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MULTIPLE SOLUTIONS FOR PERTURBED INDEFINITE SEMILINEAR ELLIPTIC EQUATIONS

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Abstract. We are looking for infinitely many weak solutions for a semilinear elliptic equation with indefinite nonlinearity. The presence of an L^2 function perturbs the symmetry of the problem. The result is obtained using the approach introduced by Rabinowitz for positive nonlinearities.

1. Introduction

We are interested in the following problem:

$$(\mathcal{P}_f) \qquad \begin{cases} -\Delta u - \lambda u = W(x)p(u) + f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ $(N \geq 3)$ with a smooth boundary. The parameter λ varies in the whole real line $\mathbb{R}$ and f is an $L^2(\Omega)$ function. The function $W(x)$ is bounded in Ω and different from zero almost everywhere ("thin" zero set; see (W)), while $p(u)$ is a continuous, subcritical (with respect to the Sobolev embedding), and superlinear function (see (p1)–(p3)). Moreover, we ask that p is an odd function (($p4$)).

Therefore, for $f \equiv 0$, problem $(\mathcal{P}_f)$ becomes symmetric and thus problems like $(\mathcal{P}_f)$ are called perturbed while the indefiniteness of the problem is due to the change of sign of $W(x)$.

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In recent years much work has been devoted to such indefinite problems (not only with symmetry): see e.g. [2], [3], [1], [9], [10], [17], [22], and [20] for an idea of the results on this subject.

The main result of the paper, that is, Theorem 3.1, assures that, under suitable assumptions, the energy functional I_f associated with problem $(\mathcal{P}_f)$ has an unbounded sequence of critical values provided

$$\theta \equiv \frac{(N+2)-(N-2)s}{N(s-1)} > \frac{s+1}{s}\,, \tag{1.1}$$

where s is the growth exponent of the function p appearing in $(p2)$.

Let us remind the reader that, if $W(x)$ is a positive function, problem $(\mathcal{P}_f)$ has been studied by Struwe [21], Bahri-Berestycki [7], and Rabinowitz [19]. The results contained in these three papers are slightly different, but the interesting aspect is due to the utilization of three different approaches and methods of proof. In particular, Rabinowitz, in [19], proves that, if $\lambda = 0$ and $W(x) > 0$, I_f has an unbounded sequence of critical values provided s satisfies (1.1).

The indefinite perturbed problem $(\mathcal{P}_f)$ was studied in 1996 by Tehrani [23] and by Badiale [5] in case where $W(x)$ is a continuous changing sign function and has a "thick" zero set. More precisely they study problems of the following type:

$$\begin{cases} -\Delta u - \lambda u = W(x)p(u) + f(x)h(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Using Struwe's approach, Tehrani proves that, if $W \equiv f$ and under suitable assumptions on W, p, h, if λ does not belong to the spectrum of the Laplacian, problem (1.2) has infinitely many solutions provided s satisfies (1.1). On the other hand, using the method of Bahri-Berestycki, Badiale achieves the same result for any $f \in L^2(\Omega)$ but only for λ less than the first eigenvalue of the Laplacian and for a smaller range of s.

We observe that the class of functions W we deal with is complementary to theirs (with respect to zero sets).

Let us explain now how we prove Theorem 3.1 following the ideas of Rabinowitz [19]. First of all, let us consider the symmetric problem

$$(\mathcal{P}) \qquad \begin{cases} -\Delta u - \lambda u = W(x)p(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where p and W are the same as in $(\mathcal{P}_f)$. Multiplicity results for this problem were obtained, for example, in [2] and [23], if λ does not belong to the spectrum of the Laplacian, under different assumptions with respect to ours; in particular the function $W(x)$ is a continuous changing sign function and has a "thick" zero set.

Using a $\mathbf{Z_2}$ version of the mountain pass theorem due to Ambrosetti and Rabinowitz [4], we prove that for any real parameter λ problem $(\mathcal{P})$ possesses an unbounded sequence of weak solutions. We underline that we prove that the energy functional I associated with $(\mathcal{P})$ has an unbounded sequence $\beta_k = I(u_k)$ of critical values and also that the sequence $\{u_k\}$ of weak solutions is unbounded.

Therefore, in order to study problem $(\mathcal{P}_f)$, we introduce an auxiliary C^1 functional J_f. We show that, for large critical levels, the critical points of J_f are the same as those of I_f, and J_f satisfies (PS) condition (3° and 4° of Proposition 3.1). So, to obtain our thesis, it is sufficient to show that J_f possesses an unbounded sequence of critical values. The main difficulty to overcome was that of proving the properties of the functional J_f in the presence of a changing sign term. For this purpose we imposed an additional integral condition involving W and P (see (pW)).

It is an interesting open question as to whether Theorem 3.1 holds weakening (1.1). In the definite case Bahri-Lions [8] and Bolle-Ghoussoub-Tehrani [11] were able to value of s to $s < n(n-2)^{-1}$.

2. The symmetric case

Let us consider the following problem:

$$(\mathcal{P}) \qquad \begin{cases} -\Delta u - \lambda u = W(x)p(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded open set of $\mathbb{R}^N$, $N \geq 3$, with smooth boundary $\partial\Omega$, $\lambda \in \mathbb{R}$. Let p be a nonlinear function satisfying the following assumptions:

(p1) p is a continuous function;

(p2) $\exists a_1,\ a_2 > 0 :\ |p(t)| \leq a_1 + a_2|t|^s, \quad s \in (1, \frac{N+2}{N-2}), \forall\ t \in \mathbb{R}$;

(p3) $\exists r > 0 :\ 0 < (s+1)P(t) \leq tp(t) \quad \forall |t| \geq r,$ where $P(t) = \int_0^t p(\tau)\, d\tau$.

Let us observe that hypothesis $(p3)$ implies that there exist two constants $a_4, a_5 > 0$ such that

$$P(t) \geq a_5|t|^{s+1} - a_4 \quad \forall t \in \mathbb{R}. \tag{2.1}$$

Therefore, there is a constant $a_3 > 0$ such that

$$\frac{1}{s+1}(tp(t) + a_3) \geq P(t) + a_4 \geq a_5|t|^{s+1} \quad \forall t \in \mathbb{R}. \tag{2.2}$$

Furthermore, let us assume that

(p4) p is an odd function.

