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MAKING MODULES FOR MOSAIC DESIGNS

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Abstract: *The creation of geometric mosaic designs has long relied on compass-straightedge geometric constructions. Nevertheless, artisans have used other methods, such as modules formed from "cutting and pasting" of single-color tiles. This article demonstrates how modules created from simple tiles may be used as a medium for designing more complex Persian mosaic patterns.*

Keywords: Modularity, Tessellation

1. INTRODUCTION

The predominance of geometry constructions of compass-straightedge in medieval Persian art exhibits both the artisans' geometry background and the direct involvement of mathematicians in the pattern making process (Jazbi, 1997). But the goal of the present article is to study other design-making approaches that are older than the domination of sophisticated geometry in making ornamental designs. The articles *Modularity in Medieval Persian Mosaics* (Sarhangi, 2008) and *Modularity in Medieval Persian Mosaics: Textual, Empirical, Analytical, and Theoretical Considerations* (Sarhangi, Jablan, and Sazdanovic, 2004) illustrate one of these methods: *modularity*. The present article is a continuation to generate more patterns using this technique. The article also includes some artworks that are based on modularity.

Section 2 presents modularity via one detailed example. This example is based on the layout of a tiling that is presented in a photograph taken from an existing Persian mosaic pattern (Figure 2.b). The photograph is from the five-volume book *Construction and Execution of Design in Persian Mosaics Design* (Maheronnaqsh, 1984).

It is important to mention that Figure 2.b and all other photographs in this article are taken from the designs on newer constructions such as mausoleums and mosques built in the past four decades. None of the mosaics in the photographs have been constructed using modularity. The techniques used for their constructions are more advanced than the methods, such as modularity, that the artisans have used in much older times. But we try to construct these patterns by modularity techniques using simple single-color square-shaped tiles to show a possibility: The possibility that the original layouts, which are much older than the existing structures in the photographs, have been discovered using modularity.

Section 3 demonstrates simple cutting of square tiles for the construction of *octagram and cross* design patterns. Note that the octagrams in the design are not equilateral, that is, the lengths of the sides of a single octagram are not congruent.

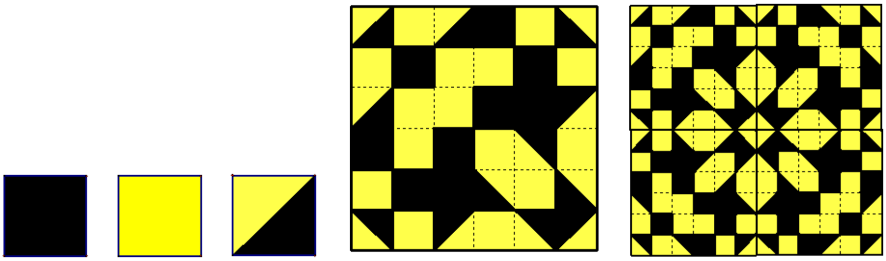
More complex cuttings of square tiles that are needed to replicate some *octagram and cross* design patterns to their exact dimensions are discussed in Section 4.

2. A BRIEF STUDY OF MODULARITY

A study of early Persian mosaic designs shows that most patterns were created by simply cutting and pasting single-color square shaped mosaics (Maheronnaqsh, 1984). In order to save energy and space for molding, casting, painting, and baking tiles, artisans used single-color tiles. The complicated and sophisticated final products on the walls or domes were from the cutting and pasting of these tiles.

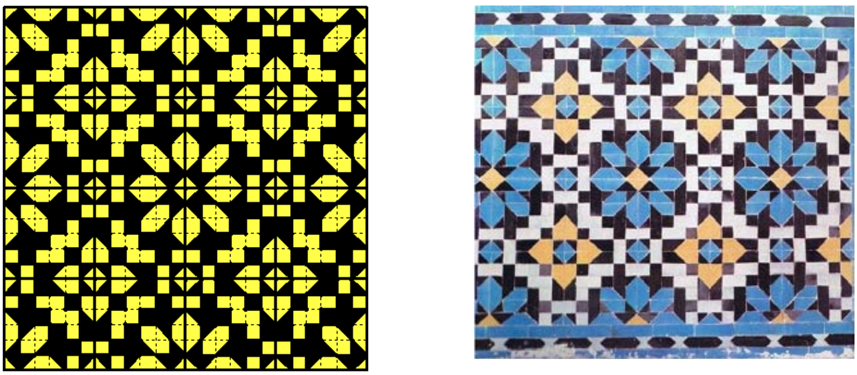
The artisans cut patterns using different formats, and then assembled them so that different colors were swapped with each other in a new arrangement (Maheronnaqsh, 1984). In this way the artisans would rely on the color contrast of cut-tiles to emphasize a design. In most cases the tiles were squares. However, other conveniently constructed shapes such as rhombi and triangles were also used.

