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Calculus of Variations

Linking solutions for quasilinear equations at critical growth involving the "1-Laplace" operator

Marco Degiovanni · Paola Magrone

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Abstract We show that the problem at critical growth, involving the 1-Laplace operator and obtained by relaxation of $-\Delta_1 u = \lambda |u|^{-1}u + |u|^{1^*-2}\,u$, admits a nontrivial solution $u \in BV(\Omega)$ for any $\lambda \geq \lambda_1$. Nonstandard linking structures, for the associated functional, are recognized.

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1 Introduction and main result

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, with Lipschitz boundary. We are interested in the existence of nontrivial solutions u to the problem which comes from the relaxation of

$$\begin{cases} -\mathrm{div}\left(\dfrac{\nabla u}{|\nabla u|}\right) = \lambda\,\dfrac{u}{|u|} + |u|^{1^*-2}u & \text{in } \Omega, \\[2ex] u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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M. Degiovanni (✉)
Dipartimento di Matematica e Fisica, Università Cattolica, Via dei Musei 41, 25121 Brescia, Italy
e-mail: m.degiovanni@dmf.unicatt.it

P. Magrone
Dipartimento di Matematica, Università di Roma Tre,
Largo San Leonardo Murialdo 1, 00146 Rome, Italy
e-mail: magrone@mat.uniroma3.it

where $\lambda \in \mathbb{R}$ and $1^* = n/(n-1)$ is the critical Sobolev exponent for the embedding of $W_0^{1,1}(\Omega)$ in $L^q(\Omega)$.

Problem (1.1) looks as the formal limit, as $p \to 1^+$, of the problem at critical growth

$$\begin{cases} -\mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right) = \lambda |u|^{p-2}\, u + |u|^{p^*-2}\, u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $p^* = np/(n-p)$. Let us set, whenever $1 \le p < n$,

$$S = S(n,p) := \inf\left\{ \frac{\int_{\mathbb{R}^n} |\nabla u|^p\, dx}{\left(\int_{\mathbb{R}^n} |u|^{p^*}\, dx\right)^{p/p^*}} :\ u \in C_c^\infty(\mathbb{R}^n)\backslash\{0\} \right\}, \tag{1.3}$$

$$\lambda_1 = \lambda_1(\Omega, p) := \inf\left\{ \frac{\int_\Omega |\nabla u|^p\, dx}{\int_\Omega |u|^p\, dx} :\ u \in C_c^\infty(\Omega)\backslash\{0\} \right\}. \tag{1.4}$$

Problem (1.2) has received much attention in the last years, starting from the celebrated paper of Brezis and Nirenberg [5], where it was shown that, for $p = 2$, problem (1.2) admits a positive solution u for every $\lambda \in]0, \lambda_1[$ and $n \ge 4$. The result has been extended by Egnell, Garcia Azorero-Peral Alonso, Guedda-Veron [19,22,25], who have proved that (1.2) admits a positive solution u for any $\lambda \in]0, \lambda_1[$, provided that $p > 1$ and $n \ge p^2$. Such a solution u can be obtained via the Mountain pass theorem of Ambrosetti and Rabinowitz [1] applied to the C^1-functional $f : W_0^{1,p}(\Omega) \longrightarrow \mathbb{R}$ defined as

$$f(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\, dx - \frac{\lambda}{p}\int_\Omega |u|^p\, dx - \frac{1}{p^*}\int_\Omega |u|^{p^*}\, dx$$

and satisfies

$$0 < f(u) < \frac{1}{n}\, S^{n/p}. \tag{1.5}$$

When $\lambda \ge \lambda_1$, it is still meaningful to look for nontrivial solutions u, but the situation is quite different in the two cases $p = 2$ and $p \ne 2$. If $p = 2$, it has been proved by Capozzi et al. [7] that problem (1.2) has a nontrivial solution u for any $\lambda \ge \lambda_1$, provided that $n \ge 5$ (see also Gazzola and Ruf [23, Corollary 1]). Such a solution can be obtained via the Linking theorem of Rabinowitz (see e.g. [31, Theorem 5.3]) applied to the functional f and still satisfies (1.5).

On the other hand, when $p \ne 2$ there is in general no direct sum decomposition of $W_0^{1,p}(\Omega)$, which allows to recognize a linking structure in a standard way, unless λ belongs to a suitable right neighborhood $[\lambda_1, \overline{\lambda}[$ of λ_1, as shown in Arioli and Gazzola [3], where it is proved that, for any $p > 1$, problem (1.2) has a nontrivial solution u for any $\lambda \in [\lambda_1, \overline{\lambda}[$, provided that $\frac{n^2}{n+1} > p^2$. Nevertheless, the result of Capozzi–Fortunato–Palmieri has been recently extended, via a nonstandard linking construction, in Degiovanni and Lancelotti [13], where it is shown that the result of Arioli–Gazzola actually holds for any $\lambda \ge \lambda_1$.