Let $W \in L^\infty(\Omega)$ be a function that assumes both positive and negative values in Ω, and let us define $W^+(x) = \sup\{W(x), 0\}$, $W^-(x) = -\inf\{W(x), 0\}$ for any $x \in \Omega$ and $W^\pm = \sup\{W^\pm(x) : x \in \overline{\Omega}\}$. Let $0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \lambda_{k+1} \leq \cdots$ be the sequence of eigenvalues of the operator $-\Delta$ with respect to the zero boundary conditions on Ω. Each eigenvalue λ_k is repeated according to its (finite) multiplicity.

Let us require

(pW1) $\exists R > 0 : W^-(p(t)t - (s+1)P(t)) \leq \gamma|t|^2$ $\forall\, t \geq R$, for some $\gamma \in (0, (\frac{s+1}{2} - 1)(\overline{\lambda} - \lambda))$, with $\overline{\lambda} = \lambda_{k+1}$ if $\lambda \in [\lambda_k, \lambda_{k+1})$, $k \in \mathbb{N}$, and $\overline{\lambda} = \lambda_2$ if $\lambda < \lambda_1$;

(W) $\mathrm{meas}\{x \in \Omega : W(x) = 0\} = 0$, $W^+(x) \not\equiv 0$.

The main result of this section is the following:

Theorem 2.1. *Suppose that the functions p and W satisfy* (p1)–(p4), (pW1), and (W). *Then problem $(\mathcal{P})$ possesses an unbounded sequence of weak solutions.*

Proof. The proof of this theorem is an application of the following $\mathbf{Z}_2$ version of the mountain pass theorem due to Ambrosetti and Rabinowitz ([4]):

Theorem 2.1. *Let E be a Hilbert space and let $I \in C^1(E, \mathbb{R})$ be even, satisfy the Palais–Smale condition, and $I(0) = 0$. Let $E^+, E^- \subset E$ be closed subspaces of E with $\dim E^- - \operatorname{codim} E^+ = 1$ and suppose there holds the following:*

(I_1) $\exists \rho, \alpha > 0 : I|_{\partial B_\rho \cap E^+} \geq \alpha$,

(I_2) $\exists R > 0$ $I \leq 0$ *on* $E^- \setminus B_R$.

Consider the following set:

$$\Gamma = \{h \in C^0(E, E) : h \text{ is odd, } h(u) = u \text{ if } u \in E^-, ||u|| \geq R\}.$$

Then

(a) $\forall \delta > 0$, $h \in \Gamma$, $\partial B_\delta \cap E^+ \cap h(E^-) \neq \emptyset$.

(b) *The number* $\beta := \inf_{h\in\Gamma} \sup_{u \in E^-} I(h(u)) \geq \alpha$ *is a critical value for* I.

Let $E = H_0^1(\Omega)$, equipped with the norm

$$||u|| = \Big(\int_\Omega |\nabla u|^2\Big)^{\frac{1}{2}},$$

and

$$I(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 - \frac{\lambda}{2}u^2 - W(x)P(u). \tag{2.3}$$

Obviously, $I \in C^1(E, \mathbb{R})$, $I(0) = 0$ and is even. Moreover, condition (PS) holds for any $\lambda \in \mathbb{R}$ by $(pW1)$ and (W), as was proved in [2, 15], if $\lambda < \lambda_1$, and in [14] if $\lambda \geq \lambda_1$.

To verify (I_1) let us suppose $\lambda \geq 0$, as the case $\lambda < 0$ can be proved in a simpler way. For any $\lambda \in \mathbb{R}^+$, $\lambda \in [\lambda_{k_0-1}, \lambda_{k_0}) \neq \emptyset$, where we assume $\lambda_0 = 0$. Let us choose $V_{k_0-1} = \langle v_1, \dots, v_{k_0-1}\rangle$ where the function v_j is an eigenfunction associated to λ_j and $E^+ = E_0^+ = V_{k_0-1}^\perp$, hence $codim E_0^+ = k_0 - 1$. By $(p2)$, for any $u \in E$,

$$I(u) \geq \frac{1}{2}\Big(\int_\Omega |\nabla u|^2 - \lambda \int_\Omega |u|^2\Big) - CW^+ \int_\Omega |u|^{s+1} - C,$$

where C (here and in the following) denotes a generic positive constant. By the Gagliardo–Nirenberg inequality (see e.g. [16], [13])

$$||u||_{L^{s+1}} \leq \alpha_s \Big(\int_\Omega |\nabla u|^2\Big)^{\frac{a}{2}} \Big(\int_\Omega |u|^2\Big)^{\frac{1-a}{2}},$$

where $a \in (0,1)$ is defined by

$$\frac{1}{s+1} = a\big(\frac{1}{2} - \frac{1}{N}\big) + (1-a)\frac{1}{2},$$

and, by the variational characterization of the eigenvalues,

$$\Big(\int_\Omega |u|^2\Big)^{\frac{1}{2}} \leq \lambda_{k_0}^{-\frac{1}{2}} \Big(\int_\Omega |\nabla u|^2\Big)^{\frac{1}{2}}, \quad \forall u \in E_0^+$$

one gets, for any $u \in E_0^+ \cap \partial B_\rho$,

$$\begin{aligned} I(u) &\geq \frac{\rho^2}{2} - CW^+\Big(\alpha_s^{s+1}\|u\|^{s+1}\lambda_{k_0}^{\frac{(s+1)(a-1)}{2}}\Big) - \frac{\lambda}{2}\lambda_{k_0}^{-1}\|u\|^2 - C \\ &\geq \rho^2\Big(\frac{1}{2}\Big(1 - \frac{\lambda}{\lambda_{k_0}}\Big) - C\ \lambda_{k_0}^{\frac{(s+1)(a-1)}{2}}\rho^{s-1}\Big) - C, \end{aligned} \tag{2.4}$$

which easily implies (I_1).