By a “modularity” approach, we mean a method that uses the cutting and pasting of two different colored tiles to come up with a set of two-color modules. Here, cutting means breaking a tile into two pieces along a single line segment with the endpoints on the edges of the tile. For example, cut both a black tile and a yellow tile along their diagonals and exchange one of the generated triangles from each. Then a set of three “modules” consisting of black, yellow, and half black-half yellow tiles are generated (Figure 1.a). Now, by using 9 black tiles, 16 yellow tiles, and 11 two-color tiles it is possible to create a square shaped “generator” (Figure 1.b). The image consisting of four copies of the generator through proper reflections of the sides recreates the repeating pattern of the design (Figure 1.c) (Sarhangi, 2008). Figure 2.a exhibits the computer-generated design by the author. Figure 2.b is a photograph of a Persian mosaic construction which is in the mausoleum of *Emamzadeh Soltanali*, Kashan, Iran (Maheronnaqsh, 1984). Both the designs have the same layout (but as is evident from the figure, the existing mosaic is not based on modularity).



(A) (B) (C)

Figure 1: The 3 module set, the generator, a two-coloring of the layout, and an image constructed from two proper reflections across the sides of the generator.



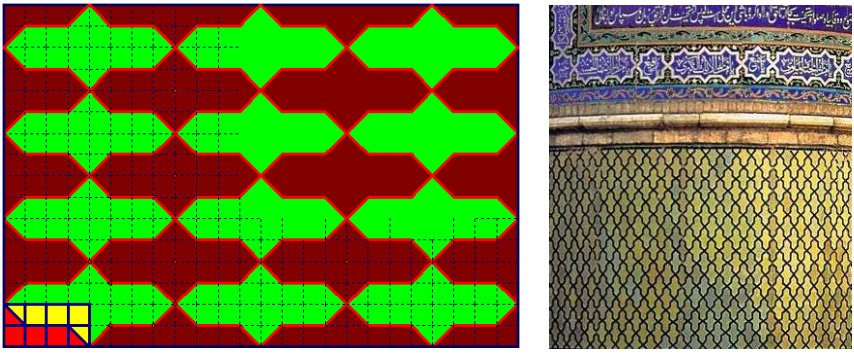
(A) (B)

Figure 2: (a) The computer generated design using modularity approach and (b) the existing Persian mosaic in the mausoleum of *Emamzadeh Soltanali*, Kashan, Iran.

3. MODULES BASED ON SQUARES

Figure 3.a illustrates another layout of a mosaic design that may be recreated using the above set of modules in Figure 1.a. Figure 3.b is a photograph of an existing tilling on the wall of *Jâme Mosque*, Herāt, Afghanistan, with the same layout as in Figure 3.a created by a different method than modularity.

It is interesting to note that *Truchet* (Smith, 1987), in an article written around 1704, that is not related to the modularity approach, laid down a mathematical framework for studying permutations based on the two-color module of this set (Truchet, 1657-1729). To honor his effort in the field of mathematics we call the set of modules in Figure 1.a as the Truchet tile set.



(A) **(B)**

Figure 3: A mosaic design using the Truchet tile set and the existing pattern on the wall of *Jâme Mosque*, Herāt, Afghanistan.

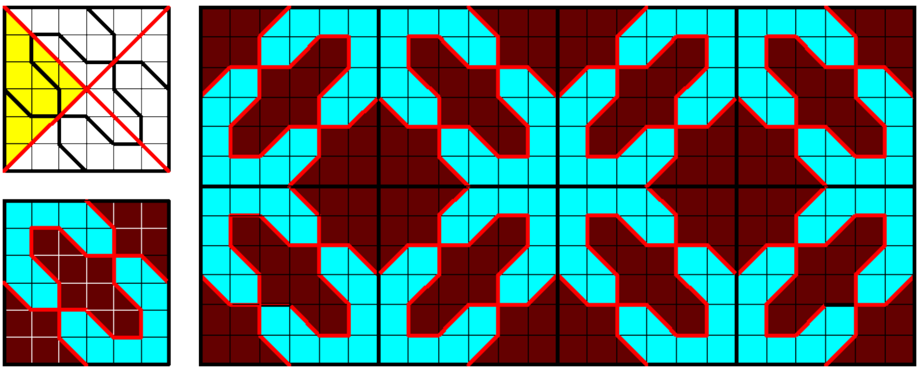


Figure 4: Another mosaic design using the Truchet tiles set.

