Bose Particles in a Box: Convergent Expansion of the Ground State in the Mean Field Limiting Regime

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References

A. P. http://arxiv.org/abs/1511.07022
A. P. http://arxiv.org/abs/1511.07025
A. P. http://arxiv.org/abs/1511.07026

Motivations and Background

- N Bose (nonrelat.) particles in a finite box of volume $|\Lambda| = L^d$

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- Thermodynamic limit: ρ fixed and $|\Lambda| \to \infty$
- Other regimes: Gross-Pitaveskii, Thomas-Fermi

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- Results towards rigorous functional integral: (B-F-K-T)

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5. Outlook

Model

- (Δ with periodic boundary conditions)

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$$+\frac{1}{2\rho}\int\int a^*(x)a^*(y)\phi(x-y)a(y)a(x)dxdy$$

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- $a^*(x)$, $a(x)$ operator-valued distributions on

$$\mathcal{F} := \Gamma \left(L^2 \left(\Lambda, \mathbb{C}; dx \right) \right) \qquad |\Lambda| = L^d$$

CCR: $\qquad [a^\#(x), a^\#(y)] = 0, \qquad [a(x), a^*(y)] = \delta(x-y) 1_{\mathcal{F}},$

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$$a(x) = \sum_{\mathbf{j}\in\mathbb{Z}^d} \frac{a_{\mathbf{j}} e^{ik_{\mathbf{j}}\cdot x}}{|\Lambda|^{\frac{1}{2}}}, \quad a^*(x) = \sum_{\mathbf{j}\in\mathbb{Z}^d} \frac{a^*_{\mathbf{j}} e^{-ik_{\mathbf{j}}\cdot x}}{|\Lambda|^{\frac{1}{2}}}$$

where $k_{\mathbf{j}} := \frac{2\pi}{L}\mathbf{j}$, $\mathbf{j} = (j_1, j_2, \ldots, j_d)$, $j_1, j_2, \ldots, j_d \in \mathbb{Z}$

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Assumptions on the two-body potential

- The pair potential $\phi(x-y)$ is a bounded, real-valued function that is periodic, i.e., $\phi(w) = \phi(w + \mathbf{j}L)$ for $\mathbf{j} \in \mathbb{Z}^d$

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- Strong interaction potential regime: The ratio $\epsilon_{\mathbf{j}}:=\frac{k_{\mathbf{j}}^2}{\phi_{\mathbf{j}}}$ is sufficiently small

Model: Particle Preserving Bogoliubov Hamiltonian

- $$H := \int (\nabla a^*)(\nabla a)(x)dx +$$
  $$+\frac{1}{2\rho}\int\int a^*(x)a^*(y)\phi(x-y)a(y)a(x)dxdy$$
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- Three-modes Bogoliubov Hamiltonian

$$\begin{aligned} H^B_{\mathbf{j}} \quad &:= (k_{\mathbf{j}}^2 + \frac{\phi_{\mathbf{j}}}{N} a^*_{\mathbf{0}} a_{\mathbf{0}})(a^*_{\mathbf{j}} a_{\mathbf{j}} + a^*_{-\mathbf{j}} a_{-\mathbf{j}}) \\ &+ \frac{\phi_{\mathbf{j}}}{N}\left\{ a^*_{\mathbf{0}} a^*_{\mathbf{0}} a_{\mathbf{j}} a_{-\mathbf{j}} + a^*_{\mathbf{j}} a^*_{-\mathbf{j}} a_{\mathbf{0}} a_{\mathbf{0}} \right\} \end{aligned}$$

Model: Particle Preserving Bogoliubov Hamiltonian

- H is restricted to
  $\mathcal{F}^N \equiv$ subspace of $\mathcal{F}$ with exactly N particles (N even)
- $H = H^B + V$
  - $V \equiv$ cubic and quartic terms in the nonzero modes
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&\quad + \underbrace{\left\{ \frac{\phi_{\mathbf{j}}}{N} a^*_{\mathbf{0}} a^*_{\mathbf{0}} a_{\mathbf{j}} a_{-\mathbf{j}} \right\}}_{W_{\mathbf{j}}} + \underbrace{\left\{ \frac{\phi_{\mathbf{j}}}{N} a^*_{\mathbf{j}} a^*_{-\mathbf{j}} a_{\mathbf{0}} a_{\mathbf{0}} \right\}}_{W^*_{\mathbf{j}}}
\end{aligned}
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Why studying the three-modes systems?