Let us prove (I_2). We choose k_0 mutually disjoint balls $B_1, \dots, B_{k_0}$, where $B_i \subset \Omega^+$ and $\Omega^+ := \{x \in \Omega : W(x) > 0\}$, in such a way that $\Omega^+ \setminus \cup_{i=1}^{k_0} B_i \neq \emptyset$, and k_0 nonnegative functions $\varphi_1, \dots, \varphi_{k_0}$, $\varphi_i \in C_0^\infty(B_i)$ for $i = 1, \dots, k_0$. Then set $E^- = E_0^- := span\{\varphi_1, \dots, \varphi_{k_0}\}$. Therefore, $dim E_0^- - codim E_0^+ =$

1. Let $u \in E_0^- \setminus B_{R_0}$ for R_0 sufficiently large; hence, $u = \sum_{i=1}^{k_0} t_i\varphi_i,\ t_i \in \mathbb{R}$. For any $\lambda \in \mathbb{R}^+$, by (2.1),

$$\begin{aligned} I(u) &= \frac{1}{2}\Big(\int_\Omega |\nabla u|^2 - \lambda \int_\Omega |u|^2\Big) - \int_{\Omega^+} W^+(x)P(u) \\ &\leq \frac{C}{2}\sum_{i=1}^{k_0} |t_i|^2 ||\varphi_i||^2 - C\sum_{i=1}^{k_0} |t_i|^{(s+1)} \int_{\Omega^+} W^+(x)|\varphi_i|^{(s+1)} + C \qquad (2.5) \\ &\leq \frac{C}{2}\sum_{i=1}^{k_0} |t_i|^2 - C\Big(\sum_{i=1}^{k_0} |t_i|^2\Big)^{\frac{s+1}{2}} + C. \end{aligned}$$

Therefore, $I(u) < 0$ as $s+1 > 2$ and $R_0 = C\sum_{i=1}^{n} |t_i|^2 \to +\infty$. Now Theorem 2.1 applies and we get a critical value

$$\overline{\beta}_0 := \inf_{h\in\Gamma_0} \sup_{u\in E_0^-} I(h(u))$$

where $\Gamma_0 = \{h \in C^0(E,E) : h \text{ is odd},\ h(u) = u \text{ if } u \in E_0^-,\ \|u| \geq R_0\}$. Next we repeat the same procedure now working with $E_1^+ = V_{k_0}^\perp$ and

$$E_1^- = span\{\varphi_1, \ldots, \varphi_{k_0+1}\},$$

with $\varphi_{k_0+1} \in C_0^\infty(B_{k_0+1})$, the ball $B_{k_0+1} \subset \Omega^+$ being disjoint from B_i for $i = 1, \ldots, k_0$ and $\Omega^+ \setminus \cup_{i=1}^{k_0+1} \neq \emptyset$. We find a second critical value

$$\overline{\beta_1} := \inf_{h\in\Gamma_1} \sup_{u\in E_1^-} I(h(u)),$$

where Γ_1 is defined as Γ_0, with R_0 replaced by $R_1 > R_0$. Continuing in this fashion, we find infinitely many critical values $\overline{\beta_k}$, $k \geq 0$.

Now we want to obtain an estimate from below for $\overline{\beta_k}$. By (a) there exists $w \in \partial B_\rho \cap E_k^+ \cap h(E_k^-)$; therefore,

$$\sup_{u\in E_k^-} I(h(u)) \geq I(w) \geq \inf_{u\in\partial B_\rho \cap E_k^+} I(u). \qquad (2.6)$$

To obtain a lower bound for the right-hand side of (2.6), we observe that by (2.4) since $(1 - \frac{\lambda}{\lambda_{k_0}}) > 0$ we can choose $\rho = \rho(k_0)$ such that the coefficient of ρ^2 is 1/4; that is,

$$\Big(\frac{1}{2}\Big(1 - \frac{\lambda}{\lambda_{k_0}}\Big) - C\lambda_{k_0}^{\frac{(s+1)(a-1)}{2}} \rho^{s-1}\Big) = \frac{1}{4}. \qquad (2.7)$$

First of all this implies $\rho(k_0)^{s-1} = C\lambda_{k_0}^{\frac{(1-a)(s+1)}{2}}$; hence, for any $k \geq 0$

$$\rho(k_0+k)^{s-1} = C\lambda_{k_0+k}^{\frac{(1-a)(s+1)}{2}}.$$

Moreover, by (2.7) and (2.4), one finds that

$$I(u) \geq \frac{1}{4}\rho(k_0+k)^2 - C$$

for any $u \in \partial B_\rho \cap E_k^+$. Since $\lambda_j \geq Cj^{\frac{2}{N}}$ for large j (see [12]), we obtain

$$\overline{\beta_k} \geq C\lambda_{k_0+k}^{\frac{(1-a)(s+1)}{s-1}} \geq Ck^{\frac{(1-a)(s+1)2}{N(s-1)}}$$

for k large. We have proved that I possesses an unbounded sequence of critical values $\overline{\beta_k} = I(u_k)$ where u_k is a weak solution of $(\mathcal{P})$.

Now we are going to prove that $\{u_k\}$ is unbounded in E. Since $I'(u_k)u_k = 0$, for any $\lambda \in \mathbb{R}$ we have

$$\int_\Omega |\nabla u_k|^2 = \int_\Omega (\lambda u_k^2 + W(x)p(u_k)u_k) \geq 0;$$

hence,

$$\int_\Omega W^+(x)p(u_k)u_k + \int_\Omega \lambda u_k^2 \geq \int_\Omega W^-(x)p(u_k)u_k. \tag{2.8}$$

Let us observe that by $(p2)$ and (2.1)

$$\begin{aligned}&\int_\Omega W^+(x)p(u_k)u_k \leq a_1\int_\Omega W^+(x)|u_k| + a_2\int_\Omega W^+(x)|u_k|^{s+1}\\&\leq a_1W^+||u_k||_{L^2}^2 + \frac{a_2}{a_5}\int_\Omega W^+(x)P(u_k) + C.\end{aligned} \tag{2.9}$$

Therefore, by (2.8), (2.9), and $(p3)$

$$\begin{aligned}&(\lambda + a_1W^+)||u_k||_{L^2}^2 + \frac{a_2}{a_5}\int_\Omega W^+(x)P(u_k)\\&\geq (s+1)\int_\Omega W^-(x)P(u_k) - C;\end{aligned} \tag{2.10}$$

that is,

$$C_1||u_k||_{L^2}^2 + \int_\Omega W(x)P(u_k) \geq -C, \tag{2.11}$$

where $C_1 = (\lambda + a_1W^+)(\max\frac{a_2}{a_5}, s+1)^{-1}$. So (2.11) and the fact that

$$\overline{\beta_k} = \frac{1}{2}\int_\Omega (W(x)p(u_k)u_k - 2W(x)P(u_k)) \to +\infty$$

yield

$$\max(1, 2C_1 - \lambda)\,||u_k||^2 \geq \int_\Omega |\nabla u_k|^2 + (2C_1 - \lambda)\int_\Omega u_k^2 = \int_\Omega W(x)p(u_k)u_k$$
$$+ \int_\Omega \lambda u_k^2 + (2C_1 - \lambda)\int_\Omega u_k^2 + \int_\Omega 2W(x)P(u_k) - \int_\Omega 2W(x)P(u_k)$$
$$\geq -C + \int_\Omega (W(x)p(u_k)u_k - 2W(x)P(u_k)) \to +\infty.$$

Hence, $\{u_k\}$ must be unbounded in E.

