# Anomalies cancellation; dark energy (sketch).

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## 1. Anomalies cancellation.

As we have seen previously, massless chiral fermions were used to derive effective actions:

$$W_\ell(A + \mathrm{d}\chi) = W_\ell(A) + \tfrac{\alpha}{16\pi^2}\chi F\tilde{F} \qquad (d = 3+1)\ . \tag{1}$$

In the effective action gauge invariance is violated, and this may give problems if we want to quantize the gauge field together with fermions. How to restore gauge invariance? There are two (related) strategies.

*1.* The first solution is to claim that we live on the edge of an higher dimensional world. Let $\Lambda$ be a bulk with dimension $d = 5$, and the manifold $\partial\Lambda = \mathbb{M}^4$ its four-dimensional edge. Let us fill the bulk with fermionic degrees of freedom (spinors), that have $2^{[D/2]} = 4$ components. If we integrate out these fermionic degrees of freedom, we obtain an effective action for the bulk. Such action $\mathcal{S}^{\text{bulk}}_{\text{eff}}(A)$ has the following form:

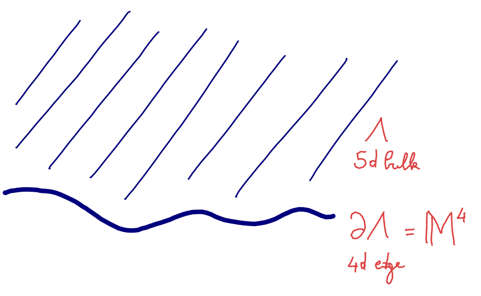


Figure 1. Five-dimensional world, with us on the four-dimensional edge.

$$\mathcal{S}^{\text{bulk}}_{\text{eff}}(A) = \tfrac{1}{lg^2}\int_\Lambda F^2 + \underbrace{\sigma\int_\Lambda A\wedge \mathrm{d}A\wedge \mathrm{d}A}_{\omega_5(A)} + \dots\ , \tag{2}$$

where the dots stand for some unimportant additional terms. Under a gauge transformation $A \mapsto A + \mathrm{d}\chi$, the Chern-Simons term $\omega_5(A)$ transforms as follows:

$$\omega_5(A) \mapsto \omega_5(A) + \underbrace{\sigma\int_\Lambda \mathrm{d}(\chi \mathrm{d}A\wedge \mathrm{d}A)}_{=\sigma\int_{\partial\Lambda}\chi \mathrm{d}A\wedge \mathrm{d}A}\ . \tag{3}$$

In other words, by Stokes' theorem the additional term of the gauge transformed $\omega_5(A)$ "lives" on the edge. In addition, tuning $\sigma = -\frac{\alpha}{16\pi^2}$ it *cancels exactly* the additional term in (1) that broke gauge invariance.

*2.* The alternative solution is to introduce a pseudo-scalar field, the *axion* $\varphi$. The axion transforms like an angle in a gauge transformation $A \mapsto A + \mathrm{d}\chi$:

$$\varphi \mapsto \varphi + \chi \,. \tag{4}$$

Then the theory given by an action

$$\mathcal{S}(\bar{\psi}_\ell, \psi_\ell; A) - \tfrac{\alpha}{16\pi^2}\varphi F\tilde{F} + \mathcal{S}_0(\varphi; A) \tag{5}$$

is gauge invariant, provided that $\mathcal{S}_0(\varphi : A)$ has a suitable form. In particular, we cannot have self-interaction potentials for the axion, since they would break the gauge invariance. The physically reasonable choice is the following

$$\mathcal{S}_0(\varphi; A) = \int \mathrm{d}t \int \mathrm{d}x \, (\partial_\mu\varphi - A_\mu)(\partial^\mu\varphi - A^\mu) \,. \tag{6}$$

The solutions *(1)* and *(2)* are essentially equivalent. Let us show that *(1)* ⇒ *(2)*. Let us suppose that the five-dimensional bulk gauge field $\hat{A}$ (to distinguish it from the usual one on the edge) has curvature independent of the fifth coordinate $x^4$. In addition, we assume the additional dimensional to be compact. Then it is possible to do dimensional reduction: we define the axion field as the line integral over the additional dimension of the gauge field

$$\varphi(x^0, \underline{x}) := \int_\gamma \hat{A} = \int_{-1}^{0} \hat{A}_4(x^0, \underline{x}, x^4)\mathrm{d}x^4 \,. \tag{7}$$

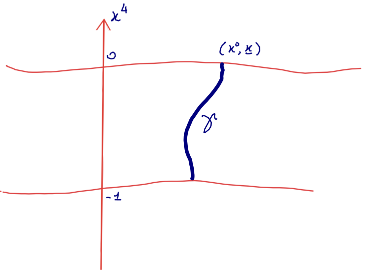


Figure 2. The compact fifth dimension and the curve $\gamma$.

With this definition, the Chern-Simons term $\omega_5(A)$ becomes $\int \varphi F \wedge F$, and defining $A = \hat{A}\big|_{\partial\Lambda}$, we reconstruct the term $\mathcal{S}_0(\varphi, A)$ as well.