- In the mean field limiting regime

$$\inf(H - C_N) \to E^B$$

with

$$E^B := \frac{1}{2} \sum_{\mathbf{j} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}} E^B_{\mathbf{j}} = -\frac{1}{2} \sum_{\mathbf{j} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}} \left[k_{\mathbf{j}}^2 + \phi_{\mathbf{j}} - \sqrt{(k_{\mathbf{j}}^2)^2 + 2\phi_{\mathbf{j}} k_{\mathbf{j}}^2} \right]$$

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- The thermodynamic limit is already nontrivial for a three-modes system: a large field problem appears

Main results

- The Feshbach flow and the ground state are well defined if
  - $\epsilon_{\mathbf{j}_*} := \frac{k_{\mathbf{j}_*}^2}{\phi_{\mathbf{j}_*}}$ is sufficiently small and for some $\nu > \frac{11}{8}$

$$\epsilon_{\mathbf{j}_*}^{\nu} \geq \frac{1}{N} \quad \Longleftrightarrow \quad \frac{k_{\mathbf{j}_*}^2}{\phi_{\mathbf{j}_*}} > \left(\frac{1}{N}\right)^{\frac{8}{11}} \quad \Longleftrightarrow \quad \left[\frac{(2\pi\mathbf{j}_*)^2}{L^2\phi_{\mathbf{j}_*}}\right]^{\nu} \geq \frac{1}{\rho L^d}$$

  - $$\rho \geq \rho_0 \left(\frac{L}{L_0}\right)^{3-d}$$

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- Conditions fulfilled at fixed (large) ρ only if $d \geq 3$ and L large enough

Ideas

- If ψ_{gs} ground state of H, $\langle\sum_{\mathbf{j}\in\mathbb{Z}^d\setminus\{\mathbf{0}\}} a^*_{\mathbf{j}} a_{\mathbf{j}}\rangle_{\psi_{gs}}$ stays bounded as $N \to \infty$
  $\Rightarrow$ Conjecture: An effective Hamiltonian in a neighborhood of E_{gs} is a multiple of the projection

$$|\eta\rangle\langle\eta| \quad , \quad \eta := \frac{1}{\sqrt{N!}} a^*_{\mathbf{0}} \dots a^*_{\mathbf{0}} \Omega$$

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- Isospectrality: 1) $\mathscr{F}(K - z)$ is bounded invertible on $\mathscr{P}\mathcal{H}$ if and only if z is in the resolvent set of K (on $\mathcal{H}$); 2) z is an eigenvalue of K if and only if 0 is an eigenvalue of $\mathscr{F}(K - z)$

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- Selection rules of H w.r.t. $\sum_{\mathbf{j}=\pm\mathbf{j}_*} a^*_{\mathbf{j}} a_{\mathbf{j}}$
  $\Rightarrow$ choose $\mathscr{P}$, $\overline{\mathscr{P}}$ associated with eigenspaces of $\sum_{\mathbf{j}=\pm\mathbf{j}_*} a^*_{\mathbf{j}} a_{\mathbf{j}}$

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- Can $\overline{\mathscr{P}}$ help to avoiding *small denominator problems*?

Three-modes system

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- For the purpose of this talk the Hilbert space $\mathcal{F}^N$ contains only the degrees of freedom $(\mathbf{0}; -\mathbf{j}_*; \mathbf{j}_*)$

Feshbach Projections for $\hat{H}^B$

- $Q^{(i,i+1)} :=$ the projection (in $\mathcal{F}^N$) onto the subspace spanned by the vectors with $N-i$ or $N-i-1$ particles in the modes $\mathbf{j}_*$ and $-\mathbf{j}_*$ $\quad\rightarrow\quad$ the operator $a^*_{\mathbf{j}_*}a_{\mathbf{j}_*} + a^*_{-\mathbf{j}_*}a_{-\mathbf{j}_*}$ has eigenvalues $N-i$ and $N-i-1$ when restricted to $Q^{(i,i+1)}\mathcal{F}^N$

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- $$\mathcal{F}^N = Q^{(0,1)}\mathcal{F}^N \oplus Q^{(2,3)}\mathcal{F}^N \oplus \cdots \oplus Q^{(N-2,N-1)}\mathcal{F}^N \oplus \{\mathbb{C}\eta\}$$

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- $Q^{(>1)} :=$ the projection onto the orthogonal complement of $Q^{(0,1)}\mathcal{F}^N$ in $\mathcal{F}^N$ $\rightarrow$ $Q^{(>1)} + Q^{(0,1)} = \mathbf{1}_{\mathcal{F}^N}$
- Iteratively, <u>for i even</u>, $2 \leq i \leq N-2$, define

$$Q^{(>i+1)} \quad \text{the projection such that} \quad Q^{(>i+1)} + Q^{(i,i+1)} = Q^{(>i-1)}$$

$$Q^{(>N-1)} \equiv |\eta\rangle\langle\eta|$$

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- Starting from $K_{-2}^B(z) := \hat{H}^B - z$

$$
\begin{aligned}
& K_i^B(z) \\
:= \; & \mathscr{P}^{(i)} K_{i-2}^B(z) \mathscr{P}^{(i)} \\
& - \mathscr{P}^{(i)} K_{i-2}^B(z) \overline{\mathscr{P}}^{(i)} \frac{1}{\overline{\mathscr{P}}^{(i)} K_{i-2}^B(z) \overline{\mathscr{P}}^{(i)}} \overline{\mathscr{P}}^{(i)} K_{i-2}^B(z) \mathscr{P}^{(i)}
\end{aligned}
$$

Flow of Feshbach Hamiltonians for $\hat{H}^B$

- Define $\mathscr{P}^{(i)} := Q^{(>i+1)}$, $\overline{\mathscr{P}^{(i)}} := Q^{(i,i+1)}$

