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On the problem of the existence for connecting trajectories under the action of gravitational and electromagnetic fields*

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Abstract: We give sufficient conditions assuring the existence of timelike trajectories connecting two prescribed events in a Lorentzian manifold. They represent the trajectories of a free falling massive particle under the action of a gravitational and electromagnetic field.

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1. Introduction and statement of the results

Let $(\mathcal{M}, g)$ be a Lorentzian manifold. In this paper we first point out how can be faced the existence of timelike trajectories joining two fixed points z_0, z_1 of a region $\{z \in \mathcal{M} : a < T(z) < b\}$ where T is a smooth time function, assuming that its boundary is convex. From a physical point of view we can interpret $\mathcal{M}$ as the space-time where the information about the gravitational field are "included" in the metric tensor g, while the action of the electromagnetic field is given by a smooth vector field A. The trajectories connecting the couple of events are the free falling trajectories of a material point z. The fundamental equation of Classical Physics related to the motion of z inside a gravitational and an electromagnetic field is the Euler–Lagrange equation related to the action functional

$$F(z) = -m_0 c \int_{t_0}^{t_1} \sqrt{-\langle \dot{z}, \dot{z} \rangle}\, \mathrm{d}t + q \int_{t_0}^{t_1} \langle A(z), \dot{z} \rangle\, \mathrm{d}t, \tag{1.1}$$

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(see [8]) where m_0 is the rest mass of the particle, q is its charge (and we shall assume $q = \pm 1$), c is the speed of light, $A(z)$ gives the action of the electromagnetic field and $\langle \cdot\,, \cdot \rangle = g(z)[\cdot\,, \cdot]$.

To obtain critical points of the functional F one can look for the critical points of the functional

$$\mathcal{S} = \frac{1}{2}\int_{\sigma_0}^{\sigma_1} \langle \dot{z}, \dot{z}\rangle\,\mathrm{d}\sigma + \int_{\sigma_0}^{\sigma_1} \langle A(z), \dot{z}\rangle\,\mathrm{d}\sigma, \tag{1.2}$$

satisfying $\langle \dot{z}(\sigma), \dot{z}(\sigma)\rangle < 0$, for any σ (cf. Remark 2.2). The functional (1.2) was introduced in [5] to study some fundamental equations in General Relativity.

The existence of critical points for (1.2) has been studied by several authors, but just in the case that $A(z) \equiv 0$ (see [4] and references therein). The presence of $A(z) \neq 0$ makes the problem more complicate. As far as we know the only existence results for critical points of $\mathcal{S}$ are on standard static manifolds (see [2]).

In this paper we assume that the manifold $\mathcal{M}$ has a smooth time function, $T : \mathcal{M} \longrightarrow \mathbb{R}$ namely satisfying

$$\langle \nabla T(z), \nabla T(z)\rangle < 0, \qquad \forall z \in \mathcal{M}.$$

Here $\nabla T(z)$ is the Lorentzian gradient of T defined by

$$dT(z)[\zeta] = \langle \nabla T(z), \zeta\rangle, \qquad \forall \zeta \in T_z\mathcal{M}.$$

The study of critical points of $\mathcal{S}$ will be done under intrinsic assumptions on the function T. Set

$$W(z) = \frac{\nabla T(z)}{\sqrt{-\langle \nabla T(z), \nabla T(z)\rangle}}. \tag{1.3}$$

By the help of $W(z)$ we can define a natural Riemannian metric on $\mathcal{M}$ (see [1]) setting

$$\langle \zeta, \zeta_1\rangle_R = \langle \zeta, \zeta_1\rangle + 2\langle W(z), \zeta\rangle\langle W(z), \zeta_1\rangle. \tag{1.4}$$

(We can easily prove that (1.4) is a Riemannian metric, using the wrong way Schwartz inequality, see [10].) For any fixed constants $a, b \in \mathbb{R}$ with $a < b$, let us consider the strip

$$\mathcal{M}_{a,b} = \big\{z \in \mathcal{M} : a < T(z) < b\big\}.$$

Our assumptions are the following:

$$\textit{the metric } \langle \cdot\,, \cdot\rangle \textit{ is complete in } \mathcal{M}_{a,b}. \tag{1.5}$$

Let

$$\beta = \beta(z) = \frac{1}{\langle \nabla^R T(z), \nabla^R T(z)\rangle_R} = -\frac{1}{\langle \nabla T(z), \nabla T(z)\rangle}$$

(see Lemma A.1) be such that

$$\exists \nu, N > 0 : \quad \nu \leqslant \beta(z) \leqslant N \qquad \forall z \in \mathcal{M}_{a,b} \tag{1.6}$$

(here ∇^R represents the gradient with respect to the Riemannian metric). Denote by H^T the hessian of T with respect to the Lorentzian metric $\big(\langle H^T(z)\zeta, \zeta\rangle = \mathrm{d}^2/\mathrm{d}s^2(T(\gamma(s)))_{/s=0}$ where γ is a geodesic such that $\gamma(0) = z$ and $\dot{\gamma}(0) = \zeta\big)$. We assume that

$$\exists K > 0 : \quad \|H^T(z)\|_R \leqslant K \qquad \forall z \in \mathcal{M}_{a,b}, \tag{1.7}$$

where $\|\cdot\|_R = \sqrt{\langle\cdot,\cdot\rangle_R}$.

$$\exists\ A_0, A_1 \in \mathbb{R} : \|A\|_R \leqslant A_0 \ \text{ and } \ \|dA\|_R \leqslant A_1 \quad \text{ on } \mathcal{M}. \tag{1.8}$$

There exists $\delta > 0$:

$$\langle H^T(z)\zeta, \zeta\rangle < 0 \quad \forall z \in T^{-1}(]a, a+\delta[), \forall \zeta \in T_z\mathcal{M}, \text{ with } \langle \zeta, \nabla T(z)\rangle = 0, \tag{1.9}$$

$$\langle H^T(z)\zeta, \zeta\rangle > 0 \quad \forall z \in T^{-1}(]b-\delta, b[), \forall \zeta \in T_z\mathcal{M}, \text{ with } \langle \zeta, \nabla T(z)\rangle = 0, \tag{1.10}$$

and

$$[dA^* - dA](z)[\zeta] \equiv 0 \quad \forall \zeta \in T_z\mathcal{M}, \forall z \in T^{-1}(]a, a+\delta[) \cup T^{-1}(]b-\delta, b[), \\ \text{with } \langle \zeta, \nabla T(z)\rangle = 0, \tag{1.11}$$

where dA is the covariant differential of A and dA^* is the adjoint operator of dA.

Finally we need the following assumption, giving the Saddle Point structure for the functional $\mathcal{S}$: there exists $\theta \in (0, 2)$ and two continuous maps $c(z)$, $d(z)$ not depending on $T(z)$ such that

$$\left|\langle\zeta,\zeta\rangle_R - \beta(z)\langle\zeta,\nabla T(z)\rangle^2\right| \leqslant \left[c(z) + d(z)\,|T(z)|^\theta\right]\langle\zeta,\zeta\rangle_R. \tag{1.12}$$

Remark 1.1. Observe that conditions (1.9) and (1.10) are equivalent to the strict convexity of the boundary of $\mathcal{M}_{a,b}$ (cf. [9])

The main result of the paper is the following

Theorem 1.2. *Assume* (1.5)–(1.12). *Then for any fixed z_0 and z_1 there exists a solution of the Euler–Lagrange equation corresponding to the functional* (1.2) *connecting z_0 with z_1.*

Remark 1.3. Whenever $A \equiv 0$, Theorem 1.2 gives the results proved in [9], under non-intrinsic hypothesis.

Remark 1.4. Using the a priori estimates in Section 4 and relative category as in [6] allows us to get, under the assumptions of Theorem 1.2, that there exists a sequence $\{z_n\}$ of critical points of $\mathcal{S}$ such that $\{z_n\} \to +\infty$. Note that the result of Theorem 1.2 has only a geometrical meaning but not yet a physical interpretation. Indeed while we are able to find critical points where $\mathcal{S}$ is strictly negative (if $|T(z_1) - T(z_0)|$ is sufficiently large), we cannot conclude that they are time-like. This is due to the particular conservation law satisfied by the critical points of $\mathcal{S}$ (see Proposition 2.1). The presence of the term $\int_0^1 \langle A(z), \dot z\rangle$ carries such difficulty, together with many others related to the a priori estimates. However the proof of Theorem 1.2 is a first step in the search of time-like critical curves for the functional $\mathcal{S}$. We hope that the techniques used in that proof will allow also to guarantee the existence of the time-like solutions.

