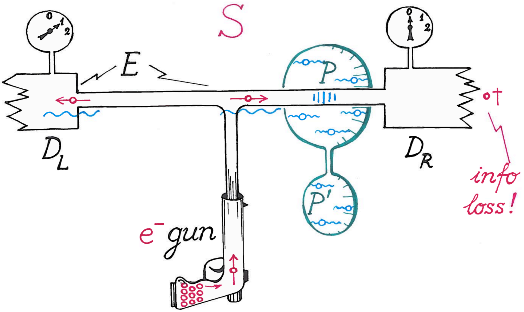
world line of J. F. $\nearrow$

All operators in $\mathcal{E}_{\geq t}$ are localized in $V^+_{P_t}$, $P_t = (t, \vec{x})$. The Figure shows that $\mathcal{E}_{\geq t_0}$ *properly contains* $\mathcal{E}_{\geq t}$, for $t > t_0$, and that

$$(\mathcal{E}_{\geq t})' \cap \mathcal{E}_{\geq t_0} \supset \mathcal{A}^{\text{out}}_{\mathcal{O}}$$

Example of a concrete (mesoscopic) qm system with only *one* instrument:

conducting "*T*-channel" ending in detectors D_L and D_R and in an electron gun, quantum dot, $P \vee P'$, with P binding up to N electrons.



Example – ctd.

$$S = E \vee (P \vee P')$$

E contains all measuring devices, including the two e^-- detectors, D_L and D_R. The *only* directly observable quantity in this system is the *click* of either D_L or D_R: A detector clicks iff an e^- is entering it. Mathematically, this quantity is represented by the linear operator

$$\widehat{X} = \mathbf{1}_{\overline{P}} \otimes \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}_E , \qquad \text{with } \overline{P} = P \vee P', \tag{5}$$

which has the (infinitely degenerate) eigenvalues $\xi = \pm 1$, with

$$\xi = +1 \leftrightarrow D_L \text{ clicks}, \qquad \xi = -1 \leftrightarrow D_R \text{ clicks}.$$

$\mathcal{O}_S$ consists of all bd. functions of $\widehat{X}$.

Direct (projective) measurements

Next, we should clarify why "Information Loss", in the sense of Eq. (4), is a fundamental property of an isolated system S featuring events; i.e., why (4) might be a basic ingredient of a *quantum theory of events and experiments*. The answer is found in the phenomenon of entanglement (with "lost"/inaccessible degrees of freedom), and it yields a theory of *"direct (projective) measurements"*!

(In the next section, we will explain what information can be reconstructed from long sequences of projective measurements of just a few quantities: Theory of *"indirect measurements"*.)

Suppose S has been prepared in a state ρ at some time t_0; ($\nearrow$ th. of state prep.!): ρ may be given by a unit ray in $\mathcal{H}_S$, i.e., it may be a *pure* state. We define ρ_t to be the state on the algebra $\mathcal{E}_{\geq t}$ obtained by restricting ρ to $\mathcal{E}_{\geq t}$:

$$\rho_t(A) := \rho(A), \quad \forall A \in \mathcal{E}_{\geq t}. \tag{6}$$

Direct measurements

By (4) ("Information Loss"), ρ_t may be a mixed state *even* if ρ is pure on $B(\mathcal{H}_S)$. This is what entanglement is all about! It is then possible that ρ_t is very close (in norm) to an incoherent superpos. of eigenstates of projections forming an instrument; e.g., to one formed by the spectral projections of a s.a. operator rep. a physical quantity, in which case this quantity has an objective value at time t. Thus, it is "Information *Loss*" that makes it possible to *gain* information about S by allowing one to measure the value of some physical quantity! One may want to call this the

"Second Law of Quantum Measurement Theory"

Let us make this story a little more precise! Given a von Neumann algebra $\mathcal{M}$ and a state ρ on $\mathcal{M}$, we define the adjoint action of an operator $X \in \mathcal{M}$ on ρ by setting

$$ad_X(\rho)(A) := \rho([X, A]), \quad \forall A \in \mathcal{M}.$$

A little algebra

We define the *"centralizer"* of a state ρ on $\mathcal{M}$ by

$$\mathcal{C}_\rho := \{X \in \mathcal{M} | ad_X(\rho) = 0\}.$$

Note that ρ is a *trace* on $\mathcal{C}_\rho$. Let $\mathcal{Z}_\rho$ denote the center of $\mathcal{C}_\rho$.
Let $X(t) = X^*(t) \in \mathcal{E}_{\geq t}$ be the operator representing a physical quantity of *S* at time t, with $X(t) = \sum_{\alpha=1}^{n} \xi_\alpha \Pi_\alpha(t)$, (spect. dec. of $X(t)$). Then this quantity has an objective value at time t iff $X(t) \in \mathcal{Z}_{\rho_t}$ $(\cdots) \Rightarrow$

$$\rho_t(A) = \sum_{\alpha=1}^{n} \rho(\Pi_\alpha(t) A \Pi_\alpha(t)), \qquad \forall A \in \mathcal{E}_{\geq t}$$