- For $i = 0$

$$
\begin{aligned}
& K_0^B(z) \\
:= \quad & Q^{(>1)}(\hat{H}^B - z)Q^{(>1)} \\
& - Q^{(>1)}\hat{H}^B Q^{(0,1)} \frac{1}{Q^{(0,1)}(\hat{H}^B - z)Q^{(0,1)}} Q^{(0,1)}\hat{H}^B Q^{(>1)}
\end{aligned}
$$

Model: Particle Preserving Bogoliubov Hamiltonian

- H is restricted to
  $\mathcal{F}^N \equiv$ subspace of $\mathcal{F}$ with exactly N particles
- $H = H^B + V$
- $H^B := \frac{1}{2}\sum_{\mathbf{j}\in\mathbb{Z}^d\setminus\{\mathbf{0}\}} H^B_{\mathbf{j}}$
- Three-modes Bogoliubov Hamiltonian

$$\hat{H}^B := \overbrace{\Big(k^2_{\mathbf{j}_*} + \frac{\phi_{\mathbf{j}_*}}{N} a^*_{\mathbf{0}} a_{\mathbf{0}}\Big)\Big(a^*_{\mathbf{j}_*} a_{\mathbf{j}_*} + a^*_{-\mathbf{j}_*} a_{-\mathbf{j}_*}\Big)}^{H^{(0)}}$$
$$+ \underbrace{\left\{\frac{\phi_{\mathbf{j}_*}}{N} a^*_{\mathbf{0}} a^*_{\mathbf{0}} a_{\mathbf{j}_*} a_{-\mathbf{j}_*}\right\}}_{W} + \underbrace{\left\{\frac{\phi_{\mathbf{j}}}{N} a^*_{\mathbf{j}_*} a^*_{-\mathbf{j}_*} a_{\mathbf{0}} a_{\mathbf{0}}\right\}}_{W^*}$$

Flow of Feshbach Hamiltonians for $\hat{H}^B$

- Define $\mathscr{P}^{(i)} := Q^{(>i+1)}$, $\overline{\mathscr{P}(i)} := Q^{(i,i+1)}$
- For $i = 0$

$$\begin{aligned} & K_0^B(z) \\ := \quad & Q^{(>1)}(\hat{H}^B - z)Q^{(>1)} \\ & - Q^{(>1)} W Q^{(0,1)} \frac{1}{Q^{(0,1)}(\hat{H}^B - z)Q^{(0,1)}} Q^{(0,1)} W^* Q^{(>1)} \end{aligned}$$

Flow of Feshbach Hamiltonians for $\hat{H}^B$

- Define $\mathscr{P}^{(i)} := Q^{(>i+1)}$, $\overline{\mathscr{P}^{(i)}} := Q^{(i,i+1)}$
- For $i = 2$

$$K_2^B(z) :=$$

$$= \quad Q^{(>3)}(\hat{H}^B - z)Q^{(>3)}$$

$$-Q^{(>3)} W \, Q^{(2,3)} \times$$

$$\times \frac{1}{Q^{(2,3)}(\hat{H}^B - WQ^{(0,1)} \frac{1}{Q^{(0,1)}(\hat{H}^B - z)Q^{(0,1)}} Q^{(0,1)} W^* - z)Q^{(2,3)}}$$

$$\times Q^{(2,3)} W^* \, Q^{(>3)}$$

Flow of Feshbach Hamiltonians for $\hat{H}^B$

- Define $\mathscr{P}^{(i)} := Q^{(>i+1)}$, $\overline{\mathscr{P}^{(i)}} := Q^{(i,i+1)}$
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$$Q^{(>3)}(\hat{H}^B - z)Q^{(>3)}$$

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$$\times \frac{1}{Q^{(2,3)}(\hat{H}^B - W \underbrace{Q^{(0,1)} \frac{1}{Q^{(0,1)}(\hat{H}^B - z)Q^{(0,1)}} Q^{(0,1)}}_{R^B_{0,0}(z)} W^* - z)Q^{(2,3)}}$$

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- For $i = 2$

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$$\times Q^{(2,3)} W^*\, Q^{(>3)}$$

General Term

- For i (even)

$$K_i^B := Q^{(>i+1)}(\hat{H}^B - z)Q^{(>i+1)}$$

$$-Q^{(>i+1)} W R_{i,i}^B(z) \sum_{l_i=0}^{\infty} \left[\Gamma_{i,i}^B(z) R_{i,i}^B(z)\right]^{l_i} W^* Q^{(>i+1)}$$

-

$$\Gamma_{i+2,i+2}^B(z) :=$$

$$= Q^{(i+2,i+3)} W R_{i,i}^B(z) \sum_{l_i=0}^{\infty} \left[\Gamma_{i,i}^B(z) R_{i,i}^B(z)\right]^{l_i} W^* Q^{(i+2,i+3)}$$