Remark 1.5. For the proof of the existence of time-like critical curves of F the situation is completely different with respect to the case $A \equiv 0$, where the global hyperbolicity assures the existence of a causal critical curve of F, namely a causal geodesic connecting two given events (see, e.g., [3]). Indeed, since both integrals in F are positively homogeneous of the same degree with respect to $\dot z$, if $A \neq 0$, global hyperbolicity is not sufficient to obtain a priori estimates for $\dot z$ even if we use the time coordinate to parameterize the admissible paths.

2. The variational principle

Denote with $H^{1,2}([0,1],\mathcal{M})$ the space

$$H^{1,2}([0,1],\mathcal{M}) = \left\{ z : [0,1] \longrightarrow \mathcal{M} : z \in AC([0,1],\mathcal{M}) \text{ and } \int_0^1 \langle \dot z, \dot z\rangle_R \, \mathrm{d}s < +\infty \right\},$$

where $AC([0,1],\mathcal{M})$ is the set of absolutely continuous curves on $\mathcal{M}$, and $\langle \cdot\,,\cdot\rangle$ is defined in (1.4). Define

$$\Omega^{1,2} = \left\{ z \in H^{1,2}([0,1],\mathcal{M}) : z(0) = z_0, z(1) = z_1 \right\}$$

and

$$\Omega^{1,2}_{a,b} = \left\{ z \in \Omega^{1,2} : z([0,1]) \subset \mathcal{M}_{a,b} \right\} \tag{2.1}$$

where $\mathcal{M}_{a,b}$ is defined in Section 1. It is well known (see, e.g., [10]) that $\Omega^{1,2}$ is a Hilbert submanifold of $H^{1,2}([0,1],\mathcal{M})$ and its tangent space at $z \in \Omega^{1,2}$ is given by

$$T_z\Omega^{1,2} = \left\{ \zeta \in H^{1,2}([0,1], T_z\mathcal{M}) : \zeta(s) \in T_{z(s)}\mathcal{M} \;\; \forall s \in [0,1], \;\; \zeta(0) = \zeta(1) = 0 \right\},$$

while the Hilbert structure is

$$\langle \zeta, \zeta\rangle_1 = \int_0^1 \langle D^R_s \zeta, D^R_s \zeta\rangle \, ds. \tag{2.2}$$

In order to prove Theorem 1.2 we need the following simple result which gives the equation satisfied by the critical points of $\mathcal{S}$.

Proposition 2.1. *If z is a critical point of $\mathcal{S}$ on $\Omega^{1,2}$, then $z \in C^2([0,1])$ and satisfies the equation*

$$D_s \dot z + dA(z)[\dot z] - dA^*(z)[\dot z] = 0. \tag{2.3}$$

Moreover $\langle \dot z, \dot z\rangle = \text{const}.$

Proof. If z is a critical point of the functional $\mathcal{S}$, then

$$\int_0^1 \langle \dot z + A(z), D_s\zeta\rangle = -\int_0^1 \big\langle (dA(z))^*[\dot z], \zeta \big\rangle \quad \forall \zeta \in T_z\Omega^{1,2}$$

and integrating by parts the right-hand side member, since $\zeta(0) = \zeta(1) = 0$, we get

$$\int_0^1 \left\langle \dot z + A(z) - \left[\int_0^s \big((dA(z(r))\big)^*[\dot z(r)]\, \mathrm{d}r \right], D_s\zeta \right\rangle = 0 \quad \forall \zeta \in T_z\Omega^{1,2}. \tag{2.4}$$

By (2.3) we deduce that $\dot z + A(z) - \left[\int_0^s \big((dA(z(r))\big)^*[\dot z(r)]\, \mathrm{d}r \right]$ is of class C^1. Then $\dot z$ is a continuous curve and applying again (2.4), $\dot z$ is of class C^1. Finally, since dA^* is the adjoint of the operator dA, multiplying (2.3) by $\dot z$, we obtain $\langle D_s \dot z, \dot z\rangle = 0$, that is

$$\langle \dot z, \dot z\rangle \equiv \text{const.} \quad \square$$

Remark 2.2. Let $z \in \Omega^{1,2}$ be a critical point of S such that $\langle \dot z(s), \dot z(s)\rangle = E_z < 0$ for any $s \in [0,1]$ and $z(0) = z_0$, $z(1) = z_1$. Suppose $\sqrt{-E_z} = m_0 c$. Then $w(s) = z(s)$ is a critical point of the functional F whenever $q = 1$, and $w(s) = z(-s)$ is a critical point of F whenever $q = -1$. In both cases w is a solution of the differential equation

$$m_0 c \frac{d}{ds}\left(\frac{\dot w}{\sqrt{-\langle \dot w, \dot w\rangle}}\right) + q\big[dA^*(w) - dA(w)\big][\dot w] = 0. \tag{2.5}$$

Indeed by Proposition 2.1, z satisfies equation (2.3). Assume $q = 1$. By the definition of F

$$F'(w)[\zeta] = -m_0 c \int_{t_0}^{t_1} \frac{1}{\sqrt{-\langle \dot w, \dot w\rangle}} \langle \dot w, D_s\zeta\rangle + \int_{t_0}^{t_1} \big\langle dA(w)[\zeta], \dot w\big\rangle + \int_{t_0}^{t_1} \langle A(w), D_s\zeta\rangle$$

that yelds (2.5) for any w critical point of class C^1 of F. Since $\sqrt{-E_z} = m_0 c$, putting $w(s) = z(s)$ in (2.5) we obtain the thesis. The same result can be obtained if $q = -1$ choosing $w(s) = z(-s)$.

3. Palais–Smale condition on a strip

For the search of critical points of F via variational methods, we need some compactness assumption on the action functional $\mathcal{S}$. The most natural one is the Palais–Smale condition.

Definition. Let X be a Hilbert manifold, Ω an open subset of X, $F : \Omega \to \mathbb{R}$ a C^1-functional, and c a real number. We say that F satisfies the Palais–Smale condition at the level c, $(P.S.)_c$, on Ω, if for every sequence $\{z_n\}_{n\in\mathbb{N}}$ in Ω satisfying:

(1) $F(z_n) \to c$,

(2) $\lim_{n\to\infty} F'(z_n) = 0$,

there exists a subsequence $\{z_{n_k}\}_{k\in\mathbb{N}}$ converging in Ω. A sequence $\{z_n\}$ in Ω satisfying (1) and (2) is called a Palais–Smale sequence at the level c.

We do not know if the functional $\mathcal{S}$ satisfies tha Palais–Smale condition, for this reason we introduce a penalizing family of functionals, denoted by $\mathcal{S}_\varepsilon$, as follows: let $\psi : [0,+\infty) \longmapsto \mathbb{R}$ be a smooth (C^2) real function having the following properties:

(1) $\psi(0) = \psi'(0) = \psi''(0) = 0$,

(2) $\psi(\sigma) > 0 \quad \forall \sigma \in \mathbb{R}^+$, $\psi'(\sigma) > 0$,

(3) $\lim_{\sigma\to+\infty} \sigma\psi'(\sigma) - \psi(\sigma) = +\infty$.