Figure 4 presents further example of the Truchet modular tiling. Note that the details on the existing design on this column in Figure 5 have been added years after the discovery of the layout of the design itself. Therefore, even though the design can be made using modularity, in later years, by copying and transferring the outlines of the design to different mediums, the artisans have used different approaches for completing the pattern.

Figure 5: An existing pattern on the wall of *Mossalâ*, Herāt, Afghanistan.

Another example of a Truchet tiling layout is given in Figure 6 that resembles an existing mosaic on the wall of *Shâhzâdeh Ebrâhim Shrine*, Kashan, Iran.

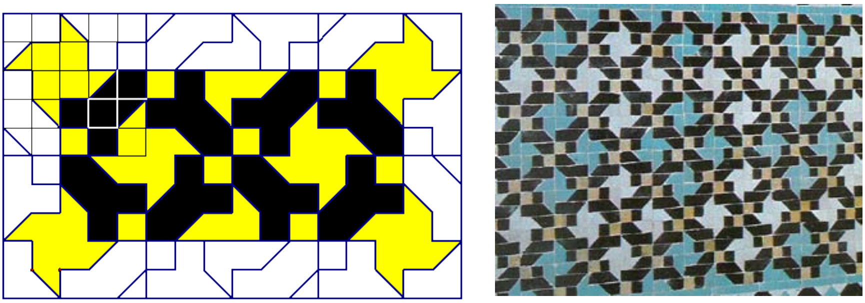


Figure 6: A modular design that resembles the tiling on the wall of *Shâhzâdeh Ebrâhim Shrine*.

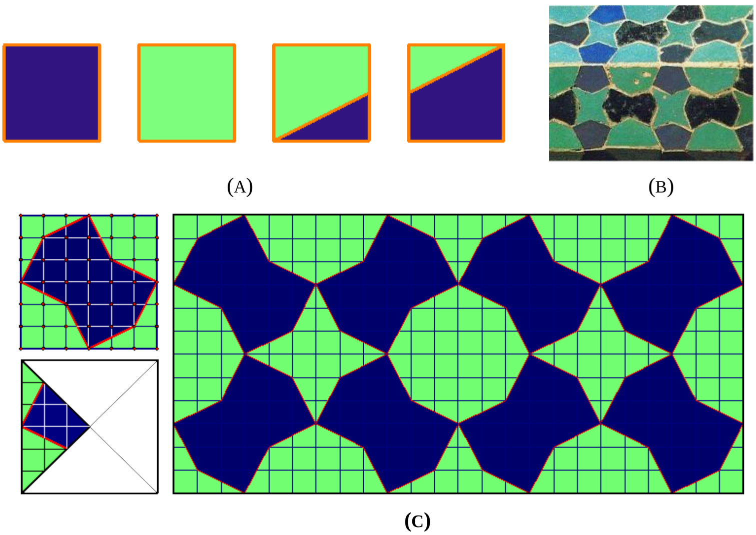
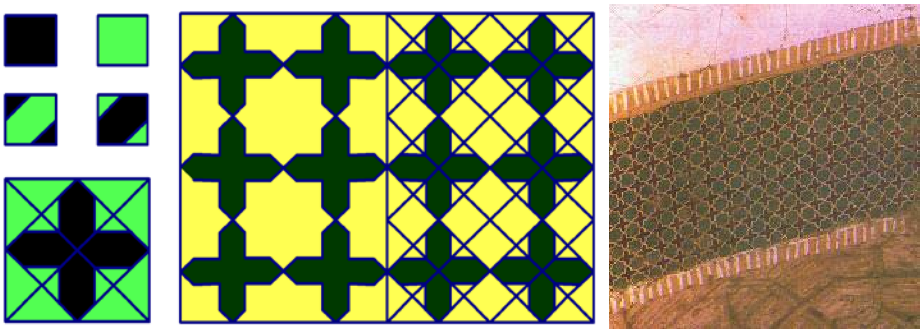


Figure 7: (a) The set of modules, (b) the existing design –not using modularity– on the wall of *Jâme Mosque, Natanz*, Iran, and (c) the details of the tilling to recreate this pattern using modularity.