Let $\{\Pi_\alpha(t)\}_{\alpha \in I_S}$ be a representation in $\mathcal{E}_{\geq t}$ of an instrument $\mathcal{I}_S$, and let

$$\Pi^\perp := \mathbf{1} - \sum_{\alpha \in I_S} \Pi_\alpha(t).$$

Suppose that $\Pi_\alpha(t) \in \mathcal{Z}_{\rho_t}(\text{or } \mathcal{C}_{\rho_t}), \forall \alpha \in I_S,$ and $\Pi^\perp \in \mathcal{Z}_{\rho_t}(\text{or } \mathcal{C}_{\rho_t})$ Then

$$\rho_t(A) = \sum_{\alpha \in I_S} \rho(\Pi_\alpha(t) A \Pi_\alpha(t)) + \rho(\Pi^\perp A \Pi^\perp), \qquad \forall A \in \mathcal{E}_{\geq t} \qquad (7)$$

Fundamental axiom of the quantum-mechanical measurement process

This means that ρ_t is an incoherent superposition of states corresponding to events $\Pi_\alpha(t), \alpha \in I_S$, and of a state $\rho_t(\Pi^\perp(\cdot)\Pi^\perp)$ not detectable by the instrument $\mathcal{I}_S$. (Amendment: Eq. (7) must hold only up to small errors!)

Axiom:

1. Given that S has been prepared in a state ρ, the first event after the preparation of S can be detected as soon as Eq. (7) holds for some instrument $\mathcal{I}_S \in \mathcal{O}_S$, provided all projections $\Pi_\alpha(t), \alpha \in I_S$, and the projection $\Pi^\perp$ belong to the center, $\mathcal{Z}_{\rho_t}$, of $\mathcal{C}_{\rho_t}/(\ldots,\ \mathcal{Z}_{\geq t}$, of $\mathcal{E}_{\geq t}$).

   In simple prose, Eq. (7) then implies that if ρ_t could be represented by a density matrix, P_t, on $\mathcal{E}_{\geq t}$ then, on the range of P_t, $\Pi_\alpha(t), \alpha \in I_S$, and $\Pi^\perp$ are functions of P_t, (up to multiplication by elements in the center of $\mathcal{E}_{\geq t}$)

2. The *probability* to detect the event $\Pi_\alpha(t), \alpha \in I_S$, is given by *Born's Rule*

$$Prob\{\Pi_\alpha(t) \text{ is detected}\} = \rho(\Pi_\alpha(t)), \tag{8}$$

Fundamental axiom – ctd.

and $\rho(\Pi^{\perp})$ is the probability that, at time t, the instrument $\mathcal{I}_S$ does not detect anything it can identify.

3. If the event corresponding to the projection $\Pi_\alpha(t)$ is detected then the state to be used for predictions after time t should be taken to be

$$\rho_{t,\alpha}(A) := \frac{\rho(\Pi_\alpha(t)\, A\, \Pi_\alpha(t))}{\rho(\Pi_\alpha(t))}, \qquad \forall A \in \mathcal{E}_{\geq t},$$

and if the instrument does not detect anything it can identify then the state

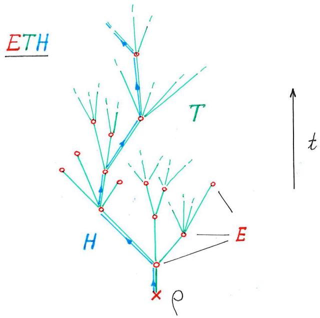
$$\rho_t^{\perp}(A) := \frac{\rho(\Pi^{\perp}\, A\, \Pi^{\perp})}{\rho(\Pi^{\perp})}, \qquad \forall A \in \mathcal{E}_{\geq t},$$

should be used.

A mathematically precise formulation of this Axiom lies beyond the scope of this lecture – debate desirable!

A *"Garden of Forking Paths"* – *ETH* Approach to QM

Item 3 of the Axiom is sometimes referred to as the "collapse of the wave function", an unfortunate expression, because the "collapse" meant here is not a physical process, but the passage to a conditional expectation.