An example of such a function is given by

$$\psi(\sigma) = e^\sigma - \big(1 + \sigma + \tfrac{1}{2}\sigma^2\big).$$

Set

$$\psi_\varepsilon(\sigma) = \begin{cases} \psi\Big(\sigma - \dfrac{1}{\varepsilon}\Big) & \text{if} \quad \sigma \geqslant \dfrac{1}{\varepsilon}, \\ 0 & \text{if} \quad \sigma < \dfrac{1}{\varepsilon}. \end{cases}$$

Now fix two real numbers $a < b$ and take, as in section 1,

$$\mathcal{M}_{a,b} = \big\{z \in \mathcal{M} : a < T(z) < b\big\}.$$

Fix $0 < \delta < \frac{1}{2}(b-a)$ and consider a C^2-map $\phi_\delta : \mathbb{R} \longmapsto \mathbb{R}$ such that

$$\phi_\delta(\sigma) = \begin{cases} b-\sigma & \text{if } \sigma \in [b-\delta, b+\delta], \\ \sigma - a & \text{if } \sigma \in [a-\delta, a+\delta]. \end{cases}$$

Take $\Phi : \overline{\mathcal{M}_{a,b}} \longmapsto \mathbb{R}$ defined as $\Phi(z) = \phi_\delta(T(z))$. By construction Φ vanishes on $\partial\mathcal{M}_{a,b}$ and it is positive on $\mathcal{M}_{a,b}$. For any $\varepsilon > 0$ we define the penalized functional

$$\mathcal{S}_\varepsilon : \Omega^{1,2}_{a,b} \longmapsto \mathbb{R}$$

as follows

$$\mathcal{S}_\varepsilon(z) = \mathcal{S}(z) - \psi_\varepsilon\left(\int_0^1 \langle \dot z, \nabla T(z)\rangle^2\right) - \varepsilon \int_0^1 \frac{1}{\Phi^2(z(s))}\,\mathrm{d}s,$$

where $\Omega^{1,2}_{a,b}$ has been defined in (2.1). To prove the Palais–Smale condition is more convenient to write $\mathcal{S}_\varepsilon$ in the following form:

$$\begin{aligned}\mathcal{S}_\varepsilon(z) = \frac{1}{2}\int_0^1 \langle \dot z, \dot z\rangle_R - \int_0^1 \langle \hat W, \dot z\rangle_R^2 + \int_0^1 \langle A(z), \dot z\rangle \\ - \psi_\varepsilon\left(\int_0^1 \langle \dot z, \nabla^R T(z)\rangle_R^2\right) - \varepsilon\int_0^1 \frac{1}{\Phi^2(z)}\,\mathrm{d}s,\end{aligned} \tag{3.1}$$

where

$$\hat W(z) = \frac{\nabla^R T(z)}{\sqrt{\langle \nabla^R T(z), \nabla^R T(z)\rangle_R}} \tag{3.2}$$

is such that $\langle \hat W, \zeta\rangle_R = \langle W, \zeta\rangle$ (cf. Lemma A.2).

We have the following

Proposition 3.1. *Assume* (1.6)–(1.8). *Let $c \in \mathbb{R}$, $\{\delta_n\}$ be an infinitesimal sequence belonging to $\mathbb{R}^+$. Let $\{z_n\} \subset \Omega^{1,2}_{a,b}$ be a sequence such that*

$$\mathcal{S}_\varepsilon(z_n) \leqslant c, \tag{3.3}$$

$$\sup_{0 \neq \zeta \in T_{z_n}\Omega^{1,2}} |\mathcal{S}'_\varepsilon(z_n)[\zeta]| \leqslant \delta_n \int_0^1 \langle D^R_s\zeta, D^R_s\zeta\rangle_R, \tag{3.4}$$

where $\mathcal{S}'_\varepsilon$ denotes the differential of $\mathcal{S}_\varepsilon$.

Then $\int_0^1 \langle \dot z_n, \nabla^R T\rangle_R^2$ is bounded and z_n is uniformly far from $\partial\mathcal{M}_{a,b}$.

Whenever z_n is uniformly far from $\partial\mathcal{M}_{a,b}$, the boundedness of $\int_0^1 \langle \dot z_n, \nabla^R T\rangle_R^2$ is useful to prove the boundedness of $\int_0^1 \langle \dot z_n, \dot z_n\rangle_R$. Indeed we have the following

Lemma 3.2. *If $\mathcal{S}_\varepsilon(z_n) \leqslant c$, z_n is uniformly far from $\partial\mathcal{M}_{a,b}$ and $\int_0^1 \langle \dot z_n, \nabla^R T\rangle_R^2 \leqslant c_1$, then $\int_0^1 \langle \dot z_n, \dot z_n\rangle_R$ is bounded.*

Proof. Since $\mathcal{S}_\varepsilon(z_n) \leqslant c$, by (3.1), (3.2) and (1.6)

$$\begin{aligned}\frac{1}{2}\int_0^1 \langle \dot z_n, \dot z_n\rangle_R &= \mathcal{S}_\varepsilon(z_n) + \int_0^1 \big\langle \hat W(z_n), \dot z_n\big\rangle^2 - \int_0^1 \langle A(z_n), \dot z_n\rangle \\ &\quad + \psi_\varepsilon\bigg(\int_0^1 \big\langle \dot z_n, \nabla^R T\big\rangle_R^2 \mathrm{d}s\bigg) + \varepsilon\int_0^1 \frac{1}{\Phi^2(z_n)}\,\mathrm{d}s \\ &\leqslant c + N\int_0^1 \langle \dot z_n, \nabla^R T\rangle_R^2\,\mathrm{d}s + \|A\|_R \int_0^1 \sqrt{\langle \dot z_n, \dot z_n\rangle_R} \\ &\quad + \psi_\varepsilon\bigg(\int_0^1 \langle \dot z_n, \nabla^R T\rangle_R^2\,\mathrm{d}s\bigg) + \varepsilon\int_0^1 \frac{1}{\Phi^2(z_n)}\,\mathrm{d}s.\end{aligned}$$

Then

$$\begin{aligned}\int_0^1 \langle \dot z_n, \dot z_n\rangle_R \leqslant 2\|A\|_R \bigg(\int_0^1 \langle \dot z_n, \dot z_n\rangle_R\bigg)^{1/2} &+ 2c + 2N\int_0^1 \langle \dot z_n, \nabla^R T\rangle_R^2\,\mathrm{d}s \\ &+ 2\psi_\varepsilon\bigg(\int_0^1 \langle \dot z_n, \nabla^R T\rangle_R^2\,\mathrm{d}s\bigg) + 2\varepsilon\int_0^1 \frac{1}{\Phi^2(z_n)}\,\mathrm{d}s.\end{aligned} \tag{3.5}$$

Since $\int_0^1 \langle \dot z_n, \nabla^R T\rangle_R^2\,\mathrm{d}s$ and $\int_0^1 1/(\Phi^2(z_n))\,\mathrm{d}s$ are bounded, (3.5) implies the boundedness of $\int_0^1 \langle \dot z_n, \dot z_n\rangle_R$. □

Proof of Proposition 3.1. For the sake of simplicity during this proof we will write z instead of z_n. By (3.1) and (3.4), for any $\zeta \in T_z\Omega^{1,2}_{a,b}$ we have that

$$\begin{aligned}0 &\leqslant \mathcal{S}'_\varepsilon(z)[\zeta] + \delta_n \int_0^1 \langle D_s^R\zeta, D_s^R\zeta\rangle_R \\ &= \int_0^1 \langle \dot z, D_s^R\zeta\rangle_R - 2\int_0^1 \langle \hat W(z), \dot z\rangle_R \big[D_\zeta^R\big(\langle \hat W(z), \dot z\rangle_R\big)\big] \\ &\quad + \int_0^1 \big\langle dA(z)[\zeta], \dot z\big\rangle + \int_0^1 \langle A(z), D_s\zeta\rangle \\ &\quad - 2\psi'_\varepsilon\bigg(\int_0^1 \big\langle \dot z, \nabla^R T(z)\big\rangle_R^2\bigg)\bigg[\int_0^1 \langle \dot z, \nabla^R T\rangle_R D_\zeta^R\big(\langle \dot z, \nabla^R T\rangle_R\big)\bigg] \\ &\quad + \frac{\varepsilon}{2}\int_0^1 \frac{1}{\Phi^3(z)}\langle \nabla^R\Phi, \zeta\rangle_R\,\mathrm{d}s + \delta_n\int_0^1 \langle D_s^R\zeta, D_s^R\zeta\rangle_R\end{aligned} \tag{3.6}$$

where $D_\zeta^R(\cdot)$ denotes the covariant derivative (with respect to (1.4)) along the direction ζ. Now take

$$\zeta(s) = \big[t_n(s) - t_*(s)\big]Y(z),$$

where

$$Y(z) = \frac{\nabla^R T(z)}{\big\langle \nabla^R T(z), \nabla^R T(z)\big\rangle_R}, \tag{3.7}$$