4. MORE MODULES BASED ON COMPLEX CUTTINGS OF SQUARES

To make a new set of modules, cut from the midpoint of a side to the midpoint of an adjacent side. That will create four modules as shown in Figure 8.a. In the following design we only need two of the modules to illustrate a close approximation of the entire "octagram and cross" design as shown in Figure 8.b. The existing tilling from a wall of *Menâr Jonbân*, Isfahan, Iran, is shown in Figure 8.c.



(A) (B) (C)

Figure 8: (a) The set of modules, (b) the details of the tilling, and (c) the existing design on the wall of *Menâr Jonbân*, Isfahan, Iran.

It is important to note that the above set of modules would not generate an octagram with congruent sides. So Figure 8.b does not demonstrate the actual existing layout in Figure 8.c. In fact, the octagram created using these modules has sides with two different lengths of 1/ 2 and $\sqrt{2}$ /2. More complex cuttings of square tiles are needed to replicate "octagram and cross" design patterns with congruent sides.

Figure 9 shows one approach for making equilateral octagrams, which has been illustrated in *Design and Execution in Persian Ceramics* (Maheronnaqsh, 1984). The construction uses intersections of lines generated from the division of each right angle of the square into four congruent angles. Obviously, this method does not create a set of modules for an equilateral "octagram and cross" tilling. There is no way that we can cut individual tiles of two colors to make new tiles like Figure 9.c (note that a cut is along a single line segment connecting a point on the side of the square to another point that lies on another side of the square). Rather, in this method, the paper copies of the final construction should be tessellated on the wall and then after the completion of the layout, the artisan should fill in the individual crosses and octagrams with small pieces of colored tiles.

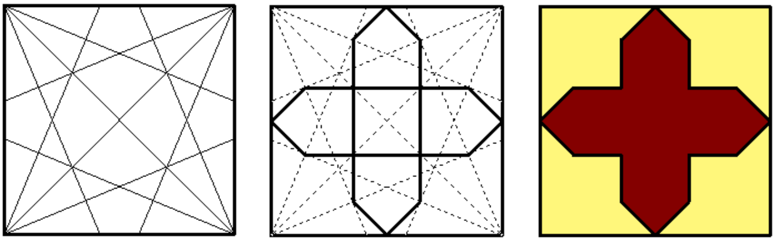


Figure 9: The compass and straightedge construction that creates an equilateral "octagram and cross" design.

The following is a solution to our modularity problem: Let $ABCD$ be a square tile with sides congruent to one unit (Figure 10). In order to create an equilateral octagram using modularity, this square should be cut in a way that the non-regular hexagon $AFGCHI$ becomes equilateral. Suppose that the sides of the hexagon are congruent to a units. Let $\overline{FB} = b$ units. Then $a + b = 1$ unit (I). Moreover, since ΔBGF is an isosceles right triangle, we obtain $a^2 = 2b^2$ (II). The above equations (I) and (II) result in $b = \sqrt{2} - 1$. So to make the correct cut, construct an arc with center A and radius $\overline{AC}$ to cut ray $\overrightarrow{AB}$ at E ($\overline{AC} = \overline{AE} = \sqrt{2}$). Construct a second arc with center B and radius $\overline{BE}$ to cut the sides of the square at F and G. To construct the rest of the module in the tile is now straightforward.

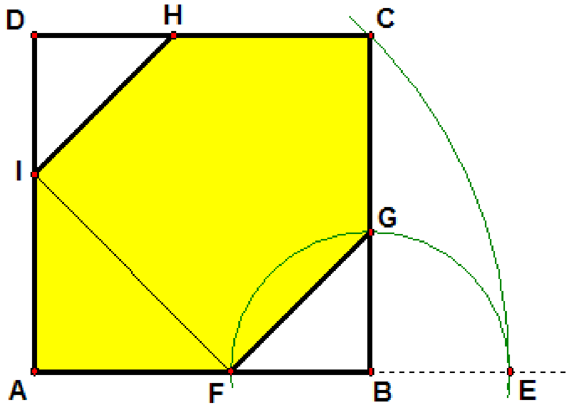
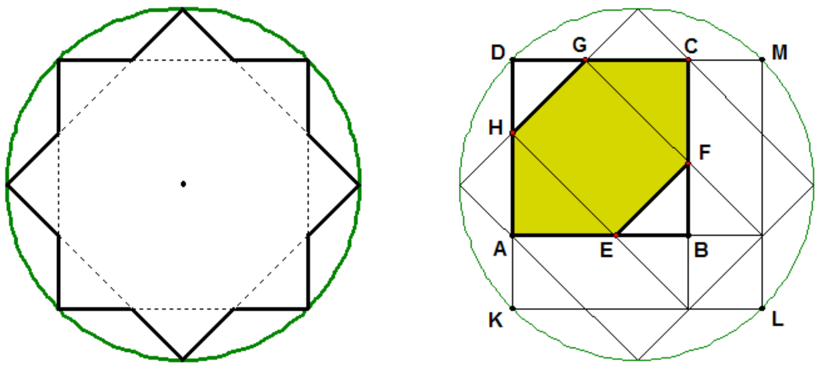