## 2. Dark energy (sketch).

It is imaginable that there is a form of matter that is not known to us yet, and that appears only in gravitational interaction.

Suppose in addition that dark energy is formed by left and right fermions $\psi_\ell$ and $\psi_r$, with corresponding gauge fields $A_\ell$ and $A_r$; with the corresponding theory preserving time and parity invariance. The corresponding effective actions $\mathcal{W}_{\ell/\mathrm{r}}(A_{\ell/\mathrm{r}}, Z_{\ell/\mathrm{r}})$ satisfy

$$\mathcal{W}_{\ell/\mathrm{r}}(A_{\ell/\mathrm{r}} + \mathrm{d}\chi_{\ell/\mathrm{r}}, Z_{\ell/\mathrm{r}}) = \pm\alpha\chi_{\ell/\mathrm{r}}[F_{A_{\ell/\mathrm{r}}}\tilde{F}_{A_{\ell/\mathrm{r}}} + \mathrm{tr}(F_{Z_{\ell/\mathrm{r}}}\tilde{F}_{Z_{\ell/\mathrm{r}}})] \ . \tag{8}$$

Now, let us define

$$\mathcal{W} = \mathcal{W}_\ell + \mathcal{W}_\mathrm{r} \ . \tag{9}$$

It follows that

$$\mathcal{W}(A_{\ell/\mathrm{r}} = \mathrm{d}\chi, Z_\ell, Z_\mathrm{r}) = \alpha\chi[\mathrm{tr}(F_{Z_\ell}\tilde{F}_{Z_\ell}) - \mathrm{tr}(F_{Z_\mathrm{r}}\tilde{F}_{Z_\mathrm{r}})] \ . \tag{10}$$

We would like to cancel the anomaly (10) with an axion.

$$\mathcal{W}(Z_\ell, Z_\mathrm{r}) - \alpha\varphi[\mathrm{tr}(F_{Z_\ell}\tilde{F}_{Z_\ell}) - \mathrm{tr}(F_{Z_\mathrm{r}}\tilde{F}_{Z_\mathrm{r}})] \ , \tag{11}$$

with $\varphi \mapsto \varphi + \chi$ under a gauge transformation. Let us assume that $\varphi$ is a scalar, and let us assume that

$$Z_\ell = X + Y \qquad\qquad Z_\mathrm{r} = X - Y \ , \tag{12}$$

$$X_0 \underset{\mathrm{T}}{\longmapsto} X_0 \qquad\qquad X_i \underset{\mathrm{T}}{\longmapsto} -X_i \ , \tag{13}$$

$$Y_0 \underset{\mathrm{T}}{\longmapsto} -Y_0 \qquad\qquad Y_i \underset{\mathrm{T}}{\longmapsto} Y_i \ . \tag{14}$$

It follows that (11) preserves time and parity invariance. The corresponding action is

$$\mathcal{S}(\psi_\ell, \psi_\mathrm{r}, X, Y) + \mathcal{S}(\varphi) + \alpha\varphi[F_X\tilde{F}_Y + F_Y\tilde{F}_X] \ . \tag{15}$$

The question now is what $\mathcal{S}(\varphi)$ should be, if we want $\varphi$ to be a good candidate for dark energy. Let us suppose that it has the form

$$\mathcal{S}(\varphi) = \int [\partial_\mu\varphi\partial^\mu\varphi + U(\varphi)] \ ; \tag{16}$$

with $U(\varphi)$ a suitable self-interaction potential. With such choice, the equation of motion for $\varphi$ yielded by (15) is

$$\Box\varphi + \frac{\delta U(\varphi)}{\delta\varphi} = \alpha(\mathbf{E}_X\mathbf{B}_Y + \mathbf{E}_Y\mathbf{B}_X) \ . \tag{17}$$

The corresponding stress-energy tensor is

$$T_{\mu\nu}(x) = \frac{\delta\mathcal{S}}{\delta g^{\mu\nu}(x)} = \underbrace{(g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\nu\alpha} - g^{\mu\nu}g^{\alpha\beta})\partial_\alpha\varphi\partial_\beta\varphi}_{\text{kinetic}} + \underbrace{g_{\mu\nu}U(\varphi)}_{\text{self-interaction}} \ . \tag{18}$$

The first problem to overcome is that the kinetic part of the stress energy tensor should vanish, or at least be very small. This can be solved choosing a suitable potential $U(\varphi)$, such as

$$U(\varphi) = \mu^4 e^{\varphi/f} \ , \tag{19}$$

where $f$ is a coupling constant that makes the exponent dimensionless. The second problem is that even if the kinetic contribution is negligible, the equation of motion (17) then yields

$$\mathbf{E}_X^2 + \mathbf{B}_X^2 + \mathbf{E}_Y^2 + \mathbf{B}_Y^2 \geq \frac{1}{\alpha} g_{\mu\nu} U(\varphi) \ . \tag{20}$$

The left hand side contributes to matter or radiation, and so it should be small compared to the dark energy contribution. The solution is to assume the coupling constant $\alpha$ to be *big enough*. If these conditions are satisfied, the model could be a potential candidate for dark energy.