$$t_n(s) = T(z(s)) \quad\text{and}\quad t_*(s) = (1-s)T(z(0)) + sT(z(1)).$$

Note that $\dot t_n = \langle \dot z, \nabla^R T\rangle_R$ and $\langle \hat W(z), \dot z\rangle_R^2 = \beta(z)\dot t_n^2$, where β is defined in (1.6). Therefore, with the above choice of ζ (see also the form of the metric g in local coordinates), (3.6) becomes

$$\begin{aligned}
0 \leqslant & \int_0^1 \langle \dot z, D^R_{\dot z} Y(z)\rangle \big[t_n(s) - t_*(s)\big] + \int_0^1 \langle \dot z, Y(z)\rangle \big[\dot t_n(s) - \dot t_*(s)\big] \\
& - \int_0^1 \langle \nabla^R \beta(z), Y(z)\rangle \big[t_n(s) - t_*(s)\big]\, \dot t_n^2 - 2\int_0^1 \beta(z) \dot t_n \big[\dot t_n(s) - \dot t_*(s)\big] \\
& + \int_0^1 \langle dA(z)[Y(z)], \dot z\rangle \big[t_n(s) - t_*(s)\big] + \int_0^1 \big\langle A(z), D_s\big[\big(t_n(s) - t_*(s)\big) Y(z)\big]\big\rangle \\
& - 2\psi_\varepsilon' \left(\int_0^1 \dot t_n^2\right) \left[\int_0^1 \dot t_n \big(\dot t_n(s) - \dot t_*(s)\big)\right] \\
& + \frac{\varepsilon}{2} \int_0^1 \frac{1}{\Phi^3(z)} \frac{\langle \nabla^R \Phi(z), \nabla^R T(z)\rangle}{\langle \nabla^R T(z), \nabla^R T(z)\rangle} \big(t_n(s) - t_*(s)\big) \\
& + \delta_n \left[\int_0^1 \langle Y, Y\rangle_R \big(\dot t_n(s) - \dot t_*(s)\big) + \int_0^1 \langle D^R_{\dot z} Y, D^R_{\dot z} Y\rangle_R \big(t_n(s) - t_*(s)\big)^2 \right. \\
& \qquad \left. + 2\int_0^1 \langle Y, D^R_{\dot z} Y\rangle_R \big(\dot t_n(s) - \dot t_*(s)\big)\big(t_n(s) - t_*(s)\big)\right].
\end{aligned} \tag{3.8}$$

An integration by parts yields

$$\int_0^1 \big\langle A(z), D_s\big[\big(t_n(s) - t_*(s)\big) Y(z)\big]\big\rangle = -\int_0^1 \langle dA(z)[\dot z], Y(z)\rangle \big[t_n(s) - t_*(s)\big]. \tag{3.9}$$

Notice that $|\dot t_*(s)| = |T(z(1)) - T(z(0))| \equiv \bar t$, where $\bar t$ is constant. Moreover, considering that $z \in \Omega^{1,2}_{a,b}$, it follows that

$$\|t_n - t_*\| \leqslant c_*. \tag{3.10}$$

Then, since $\|Y\|_R \leqslant \sqrt{N}$ (see (1.6)) using Proposition A.3 and assumptions (1.6) and (1.8), combining (3.8)–(3.10) gives

$$\begin{aligned}
0 \leqslant\ & c_* M_1 \int_0^1 \langle \dot z, \dot z\rangle_R + \sqrt{N} \int_0^1 \sqrt{\langle \dot z, \dot z\rangle_R}\, \big(\dot t_n(s) - \dot t_*(s)\big) + M_3 c_* \int_0^1 \dot t_n^2 \\
& + 2N \int_0^1 \big(\dot t_n^2 + |\dot t_n \dot t_*|\big) + 2A_1 \sqrt{N} c_* \int_0^1 \sqrt{\langle \dot z, \dot z\rangle_R} + 2\bar t^2 \psi_\varepsilon' \left(\int_0^1 \dot t_n^2\right) \\
& - 2\psi_\varepsilon' \left(\int_0^1 \dot t_n^2\right) \left[\int_0^1 \dot t_n^2\right] + \frac{\varepsilon}{2} \int_0^1 \frac{1}{\Phi^3(z)} \frac{\langle \nabla^R \Phi(z), \nabla^R T(z)\rangle}{\langle \nabla^R T(z), \nabla^R T(z)\rangle} \big(t_n(s) - t_*(s)\big) \\
& + \delta_n N \int_0^1 \big(\dot t_n(s) - \dot t_*(s)\big)^2 + \delta_n M_2 c_*^2 \int_0^1 \langle \dot z, \dot z\rangle_R + 2\delta_n c_* \sqrt{N} M_2 \int_0^1 |\dot t_n \dot t_*|.
\end{aligned}$$

Then assuming by contradiction that $\int_0^1 \dot t_n^2 \to +\infty$ (and using the properties of ψ_ε) gives the existence of constants $D_0, D_1 > 0$ such that

$$\begin{aligned}
0 \leqslant D_0 + D_1 \int_0^1 \langle \dot z, \dot z\rangle_R - 2\psi_\varepsilon' \left(\int_0^1 \dot t_n^2\right) \left[\int_0^1 \dot t_n^2\right] \\
+ \frac{\varepsilon}{2} \int_0^1 \frac{1}{\Phi^3(z)} \frac{\langle \nabla^R \Phi(z), \nabla^R T(z)\rangle}{\langle \nabla^R T(z), \nabla^R T(z)\rangle} \big(t_n(s) - t_*(s)\big).
\end{aligned} \tag{3.11}$$

By (3.5) we deduce the existence of constants $D_2, D_3 > 0$ such that

$$\int_0^1 \langle \dot z, \dot z\rangle_R \leqslant D_3 + D_4 \left[\psi_\varepsilon\left(\int_0^1 \dot t_n^2\right) + \varepsilon \int_0^1 \frac{1}{\Phi^2(z)}\right]. \tag{3.12}$$

Finally, combining (3.11) and (3.12), using the properties of ψ_ε and the sign of $\langle \nabla^R \Phi(z), \nabla^R T(z)\rangle_R\,(t_n(s) - t_*(s))$ near by $\partial\mathcal{M}_{a,b}$ (see the definition of Φ) allows to conclude that $\int_0^1 \dot t_n^2$ is bounded. Now, as $\Phi(z) = \phi_\delta(T(z))$ using once again (3.11) and (3.12) gives the existence of constants $D_5, D_6 > 0$ such that

$$\int_0^1 \frac{1}{\phi_\delta^3(t_n)} \leqslant D_5 \int_0^1 \frac{1}{\phi_\delta^2(t_n)} + D_6.$$

By the definition of ϕ_δ we deduce the existence of $D_7 > 0$ for which

$$\frac{1}{\phi_\delta^3(t)} \geqslant \frac{2D_5}{\phi_\delta^2(t)} - D_7 \qquad \text{for any } t \in\ ]a, b[.$$

Then $\int_0^1 1/(\phi_\delta^2(t_n))$ must be bounded. Since $\int_0^1 \dot t_n^2$ is bounded, we have that t_n is uniformly far from $\partial\mathcal{M}_{a,b}$. $\square$

Proposition 3.3. *Assume* (1.6)–(1.8). *Then* $\mathcal{S}_\varepsilon$ *satisfies* $(P.S.)_c$ *for every* $c \in \mathbb{R}$.