Figure 10: The cutting instruction for an equilateral "octagram and cross" tessellation using modularity.

Note that even though the cut has been generated using a compass, the construction is very elementary, suggesting the advantage of the modularity over the method presented in Figure 9.

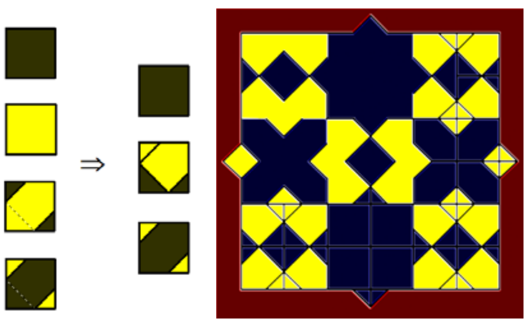
Suppose that a circle of one unit radius with eight equidistance points on it is given. Starting from one of the points connect every other point with a segment to make a square. Another square can be constructed in a similar manner using the remaining points. The segments create an equilateral octagram inscribed inside of the circle (Figure 11.a). In Figure 11.b, the intersections of the segments also make the square *ABCD* with correct cuts for equilateral hexagons *AEFCGH* for constructing another equilateral octagram. We can prove that the dimension of this square is one unit. For this, from ΔKDM we obtain $\overline{DM} = \sqrt{2}$. Also, from ΔHDG we conclude that $\overline{HG} = \sqrt{2}\ \overline{DG}$. Therefore, $\overline{DM} = \overline{DG} + \overline{GC} + \overline{CM} = \overline{DG} + \overline{HG} + \overline{CM} = \overline{DG} + \sqrt{2}\ \overline{DG} + \overline{DG} = \sqrt{2}$. This shows that $\overline{DG} = \sqrt{2} - 1$ and $\overline{GC} = \sqrt{2}\ \overline{DG} = 2 - \sqrt{2}$ and therefore, *ABCD* is a square unit. Hence, this construction can be considered as another method for finding correct cuts on a square with unit sides.



(A) (B)

Figure 11: (a) Constructing an equilateral octagram using two squares, (b) The correct cut on the square *ABCD.*

If we introduce some new cuts to the two two-colored tiles of the “octagram and cross” modules (Figure 12.a), and replace some of the triangular pieces, we will obtain a set of three modules (Figure 12.b) for another interesting and more complicated tessellation (Figure 12.c).

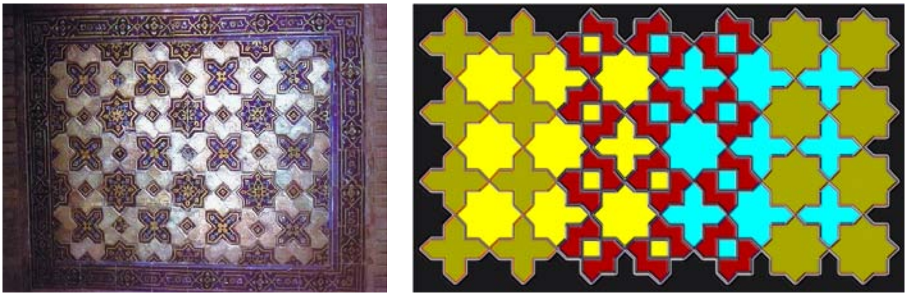


(a) (b) (c)

Figure 12: (a) The set of modules with extra cuts, (b) the new set of modules, (c) and the computer created design.

The mosaic on the wall of *Goharshâd Mosque*, Mash-had, Iran (Figure 13.a), constructed using a different method than modularity, illustrates the same layout as is created from using the above set of modules.