3. The nonsymmetric case

This section deals with $(\mathcal{P}_f)$ where $f \in L^2(\Omega)$, $f \not\equiv 0$. The corresponding Euler functional is

$$I_f(u) = \int_\Omega \Big(\frac{1}{2}|\nabla u|^2 - \frac{\lambda}{2}u^2 - W(x)P(u) - f(x)u\Big), \; u \in H_0^1(\Omega). \tag{3.1}$$

Let us define the set $\mathcal{I}_\varepsilon = \{u \in H_0^1(\Omega) : I_f'(u) = 0,\ I_f(u) \geq \varepsilon > 0\}$, where ε is a sufficiently small positive number. We assume the following hypothesis:

(pW) $\exists \delta > 0 : 0 < \frac{\int_\Omega W^-(x)h(u_k)}{\int_\Omega W^+(x)h(u_k)} \leq 1 - \delta$, definitively $\forall \; \{u_k\} \in \mathcal{I}_\varepsilon$ where $h(u_k) = u_k p(u_k) - 2P(u_k)$.

Lemma 3.1. *If (pW) holds, there exists a constant $\tilde{C} > 0$ such that*

$$\tilde{C}\Big|\int_\Omega W(x)h(u)\Big| \geq \int_\Omega |W(x)|h(u) \quad \forall u \in \mathcal{I}_\varepsilon. \tag{3.2}$$

Obviously the constant $\tilde{C} \geq 1$.

Proof. By (pW)

$$0 < \limsup_{\{k: u_k \in \mathcal{I}_\varepsilon\}} \frac{\int_\Omega W^-(x)h(u_k)}{\int_\Omega W^+(x)h(u_k)} = K < 1.$$

So

$$\limsup_{\{k: u_k \in \mathcal{I}_\varepsilon\}} \frac{\int_\Omega W^+(x)h(u_k) + \int_\Omega W^-(x)h(u_k)}{\left|\int_\Omega W^+(x)h(u_k) - \int_\Omega W^-(x)h(u_k)\right|}$$
$$\leq \frac{1 + \limsup_{\{k: u_k \in \mathcal{I}_\varepsilon\}} \frac{\int_\Omega W^-(x)h(u_k)}{\int_\Omega W^+(x)h(u_k)}}{1 + \liminf_{\{k: u_k \in \mathcal{I}_\varepsilon\}} \Big(-\frac{\int_\Omega W^-(x)h(u_k)}{\int_\Omega W^+(x)h(u_k)}\Big)} = \frac{1+K}{1-K}. \qquad \square$$

We are going to prove the following theorem, which extends the result by Rabinowitz (see Theorem 10.4 of [18]) to the case of indefinite nonlinearities, following the same spirit:

Theorem 3.1. *If p and W satisfy* ($p1$)–($p4$), (pW), ($pW1$), (W), *and* $f \in L^2(\Omega)$, $f \not\equiv 0$, *then problem* $(\mathcal{P}_f)$ *possesses infinitely many solutions, provided that s in* ($p2$) *is further restricted by the condition* (1.1).

In order to prove Theorem 3.1 we will need some additional lemmas and propositions.

Lemma 3.1. *Under the hypotheses of Theorem* 3.1, *there exists a constant A depending on $||f||_{L^2}$ such that, if u belongs to $\mathcal{I}_\varepsilon$, then*

$$\int_\Omega |W(x)| \left(P(u) + a_4\right) \leq A \left(I_f(u)^2 + 1\right)^{\frac{1}{2}}. \tag{3.3}$$

Remark 3.1. If $f \equiv 0$ and P is a homogeneous function, one can show, without hypothesis (pW), that

$$\int_\Omega W(x)|u|^{s+1} \leq A \left(I_f(u)^2 + 1\right)^{\frac{1}{2}}. \tag{3.4}$$

This suggests that the estimate (3.3) may be too strong. However, the proofs carried out in the sequel will highlight that an estimate like (3.4) is not useful, because W changes sign.

Proof. Suppose u is a critical point of I_f; then by ($p3$) and by Remark 3.1,

$$\begin{aligned}
|I_f(u)| &= \Big|I_f(u) - \frac{1}{2}I_f'(u)u\Big| = \Big|\int_\Omega \frac{1}{2}W(x)p(u)u - W(x)P(u) - \frac{1}{2}fu\Big| \\
&\geq \frac{1}{2}\Big|\int_\Omega W(x)(p(u)u - 2P(u))\Big| - \frac{1}{2}\Big|\int_\Omega fu\Big| \\
&\geq \frac{1}{2\tilde{C}}\int_\Omega |W(x)|p(u)u - \frac{1}{\tilde{C}}\int_\Omega |W(x)|P(u) - \frac{1}{2}||f||_{L^2}||u||_{L^2} \\
&\geq \frac{s-1}{2\tilde{C}}\int_\Omega |W(x)|P(u) - \frac{1}{2}||f||_{L^2}||u||_{L^2} \\
&\geq \frac{s-1}{2\tilde{C}}\int_\Omega |W(x)|(P(u) + a_4) - C||u||^2_{L^2} - C.
\end{aligned} \tag{3.5}$$

Let us observe now that by the Hölder inequality,

$$||u||^2_{L^2} = \int_{supp\ W} \frac{1}{|W(x)|^{\frac{2}{s+1}}}|W(x)|^{\frac{2}{s+1}}|u|^2 \leq C\Big(\int_\Omega |W(x)||u|^{s+1}\Big)^{\frac{2}{s+1}}. \tag{3.6}$$

By the Young inequality, since $s+1>2$, for $\varepsilon>0$,

$$C\Big(\int_\Omega |W(x)||u|^{s+1}\Big)^{\frac{2}{s+1}} \le C(\varepsilon)+\varepsilon\int_\Omega |W(x)||u|^{s+1},$$

where $C(\varepsilon)\to\infty$ as $\varepsilon\to 0$. Hence by (3.5) and using (2.2), one finds that

$$|I_f(u)| \ge \left(\frac{s-1}{2\tilde{C}}-\frac{\varepsilon}{a_5}\right)\int_\Omega |W(x)|(P(u)+a_4)-C(\varepsilon)-C \tag{3.7}$$

so, choosing ε such that $(\frac{s-1}{2\tilde{C}}-\frac{\varepsilon}{a_5})>0$, the thesis follows. $\square$