Proof. $\mathcal{S}'_\varepsilon(z)$ is a linear and continuous operator in the space $\Omega^{1,2}_{a,b}$ endowed with the Hilbert structure (2.2). So, if $\{z_n\}$ is a Palais–Smale sequence, for every $n \in \mathbb{N}$ we can write

$$\mathcal{S}'_\varepsilon(z_n)[\zeta] = \int_0^1 \langle A_n, D_s^R \zeta\rangle_R,$$

where A_n goes to 0 as $n \to +\infty$ with respect to L^2-norm. Therefore, by construction,

$$\begin{aligned}
&\int_0^1 \langle \dot z_n, D_s\zeta\rangle + \int_0^1 \big\langle dA(z_n)[\zeta], \dot z_n\big\rangle + \int_0^1 \langle A(z_n), D_s\zeta\rangle \\
&\quad - 2\psi'_\varepsilon\left(\int_0^1 \langle \dot z_n, \nabla T\rangle^2\right)\left[\int_0^1 \langle \dot z_n, \nabla T\rangle\big(\langle D_s\zeta, \nabla T\rangle + \langle \dot z_n, H^T(z_n)[\zeta]\rangle\big)\right] \\
&\quad + 2\varepsilon \int_0^1 \frac{1}{\Phi^3(z_n)} \langle \Phi'(z_n), \zeta\rangle \\
&= \int_0^1 \langle A_n, D_s^R\zeta\rangle_R.
\end{aligned} \tag{3.13}$$

By Proposition 3.1 and assumption (1.5), unless to consider a subsequence, $\{z_n\}$ converges to $z \in \Omega^{1,2}_{a,b}$ uniformly and weakly in $H^{1,2}$. We have just to prove that the convergence in $H^{1,2}$ is strong. In order to isolate $D_s\zeta$ in (3.13), we shall integrate by parts the terms that contain ζ. Using the same techniques of [7] we can state that the covariant integrals appearing in the integration by parts are bounded in $H^{1,2}$. Moreover

$$D_s^R\zeta = D_s\zeta + \Gamma(z_n)[\dot z_n, \zeta],$$

where Γ is bilinear form depending continuously on z_n. So (3.13) becomes

$$\int_0^1 \langle \dot{z}_n + \sigma_n, D_s\zeta\rangle - 2\psi_\varepsilon'\left(\int_0^1 \langle \dot{z}_n, \nabla T\rangle^2\right)\int_0^1 \langle \dot{z}_n, \nabla T\rangle\langle D_s\zeta, \nabla T\rangle = \int_0^1 \langle B_n, D_s\zeta\rangle, \tag{3.14}$$

where (unless to consider a subsequence) σ_n converges uniformly and B_n goes to 0 in L^2. By (3.14) there exists a sequence k_n uniformly bounded, such that $D_s k_n = 0$ and

$$\dot{z}_n + \sigma_n - 2\psi_\varepsilon'\left(\int_0^1 \langle \dot{z}_n, \nabla T\rangle^2\right)\langle \dot{z}_n, \nabla T\rangle\nabla T = B_n + k_n. \tag{3.15}$$

Then multiplying both terms by ∇T we have

$$\langle \dot{z}_n, \nabla T\rangle\left[1 - 2\psi_\varepsilon'\left(\int_0^1 \langle \dot{z}_n, \nabla T\rangle^2\right)\langle \nabla T, \nabla T\rangle\right] = \langle B_n + k_n - \sigma_n, \nabla T\rangle. \tag{3.16}$$

Then by (3.15) and (3.16) we can write

$$\dot{z}_n = a_n + b_n,$$

where a_n converges uniformly and $b_n \to 0$ in L^2, showing that $\{z_n\}$ converges strongly to z with respect to the $H^{1,2}$ norm. $\square$

4. A priori estimates for the critical points of $\mathcal{S}_\varepsilon$

Let us consider a family of curves $\{z_\varepsilon\}_{\varepsilon>0}$ such that any z_ε is a critical point of $\mathcal{S}_\varepsilon$. Arguing as in the proof of Lemma 2.1 and using (3.16) with $B_n = 0$, shows that any z_ε is of class C^2. Moreover putting $A_n = 0$ in (3.13) and integrating by parts (with z_n replaced by z_ε) gives the differential equation satisfied by z_ε

$$-D_s\dot{z}_\varepsilon + \left[dA^*(z_\varepsilon) - dA(z_\varepsilon)\right][\dot{z}_\varepsilon] + 2\varepsilon\frac{\nabla\Phi(z_\varepsilon)}{\Phi^3(z_\varepsilon)} + 2\psi_\varepsilon'\left(\int_0^1 \langle \dot{z}_\varepsilon, \nabla T(z_\varepsilon)\rangle^2\right) \cdot \left[\langle D_s\dot{z}_\varepsilon, \nabla T(z_\varepsilon)\rangle\nabla T(z_\varepsilon) + \langle \dot{z}_\varepsilon, H^T(z_\varepsilon)[\dot{z}_\varepsilon]\rangle\nabla T(z_\varepsilon)\right] = 0. \tag{4.1}$$

Proposition 4.1. *Fix $c \in \mathbb{R}$ and assume* (1.6), (1.7) *and* (1.8). *Let z_ε be a critical point of $\mathcal{S}_\varepsilon$ such that*

$$\mathcal{S}_\varepsilon(z_\varepsilon) < c \qquad \text{for any } \varepsilon \in \,]0,1]. \tag{4.2}$$

Then $\int_0^1 \langle \dot{z}_\varepsilon, \dot{z}_\varepsilon\rangle_R$ is bounded independently of $\varepsilon \in \,]0,1]$.

Proof. Since z_ε is a critical point of $\mathcal{S}_\varepsilon$, we have

$$\mathcal{S}_\varepsilon'(z_\varepsilon)[\zeta] = 0 \qquad \forall \zeta \in T_{z_\varepsilon}\Omega^{1,2}.$$

Choose $\zeta = (\nabla T(z_\varepsilon))/(\langle \nabla T(z_\varepsilon), \nabla T(z_\varepsilon)\rangle)\tau$, where $\tau \in H_0^{1,2}([0,1],\mathbb{R})$. Set $t_\varepsilon = T(z_\varepsilon)$, so

$\dot{t}_\varepsilon = \langle \nabla T(z_\varepsilon), \dot{z}_\varepsilon\rangle$. A straightforward computation gives the existence of $C > 0$ such that

$$\begin{aligned} 0 \leqslant C \left(\int_0^1 \tfrac{1}{2}\langle \dot{z}_\varepsilon, \dot{z}_\varepsilon\rangle_R + 1\right) |\tau| - \nu \int_0^1 \dot{t}_\varepsilon \dot{\tau} + C \int_0^1 |\dot{\tau}| \\ + 2\varepsilon \int_0^1 \frac{1}{\Phi^3(z_\varepsilon)} \frac{\langle \nabla\Phi(z_\varepsilon), \nabla T(z_\varepsilon)\rangle}{\langle \nabla T(z_\varepsilon), \nabla T(z_\varepsilon)\rangle} \tau - \psi_\varepsilon' \left(\int_0^1 \dot{t}_\varepsilon^2\right) \int_0^1 \dot{t}_\varepsilon \dot{\tau}, \end{aligned} \tag{4.3}$$

where ν is defined by (1.6). Now multiplying by $\dot{z}_\varepsilon$ both sides of (4.1) gives the existence of a constant $E_\varepsilon \in \mathbb{R}$ such that

$$\tfrac{1}{2}\langle \dot{z}_\varepsilon, \dot{z}_\varepsilon\rangle - \psi_\varepsilon' \left(\int_0^1 \dot{t}_\varepsilon^2\right) \langle \dot{z}_\varepsilon, \nabla T(z_\varepsilon)\rangle^2 + \frac{\varepsilon}{\Phi^2(z_\varepsilon)} \equiv E_\varepsilon. \tag{4.4}$$

Then integrating in $[0, 1]$ and recalling the definition of $\mathcal{S}_\varepsilon$ gives

$$\begin{aligned} E_\varepsilon &= \frac{1}{2}\int_0^1 \langle \dot{z}_\varepsilon, \dot{z}_\varepsilon\rangle - \psi_\varepsilon' \left(\int_0^1 \dot{t}_\varepsilon^2\right) \int_0^1 \dot{t}_\varepsilon^2 + \varepsilon \int_0^1 \frac{1}{\Phi^2(z_\varepsilon)} \\ &= \mathcal{S}_\varepsilon(z_\varepsilon) - \int_0^1 \langle A(z_\varepsilon), \dot{z}_\varepsilon\rangle + 2\varepsilon \int_0^1 \frac{1}{\Phi^2(z_\varepsilon)} + \psi_\varepsilon\left(\int_0^1 \dot{t}_\varepsilon^2\right) - \psi_\varepsilon' \left(\int_0^1 \dot{t}_\varepsilon^2\right) \int_0^1 \dot{t}_\varepsilon^2. \end{aligned} \tag{4.5}$$