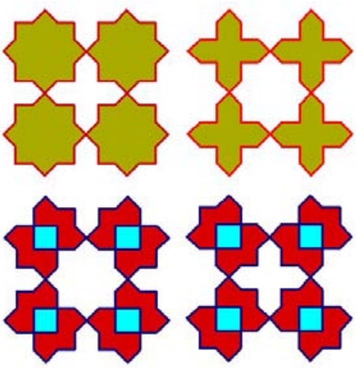
Figure 13.b presents a tessellation that combines the previous two tessellations in such a way that from left to right starts with the "octagram and cross" tessellation, then transforms to the other tessellation in Figure 12.c, and finally goes back to the "octagram and cross" tessellation, but in a fashion that the crosses and octagrams have been replaced.



(a) (b)

Figure 13: (a) The design with the details in an existing mosaic on the wall of *Goharshâd Mosque*, Mash-had, Iran, (b) A metamorphosis made from the two previously mentioned tessellation.

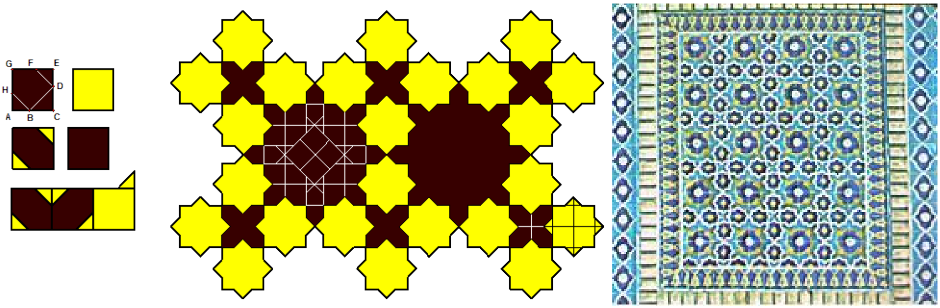
We note that the two top images in Figure 14 present the relationships between a cross and an octagram. The blank space between four octagrams is a cross, and on the other hand, the four crosses make an octagram. So in some sense we may say that the cross and the octagram are each other's duals.



An interesting observation about the butterfly-shape element in this figure is that the space between each four of them could be either a cross or an octagram depending on their orientations.

Figure 14*:* The details in Figure 13.B.

A pleasing and detailed pattern is found on a wall of *Mazâr Sharif*, Afghanistan (Maheronnaqsh, 1984), as shown in Figure 15.c. The layout of this mosaic is more elaborate than the layouts of previous two mosaics shown in Figures 8.c and 13.a. Figure 15.a shows the set of modules and Figure 15.b is the final tessellation using this set. The fine white lines, as well as the black lines dividing an octagram on the far right in Figure 15.b, shows how we use the pieces of the modules to construct this tessellation.



(a) (b) (c)

Figure 15: (a) The set of modules with extra cuts, (b) the constructed design using the Geometers' Sketchpad as a solution to the problem of tiling the design, (c) and the existing mosaic pattern on the wall of *Mazâr Sharif*, Afghanistan.

6. CONCLUSION

The geometric constructions on the walls of medieval Persian structures are rich sources of inspiration that can provide many ideas on how the layouts for those mosaics were originally made. This paper has examined some of these layouts that can be created from simple single-color tiles, cut and pasted together into modules having square symmetry.

REFERENCES

Jazbi, S. A. ed. (1997), *Applied Geometry*, Soroush Press, ISBN 964 435 201 7, Tehran.

Maheronnaqsh, M. (1984), *Design and Execution in Persian Ceramics*, Reza Abbasi Museum Press, Tehran.

Sarhangi, R. (2008), Modules and Modularity in Mosaic Patterns, *Symmetry: Culture and Science*, Raymond Tennant and Gyorgy Darvas, Editors, 19, 2-3, 2008, 153-163.

Sarhangi, R., Jablan S., and Sazdanovic R. (2004), Modularity in Medieval Persian Mosaics: Textual Empirical, Analytical, and Theoretical Considerations, *Bridges Conference Proceedings*, Central Book Manufacturing, Kansas, pp. 281-292.

Smith, D. E. (1987), The Tiling Patterns of Sébastien Truchet and the Topology of Structural Hierarchy, *Leonardo* 20, 4, 1987, pp. 373-385.

Truchet, S. (1657-1729) was an eclectic Dominican Father known for being active in areas such as mathematics and graphics. As mathematician and designer he worked on plane tessellation of a single tile of half black-half white and rotating it in the four main directions (Please see http://en.wikipedia.org/wiki/Sebastien_Truchet).