Now let $\chi\in C^\infty(\mathbb{R},\mathbb{R})$ be such that $\chi(\xi)\equiv 1$ for $\xi\le 1$, $\chi(\xi)\equiv 0$ for $\xi\ge 2$, and $\chi'(\xi)\in(-2,0)$ for $\xi\in(1,2)$. Moreover, let us define $Q(u)\equiv 2A\left(I_f(u)^2+1\right)^{\frac{1}{2}}$ and

$$\psi(u)\equiv\chi\Big(Q(u)^{-1}\int_\Omega |W(x)|\left(P(u)+a_4\right)\Big).$$

Observe that, by Lemma 3.1, if $u\in\mathcal{I}_\varepsilon$, the argument of χ lies in $\left[0,\frac{1}{2}\right]$; therefore, $\psi(u)\equiv 1$. Finally, set

$$J_f(u)=\int_\Omega\Big(\frac{1}{2}|\nabla u|^2-\frac{\lambda}{2}u^2-W(x)P(u)-\psi(u)f(x)u\Big). \tag{3.8}$$

Then $J_f(u)=I_f(u)$ if u is a critical point of I_f with $I_f(u)\ge\varepsilon>0$. In the next proposition we will show the main properties of J_f.

Proposition 3.1. *Under the hypotheses of Theorem* 3.1 *the following properties of J_f hold:*

1° $J_f\in C^1(E,\mathbb{R})$;
2° *There exists a constant D depending on $\|f\|_{L^2}$ such that*

$$|J_f(u)-J_f(-u)|\le D(|J_f(u)|^{\frac{1}{s+1}}+1)\quad \textit{for all } u\in E;$$

3° *There exists a constant $M_0>0$ such that if $J_f(u)\ge M_0$ and $J_f'(u)=0$, then $J_f(u)=I_f(u)$ and $I_f'(u)=0$;*
4° *There exists a constant $M_1\ge M_0$ such that, for any $c>M_1$, J_f satisfies $(PS)_{loc}$ at c.*

Proof. 1° It is true by standard arguments.

2° Observe that

$$\Big|\int_\Omega fu\Big|\le\alpha_1\left(|I_f(u)|^{\frac{1}{s+1}}+1\right). \tag{3.9}$$

Indeed, since (W) holds, by (2.2), one gets

$$\begin{aligned}\Big|\int_\Omega fu\Big| &= \Big|\int_{supp\ W} \frac{f}{W(x)^{\frac{1}{s+1}}} W(x)^{\frac{1}{s+1}} u\Big| \le \Big\|fW(x)^{-\frac{1}{s+1}}\Big\|_{L^2}\Big\|W(x)^{\frac{1}{s+1}}u\Big\|_{L^2}\\ &\le C\Big\|W(x)^{\frac{1}{s+1}}u\Big\|_{L^{s+1}} \le C\Big(\int_\Omega |W(x)|(P(u)+a_4)\Big)^{\frac{1}{s+1}}. \qquad (3.10)\end{aligned}$$

Now, if $u \in$ *supp* ψ, one has

$$\int_\Omega |W(x)|\left(P(u)+a_4\right) \le 2Q(u) = 4A\left(I_f(u)^2+1\right)^{\frac{1}{2}} \le C\left(|I_f(u)|+1\right),$$

so the last inequality and (3.10) yield (3.9).
Now let us estimate the "lack" of symmetry of J_f.

$$|J_f(u)-J_f(-u)| = \Big|\int_\Omega (-\psi(u)fu+\psi(-u)fu)\Big| \le (\psi(-u)+\psi(u))\Big|\int_\Omega fu\Big|. \qquad (3.11)$$

Observe that, by (3.9), one has

$$\psi(u)\Big|\int_\Omega fu\Big| \le \psi(u)\alpha_1\left(|I_f(u)|^{\frac{1}{s+1}}+1\right), \qquad (3.12)$$

so, by the definitions of $I_f,\ J_f$, and by (3.12), we deduce

$$\begin{aligned}|I_f(u)| &= \Big|\int_\Omega \frac{1}{2}|\nabla u|^2 - \frac{\lambda u^2}{2} - W(x)P(u) - fu - \psi fu + \psi fu\Big|\\ &\le |J_f(u)| + |\psi-1|\Big|\int_\Omega fu\Big| \le |J_f(u)| + \Big|\int_\Omega fu\Big|.\end{aligned} \qquad (3.13)$$

So, by (3.12), (3.13), and using the Young inequality,

$$\psi(u)\Big|\int_\Omega fu\Big| \le C\psi(u)\Big(|J_f(u)|^{\frac{1}{s+1}} + \Big|\int_\Omega fu\Big|^{\frac{1}{s+1}}+1\Big) \le C\Big(|J_f(u)|^{\frac{1}{s+1}}+1\Big). \qquad (3.14)$$

Hence, using (3.14) and a similar estimate for $\psi(-u)$, together with (3.11), yields $2°$.

To prove $3°$ it suffices to prove that for M_0 large, if u is critical for J_f with $J_f(u) \ge M_0$, then

$$Q(u)^{-1}\int_\Omega |W(x)|\left(P(u)+a_4\right) < 1. \qquad (3.15)$$

The definition of ψ then implies $\psi(v) \equiv 1$ for v near u. Hence, $\psi'(u) = 0$, so $J_f(u) = I_f(u),\ J'_f(u) = I'_f(u)$, and $3°$ will follow. Actually by the definition of J_f one gets

$$J_f'(u)u = \int_\Omega \left[|\nabla u|^2 - \lambda u^2 - W(x)p(u)u - (\psi(u) + \psi'(u)u)fu\right]. \qquad (3.16)$$

Setting $\theta(u) = Q(u)^{-1} \int_\Omega |W(x)| \, (P(u) + a_4)$, one has

$$\begin{aligned}\psi'(u)u &= \chi'(\theta(u))\theta'(u)u \\ &= \chi'(\theta(u))Q(u)^{-2}\Big[Q(u)\int_\Omega |W(x)|p(u)u - uQ'(u)\int_\Omega |W(x)|(P(u)+a_4)\Big] \\ &= \chi'(\theta(u))Q(u)^{-2}\Big[Q(u)\int_\Omega |W(x)|p(u)u - (2A)^2 I_f(u)I_f'(u)u\theta(u)\Big].\end{aligned}$$