Since $\frac{1}{2}\langle \dot{z}, \dot{z}\rangle_R = \frac{1}{2}\langle \dot{z}, \dot{z}\rangle + \beta(z)\dot{t}^2$ and $\nu \leqslant \beta(z) \leqslant N$ for any $z \in \mathcal{M}_{a,b}$, combining (4.2)–(4.5) gives the existence of $C_1 > 0$ such that

$$\begin{aligned} 0 \leqslant C_1 \Bigg[\int_0^1 |\tau| + \int_0^1 \sqrt{\langle \dot{z}_\varepsilon, \dot{z}_\varepsilon\rangle_R}\, |\tau| + 2\varepsilon \int_0^1 \frac{1}{\Phi^2(z_\varepsilon)} \int_0^1 |\tau| \\ + \psi_\varepsilon\left(\int_0^1 \dot{t}_\varepsilon^2\right) \int_0^1 |\tau| + \psi_\varepsilon'\left(\int_0^1 \dot{t}_\varepsilon^2\right) \int_0^1 \dot{t}_\varepsilon^2 |\tau| + \int_0^1 \dot{t}_\varepsilon^2 |\tau|\Bigg] - \nu \int_0^1 \dot{t}_\varepsilon \dot{\tau} \\ + C_1 \int_0^1 |\dot{\tau}| + 2\varepsilon \int_0^1 \frac{1}{\Phi^3(z_\varepsilon)} \frac{\langle \nabla\Phi(z_\varepsilon), \nabla T(z_\varepsilon)\rangle}{\langle \nabla T(z_\varepsilon), \nabla T(z_\varepsilon)\rangle} \tau - \psi_\varepsilon'\left(\int_0^1 \dot{t}_\varepsilon^2\right) \int_0^1 \dot{t}_\varepsilon \dot{\tau}. \end{aligned} \tag{4.6}$$

Choose $\tau = \sinh(\omega(t_\varepsilon - t_*))$, where $t_*(s) = (1-s)T(z(0)) + sT(z(1))$. If $T(z) \in\ ]b-\delta, b[$, $\nabla\Phi(z) = -\nabla T(z)$ and $\tau > 0$, while if $T(z) \in\ ]a, a+\delta[$, $\nabla\Phi(z) = \nabla T(z)$ and $\tau < 0$. Then there exists $\theta_0 > 0$ (independent of ε) such that

$$2\varepsilon \frac{1}{\Phi^3(z_\varepsilon)} \frac{\langle \nabla\Phi(z_\varepsilon), \nabla T(z_\varepsilon)\rangle}{\langle \nabla T(z_\varepsilon), \nabla T(z_\varepsilon)\rangle} \tau \leqslant -\frac{\varepsilon\theta_0}{\Phi^3(z_\varepsilon)} \tag{4.7}$$

for any $s \in\ ]a, a+\delta[\ \cup\ ]b-\delta, b[$. Fix $\omega > 1$ such that $1 - \nu\omega > 0$. Since $T(z_\varepsilon)$ is uniformly bounded, using (4.6), (4.7) and the definition of ψ_ε allows to deduce the existence of a constant $D > 0$ such that

$$\int_0^1 \dot{t}_\varepsilon^2 \leqslant D\left(1 + \int_0^1 \sqrt{\langle \dot{z}_\varepsilon, \dot{z}_\varepsilon\rangle_R}\right). \tag{4.8}$$

As

$$\begin{aligned} \mathcal{S}_\varepsilon(z_\varepsilon) = \frac{1}{2}\int_0^1 \langle \dot{z}_\varepsilon, \dot{z}_\varepsilon\rangle_R + \int_0^1 \langle A(z_\varepsilon), \dot{z}_\varepsilon\rangle_R - \int_0^1 \beta(z_\varepsilon)\dot{t}_\varepsilon^2 \\ - \varepsilon \int_0^1 \frac{1}{\Phi^2} - \psi_\varepsilon\left(\int_0^1 \dot{t}_\varepsilon^2\right), \end{aligned} \tag{4.9}$$

$\mathcal{S}_\varepsilon(z_\varepsilon) \leqslant c$, $|\langle A(z_\varepsilon), \dot z_\varepsilon\rangle| \leqslant A_0\sqrt{\langle \dot z_\varepsilon, \dot z_\varepsilon\rangle_R}$ and $\beta \geqslant \nu$, by (4.8) we obtain the existence of $D_0 > 0$ such that

$$\frac{1}{2}\int_0^1 \langle \dot z_\varepsilon, \dot z_\varepsilon\rangle_R \leqslant D_0\left[1 + \varepsilon\int_0^1 \frac{1}{\Phi^2} + \psi_\varepsilon\left(\int_0^1 \dot t_\varepsilon^2\right)\right]. \tag{4.10}$$

Finally, setting (4.10) in (4.6) with τ as above allows to get that $\int_0^1 \dot t_\varepsilon^2$ and $\varepsilon\int_0^1 1/(\Phi^2(z_\varepsilon))$ are bounded independently of ε. $\square$

Remark 4.2. Under the assumption of Proposition 4.1, if z_ε is a critical point of $\mathcal{S}_\varepsilon$ and (4.2) holds (thanks to the definition of ψ_ε) we have

$$\psi_\varepsilon'\left(\int_0^1 \dot t_\varepsilon^2\right) = 0$$

for all ε sufficiently small. Therefore z_ε satisfies

$$-D_s\dot z_\varepsilon + \left[dA^*(z_\varepsilon) - dA(z_\varepsilon)\right][\dot z_\varepsilon] + 2\varepsilon\frac{\nabla\Phi(z_\varepsilon)}{\Phi^3(z_\varepsilon)} = 0. \tag{4.11}$$

Lemma 4.3. *Fix $c \in \mathbb{R}$ and assume that* (4.2) *holds. Suppose that* (1.6)–(1.11) *are satisfied. Then there exist $\delta(c) > 0$ and $\varepsilon(c) > 0$ such that*

$$\Phi(z_\varepsilon(s)) \geqslant \delta(c) \quad \textit{for any} \quad \varepsilon \in\ ]0, \varepsilon(c)] \quad \textit{and} \quad s \in [0,1]. \tag{4.12}$$

Proof. Take $\rho_\varepsilon(s) = \Phi(z_\varepsilon(s))$. If, by contradiction, (4.12) is not satisfied (since $\rho_\varepsilon(0) = \Phi(z_0)$ and $\rho_\varepsilon(1) = \Phi(z_1)$ for any ε) there exists $s_\varepsilon \in\ ]0,1[$ minimum point for ρ_ε such that

$$\lim_{\varepsilon\to 0}\Phi(z_\varepsilon(s_\varepsilon)) = 0.$$

By the construction of Φ, $T(z_\varepsilon(s_\varepsilon))$ is an element of interval $]a, a+\delta[\ \cup\ ]b-\delta, b[$ for any ε sufficiently small and

$$\rho_\varepsilon'(s_\varepsilon) = \langle \nabla T(z_\varepsilon), \dot z_\varepsilon\rangle = 0. \tag{4.13}$$

It will be enough to consider the case that $T(z_\varepsilon(s_\varepsilon)) \in\ ]a\,,\ a+\delta[$ because when $T(z_\varepsilon(s_\varepsilon)) \in\ ]b-\delta, b[$ can be dealt in the same way. Since $s_\varepsilon \in\ ]0,1[$ is a minimum point for ρ_ε we have

$$\rho_\varepsilon''(s_\varepsilon) \geqslant 0. \tag{4.14}$$

Moreover by the construction of Φ,

$$\rho_\varepsilon''(s_\varepsilon) = \left\langle H^T(z_\varepsilon)[\dot z_\varepsilon], \dot z_\varepsilon\right\rangle + \langle \nabla T(z_\varepsilon), D_s\dot z_\varepsilon\rangle. \tag{4.15}$$

Then, combining (4.13)–(4.15) and (4.11), and recalling the construction of Φ gives

$$0 \leqslant \left\langle H^T(z_\varepsilon)[\dot z_\varepsilon], \dot z_\varepsilon\right\rangle + \left\langle \nabla T(z_\varepsilon), (dA^*(z_\varepsilon) - dA(z_\varepsilon))[\dot z_\varepsilon]\right\rangle + 2\varepsilon\frac{\langle \nabla\Phi(z_\varepsilon), \nabla T(z_\varepsilon)\rangle}{\Phi^3(z_\varepsilon)}.$$