-

$$\Gamma_{2,2}^B(z) := Q^{(2,3)} W R_{0,0}^B(z) W^* Q^{(2,3)}$$

Range of the spectral parameter z

- Spectrum of $H^B_{\mathbf{j}_*}$ as $N \to \infty$ (Seiringer):
  - the ground state energy tends to

$$E^B_{\mathbf{j}_*} := -\left[k^2_{\mathbf{j}_*} + \phi_{\mathbf{j}_*} - \sqrt{(k^2_{\mathbf{j}_*})^2 + 2\phi_{\mathbf{j}_*} k^2_{\mathbf{j}_*}}\right]$$

$$E^B_{\mathbf{j}_*} \to -\phi_{\mathbf{j}_*} \quad \text{as} \quad \epsilon_{\mathbf{j}_*} \to 0$$

  - the first excited eigenvalue tends to

$$E^B_{\mathbf{j}_*} + \sqrt{(k^2_{\mathbf{j}_*})^2 + 2\phi_{\mathbf{j}^*} k^2_{\mathbf{j}_*}}$$

Range of the spectral parameter z

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  - the first excited eigenvalue tends to

$$E^B_{\mathbf{j}_*} + \sqrt{(k^2_{\mathbf{j}_*})^2 + 2\phi_{\mathbf{j}^*} k^2_{\mathbf{j}_*}}$$

- Question:
  Can we control the flow for $z < E^B_{\mathbf{j}_*} + \sqrt{(k^2_{\mathbf{j}_*})^2 + 2\phi_{\mathbf{j}^*} k^2_{\mathbf{j}_*}}$?

General Term

- Recursive relation

$$\Gamma^B_{i+2,i+2}(z) :=$$

$$= Q^{(i+2,i+3)} W \, (R^B_{i,i}(z))^{\frac{1}{2}} \sum_{l_i=0}^{\infty} \left[(R^B_{i,i}(z))^{\frac{1}{2}} \Gamma^B_{i,i}(z) (R^B_{i,i}(z))^{\frac{1}{2}} \right]^{l_i} \times$$

$$\times (R^B_{i,i}(z))^{\frac{1}{2}} W^* Q^{(i+2,i+3)}$$

- Initial term

$$\Gamma^B_{2,2}(z) := Q^{(2,3)} W \, R^B_{0,0}(z) W^* Q^{(2,3)}$$

Key estimates to control the Feshbach flow

- The flow is well defined if for each $i \leq N-2$ and even

$$\sum_{l=0}^{\infty}\left[(R^{B}_{i,i}(z))^{\frac{1}{2}}\Gamma^{B}_{i,i}(z)(R^{B}_{i,i}(z))^{\frac{1}{2}}\right]^{l}$$

  is well defined

- Main Theorem:

$$\|\sum_{l=0}^{\infty}\left[(R^{B}_{i,i}(z))^{\frac{1}{2}}\Gamma^{B}_{i,i}(z)(R^{B}_{i,i}(z))^{\frac{1}{2}}\right]^{l}\| \leq \frac{1}{X_i}$$

  where $X_0 \equiv 1$ and

$$X_{i+2} := 1 - \frac{1}{4(1 + a_{\epsilon_{\mathbf{j}_*}} - \frac{b_{\epsilon_{\mathbf{j}_*}}}{N-i+1} - \frac{1-c_{\epsilon_{\mathbf{j}_*}}}{(N-i+1)^2})}\frac{1}{X_i}$$

Key estimates to control the Feshbach flow

- $z \leq E^B_{\mathbf{j}_*} + (\delta - 1)\phi_{\mathbf{j}^*}\sqrt{\epsilon^2_{\mathbf{j}_*} + 2\epsilon_{\mathbf{j}_*}}$, $\delta < 2$

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- $\epsilon_{\mathbf{j}_*} := \frac{k^2_{\mathbf{j}_*}}{\phi_{\mathbf{j}_*}}$ small but $\epsilon^{\nu}_{\mathbf{j}_*} \geq \frac{1}{N}$ for some $\nu > \frac{11}{8}$

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- key estimate

$$\| \left[R^B_{i,i}(z)\right]^{\frac{1}{2}} W \left[R^B_{i-2,i-2}(z)\right]^{\frac{1}{2}} \| \, \| \left[R^B_{i-2,i-2}(z)\right]^{\frac{1}{2}} W^* \left[R^B_{i,i}(z)\right]^{\frac{1}{2}} \|$$

$$\leq \frac{1}{4(1 + a_{\epsilon_{\mathbf{j}_*}} - \frac{2b_{\epsilon_{\mathbf{j}_*}}}{N-i+1} - \frac{1-c_{\epsilon_{\mathbf{j}_*}}}{(N-i+1)^2})}$$

where $a_{\epsilon_{\mathbf{j}_*}} := \mathcal{O}(\epsilon_{\mathbf{j}_*})$, $b_{\epsilon_{\mathbf{j}_*}} := \mathcal{O}(\sqrt{\epsilon_{\mathbf{j}_*}})$, $c_{\epsilon_{\mathbf{j}_*}} := \mathcal{O}(\epsilon_{\mathbf{j}_*})$