Set now

$$T_1(u) = \chi'(\theta(u))(2A)^2 Q(u)^{-2} I_f(u)\theta(u)\int_\Omega fu,$$

$$T_2(u) = \chi'(\theta(u))Q(u)^{-1}\int_\Omega fu.$$

One gets

$$\begin{aligned}J_f'(u)u &= (1+T_1(u))\int_\Omega |\nabla u|^2 - \lambda(1+T_1(u))\int_\Omega u^2 \qquad (3.17)\\ &-(1+T_1(u))\int_\Omega W(x)p(u)u - T_2\int_\Omega |W(x)|p(u)u \quad -(\psi(u)+T_1(u))\int_\Omega fu.\end{aligned}$$

Consider now

$$J_f(u) - \frac{1}{2(1+T_1(u))}J_f'(u)u. \qquad (3.18)$$

If $\psi(u) = 1$ and $T_1(u) = T_2(u) = 0$ then (3.18) reduces to the left-hand side of (3.5), so (3.15) is a consequence of (3.3). Since $0 \le \psi(u) \le 1$, if $T_1(u)$ and $T_2(u)$ are small enough, the calculation made in (3.5) can be used in (3.18) to prove a similar estimate, with A replaced by a larger constant, still smaller than $2A$, so again (3.15) holds. Therefore, it suffices to show that $T_1(u),\ T_2(u) \to 0$ as $M_0 \to \infty$. If $u \notin$ *supp* ψ then $T_1(u) = T_2(u) = 0$, so we assume $u \in$ *supp* ψ. By (3.9) and the definition of T_1 and χ,

$$\begin{aligned}|T_1(u)| &\le 2(2A)^2\frac{|I_f(u)|}{(2A)^2|I_f(u)^2+1|}\theta(u)\Big|\int_\Omega fu\Big| \\ &\le 4\alpha_1|I_f(u)|^{-1}\big(|I_f(u)|^{\frac{1}{s+1}}+1\big). \qquad (3.19)\end{aligned}$$

Now, if $u \in \ supp\psi$, by the definition of J_f and I_f one gets

$$I_f(u) \geq J_f(u) - \Big| \int_\Omega fu \Big|.$$

Thus, by (3.9)

$$I_f(u) + \alpha_1 |I_f(u)|^{\frac{1}{s+1}} \geq J_f(u) - \alpha_1 \geq \frac{M_0}{2} \tag{3.20}$$

for M_0 to be chosen large enough. Therefore, if $I_f(u) \leq 0$, (3.20) and the Young inequality with $(s+1)' = \nu$ imply

$$\frac{\alpha_1^\nu}{\nu} + \frac{|I_f(u)|}{s+1} \geq \frac{M_0}{2} + |I_f(u)|,$$

which is impossible if we choose $M_0 \geq \frac{2\alpha_1^\nu}{\nu}$. Hence, $I_f(u) > 0$. Therefore, (3.20) yields $I_f(u) \geq \frac{M_0}{4}$ or $I_f(u) \geq (\frac{M_0}{4\alpha_1})^{s+1}$. So in both cases $I_f(u) \to +\infty$ as $M_0 \to +\infty$, which together with (3.19) shows $T_1(u) \to 0$ as $M_0 \to +\infty$, and using the same argument $T_2(u) \to 0$ as $M_0 \to +\infty$. This concludes the proof of $3°$.

The check of $4°$ follows the same lines as the previous proof. It suffices to show that there exists $M_1 > M_0$ such that if $\{u_m\} \in E = H_0^1(\Omega)$, $M_1 \leq J_f(u_m) \leq K$, and $J_f'(u_m)u_m \to 0$, then $\{u_m\}$ is bounded. Let us fix $m \in \mathbb{N}$ such that $u_m \in$ *supp* ψ. Indeed, if $u_m \notin$ *supp* ψ, then $||u_m||$ is bounded since the functional I_f is symmetric and the (PS) condition was proved in [14]. For large m and $\rho > 0$ one has

$$\begin{aligned} &\rho ||u_m|| + K \geq J_f(u_m) - \rho J_f'(u_m)u_m \\ &= \Big(\frac{1}{2} - \rho(1+T_1(u_m))\Big)\left(||u_m||^2 - \lambda ||u_m||_{L^2}^2\right) \\ &+ [\rho(\psi(u_m) + T_1(u_m)) - \psi(u_m)] \int_\Omega f u_m - \int_\Omega P(u_m)W(x) \\ &+ \rho(1+T_1(u_m)) \int_\Omega u_m p(u_m)W(x) + \rho T_2(u_m) \int_\Omega u_m p(u_m)|W(x)|. \end{aligned} \tag{3.21}$$

For M_1 sufficiently large, and therefore T_1, T_2 small, we can choose $\rho \in ((s+1)^{-1}, 2^{-1})$ and $\varepsilon > 0$ such that

$$\frac{1}{2(1+T_1(u_m))} > \rho + \varepsilon > \rho - \varepsilon > \frac{1}{(s+1)(1+T_2(u_m))}. \tag{3.22}$$

Hence, one finds that

$$\Big(\frac{1}{2} - \rho(1+T_1(u_m))\Big)\left(||u_m||^2 - \lambda ||u_m||_{L^2}^2\right) \geq \frac{\varepsilon}{2}||u_m||^2 - C\lambda ||u_m||_{L^2}^2 \tag{3.23}$$

and

$$[\rho(\psi(u_m)+T_1(u_m))-\psi(u_m)]\int_\Omega fu_m \geq -C||u_m||^2_{L^2}. \tag{3.24}$$

We are going to estimate the last three integrals in (3.21).

$$\begin{aligned}
&-\int_\Omega P(u_m)W(x)+\rho(1+T_1(u_m))\int_\Omega u_m p(u_m)W(x)\\
&+\rho T_2(u_m)\int_\Omega u_m p(u_m)|W(x)|\\
&\geq -\int_\Omega P(u_m)|W(x)|+\rho(1+T_1(u_m))\int_\Omega u_m p(u_m)W(x)\\
&+\rho T_2(u_m)\int_\Omega u_m p(u_m)|W(x)|+\rho\int_\Omega u_m p(u_m)|W(x)|\\
&-\rho\int_\Omega u_m p(u_m)|W(x)|-\rho(1+T_2(u_m))(s+1)\int_\Omega |W(x)|P(u_m)\\
&+\rho(1+T_2(u_m))(s+1)\int_\Omega |W(x)|P(u_m)\\
&=\rho(1+T_2(u_m))\int_\Omega (u_m p(u_m)-(s+1)P(u_m))|W(x)|\\
&+(\rho(1+T_2(u_m))(s+1)-1)\int_\Omega |W(x)|P(u_m)\\
&+\rho(1+T_1(u_m))\int_\Omega u_m p(u_m)W(x)-\rho\int_\Omega u_m p(u_m)|W(x)|.
\end{aligned} \tag{3.25}$$