E: "events" (proj. measnts.), *T*: "trees" (of states),
H: "histories" ; probs. of "histories" are det. by QM

Projective measurements – *summary*

(1) Given the *initial state* of the system *S*, *time evolution*, {*U(t,s)*}, *determines* which pot. prop. $a \in \mathcal{O}_S$ will first become empirical (objective, measureable), and around which time!

(2) Measnt. of a_2 is *independent* of an *earlier* measnt. of a_1 iff a_2 becomes empirical/objective *after* time of measnt. of a_1 , no matter what the outcome of measnt. of a_1 was, i.e., *for all states* $\rho_j^{a_1}(\cdot)$ *, j= 1, ... , k,* with $\rho_j^{a_1}(\cdot)$ as in (9).

$\Rightarrow$ *Decoherence, "consistent histories".*

(3) Time of measurement*:* Time, t_*, of observation of a det. by minimizing in t the fu. $\| a(t)|_{RangeP_t} - F(P_t) \cdot z \|$, where $F(P_t) \cdot z$ is the "*cond. exp.*" of $a(t)$ onto $\mathcal{Z}_{\geq t}$.

(4) General theory of repeated measurements: "POVM's".

Projective measurements – *summary*

(5) A state is called *"passive"* iff the center, $\mathcal{Z}_{\geq t}$, of the centralizer of $\mathcal{E}_{\geq t}$ is *time-independent*. There are plenty of examples of passive states:

- Equilibrium (KMS) states at positive temperature in QFT; KMS states of a QFT in the space-time of a static black hole.
- Perturbations of the vacuum state by coherent clouds of massless particles (e.g., of photons).

Passive states have the property that they do *not admit any projective measurements/observations* of any physical quantities – besides measurements of *time-independent* parameters characteristic of the state in question, e.g., the temperature or a chemical potential of an equilibrium state, which, indeed, are time-independent quantities.

We have to learn more about which states and which types of time evolutions of isolated systems admit non-trivial measurements!

3. Indirect (Kraus) measurements

Assume, $\mathcal{O}_S$ consists of a single finite-dimensional, commutative algebra with spectrum $\mathcal{X}_S$ = a finite set of points, $\{1, ..., N\}$. In the concrete model considered in Section 2, $\mathcal{O}_S$ consists of all functions of the operator

$$\widehat{X} = \mathbf{1}_{\overline{P}} \otimes \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}_E,$$

hence $\mathcal{X}_S = \{-1, +1\}$.
With each point $\xi \in \mathcal{X}_S$ we associate an orthogonal projection $\pi_\xi \in \mathcal{O}_S$, and all operators in $\mathcal{O}_S$ are linear combinations of the π_ξ's. We suppose that successive projective measurements/ observations of quantities corresponding to operators in $\mathcal{O}_S$ at times $\approx t_1, ..., \approx t_k$ have yielded a sequence of measurement results,

$$\underline{\xi}^{(k)} := \{\xi_1, ..., \xi_k\} \in \mathcal{X}_S^{\times k}. \tag{9}$$

Kraus measurements – ctd.

We assume that S has been prepared in the state ρ before measnts./observations of quantities corresponding to ops. in $\mathcal{O}_S$ have started. QM predicts that the probability (frequency) of a measurement protocol $\underline{\xi}^{(k)}$ is given by

$$\mu_\rho(\xi_1, ..., \xi_k) = \mathrm{Tr}(\pi_{\xi_k}(t_k) \cdots \pi_{\xi_1}(t_1)\ P\ \pi_{\xi_1}(t_1) \cdots \pi_{\xi_k}(t_k)), \quad (10)$$

where P is the density matrix coresponding to the state ρ; (*LSW - formula*).Obviously,

$$\sum_{\xi_k \in \mathcal{X}_S} \mu_\rho(\underline{\xi}^{(k-1)}, \xi_k) = \mu_\rho(\underline{\xi}^{(k-1)}), \qquad \mu_\rho(\emptyset) = 1. \qquad (11)$$

It follows that μ_ρ extends to a probability measure on the space, Ξ, of infinitely long measurement protocols; (equipped with the σ-algebra of cylinder sets).

Kraus measurements – ctd.

Let $\mathcal{O}_S[\infty]$ be the algebra of functions in $L^\infty(\Xi, [\mu_\rho])$ that do not depend on any finite set of measurement outcomes: *"Observables at infinity"*; (can be identified with $\mathcal{E}_\infty$!)
Let $\Xi[\infty]$ be the spectrum of $\mathcal{O}_S[\infty]$. Then the measure μ_ρ can be decomposed into a convex combination of *"extremal measures"*:

$$\mu_\rho(\cdot) = \int_{\Xi[\infty]} \mathrm{d}P(\nu)\mu_\rho(\cdot|\nu). \tag{12}$$

The measures $\mu_\rho(\cdot|\nu)$ come from *states*, ρ_ν, of S; for different points ν, they are *mutually singular*. Thus, a very long measurement protocol $\underline{\xi}^{(k)}$ determines a point $\nu \in \Xi[\infty]$ (called a "fact") with an error likelihood that tends to 0, as $k \to \infty$, and ν then determines the values of all "observables at infinity".