Key estimates to control the Feshbach flow

- Artificial $\phi_{\mathbf{j}_*}-$dependent Gap

$$R^{B}_{i,i}(z)$$

$$= Q^{(i,i+1)} \frac{1}{Q^{(i,i+1)}(\hat{H}^{B} - z)Q^{(i,i+1)}} Q^{(i,i+1)}$$

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$$= Q^{(i,i+1)} \frac{1}{Q^{(i,i+1)}(H^{(0)} - z)Q^{(i,i+1)}} Q^{(i,i+1)}$$

$$H^{(0)} \geq 0 \quad \text{and} \quad z \simeq -\phi_{\mathbf{j}_*}$$

Key estimates to control the Feshbach flow

- Control of the sequence

$$X_{i+2} := 1 - \frac{1}{4(1 + a_{\epsilon_{\mathbf{j}_*}} - \frac{b_{\epsilon_{\mathbf{j}_*}}}{N-i+1} - \frac{1-c_{\epsilon_{\mathbf{j}_*}}}{(N-i+1)^2})} \frac{1}{X_i}$$

$X_0 \equiv 1$ and, $0 \leq i \leq N-2$ and even

Key estimates to control the Feshbach flow

- For $\epsilon_{\mathbf{j}_*} = 0$

$$X_{i+2} := 1 - \frac{1}{4(1 - \frac{1}{(N-i+1)^2})X_i}$$

from $X_0 \equiv 1$ up to X_{N-2}

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- Exact Solution

$$X_i = \frac{1}{2}(1 - \frac{1}{N-i})$$

  but for $X_0 = \frac{1}{2}(1 - \frac{1}{N})$

Key estimates to control the Feshbach flow

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$$X_i = \frac{1}{2}(1 - \frac{1}{N-i})$$

but for $X_0 = \frac{1}{2}(1 - \frac{1}{N})$

- By induction (for $\epsilon_{\mathbf{j}_*} > 0$)

$$X_i \geq \frac{1}{2}(1 - \frac{1}{N-i}) + o(1)$$

for $\delta < 1 + \sqrt{\epsilon_{\mathbf{j}_*}} \quad \Longleftrightarrow \quad z < E^B_{\mathbf{j}_*} + \sqrt{\epsilon_{\mathbf{j}_*}}\sqrt{(k^2_{\mathbf{j}_*})^2 + 2\phi_{\mathbf{j}_*} k^2_{\mathbf{j}_*}}$

Final Feshbach Hamiltonian, Fixed Point, and GS Energy

- $$\begin{aligned} & K^{B}_{N-2}(z) = \\ = & \, Q^{(>N-1)}(\hat{H}^{B} - z)Q^{(>N-1)} \\ & - Q^{(>N-1)} W \times \\ & \quad \times R^{B}_{N-2,N-2}(z) \sum_{l=0}^{\infty} \left[\Gamma^{B}_{N-2,N-2}(z) R^{B}_{N-2,N-2}(z) \right]^{l} \\ & \quad \times W^{*} \, Q^{(>N-1)} \end{aligned}$$

Final Feshbach Hamiltonian, Fixed Point, and GS Energy

- $Q^{(>N-1)} = P_\eta = |\eta\rangle\langle\eta|$, hence

$$K^B_{N-2}(z) =$$

$$= P_\eta(\hat{H}^B - z)P_\eta$$

$$-P_\eta W\, R^B_{N-2,N-2}(z) \sum_{l=0}^{\infty} \left[\Gamma^B_{N-2,N-2}(z) R^B_{N-2,N-2}(z)\right]^l W^*\, P_\eta$$

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- $Q^{(>N-1)} = P_\eta = |\eta\rangle\langle\eta|$, hence

$$K^B_{N-2}(z) =$$

$$= -zP_\eta$$

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- $Q^{(>N-1)} = P_\eta = |\eta\rangle\langle\eta|$, hence

$$K^B_{N-2}(z) = f(z)P_\eta$$

$$f(z) = -z$$

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- $f(z)$ is decreasing and there is (only) one point z_* in the interval

$$(-\infty, E^B_{\mathbf{j}_*} + \sqrt{\epsilon_{\mathbf{j}_*}}\sqrt{(k^2_{\mathbf{j}_*})^2 + 2\phi_{\mathbf{j}_*}k^2_{\mathbf{j}_*}})$$

such that $f(z_*) = 0$

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such that $f(z_*) = 0$
- $f(z_*) = 0 \quad \Rightarrow \quad z_*$ is the ground state energy of $\hat{H}^B$

GS vector

- Feshbach theory: If φ eigenvector of $\mathscr{F}(K - z_*)$ with eigenvalue 0

$$[\mathscr{P} - \frac{1}{\overline{\mathscr{P}}(K - z_*)\overline{\mathscr{P}}}\overline{\mathscr{P}}K\mathscr{P}]\varphi$$

is eigenvector of K with eigenvalue z_*

GS vector

- Convergent expansion (up to any desired precision)

$$\psi^B =$$

$$= \eta$$

$$- \frac{1}{Q^{(N-2,N-1)} K^B_{N-4}(z_*) Q^{(N-2,N-1)}} Q^{(N-2,N-1)} W^* \eta$$

$$- \sum_{j=2}^{N/2} \prod_{r=2j}^{4} \left[- \frac{1}{Q^{(N-r,N-r+1)} K^B_{N-r-2}(z_*) Q^{(N-r,N-r+1)}} W^*_{N-r,N-r+2} \right] \times$$

