Chiral anomalies.

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1. Preliminaries.

Let M be a $2n$-dimensional Eculidean or Minkowski space; and let

$$\left\{\gamma^0, \gamma^1, \ldots, \gamma^{2n-1}\right\} \subset M_{2^n}(\mathbb{C})$$

be the corresponding representation of the Clifford algebra $\mathrm{Cl}_{1,2n-1}(\mathbb{R})$. It is customary to define the $2n+1$-th gamma matrix γ as

$$\gamma = i\gamma^0 \cdots \gamma^{2n-1} \ .$$

If M is four dimensional, γ is often denoted as γ^5. The matrix γ anticommutes with every γ^μ, $\mu \in \{0, 2n-1\}$. We adopt the convention of summing over repeated indices; in addition an index can be lowered or raised using the metric. The Dirac operator D is defined, in local coordinates, by

$$D = i\gamma^\mu(\partial_\mu - iA_\mu) \ , \tag{1}$$

where A is a (external) gauge field. A Fermi field of mass m is denoted by $\psi(x)$, $x \in M$, and its adjoint is $\bar{\psi}(x) = \psi^*(x)\gamma_0$. The field $\psi(x)$ is a 2^n component spinor field that obeys the Dirac equation

$$D\psi = 0 \ . \tag{2}$$

There are two currents associated with the Dirac equation (2):

$$\mathscr{J}^\mu = \bar{\psi}\gamma^\mu\psi \ , \qquad \text{(standard current)} \ ; \tag{3}$$

$$\tilde{\mathscr{J}}^\mu = \bar{\psi}\gamma^\mu\gamma\psi \ , \qquad \text{(chiral or axial current)} \ . \tag{4}$$

The conservation of the standard current comes from the global $U(1)$ invariance of the action for the Fermi and gauge field; in particular, $\partial_\mu \mathscr{J}^\mu = 0$ and $\partial_\mu \tilde{\mathscr{J}}^\mu \propto m$ when $A = 0$ (considered as an external field). Therefore if the Fermi field is massless, the chiral current is also conserved: $\partial_\mu \mathscr{J}^\mu = 0 = \partial_\mu \tilde{\mathscr{J}}^\mu$ for $m = 0$ and $A = 0$. Let us denote by

$$Q = \int \mathscr{J}^0(\underline{x}, t)\mathrm{d}\underline{x} \ , \tag{5}$$

$$\tilde{Q} = \int \tilde{\mathscr{J}}^0(\underline{x}, t)\mathrm{d}\underline{x} \tag{6}$$

the two conserved charges.

Remark 1.1. It is possible to use a coordinate free language to define the currents and charges. Let I^μ be a conserved current, $\partial_\mu I^\mu = 0$. The $2n-1$ dual form associated to I^μ is defined by

$$i = i_{\mu_1\cdots\mu_{2n-1}} \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_{2n-1}} \ ,$$

with $i_{\mu_1\cdots\mu_{2n-1}} = \varepsilon_{\mu_1\cdots\mu_{2n}} I^{\mu_{2n}}$. The conservation $\partial_\mu I^\mu = 0$ is true iff $\mathrm{d}i = 0$. In addition, $\mathrm{d}i = 0$ yields $i = \mathrm{d}\beta$, with β a $2n-2$ form (the potential of the conserved current). The conserved charge Q can be written as the integral

$$Q = \int_\Sigma i$$

of the form i in the space hypersurface Σ.

2. n = 1.

By Poincaré's lemma, $\partial_\mu \mathscr{J}^\mu = 0 \Leftrightarrow \mathrm{d}i = 0 \Rightarrow i = \frac{Q}{2\pi}\mathrm{d}\varphi$, where $\frac{Q}{2\pi}$ is a normalization factor and φ a scalar field. Therefore

$$\mathscr{J}^\mu = \tfrac{Q}{2\pi}\varepsilon^{\mu\nu}\partial_\nu\varphi \ . \tag{7}$$