Coming to the case $p = 1$, let us first give a precise relaxed formulation of (1.1). First of all, denote by $\|\ \|_p$ the usual norm in L^p and by $\mathscr{H}^k$ the k-dimensional Hausdorff measure. For every $u \in BV(\Omega)$ (see e.g. [2,24]), let us set

$$|Du|(\Omega) := \sup\left\{ \int_\Omega u\, \mathrm{div}\, v\, dx :\ v \in C_c^\infty(\Omega; \mathbb{R}^n),\ \|v\|_\infty \le 1 \right\}.$$

Then, according to Kawohl and Schuricht [28], we mean that we are looking for $u \in BV(\Omega)$ such that

$$\begin{cases} \text{there exist } z \in L^\infty(\Omega; \mathbb{R}^n) \text{ and } \gamma \in L^\infty(\Omega) \text{ such that} \\ \|z\|_\infty \le 1, \ \operatorname{div} z \in L^n(\Omega), \ -\int_\Omega u \operatorname{div} z \, dx = |Du|(\Omega) + \int_{\partial\Omega} |u| \, d\mathscr{H}^{n-1}, \\ \|\gamma\|_\infty \le 1, \ \gamma |u| = u \qquad \text{a.e. in } \Omega, \\ -\operatorname{div} z = \lambda\gamma + |u|^{1^*-2} u \qquad \text{a.e. in } \Omega, \end{cases} \tag{1.6}$$

(n is the exponent conjugate to 1^*). Other equivalent formulations can be obtained applying the next Proposition 3.1. Since $u = 0$ is a solution for any λ (take $(z, \gamma) = (0, 0)$), we say that $u = 0$ is the *trivial* solution of (1.6). Let us also define a locally Lipschitz functional $f : BV(\Omega) \longrightarrow \mathbb{R}$ by

$$f(u) = |Du|(\Omega) + \int_{\partial\Omega} |u| \, d\mathscr{H}^{n-1} - \lambda \int_\Omega |u| \, dx - \frac{1}{1^*} \int_\Omega |u|^{1^*} \, dx.$$

The resul of Brezis–Nirenberg has been extended also to this setting by Demengel [17], who has proved that (1.6) admits a nonnegative, nontrivial solution u satisfying

$$0 < f(u) < \frac{1}{n} S^n \tag{1.7}$$

for any $\lambda \in]0, \lambda_1[$. The argument is based on an approximation procedure from the case $p > 1$.

Our purpose is to cover the case $\lambda \ge \lambda_1$, in the line of the result of Capozzi–Fortunato–Palmieri, by a direct approach. Our result is the following

Theorem 1.1 *Let Ω be a bounded open subset of $\mathbb{R}^n$, $n \ge 2$, with Lipschitz boundary. Then, for every $\lambda \ge \lambda_1$, problem* (1.6) *admits a nontrivial solution $u \in BV(\Omega) \cap L^\infty(\Omega)$ satisfying* (1.7)*.*

For the proof, we will apply (nonsmooth) variational methods to the functional f. A first idea could be to apply the approach of Chang [8] to the locally Lipschitz functional f defined on $BV(\Omega)$. However, it has been already observed that, in such a setting, the Palais–Smale condition fails even in the subcritical case, as the norm-convergence of BV cannot be usually obtained for a Palais–Smale sequence (see Marzocchi [29] and Degiovanni et al. [15]). For this reason, it is more convenient to extend the functional f to $L^{1^*}(\Omega)$ with value $+\infty$ outside $BV(\Omega)$. In this setting, the nonsmoothness increases, as f is only lower semicontinuous, but the techniques of Corvellec–Degiovanni–Marzocchi, Ioffe–Schwartzman, Katriel [11,26,27] can be applied, in particular as specified in Degiovanni and Schuricht [16]. On the other hand, we have more compactness and in Theorem 5.3 we will show that f satisfies $(PS)_c$ whenever $c < (1/n)S^n$, as one may expect from the case $p > 1$ (see [25, Theorem 3.4]).

A second difficulty, typical in the case $p \ne 2$ when $\lambda \ge \lambda_1$, is that there is no direct sum decomposition which allows to recognize a linking structure in a standard way. Therefore, as in [13], we will apply the Linking theorem of [12], in which linear subspaces are substituted by cones.

In the next section we recall mainly from [16] some tools of nonsmooth analysis. In Sect. 3 we specify our functional framework, taking advantage of the results of [28]. In Sect. 4 we build the cones which have to substitute linear subspaces in the linking structure. Sect. 5 is devoted to the Palais–Smale condition, while in the last section we prove the main result.