$$\times \frac{1}{Q^{(N-2,N-1)} K^B_{N-4}(z_*) Q^{(N-2,N-1)}} Q^{(N-2,N-1)} W^* \eta$$

where $W^*_{N-r,N-r+2} := Q^{(N-r,N-r+1)} W^* Q^{(N-r+2,N-r+3)}$

Regimes and dimensions

- The flow is well defined if $\epsilon_{\mathbf{j}_*} := \frac{k^2_{\mathbf{j}_*}}{\phi_{\mathbf{j}_*}}$ is sufficiently small and for some $\nu > \frac{11}{8}$

$$\epsilon^{\nu}_{\mathbf{j}_*} \geq \frac{1}{N} \quad \Longleftrightarrow \quad \frac{k^2_{\mathbf{j}_*}}{\phi_{\mathbf{j}_*}} > \left(\frac{1}{N}\right)^{\frac{8}{11}}$$

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- When is this condition fulfilled?

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- When is this condition fulfilled?
  - mean field limiting regime
  - at fixed ρ only if

$$\left[\frac{(2\pi \mathbf{j}_*)^2}{L^2 \phi_{\mathbf{j}_*}}\right]^{\nu} \geq \frac{1}{\rho L^d}$$

$\Rightarrow \quad d \geq 3$ and L large enough

Regimes and dimensions

- Existence of the fixed point if

$$\rho \geq \rho_0 \left(\frac{L}{L_0}\right)^{3-d}$$

with ρ_0 sufficiently large ($L_0 \equiv 1$)

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- In the mean field limiting regime, $z_* \to E^{Bog}_{\mathbf{j}_*}$ as $N \to \infty$
- For $d = 3$ and $\rho \geq \rho_0 (\frac{L}{L_0})^s$ with $s > 0$, $z_* \to E^{Bog}_{\mathbf{j}_*}$ as $L \to \infty$

Bogoliubov Hamiltonian for two couples of modes

- $$H^{B}_{\mathbf{j}_1,\mathbf{j}_2} := \sum_{\mathbf{j}\in\mathbb{Z}^d\setminus\{\pm\mathbf{j}_1\,;\,\pm\mathbf{j}_2\}} k_{\mathbf{j}}^2 a_{\mathbf{j}}^* a_{\mathbf{j}} + \sum_{l=1}^{2} \hat{H}^{B}_{\mathbf{j}_l}$$

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- Feshbach Hamiltonian at step $i \leq N-2$

$$\begin{aligned}\mathscr{K}^{B\,(i)}_{\mathbf{j}_1,\mathbf{j}_2}(z^B_{\mathbf{j}_1}+z) &= Q^{(>i+1)}_{\mathbf{j}_2}(H^B_{\mathbf{j}_1,\mathbf{j}_2} - z^B_{\mathbf{j}_1} - z)Q^{(>i+1)}_{\mathbf{j}_2} \\ &- \sum_{l_i=0}^{\infty} Q^{(>i+1)}_{\mathbf{j}_2} W_{\mathbf{j}_2}\, R^B_{\mathbf{j}_1,\mathbf{j}_2\,;\,i,i}(z^B_{\mathbf{j}_1}+z) \times \\ &\times \Big[\Gamma^B_{\mathbf{j}_1,\mathbf{j}_2\,;\,i,i}(z^B_{\mathbf{j}_1}+z) R^B_{\mathbf{j}_1,\mathbf{j}_2\,;\,i,i}(z^B_{\mathbf{j}_1}+z)\Big]^{l_i} W^*_{\mathbf{j}_2} Q^{(>i+1)}_{\mathbf{j}_2}\end{aligned}$$

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$$- \sum_{l_i=0}^{\infty} Q^{(>i+1)}_{\mathbf{j}_2} W_{\mathbf{j}_2} R^B_{\mathbf{j}_1,\mathbf{j}_2\,;\,i,i}(z^B_{\mathbf{j}_1}+z) \times$$
$$\times \left[\Gamma^B_{\mathbf{j}_1,\mathbf{j}_2\,;\,i,i}(z^B_{\mathbf{j}_1}+z) R^B_{\mathbf{j}_1,\mathbf{j}_2\,;\,i,i}(z^B_{\mathbf{j}_1}+z)\right]^{l_i} W^*_{\mathbf{j}_2} Q^{(>i+1)}_{\mathbf{j}_2}$$
- $$R^B_{\mathbf{j}_1,\mathbf{j}_2\,;\,i,i}(z^B_{\mathbf{j}_1}+z) := Q^{(i,i+1)}_{\mathbf{j}_2} \frac{1}{Q^{(i,i+1)}_{\mathbf{j}_2}(H^B_{\mathbf{j}_1,\mathbf{j}_2} - z^B_{\mathbf{j}_1} - z)Q^{(i,i+1)}_{\mathbf{j}_2}} Q^{(i,i+1)}_{\mathbf{j}_2}$$