By ($p3$) the first integral is positive, while by (2.1) the second integral is greater than

$$-C+\frac{\varepsilon}{2}a_5\int_\Omega |W(x)||u_m|^{s+1}. \tag{3.26}$$

For what concerns the third integral one finds, by ($p3$) and ($pW1$),

$$\begin{aligned}
&\rho(1+T_1(u_m))\int_\Omega u_m p(u_m)(W^+(x)-W^-(x))\\
&\rho(1+T_1(u_m))\int_\Omega (s+1)P(u_m)W^+(x)-\gamma\rho(1+T_1(u_m))\int_\Omega u_m^2-C\\
&-\rho(1+T_1(u_m))\int_\Omega (s+1)P(u_m)W^-(x)\\
&\rho(1+T_1(u_m))\int_\Omega (s+1)P(u_m)W(x)-C\ ||u_m||^2_{L^2}-C
\end{aligned}$$

$$+C||u_m||^2 - C\ ||u_m||^2_{L^2} - C. \tag{3.27}$$

It remains to estimate the last integral in (3.25). By ($p2$) and ($p3$) one has

$$\begin{aligned}-\rho \int_\Omega u_m p(u_m)|W(x)| &\geq -C \int_\Omega |u_m| - C \int_\Omega P(u_m)|W(x)| - C \\ -C\ ||u_m||_{L^2} &- 4AC(|I_f(u_m)|^2 + 1)^{\frac{1}{2}} - C,\end{aligned} \tag{3.28}$$

where we have used the fact that $u_m \in supp\ \psi$. Let us observe now that

$$I_f(u_m) = J_f(u_m) + (\psi - 1) \int_\Omega f u_m \leq K + \Big| \int_\Omega f u_m \Big| \leq K + C\|u_m\|^2_{L^2}; \tag{3.29}$$

hence,

$$-\rho \int_\Omega u_m p(u_m)|W(x)| \geq -C\ \ - C\|u_m\|^2_{L^2}. \tag{3.30}$$

So from (3.21), using (3.23), (3.24), (3.26), (3.27), and (3.30), we get

$$\rho||u_m|| + K \geq C||u_m||^2 + C \int_\Omega |W(x)||u_m|^{s+1} - C\ ||u_m||^2_{L^2} - C. \tag{3.31}$$

Let us observe now that by the Hölder inequality,

$$||u||^2_{L^2} = \int_{supp\ W} \frac{1}{|W(x)|^{\frac{2}{s+1}}} |W(x)|^{\frac{2}{s+1}} |u|^2 \leq C\Big(\int_\Omega |W(x)||u|^{s+1}\Big)^{\frac{2}{s+1}}. \tag{3.32}$$

By the Young inequality, since $s + 1 > 2$, for $\varepsilon > 0$,

$$C\Big(\int_\Omega |W(x)||u|^{s+1}\Big)^{\frac{2}{s+1}} \leq C(\varepsilon) + \varepsilon \int_\Omega |W(x)||u|^{s+1}, \tag{3.33}$$

where $C(\varepsilon) \to \infty$ as $\varepsilon \to 0$. By (3.31)–(3.33), one has

$$\rho||u_m|| + C \geq C||u_m||^2 + C \int_\Omega |W(x)||u_m|^{s+1} - C(\varepsilon) - \varepsilon \int_\Omega |W(x)||u|^{s+1}.$$

Choosing ε small enough, one finds

$$\rho||u_m|| + C \geq C||u_m||^2;$$

that is, $\{u_m\}$ is bounded in E. Then by standard arguments $4°$ holds. □

In order to obtain an unbounded sequence of critical values for the functional I_f, by $3°$ of Proposition 3.1, it suffices to prove that J_f possesses multiple critical values. Let us introduce the following minimax values:

$$b_k := \inf_{h \in \Gamma_k} \sup_{u \in E_k^-} J_f(h(u)),$$

where Γ_k and E_k^- are defined as in the proof of Theorem 2.1. These minimax values will not in general be critical for J_f unless $f \equiv 0$. Since the presence of the ψ term in J_f does not affect the verification of $(I_1), (I_2)$, arguing as in the proof of Theorem 2.1, we obtain the following lower bound for b_k: there exists a constant C and $\tilde{k} \in \mathbb{N}$ such that

$$b_k \geq C\ k^\theta \quad \text{for all } k \geq \tilde{k}, \tag{3.34}$$

where θ was defined in (1.1).

To get critical values of J_f from the sequence $\{b_k\}$, another set of minimax values must be introduced. Define

$$U_k = \{u = t\varphi_{k+1} + w : t \geq 0,\ w \in E_k^-\},$$

$$\Lambda_k = \{h \in C(U_k, E),\ h \text{ is odd},\ h = \text{id on } Q_k \equiv (E_k^- \setminus \mathring{B}_{R_k}) \cup (E_{k+1}^- \setminus \mathring{B}_{R_{k+1}})\}.$$

Set

$$c_k := \inf_{h \in \Lambda_k} \sup_{u \in U_k} J_f(h(u)).$$

Comparing the definition of c_k and b_k it is easy to check that $c_k \geq b_k$.

Proposition 3.1. *Assume $c_k > b_k \geq M_1$. For $\delta \in (0, c_k - b_k)$ define*

$$\Lambda_k(\delta) := \{h \in \Lambda_k :\ J_f(h(u)) \leq b_k + \delta \text{ for } u \in E_k^-\}$$

and

$$c_k(\delta) := \inf_{h \in \Lambda_k(\delta)} \sup_{u \in U_k} J_f(h(u)). \tag{3.35}$$

Then $c_k(\delta)$ is a critical value of J_f.

Proof. This proof follows the same lines as Proposition 10.43 of Rabinowitz [18], using 1° and 4° of Proposition 3.1. □

Now, if $c_k > b_k$ for a sequence of $k \to \infty$, then by Proposition 3.1 and by (3.34), J_f has an unbounded sequence of critical values and the proof is complete. It remains to show that the relation $c_k = b_k$ is impossible for all large k.