Exchangeable measures

If the order in which the measurement results $\xi_1, ..., \xi_k$ are obtained does not matter, for any k, (i.e., if successive measurements commute with each other) then $\mu_\rho(\xi_{\sigma(1)}, ..., \xi_{\sigma(k)})$ is *independent* of the permutation σ, $\forall\sigma$ and all k. Then Eq. (12) follows from *de Finetti's theorem*, which also says that the measures $\mu_\rho(\cdot|\nu)$ are product measures:

$$\mu_\rho(\xi_1, ..., \xi_k|\nu) = \prod_{j=1}^{k} p(\xi_j|\nu), \tag{13}$$

with $p(\xi|\nu) \geq 0$ and $\sum_\xi p(\xi|\nu) = 1$.
A simple example of this situation is a model of the system described in Section 2, for which $\mathcal{O}_S = \langle X \rangle$ and $\mathcal{X}_S = \{-1, +1\}$. (Assuming that the electrons moving through the T-shaped wires are entirely independent of each other and that the detectors D_L and D_R return to the same state after each measurement, and before the next electron travels through the T-shaped wires, one concludes that the measures μ_ρ are *exchangeable*.)

Exchangeable measures – ctd.

Let $\nu \in \{0, ..., N\} =: \Xi[\infty]$ be the number of e^- in the quantum dot P. Let us assume, for the time being, that ν is *time-independent*, i.e., we consider a *non-demolition measurement* of ν. Because μ_ρ is exchangeable, we have that

$$\mu_\rho(\underline{\xi}^{(k)}) = \sum_{\nu=1}^{N} P_\rho(\nu)\mu(\underline{\xi}^{(k)}|\nu), \tag{14}$$

with

$$\mu_\rho(\underline{\xi}^{(k)}|\nu) = \prod_{j=1}^{k} p(\xi_j|\nu),$$

where:

$P_\rho(\nu)$: Born probability for ν e^- bound by P, as predicted by ρ;
$p(\xi|\nu)$: QM probability for an e^- in the "T-channel" to be scattered into $D_\xi, \xi = -1(R), +1(L)$, given that there are ν electrons bound to P.

Frequencies of "events"

An example of an "observable at infinity" that is usually well defined is the *"asymptotic frequency"*, $p(\xi|\cdot)$, of an event $\xi \in \mathcal{X}_S$. We define

$$f_{\xi}^{(l,l+k)}(\underline{\xi}) := \frac{1}{k}\left(\sum_{j=l+1}^{l+k} \delta_{\xi,\xi_j}\right), \quad \text{with } \sum_{\xi} f_{\xi}^{(l,l+k)}(\underline{\xi}) = 1. \tag{15}$$

One expects that, for "most" states ρ,
(1) *The Law of Large Numbers*

$$\lim_{k\to\infty} f_{\xi}^{(l,l+k)}(\underline{\xi}) =: p(\xi|\nu), \tag{16}$$

for some point (or "fact") $\nu \in \Xi[\infty]$, holds. This is indeed the case for the simple model described above.

Hypothesis: We assume that $\Xi[\infty] = \{0, ..., N\}$, $N < \infty$, with

$$\min_{\nu_1 \neq \nu_2} |p(\xi|\nu_1) - p(\xi|\nu_2)| \geq \kappa > 0, \quad \text{for some } \xi \in \mathcal{X}_S \tag{17}$$

“q-hypothesis testing”

With each $\nu \in \Xi[\infty]$ we associate a subset

$$\Xi_\nu(l, k; \underline{\xi}) := \{\underline{\xi} \mid |f_{\xi}^{(l,l+k)}(\underline{\xi}) - p(\xi|\nu)| < \epsilon_k\}, \tag{18}$$

where

$$\epsilon_k \to 0, \sqrt{k}\ \epsilon_k \to \infty, \quad \text{as } k \to \infty$$

Main Results:

(2) *Disjointness:* It follows from Hyp. (17) and definition (18) that, for k so large that $\epsilon_k < \kappa/2$,

$$\Xi_{\nu_1}(l, k; \underline{\xi}) \cap \Xi_{\nu_2}(l, k; \underline{\xi}) = \emptyset, \quad \nu_1 \neq \nu_2$$

(3) *Central Limit Theorem:* $\Rightarrow$ Under suitable hypotheses on the states ρ,

$$\mu_\rho\left(\bigcup_\nu \Xi_\nu(l, k; \underline{\xi})\right) \to 1, \quad k \to \infty$$

hypothesis testing – ctd.