From (1.11) and (4.13) it follows that $(dA^*(z_\varepsilon) - dA(z_\varepsilon))[\dot z_\varepsilon] = 0$. Then in s_ε,

$$-2\varepsilon\frac{\langle \nabla\Phi(z_\varepsilon), \nabla T(z_\varepsilon)\rangle}{\Phi^3(z_\varepsilon)} \leqslant \langle H^T(z_\varepsilon)[\dot z_\varepsilon], \dot z_\varepsilon\rangle.$$

If $T(z_\varepsilon(s_\varepsilon)) \in \,]a, a+\delta[$, $\nabla\Phi(z_\varepsilon) = \nabla T(z_\varepsilon)$, therefore

$$-\langle \nabla\Phi(z_\varepsilon), \nabla T(z_\varepsilon)\rangle = -\langle \nabla T(z_\varepsilon), \nabla T(z_\varepsilon)\rangle > 0,$$

while $\langle H^T(z_\varepsilon(s_\varepsilon))\dot z_\varepsilon(s_\varepsilon), \dot z_\varepsilon(s_\varepsilon)\rangle < 0$ by assumption (1.9). Such a contradiction allows to conclude the proof. □

Remark 4.4. Under the assumptions of Lemma 4.3, going to the limit as $\varepsilon \to 0$ allows to obtain a sequence $\{z_{\varepsilon_n}\}$ that converges (with respect to the C^2-norm) to a critical point of the functional $\mathcal{S}$.

5. Existence of critical points of $\mathcal{S}$

In this section we will prove the main result of this paper.

Remark 5.1. By (1.5) and (1.6), using the flow $\eta(s, z)$ associated to the vector field ∇T, allows easily to obtain an orthogonal splitting structure for $\mathcal{M}$. More precisely, set $\mathcal{M}_0 = T^{-1}(a+b/2)$ and denote by π the projection of $\mathcal{M}$ on $\mathcal{M}_0$ obtained by means of the flow η. The map $z \longmapsto (\pi(z), T(z))$ allows to construct an isometry between $\mathcal{M}$ and the manifold $\mathcal{M}_0 \times \mathbb{R}$ endowed with the metric

$$ds^2 = \langle \alpha(x,t)\xi, \xi\rangle\, dx^2 - \beta(x,t)\tau^2\, dt^2,$$

where $x \in \mathcal{M}_0$, $t \in \mathbb{R}$, $\zeta = (\xi, \tau) \in T_x\mathcal{M}_0 \times \mathbb{R}$, α is a positive linear operator and β a positive scalar field. With the above notations we can assume that the space $\Omega^{1,2}_{a,b}$ can be written as

$$\Omega^{1,2}_{a,b} = \Lambda(x_0, x_1) \times H^{1,2}_{a,b}\big(T(z_0), T(z_1); \mathbb{R}\big),$$

with

$$\Lambda(x_0, x_1) = \big\{x \in H^{1,2}\big([0,1]; \mathcal{M}_0\big) : x(0) = x_0, x(1) = x_1\big\}$$

and

$$H^{1,2}_{a,b}\big(T(z_0), T(z_1); \mathbb{R}\big) \\ = \big\{t \in H^{1,2}([0,1], \mathbb{R}) : \ a < t(s) < b \ \ \forall s, t(0) = T(z_0), t(1) = T(z_1)\big\}.$$

Now set $H^{1,2}_k = t_* + H_{k,0}$, where

$$H_{k,0} = \mathrm{span}\big\{\sin(j\pi s), \ \ j = 1, 2, \ldots, k\big\},$$

and t_* is the segment joining $t_0 = T(z_0)$ and $t_1 = T(z_1)$.

In order to prove our result we need to use the Saddle Point Theorem (see [12]) and for this aim we have to introduce a Galerkin approximation argument in the variable t, constructing, for any $k \in \mathbb{N}$, the spaces

$$\Omega^{1,2}_{a,b,k} = \Lambda(x_0, x_1) \times \big(H^{1,2}_k \cap H^{1,2}_{a,b}(T(z_0), T(z_1); \mathbb{R})\big).$$

Observe that the same proof of Proposition 3.3 implies that the restriction $\mathcal{S}_{\varepsilon,k}$ of $\mathcal{S}_\varepsilon$ to the space $\Omega^{1,2}_{a,b,k}$ satisfies Palais–Smale condition for every $k \in \mathbb{N}$.

Proof of theorem 1.2. Define

$$\Sigma_* = \left\{(x,t) \in \Omega^{1,2}_{a,b,k} : t = t_*\right\}.$$

For any $z = (x, t_*)$, using the Riemannian structure, and recalling that

$$\langle \nabla^R T, \dot z\rangle_R = \dot t_* = t_1 - t_0,$$

we easily get the existence of $c_* = c_*(|t_1 - t_0|) > 0$ such that for any $\varepsilon \in]\,0,1]$ and any $k \in \mathbb{N}$

$$\mathcal{S}_{\varepsilon,k} \geqslant -c_*. \tag{5.1}$$

Since $\mathcal{M}_0$ is connected, there always exists a C^1-curve x_* joining x_0 and x_1. Put

$$Q(R) = \left\{(x_*,t) \in \Omega^{1,2}_{a,b} : \|t - t_*\|_{H^{1,2}} < R\right\}$$

and the corresponding finite-dimensional set

$$Q_k(R) = \left\{(x_*,t) \in \Omega^{1,2}_{a,b,k} : \|t - t_*\|_{H^{1,2}_k} \leqslant R\right\}.$$

By (1.8), for any $z = (x_*, t) \in Q(R)$ we have

$$\begin{aligned}\mathcal{S}_\varepsilon(x_*,t) = \frac{1}{2}\int_0^1 \langle \alpha(x,t)\dot x_*, \dot x_*\rangle - \frac{1}{2}\int_0^1 \beta(x_*,t)\dot t^2 \\ + \|A\|_R \int_0^1 \left(\langle \alpha(x_*,t)\dot x_*, \dot x_*\rangle + \beta(x_*,t)\dot t^2\right)^{1/2}.\end{aligned} \tag{5.2}$$

Moreover by (1.6) and (1.12) there exist two positive constants d_1 and d_2 such that

$$\begin{aligned}\mathcal{S}_\varepsilon(x_*,t) \leqslant d_1 + d_2\int_0^1 |t|^\theta - \frac{\nu}{2}\int_0^1 \dot t^2 + \|A\|_R \int_0^1 \sqrt{1 + d_2|t|^\theta} \\ + \|A\|_R\sqrt{N}\int_0^1 |\dot t|.\end{aligned} \tag{5.3}$$

Since $\theta \in]\,0,2[$, for any $\overline{R}$ sufficiently large and for any $\varepsilon \in]\,0,1]$,

$$\sup \mathcal{S}_\varepsilon(\partial Q(\bar R)) < \inf \mathcal{S}_\varepsilon(\Sigma_*). \tag{5.4}$$

So

$$c_{k,\varepsilon} = \inf_{h\in\Gamma_k} \sup \mathcal{S}_\varepsilon\big(h(Q_k(R))\big).$$

Take $\Gamma_k = \{h \in C(\Omega^{1,2}_{a,b,k}, \Omega^{1,2}_{a,b,k}) / h(z) = z \ \ \forall z \in \partial Q_k(R)\}$, and set

$$c_{k,\varepsilon} = \inf_{h\in\Gamma_k} \sup \mathcal{S}_\varepsilon\big(h(Q_k(R))\big),$$

we have that $c_{k,\varepsilon} \in]\inf \mathcal{S}_\varepsilon(\Sigma_*), \sup \mathcal{S}_\varepsilon(Q(R))[$. By the Saddle Point Theorem (see [12]) it is a critical value of $\mathcal{S}_{k,\varepsilon}$. If z^ε_k is a critical point of $\mathcal{S}_{\varepsilon,k}$ we have in particular

$$\mathcal{S}_{\varepsilon,k}(z_\varepsilon)[(T(z_\varepsilon) - t_*)Y(z_\varepsilon)] = 0 \quad \text{for any } k.$$

Therefore the same proof of Proposition 3.1 allows to obtain that $\|\dot z^\varepsilon_k\|_{L^2}$ is bounded independently of k. Moreover, a slight change in the proof of Proposition 3.3 gives that

$$z^\varepsilon_k \to z^\varepsilon$$

in $H^{1,2}$ (up to a subsequence). Clearly z^ε is a critical point of $\mathcal{S}_\varepsilon$ such that

$$\mathcal{S}_\varepsilon(z^\varepsilon) \in \left]\inf \mathcal{S}_\varepsilon(\Sigma_*), \sup \mathcal{S}_\varepsilon(Q(R))\right[.$$

By Proposition 4.1, if ε is sufficiently small, z^ε is a critical point of $\mathcal{S}$. □

Appendix

In this section we prove some useful properties of $\langle \cdot\,, \cdot\rangle$.