Bogoliubov Hamiltonian for two couples of modes

- Last implementation (step N) of the Feshbach map, projections

$$\mathscr{P}_{\psi^B_{\mathbf{j}_1}} := |\frac{\psi^B_{\mathbf{j}_1}}{\|\psi^B_{\mathbf{j}_1}\|}\rangle\langle\frac{\psi^B_{\mathbf{j}_1}}{\|\psi^B_{\mathbf{j}_1}\|}| \quad , \quad \overline{\mathscr{P}_{\psi^B_{\mathbf{j}_1}}} := \mathbf{1}_{Q^{(>N-2)}_{\mathbf{j}_2}\mathcal{F}^N} - \mathscr{P}_{\psi^B_{\mathbf{j}_1}}$$

Bogoliubov Hamiltonian for two couples of modes

- Last implementation (step N) of the Feshbach map, projections

$$\mathscr{P}_{\psi^B_{\mathbf{j}_1}} := \Big|\frac{\psi^B_{\mathbf{j}_1}}{\|\psi^B_{\mathbf{j}_1}\|}\Big\rangle\Big\langle\frac{\psi^B_{\mathbf{j}_1}}{\|\psi^B_{\mathbf{j}_1}\|}\Big| \quad , \quad \overline{\mathscr{P}_{\psi^B_{\mathbf{j}_1}}} := \mathbf{1}_{Q^{(>N-2)}_{\mathbf{j}_2}\mathcal{F}^N} - \mathscr{P}_{\psi^B_{\mathbf{j}_1}}$$

- Final Feshbach Hamiltonian

$$\mathscr{K}^{B\,(i)}_{\mathbf{j}_1,\,\mathbf{j}_2}(z^B_{\mathbf{j}_1} + z) = f^B_{\mathbf{j}_1,\mathbf{j}_2}(z)\mathscr{P}_{\psi^B_{\mathbf{j}_1}}$$

- Fixed point equation

$$\begin{aligned} 0 &= f^B_{\mathbf{j}_1,\mathbf{j}_2}(z^{(2)}) \\ &= -z^{(2)} - \langle \frac{\psi^B_{\mathbf{j}_1}}{\|\psi^B_{\mathbf{j}_1}\|}, \Gamma^B_{\mathbf{j}_1,\mathbf{j}_2;\,N,N}(z^B_{\mathbf{j}_1} + z^{(2)})\frac{\psi^B_{\mathbf{j}_1}}{\|\psi^{Bog}_{\mathbf{j}_1}\|}\rangle \\ &\quad +\mathcal{O}(1/(\ln N)^{\frac{1}{4}}) \end{aligned}$$

Bogoliubov Hamiltonian for two couples of modes

- Key ingredients to solve the fixed point equation

$$z^{(2)} = -\langle \frac{\psi^{B}_{\mathbf{j}_1}}{\|\psi^{B}_{\mathbf{j}_1}\|}, \Gamma^{B}_{\mathbf{j}_1,\mathbf{j}_2;\,N,N}(z^{B}_{\mathbf{j}_1} + z^{(2)})\frac{\psi^{B}_{\mathbf{j}_1}}{\|\psi^{Bog}_{\mathbf{j}_1}\|}\rangle + \mathcal{O}(1/(\ln N)^{\frac{1}{4}})$$

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-

$$R^B_{\mathbf{j}_1,\mathbf{j}_2\,;\,i,i}(z^B_{\mathbf{j}_1}+z) := Q^{(i,i+1)}_{\mathbf{j}_2} \frac{1}{Q^{(i,i+1)}_{\mathbf{j}_2}(H^B_{\mathbf{j}_1,\mathbf{j}_2} - z^B_{\mathbf{j}_1} - z) Q^{(i,i+1)}_{\mathbf{j}_2}} Q^{(i,i+1)}_{\mathbf{j}_2}$$

$$\Rightarrow\ \tilde{R}^B_{\mathbf{j}_1,\mathbf{j}_2\,;\,i,i}(z) := Q^{(i,i+1)}_{\mathbf{j}_2} \frac{1}{Q^{(i,i+1)}_{\mathbf{j}_2}(\hat{H}^B_{\mathbf{j}_2} - z) Q^{(i,i+1)}_{\mathbf{j}_2}} Q^{(i,i+1)}_{\mathbf{j}_2}$$

$$\Gamma^B_{\mathbf{j}_1,\mathbf{j}_2;\,N,N}(z^B_{\mathbf{j}_1} + z) \Rightarrow\ \tilde{\Gamma}^B_{\mathbf{j}_1,\mathbf{j}_2;\,N,N}(z)$$

Bogoliubov Hamiltonian for two couples of modes

- Ground state vector

$$\psi^B_{\mathbf{j}_1,\mathbf{j}_2} :=$$

$$\Big\{\sum_{j=2}^{N/2}$$

$$\Big[\check{\prod}^{4}_{s=N-2j}\Big(-\frac{1}{Q^{(s,s+1)}_{\mathbf{j}_2}\mathscr{K}^{B\,(s-2)}_{\mathbf{j}_1,\mathbf{j}_2}(z^B_{\mathbf{j}_1,\mathbf{j}_2})Q^{(s,s+1)}_{\mathbf{j}_2}}W^*_{\mathbf{j}_2;s,s+2}\Big)\Big]+1\Big\}$$