In addition, when $n = 1$ and $m = 0$ the following identity is satisfied: $\tilde{\mathscr{J}}^\mu = \varepsilon^{\mu\nu}\mathscr{J}_\nu$. It follows that

$$\tilde{\mathscr{J}}^\mu = \tfrac{Q}{2\pi}\partial^\mu\varphi \ . \tag{8}$$

If $F_{\mu\nu} = 0$, then $\partial_\mu \tilde{\mathscr{J}}^\mu = 0$; hence $\Box\varphi = 0$, and its action is that of a free massless field

$$\mathcal{S}(\varphi) = \tfrac{1}{4\pi}\int \partial_\mu\varphi\partial^\mu\varphi \mathrm{d}x. \tag{9}$$

The conjugate momentum field $\pi(x)$ is defined in the usual fashion

$$\pi(x) = \frac{\delta\mathcal{S}(\varphi)}{\delta\dot{\varphi}} = \tfrac{1}{2\pi}\dot{\varphi} = -\tfrac{1}{Q}\mathscr{J}^1(x) \ . \tag{10}$$

Quantizing the field and its conjugate momentum canonically, we obtain the well known equal-time commutation relation

$$[\pi(\underline{x},t),\varphi(\underline{y},t)] = -i\delta(\underline{x}-\underline{y}) \qquad\qquad (\hbar = 1 = c)\ . \tag{11}$$

The axial current is given by $\tilde{\mathscr{J}}^0 = Q\pi$, $\tilde{\mathscr{J}}^1 = \frac{Q}{2\pi}\varphi'$; and yields the anomalous commutator

$$[\mathscr{J}^0(\underline{x},t), \tilde{\mathscr{J}}^0(\underline{y},t)] = i\tfrac{Q^2}{2\pi}\delta'(\underline{x}-\underline{y}) \ . \tag{12}$$

Let us now introduce the left and right currents $\mathscr{J}^{\mu}_{\ell/\mathrm{r}} = \mathscr{J}^{\mu} \pm \tilde{\mathscr{J}}^{\mu}$. The corresponding left and right chiral spinors $\psi_{\ell/\mathrm{r}} = \frac{(1\mp\gamma)}{2}\psi$ satisfy

$$
\begin{aligned}
\psi^{(q)}_{\ell/\mathrm{r}}(\underline{x},t) &= :\ \exp 2\pi i\,\tfrac{q}{Q}\int_{\underline{x}}^{\infty} \mathscr{J}^{1}_{\ell/\mathrm{r}}(\underline{y},t)\mathrm{d}\underline{y}\ : \\
&= :\ \exp \pi i q\Big[\pm\varphi(\underline{x},t) + \int_{\underline{x}}^{\infty}\pi(\underline{y},t)\mathrm{d}\underline{y}\Big]\ :
\end{aligned} \tag{13}
$$

where q is a parameter yet to be defined. Using the commutation relations (11), it is possible to calculate the commutator between the charge operator $\hat{Q} = \frac{Q}{2\pi}\int\varphi'(\underline{x},t)\mathrm{d}\underline{x}$ and the chiral field operator

$$
[\hat{Q},\psi_{\ell/\mathrm{r}}(x)] = Qq\,\psi_{\ell/\mathrm{r}}(x)\ . \tag{14}
$$

It is therefore necessary that $Qq = -1$, for the electron to be observable. In addition,

$$
\psi_{\ell}(\underline{x},t)\psi_{\ell}(\underline{y},t) = e^{i\pi q^2}\psi_{\ell}(\underline{y},t)\psi_{\ell}(\underline{x},t)\ ; \tag{15}
$$

hence $q = 1$ to preserve the anticommutation relations.

In the presence of an external electromagnetic field $E = \varepsilon^{\mu\nu}\partial_\mu A_\nu$ the action reads

$$
\mathcal{S}(\varphi) \to \mathcal{S}(\varphi, A) = \mathcal{S}(\varphi) + \int \mathscr{J}^{\mu}(x)A_\mu(x)\mathrm{d}^2x = \tfrac{1}{4\pi}\int[\partial_\mu\varphi\partial^\mu\varphi + 2Q\varphi E]\mathrm{d}^2x\ .
$$

The corresponding equation of motion is

$$
\Box\varphi = QE = \frac{2\pi}{Q}\partial_\mu\tilde{\mathscr{J}}^{\mu}\ ; \tag{16}
$$

and it yields the form of the chiral anomaly in $1+1$ dimensions. It is also possible to define a conserved (but *not* gauge invariant) current

$$
\hat{\mathscr{J}}^{\mu}_{\ell/\mathrm{r}} = \mathscr{J}^{\mu}_{\ell/\mathrm{r}} \mp \tfrac{Q^2}{2\pi}\varepsilon^{\mu\nu}A_\nu\ . \tag{17}
$$