(4) *Theorem of Boltzmann-Sanov* $\Rightarrow$ If the measures μ_ρ are exchangeable one has that

$$\mu\left(\Xi_{\nu_1}(l, k; \underline{\varepsilon})|\nu_2\right) \leq C\ e^{-k\sigma(\nu_1\|\nu_2)}$$

where σ is a relative entropy.

(5) *Theorem of Maassen and Kümmerer* $\Rightarrow$ In the simple model described above, the state of S, restricted to $B(\mathcal{H}_P)$ approaches a state with a *fixed number of electrons* in the quantum dot P ("purification") – for any initial state.

The theory of indirect measurements outlined here only concerns measurements of time-independent "facts", which correspond to points in $\Xi[\infty]$ (*non-demolition measurements!*). However, most interesting "facts" depend on time! Thus, one must ask how one can acquire information concerning *time-dependent facts* indirectly, through repeated, successive direct measnts. of quantities corresponding to operators in $\mathcal{O}_S$.

We consider the simple model introduced above. We assume that electrons can enter into, or tunnel out of the component P of the quantum dot $\overline{P}$, i.e., the number of electrons, ν, in P may slowly vary in time. We define

$$\Xi_{\nu_1,\ldots,\nu_r}(k;\underline{\varepsilon}) := \{\underline{\xi} \mid |f_L^{(\ell k-k,\ell k)}(\underline{\xi}) - p(L|\nu_\ell)| < \varepsilon_k, \forall \ell = 1, \ldots, r\}$$

and

$$\mathcal{P}_\rho(\nu_1, \ldots, \nu_r) := \mu_\rho(\Xi_{\nu_1,\ldots,\nu_r}(k;\underline{\varepsilon}))$$

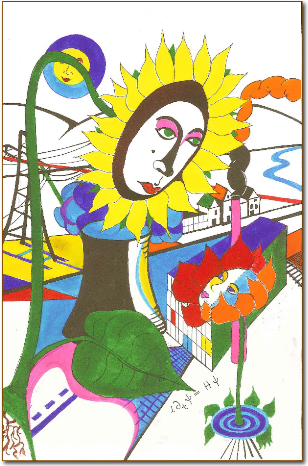
(6) *Theorem on quantum jumps:* For each $r < \infty$,

$$\sum_{\nu_1,\ldots,\nu_r} \mathcal{P}_\rho(\nu_1, \ldots, \nu_r) \to 1,$$

in the limit where the temporal variation of the number of electrons in P tends to 0 and $k \to \infty$.

Remark. In suitable limiting regimes, $\mathcal{P}_\rho(\nu_1, \ldots, \nu_r)$ is the path-space measure of a *Markov chain* with state space $= \{1, \ldots, N\}$.

4. Conclusions – discussion



"In all my films, I have been faithful to these suspension points in the conclusions. Besides, I have never written the word 'END' on the screen."
(Federico Fellini)

"Everyone wants to understand art (physics). Why don't we try to understand the song of a bird? Why do we love the night, the flowers, everything around us, without trying to understand them? But in the case of a painting (result in physics), people think they have to understand." (Pablo Picasso)

Thanks for your attantion!

References to some of our papers

1. J. Fröhlich and B. Schubnel, "Do we understand quantum mechanics – finally?", in: Wolfgang Reiter et al. (eds.), *Erwin Schrödinger – 50 years after*, Zurich: European Math. Soc. Publ. 2013, pages 37 - 84.
2. J. Fröhlich and B. Schubnel, "Quantum Probability Theory and the Foundations of Quantum Mechanics", in: Philippe Blanchard and Jürg Fröhlich (eds.), *The Message of Quantum Science – Attempts Towards a Synthesis*, Lecture Notes in Physics vol. **899**, Berlin-Heidelberg: Springer-Verlag 2015, pages 131 - 193
3. M. Ballesteros, M. Fraas, J. Fröhlich and B. Schubnel, "Indirect retrieval of information and the emergence of facts in quantum mechanics", arXiv:1506.01213; and refs. given there.
4. J. Faupin, J. Fröhlich and B. Schubnel, "On the probabilistic nature of quantum mechanics and the notion of closed systems", to appear in Ann. Henri Poincar, 2015.

Please, take a look at some of the excellent papers by numerous colleagues that we have quoted in the works listed above.