Lemma A.1.

$$\langle \nabla T(z), \nabla T(z)\rangle = -\langle \nabla^R T(z), \nabla^R T(z)\rangle_R \quad \textit{and} \quad \nabla T(z) = -\nabla^R T(z) \tag{A.1}$$

where ∇^R represents the gradient of T with respect to the metric (1.4)*, while ∇ is the one with respect to the Lorentzian metric.*

Proof. As the differentiation is invariant with respect to the choice of the metric structure on $\mathcal{M}$, we have that

$$dT(z)[\zeta] = \langle \nabla^R T(z), \zeta\rangle_R = \langle \nabla T(z), \zeta\rangle \quad \forall \zeta \in T_z\Omega^{1,2}. \tag{A.2}$$

In particular, if $\zeta = \nabla^R T(z)$, from (A.2) we get

$$\langle \nabla^R T(z), \nabla^R T(z)\rangle = \langle \nabla T(z), \nabla^R T(z)\rangle. \tag{A.3}$$

By (1.4), (A.2) can be written as

$$\langle \nabla T(z), \zeta\rangle = \langle \nabla^R T(z), \zeta\rangle_R = \langle \nabla^R T(z), \zeta\rangle + 2\langle W, \nabla^R T(z)\rangle\langle W, \zeta\rangle \tag{A.4}$$

for all $\zeta \in T_z\Omega^{1,2}$. Then

$$\nabla T(z) = \nabla^R T(z) - 2\,\frac{\langle \nabla T(z), \nabla^R T(z)\rangle}{\langle \nabla T(z), \nabla T(z)\rangle}\,\nabla T(z). \tag{A.5}$$

Multiplying (A.5) by $\nabla T(z)$, with respect to the Lorentzian metric, we have that

$$\langle \nabla T(z), \nabla T(z)\rangle = \langle \nabla^R T(z), \nabla T(z)\rangle - 2\langle \nabla T(z), \nabla^R T(z)\rangle = -\langle \nabla^R T(z), \nabla T(z)\rangle,$$

so the thesis follows by (A.3) and (A.5). □

Lemma A.2. *Let $\hat{W}(z) = \nabla^R T(z)/\sqrt{\langle \nabla^R T(z), \nabla^R T(z)\rangle_R}$. Then*

$$\langle \hat{W}(z), \zeta\rangle_R = \langle W(z), \zeta\rangle \qquad \forall \zeta \in T_z\mathcal{M},$$

where W is defined by (1.3).

Proof. Follows by straightforward calculations. □

Proposition A.3. *Assume* (1.6) *and* (1.7)*. Then there exist constants M_1, M_2 and M_3 such that*

(1) $|\langle \zeta, D^R_\zeta Y(z)\rangle| \leqslant M_1\langle \zeta, \zeta\rangle_R$,
(2) $\|D^R_\zeta Y(z)\|_R \leqslant M_2\langle \zeta, \zeta\rangle_R^{1/2}$,
(3) $|\langle \nabla^R \beta(z), Y(z)\rangle_R| \leqslant M_3$,

for any $z \in \mathcal{M}$, *and* $\zeta \in T_z\mathcal{M}$, $Y(z) = \sqrt{\beta(z)}\hat{W}(z)$, $\hat{W}$ *as in Lemma* A.2, $\beta = \beta(z) = \big(1/\langle \nabla^R T(z), \nabla^R T(z)\rangle_R\big)$, *and* D^R_ζ *is the covariant derivative with respect to the metric* (1.4) *along the direction* ζ.

Proof. First of all we need to prove that

$$\exists\, h_1 > 0: \quad \|H^T_R(z)\|_R \leqslant h_1 \qquad \forall z \in \mathcal{M}_{a,b}. \tag{A.6}$$

Since we can consider Riemannian geodesics as critical points of the functional $\int_0^1 \langle \dot z, \dot z\rangle + \langle W(z), \dot z\rangle^2$, it is easy to prove that they satisfy the equation

$$-D_s\dot z + 2\langle W(z), \dot z\rangle[\mathrm{d}W(z)]^T\dot z - 2\,\frac{\mathrm{d}}{\mathrm{d}s}\big(\langle W(z), \dot z\rangle W(z)\big) = 0 \tag{A.7}$$

where $[dW(z)]^T$ represents the transpose of the differential of W. Let us define the real function $r(s) = T(z(s))$. By construction

$$r''(\zeta) = H^T_R(z)[\dot\zeta, \dot\zeta]. \tag{A.8}$$

On the other hand, differentiating r with respect to the Lorentzian metric, we have that

$$r''(s) = \langle H^T(z)\dot z, \dot z\rangle + \langle \nabla T(z), D_s\dot z\rangle. \tag{A.9}$$

Therefore, substituing (A.7) in (A.9) and comparing (A.8) and (A.9), from (1.3) we obtain

$$\begin{aligned} H^T_R(z)[\dot\zeta, \dot\zeta] = \langle H^T(z)\dot\zeta, \dot\zeta\rangle - 2\langle W(z), \dot\zeta\rangle\langle \mathrm{d}W(z)[\nabla T], \dot\zeta\rangle \\ - 2\sqrt{-\langle \nabla T, \nabla T\rangle}\langle \mathrm{d}W(z)[\dot\zeta], \dot\zeta\rangle. \end{aligned} \tag{A.10}$$

By construction, for any $\zeta \in T_z\Omega^{1,2}$ we have that

$$dW(z)[\zeta] = \frac{H^T(z)[\zeta]\langle \nabla T, \nabla T\rangle + \nabla T\langle H^T(z)\zeta, \nabla T\rangle}{-\langle \nabla T, \nabla T\rangle\sqrt{-\langle \nabla T, \nabla T\rangle}}.$$

Then, taking $\zeta = \nabla T$, from (A.1) it follows

$$dW(z)[\nabla T] = \frac{H^T(z)[\nabla T]}{\sqrt{-\langle \nabla T, \nabla T\rangle_R}} + \frac{\langle H^T(z)\nabla T, \nabla T\rangle}{\sqrt{-\langle \nabla T, \nabla T\rangle_R}^{\,3}}, \tag{A.11}$$

so, being

$$\|G(z)\|_R = \sup_{\langle \zeta, \zeta\rangle_R = 1} |\langle G(z)\zeta, \zeta\rangle|$$

for any bilinear operator G, (1.6), (1.7), (A.10) and (A.11) imply (A.6).

By construction

$$D^R_\zeta Y = \frac{H^T_R(z)[\zeta]}{\langle \nabla T, \nabla T\rangle_R} - \nabla^R T\left(\frac{2}{\langle \nabla T, \nabla T\rangle_R}\langle \nabla^R T, H^T_R(z)[\zeta]\rangle_R\right).$$

Then (1.6), (1.7) and (A.6) yeld

$$|\langle \zeta, D^R_\zeta Y\rangle_R| \leqslant \left(h_1 N + \frac{2h_1 N^2}{\nu}\right)\langle \zeta, \zeta\rangle_R.$$

Taking $M_1 = h_1 N + (2h_1 N^2)/\nu$ we obtain (1). By construction (2) is an obvious consequence of (1). To finish the proof we have to prove that $\nabla^R \beta(z)[Y(z)]$ is bounded. Assumptions (1.6), $(A.6)$ yield $\nabla^R \beta(z)[Y(z)] = -2\beta^2 \langle H_R^T(z)[Y(z)], \nabla T(z)\rangle_R \leqslant 2\beta^{3/2} \|H_R^T(z)\|_R \|Y(z)\|_R \leqslant 2N^2 h_1$, from which the thesis follows. □

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