In fact, since the standard current is conserved, $\partial_\mu\hat{\mathscr{J}}^{\mu}_{\ell/\mathrm{r}} = \partial_\mu\tilde{\mathscr{J}}^{\mu} \mp \frac{Q^2}{2\pi}E = 0$. The corresponding charge $\int\hat{\mathscr{J}}^{0}_{\ell/\mathrm{r}}(\underline{x},t)\mathrm{d}\underline{x}$ is gauge invariant and conserved in time.

3. Arbitrary $\mathbf{n} \in \mathbb{N}_*$.

In arbitrary dimensions it is not possible to bosonize the Fermi fields as it was done in (13). Therefore we introduce two gauge fields: the vector A_μ as before, and the *axial* vector Z_μ. The Dirac operator becomes

$$
D_{A,Z} = i\gamma^\mu(\partial_\mu - iA_\mu - iZ_\mu\gamma)\ . \tag{18}
$$

The corresponding action for the Fermi fields is

$$\mathcal{S}(\bar{\psi}, \psi; A, Z) = \int \bar{\psi}(x) D_{A,Z} \psi(x) \mathrm{d}^{2n} x \;, \tag{19}$$

that yields an effective action

$$e^{i\mathcal{S}_{\mathrm{eff}}(A,Z)} = \text{const.} \int_{\text{Berezin}} e^{i\mathcal{S}(\bar{\psi},\psi;A,Z)} \mathcal{D}\psi \mathcal{D}\bar{\psi} = \det{}_{\mathrm{ren}} D_{A,Z} \;. \tag{20}$$

The constant in (20) is chosen such that $\mathcal{S}_{\mathrm{eff}}(0,0) = 0$. The functional integral in (20) is equivalent to a renormalized determinant with the condition $\det_{\mathrm{ren}} D_{0,0} = 1$. The correlation functions with respect to the effective action are calculated in the usual way:

$$\langle \mathscr{J}^{\mu_1}(x_1) \cdots \tilde{\mathscr{J}}^{\nu_1}(y_1) \cdots \rangle^{\mathrm{corr}}_{A,Z} = (-i) \frac{\delta}{\delta A_{\mu_1}(x_1)} \cdots (-i) \frac{\delta}{\delta Z_{\nu_1}(y_1)} \cdots \mathcal{S}_{\mathrm{eff}}(A,Z) \;. \tag{21}$$

Now if we perform a gauge and axial transformation

$$A_\mu \mapsto A_\mu + \partial_\mu \chi \;, \tag{22}$$

$$Z_\mu \mapsto Z_\mu + \partial_\mu \alpha \;, \tag{23}$$

a transformation is induced on the Fermi fields:

$$\psi \mapsto \psi_{\chi,\alpha} = e^{i(\chi + \alpha\gamma)} \psi \;, \tag{24}$$

$$\bar{\psi} \mapsto \bar{\psi}_{\chi,\alpha} = \bar{\psi} e^{i(-\chi + \alpha\gamma)} \;. \tag{25}$$

Therefore, at the classical level $\mathcal{S}(\bar{\psi}, \psi; A, Z) = \mathcal{S}(\bar{\psi}_{\chi,\alpha}, \psi_{\chi,\alpha}; A + \mathrm{d}\chi, Z + \mathrm{d}\alpha)$. The quantum anomalies emerge if the Fermi fields' transformation has a nontrivial Jacobian: the latter modifies the Berezin integral, thus breaking the classical invariance. More explicitly, we get

$$\mathcal{D}\psi \mathcal{D}\bar{\psi} = (J^{-1}) \mathcal{D}\psi_{\chi,\alpha} \mathcal{D}\bar{\psi}_{\chi,\alpha} \;, \tag{26}$$

where, at least formally,

$$J = \exp[2i \underbrace{\mathrm{Tr}(\alpha\gamma)}_{\infty}] \;. \tag{27}$$

To regularize the Jacobian, it is possible to proceed as follows. For simplicity, let us consider the concrete case of a compact toric manifold $M = \mathbb{T}^{2n}$ with imaginary time. Then the Dirac operator has discrete spectrum $\sigma(D_{A,0}) = \{i\lambda_m \in i\mathbb{R}, m \in \mathbb{Z}\}$, with the λ_m symmetrically distributed with respect to zero. It follows that