Proposition 3.1. *If $c_k = b_k$ for all $k \geq k^*$, there exists a constant $C > 0$, and $\hat{k} \geq k^*$ such that*

$$b_k \leq C k^{\frac{s+1}{s}} \tag{3.36}$$

for all $k > \hat{k}$.

Comparing (3.36) to (3.34) and (1.1), one gets a contradiction, and the proof of Theorem 3.1 is complete.

Proof of Proposition 3.1. Let $\varepsilon > 0$ and $k \geq k^*$. Choose $h \in \Lambda_k$ such that

$$\sup_{u \in U_k} J_f(h(u)) \leq b_k + \varepsilon. \tag{3.37}$$

Since $U_k \cup (-U_k) = E^-_{k+1}$, h can be continuously extended to E^-_{k+1} as an odd function. Therefore, by definition of b_k

$$b_{k+1} \leq \sup_{E^-_{k+1}} J_f(h(u)) = \max_{B_{R_{k+1}} \cap E^-_{k+1}} J_f(h(u)) = J_f(h(w)) \tag{3.38}$$

for some $w \in B_{R_{k+1}} \cap E^-_{k+1}$. If $w \in U_k$ by (3.37) and (3.38),

$$J_f(h(w)) \leq b_k + \varepsilon. \tag{3.39}$$

Suppose $w \in -U_k$. Then, since by (3.34) $b_k \to \infty$ as $k \to \infty$, (3.38) and 2° of Proposition 3.1 imply $J_f(-h(w)) > 0$ if k is large, say $k \geq \hat{k}$. By 2°, the oddness of h and (3.37), one gets

$$\begin{aligned} J_f(h(w)) = J_f(-h(-w)) &\leq J_f(h(-w)) + D\left((J_f(h(-w)))^{\frac{1}{s+1}} + 1\right) \\ &\leq b_k + \varepsilon + D((b_k + \varepsilon)^{\frac{1}{s+1}} + 1). \end{aligned} \tag{3.40}$$

Combining (3.38)–(3.40) one finds

$$b_{k+1} \leq b_k + \varepsilon + D((b_k + \varepsilon)^{\frac{1}{s+1}} + 1). \tag{3.41}$$

Since ε is arbitrary, (3.41) implies

$$b_{k+1} \leq b_k + D((b_k)^{\frac{1}{s+1}} + 1) \tag{3.42}$$

for all $k \geq \hat{k}$. Arguing as in Proposition 10.46 of Rabinowitz [18], (3.42) implies the thesis by induction. □

References

[1] S. Alama and M. del Pino, *Solutions of elliptic equations with indefinite nonlinearities via Morse theory and linking,* Ann. I.H.P. Analyse non lineaire, 13 (1996), 95–115.

[2] S. Alama and G. Tarantello, *On semilinear elliptic equations with indefinite nonlinearities,* Calc. Var., 1 (1993), 439–475.

[3] S. Alama and G. Tarantello, *Elliptic problems with nonlinearities indefinite in sign,* J. Funct. An., 141 (1996), 159–215.

[4] A. Ambrosetti and P.H. Rabinowitz, *Dual variational methods in critical point theory and applications,* J. Funct. An., 14 (1973), 349–381.

[5] M. Badiale, *Infinitely many solutions for some indefinite nonlinear elliptic problems,* Comm. on Appl. Non. Anal., 3 (1996), 61–76.

[6] M. Badiale and E. Nabana, *A remark on multiplicity solutions for semilinear elliptic problems with indefinite nonlinearities*, C.R. Acad. Sc. Paris, 323, serie 1 (1996), 151–156.

[7] A. Bahri and H. Berestycki, *A perturbation method in critical point theory and applications*, Trans. A.M.S., 267 (1981), 1–32.

[8] A. Bahri and P.L. Lions, *Morse index of some min-max critical points I. Application to multiplicity results*, Commun. Pure & App. Math., 41 (1988), 1027–1037.

[9] H. Berestycki, I. Capuzzo Dolcetta, and L. Nirenberg, *Variational methods for indefinite superlinear homogeneous elliptic systems,* NoDEA, 2 (1995), 533–572.

[10] H. Berestycki, I. Capuzzo Dolcetta, and L. Nirenberg, *Superlinear indefinite elliptic problems and nonlinear Liouville theorems,* Topological Methods in Nonlinear Analysis, 4 (1994), 59–78.

[11] P. Bolle, N. Ghoussoub, and H. Tehrani, *The multiplicity of solutions in nonhomogeneous boundary value problems*, Manuscripta Math., 101 (2000), 325–350.

[12] R. Courant and D. Hilbert, "Methods of Mathematical Physics," Vols. I, II, Interscience, New York, 1953 and 1962.

[13] A. Friedman, "Partial Differential Equations," Holt, Rinehart, and Winston, Inc., New York, 1969.

[14] M. Grossi, P. Magrone, and M. Matzeu, *Linking type solutions for elliptic equations with indefinite nonlinearities up to the critical growth*, Disc. Cont. Dyn. Sys., 7 (2001), 703–718.

[15] P. Magrone, *On a class on semilinear elliptic equations with potential changing sign*, Dynamical Systems and Appl., 9 (2000), 459–468.

[16] L. Nirenberg, *On elliptic partial differential equations*, Ann. Sc. Norm. Sup. Pisa., 13 (1959), 1–48.

[17] T. Ouyang, *On positive solutions of semilinear equations* $\Delta u + \lambda u + hu^p = 0$ *on compact manifolds*, part II, Indiana Univ. Math. J., 40 (1991), 1083–1141.

[18] P. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations,* Regional Conference Series in Mathematics, 65 (1984).

[19] P. Rabinowitz, *Multiple critical points of perturbed symmetric functionals*, Trans. A.M.S., 272 (1982), 753–770.

[20] M. Ramos, S. Terracini, and C. Troestler, *Superlinear indefinite elliptic problems and Pohozaev type identities,* J. Funct. Anal., 159 (1998), 596–628.

[21] M. Struwe, *Infinitely many critical points for functionals which are not even and applications to superlinear boundary value problems*, Arch. Math., 36 (1981), 360–369.

[22] H.T. Tehrani, *On indefinite superlinear elliptic equations*, Calc. Var. P.D.E. 4, 2 (1996), 139–153.

[23] H.T. Tehrani, *Infinitely many solutions for indefinite semilinear elliptic equations without symmetries*, Comm. in PDE, 21 (1996), 541–557.