$$\times\Big[Q^{(>N-1)}_{\mathbf{j}_2}-\frac{Q^{(N-2,N-1)}_{\mathbf{j}_2}}{Q^{(N-2,N-1)}_{\mathbf{j}_2}\mathscr{K}^{B\,(N-4)}_{\mathbf{j}_1,\mathbf{j}_2}(z^B_{\mathbf{j}_1,\mathbf{j}_2})Q^{(N-2,N-1)}_{\mathbf{j}_2}}W^*_{\mathbf{j}_2}\Big]\times$$

$$\times\Big[\mathscr{P}_{\psi^{Bog}_{\mathbf{j}_1}}-\frac{\overline{\mathscr{P}_{\psi^B_{\mathbf{j}_1}}}}{\overline{\mathscr{P}_{\psi^B_{\mathbf{j}_1}}}\mathscr{K}^{B\,(N-2)}_{\mathbf{j}_1,\mathbf{j}_2}(z^B_{\mathbf{j}_1,\mathbf{j}_2})\overline{\mathscr{P}_{\psi^B_{\mathbf{j}_1}}}}\mathscr{K}^{B\,(N-2)}_{\mathbf{j}_1,\mathbf{j}_2}(z^B_{\mathbf{j}_1,\mathbf{j}_2})\Big]\psi^B_{\mathbf{j}_1}$$

Complete Hamiltonian: Control of the cubic and quartic terms

- New first step in each Feshbach flow

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- *Short range property* of the interaction Hamiltonian in the particle states occupation numbers

Complete Hamiltonian: Control of the cubic and quartic terms

- New first step in each Feshbach flow
- *Short range property* of the interaction Hamiltonian in the particle states occupation numbers
- Semigroup property of the Feshbach map

General Term

- Recursive relation

$$\Gamma^{B}_{i+2,i+2}(z) :=$$

$$= Q^{(i+2,i+3)} W \, (R^{B}_{i,i}(z))^{\frac{1}{2}} \sum_{l_i=0}^{\infty} \left[(R^{B}_{i,i}(z))^{\frac{1}{2}} \Gamma^{B}_{i,i}(z) (R^{B}_{i,i}(z))^{\frac{1}{2}} \right]^{l_i} \times$$

$$\times (R^{B}_{i,i}(z))^{\frac{1}{2}} W^{*} Q^{(i+2,i+3)}$$

- Initial term

$$\Gamma^{B}_{2,2}(z) := Q^{(2,3)} W \, R^{B}_{0,0}(z) W^{*} Q^{(2,3)}$$

Outlook / Thomas-Fermi + Gross Pitaveskii limit

- N Bose (nonrelat.) particles in a finite box of volume $|\Lambda| = 1$

$$H = -\sum_{i} \Delta_i^{(x)} + g\, N^2 \sum_{i<j} \phi(N(x_i - x_j))$$

with $N, g \to +\infty$

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- Three-modes Hamiltonian

$$H^B_{\mathbf{j}_*} = \sum_{\pm \mathbf{j}_*} (N^2 k_{\mathbf{j}}^2 + g \frac{\phi_{\mathbf{j}_*}}{N} a_{\mathbf{0}}^* a_{\mathbf{0}}) a_{\mathbf{j}}^* a_{\mathbf{j}} + g \frac{\phi_{\mathbf{j}_*}}{N} \left\{ a_{\mathbf{0}}^* a_{\mathbf{0}}^* a_{\mathbf{j}} a_{-\mathbf{j}} + a_{\mathbf{j}_*}^* a_{-\mathbf{j}_*}^* a_{\mathbf{0}} a_{\mathbf{0}} \right\}$$

  where $k_{\mathbf{j}}^2 \gtrsim N^{-2}$

Outlook / Thomas-Fermi + Gross Pitaveskii limit

- N Bose (nonrelat.) particles in a finite box of volume $|\Lambda| = 1$

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with $N, g \to +\infty$

- Rescaling: $y = Nx \Rightarrow N$ particles in a box of volume $|\Lambda| = N^3$

$$H = -N^2 \sum_i \Delta_i^{(y)} + g\, N^2 \sum_{i<j} \phi(y_i - y_j)$$

- Three-modes Hamiltonian

$$H^B_{\mathbf{j}_*} = g\phi_{\mathbf{j}_*} \Big[\sum_{\pm \mathbf{j}_*} \Big(\frac{N^2 k^2_{\mathbf{j}}}{g\phi_{\mathbf{j}_*}} + \frac{1}{N} a^*_{\mathbf{0}} a_{\mathbf{0}} \Big) a^*_{\mathbf{j}} a_{\mathbf{j}} + \frac{1}{N} \Big\{ a^*_{\mathbf{0}} a^*_{\mathbf{0}} a_{\mathbf{j}} a_{-\mathbf{j}} + a^*_{\mathbf{j}_*} a^*_{-\mathbf{j}_*} a_{\mathbf{0}} a_{\mathbf{0}} \Big\} \Big]$$

where $k^2_{\mathbf{j}} \gtrsim N^{-2} \quad \Rightarrow \quad \frac{N^2}{g} \frac{k^2_{\mathbf{j}}}{\phi_{\mathbf{j}_*}} > N^{-\frac{8}{11}} \quad$ for $\quad g \lesssim N^{\frac{8}{11}}$

THANK YOU