$$\mathrm{Tr}(\alpha\gamma) = \sum_{m \in \mathbb{Z}} \int \psi^*_m(x)(\alpha\gamma\psi_m)(x) \mathrm{d}^{2n} x = \lim_{M \to \infty} \sum_{m=-M}^{M} e^{-\frac{\lambda_m^2}{M^2}} \int \psi^*_m(x)(\alpha\gamma\psi_m)(x) \mathrm{d}^{2n} x \;. \tag{28}$$

We suppose that $[\alpha\gamma, D_A]_+ = 0$; and denote $M^{-2} \equiv \beta$, $D_A \equiv D$. Then

$$\frac{d}{d\beta}\,\mathrm{Tr}(\alpha\gamma e^{\beta D^2}) = \mathrm{Tr}(\alpha\gamma e^{\beta D^2} D\cdot D) = \mathrm{Tr}(D\alpha\gamma e^{\beta D^2} D)$$
$$= -\,\mathrm{Tr}(\alpha\gamma D e^{\beta D^2} D) = -\frac{d}{d\beta}\,\mathrm{Tr}(\alpha\gamma e^{\beta D^2})\ ,$$

therefore $\mathrm{Tr}(\alpha\gamma e^{\beta D^2})$ is independent of β. In addition, $\mathrm{Tr}(\alpha\gamma e^{D_A^2/M^2}) = -\int \alpha(x)\mathcal{A}(x)\mathrm{d}^{2n}x$, with $\mathcal{A}$ the index density. For $n = 2$, $\mathcal{A}$ has the following form:

$$\mathcal{A}(x) = -\tfrac{1}{32\pi^2} F_{\mu\nu}(x)\,\underbrace{\tilde{F}^{\mu\nu}(x)}_{\varepsilon^{\mu\nu\lambda\delta}F_{\lambda\delta}} = -\tfrac{1}{8\pi^2}\mathbf{E}(x)\cdot\mathbf{B}(x)\ . \tag{29}$$

In general dimensions, it is a polynomial of degree n. The index density is calculated by means of a Dyson expansion, in which all but one term vanish either because of the properties of the gamma matrices or in the limit $\beta \to 0$. The only surviving term is the one of degree n, that yields the result.

Now, it is possible to compute the transformed effective action.

$$\mathcal{S}_{\mathrm{eff}}(A + \mathrm{d}\chi, Z + \mathrm{d}\alpha) = \mathcal{S}_{\mathrm{eff}}(A, Z) + 2i\int \alpha(x)\mathcal{A}(x)\mathrm{d}^{2n}x\ . \tag{30}$$

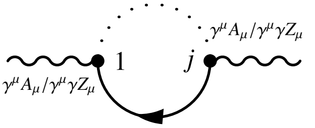
By means of equation (30), the chiral anomaly is obtained in a straightforward way:

$$\partial_\mu\langle \mathscr{J}^\mu(x)\rangle_{A,Z} = -i\frac{\delta\mathcal{S}_{\mathrm{eff}}(A + \mathrm{d}\chi, Z + \mathrm{d}\alpha)}{\delta\chi(x)}\bigg|_{\chi,\alpha=0} = 0\ , \tag{31}$$

$$\partial_\mu\langle \tilde{\mathscr{J}}^\mu(x)\rangle_{A,Z} = -i\frac{\delta\mathcal{S}_{\mathrm{eff}}(A + \mathrm{d}\chi, Z + \mathrm{d}\alpha)}{\delta\alpha(x)}\bigg|_{\chi,\alpha=0} = 2\mathcal{A}(x)\ . \tag{32}$$

Alternatively, it is possible to compute the effective action perturbatively:

$$\mathcal{S}_{\mathrm{eff}}(A, Z) = \Big[\mathrm{tr}(\ln D_{A,Z})\Big]_{\mathrm{renormalized}}$$

$$\underset{(\ln(1+x)=\sum_j \frac{(-1)^{j+1}}{n}x^j)}{=}\ \sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j}$$ 

.

For $n = 2$, the relevant diagram is the one with $j = 3$ vertices (proportional to $\alpha\mathbf{E}\cdot\mathbf{B}$).

Finally, we review how to compute the anomalous commutators. Let $\mathscr{J}^\mu_{\ell/\mathrm{r}}$ be defined by (17), and define the new currents

$$\hat{\mathscr{J}}^\mu_{\ell/\mathrm{r}}(\cdot) = \mathscr{J}^\mu_{\ell/\mathrm{r}}(\cdot) \mp 2\omega^\mu(\cdot; A)\ , \tag{33}$$

where ω is the Chern-Simons form. Then $\partial_\mu\hat{\mathscr{J}}^\mu_{\ell/\mathrm{r}} = 0$, and the corresponding conserved charges are gauge invariant. However, the currents $\hat{\mathscr{J}}^\mu_{\ell/\mathrm{r}}$ themselves *are not* gauge invariant.

In $n = 2$, ω is a 3-form that has the form $\omega = A \wedge \mathrm{d}A$ (for an Abelian gauge theory). Therefore $\mathrm{d}\omega = \mathrm{d}A \wedge \mathrm{d}A = F \wedge F$, that does not vanish since F is a 2-form. The Hodge dual $*(F \wedge F)$ of $F \wedge F$ is the scalar $F_{\mu\nu}\tilde{F}^{\mu\nu}$ that we already encountered in the definition of $\mathcal{A}$, see equation (29).

Let us consider the general framework of an Hamiltonian field theory of free massless chiral fermions in an external electromagnetic field. Denote by $\mathscr{V}$ the affine space of vector potentials A, and $\mathscr{F}_A$ the corresponding Fock space. The Hilbert bundle $\mathscr{H}$ is the bundle with base space $\mathscr{V}$ and fibre $\mathscr{F}_A$ over $A \in \mathscr{V}$. On $\mathscr{V}$, the gauge transformations χ induce orbits of vector potentials. Let $g^\chi(\underline{x}) = e^{i\chi(\underline{x})}$, and $\mathcal{G} = \{g^\chi(\underline{x})\}$ be the group of time-independent gauge transformations. It is possible to give a projective representation U of $\mathcal{G}$ on $\mathscr{H}$ that satisfies

$$U(g_1)U(g_2) = \lambda(g_1, g_2)U(g_1 g_2) \ , \tag{34}$$

with $\lambda(g_1, g_2)$ a phase factor. The generator of $U(e^{i\chi})$ is denoted by $\int \chi(\underline{x})G(\underline{x})\mathrm{d}\underline{x}$. $G(\underline{x})$ has the following form for left-handed fermions:

$$G(\underline{x}) = -i\underline{\nabla}\frac{\delta}{\delta A(\underline{x})} + \mathscr{J}^0_\ell(\underline{x}, A) \ . \tag{35}$$

Locally, it is always possible to use $\hat{\mathscr{J}}^0_\ell$ instead of $\mathscr{J}^0_\ell$, obtaining

$$\hat{G}(x) = -i\underline{\nabla}\frac{\delta}{\delta A(\underline{x})} + \hat{\mathscr{J}}^0_\ell(\underline{x}, A) \ . \tag{36}$$

Since $\mathcal{G}$ is an abelian group, $[\hat{G}(\underline{x}), \hat{G}(\underline{y})] = 0$. This gives the anomalous commutator by a direct computation

$$0 = [\hat{G}(\underline{x}), \hat{G}(\underline{y})] = [\mathscr{J}^0_\ell(\underline{x}), \mathscr{J}^0_\ell(\underline{y})] - 2i\underline{\nabla}\frac{\delta}{\delta A(\underline{x})}\omega^0(\underline{y}, A) + 2i\underline{\nabla}\frac{\delta}{\delta A(\underline{y})}\omega^0(\underline{x}, A) \ . \tag{37}$$

For $n = 1$, equation (37) takes the well-known form

$$[\mathscr{J}^0_{\ell/\mathrm{r}}(\underline{x}), \mathscr{J}^0_{\ell/\mathrm{r}}(\underline{y})] = \pm\tfrac{i}{4\pi^2}(B(\underline{x}) \cdot \underline{\nabla})\delta(\underline{x} - \underline{y}) \ . \tag{38}$$