2 Tools of nonsmooth analysis

Let Y be a metric space endowed with the distance d and let $f : Y \to [-\infty, +\infty]$ be a function. We set

$$\mathrm{dom}(f) = \{u \in Y : \ |f(u)| < +\infty\}$$

and consider

$$\mathrm{epi}\,(f) = \{(u, s) \in Y \times \mathbb{R} : \ f(u) \leq s\}$$

endowed with the topology induced by $Y \times \mathbb{R}$. The next definition, equivalent to that of [14], is taken from [6].

Definition 2.1 For every $u \in \mathrm{dom}(f)$, we denote by $|df|\,(u)$ the supremum of the σ's in $[0, +\infty[$ such that there exist a neighborhood W of $(u, f(u))$ in $\mathrm{epi}\,(f)$, $\delta > 0$ and a continuous map $\mathscr{H} : W \times [0, \delta] \to Y$ satisfying

$$d(\mathscr{H}((v, s), t), v) \leq t, \quad f(\mathscr{H}((v, s), t)) \leq s - \sigma t,$$

whenever $(v, s) \in W$ and $t \in [0, \delta]$.

The extended real number $|df|\,(u)$ is called the *weak slope* of f at u.

The idea is to look for local deformations $\mathscr{H}$, along which the function f can be decreased with a certain rate σ with respect to the displacement $d(\mathscr{H}((v, s), t), v)$, and then optimize σ.

In particular, if Y is an open subset of a normed space and f is of class C^1, then $|df|\,(u) = \|f'(u)\|$ for every $u \in Y$ (see [14, Corollary 2.12]).

Moreover, it is easily seen that $|df|$ is lower semicontinuous with respect to the graph topology: if (u_k) is a sequence convergent to u in $\mathrm{dom}(f)$ with $f(u_k) \to f(u)$, then

$$\liminf_k \ |df|\,(u_k) \geq |df|\,(u).$$

Definition 2.2 An element $u \in Y$ is said to be a (*lower*) *critical point* of f, if $|f(u)| < +\infty$ and $|df|\,(u) = 0$. A real number c is said to be a (*lower*) *critical value* of f, if there exists a (lower) critical point u of f with $f(u) = c$.

Definition 2.3 A *Palais–Smale sequence* (*(PS)-sequence*, for short) for f is a sequence (u_k) in Y such that

$$\sup_k |f(u_k)| < +\infty$$

and such that $|df|\,(u_k) \to 0$.

Given a real number c, a *Palais–Smale sequence at level c* (*$(PS)_c$-sequence*, for short) is a (PS)-sequence (u_k) such that $f(u_k) \to c$.

The function f is said to satisfy $(PS)_c$, if every $(PS)_c$-sequence admits a convergent subsequence in Y.

Assume now that X is a real Banach space, whose dual space will be denoted by X'. In the following, $\partial f(u)$ will denote the Clarke–Rockafellar subdifferential and $f^0\,(u; v)$ the associated generalized directional derivative [10,32].

Let $f_0 : X \longrightarrow]-\infty, +\infty]$ be a convex, lower semicontinuous function and $f_1, g : X \longrightarrow \mathbb{R}$ two locally Lipschitz continuous functions. Let also $f = f_0 + f_1$ and

$$M = \{u \in X : \ g(u) = 0\}.$$

In such a case, according to the results of [16], we have that the functions

$$|df| : \mathrm{dom}(f) \longrightarrow [0, +\infty], \quad \left|d\left(f\big|_M\right)\right| : \mathrm{dom}(f) \cap M \longrightarrow [0, +\infty]$$

are lower semicontinuous with respect to the topology induced by X.

We are first interested in a (nonsmooth) extension of the Linking theorem, in which linear subspaces are substituted by symmetric cones. If $A \subseteq X\setminus\{0\}$ is symmetric, we denote by Index (A) the $\mathbb{Z}_2$-cohomological index of Fadell and Rabinowitz [20,21]. Let us recall that $\gamma^+(A) \leq \text{Index}\,(A) \leq \gamma^-(A)$, where, according to [9],

$$\gamma^+(A) = \sup\left\{m \in \mathbb{N} : \text{ there exists an odd continuous map } \psi : \mathbb{R}^m\setminus\{0\} \longrightarrow A\right\},$$
$$\gamma^-(A) = \inf\left\{m \in \mathbb{N} : \text{ there exists an odd continuous map } \psi : A \longrightarrow \mathbb{R}^m\setminus\{0\}\right\}.$$

Theorem 2.4 *Let X_-, X_+ be two symmetric cones in X such that X_+ is closed in X,*

$$X_- \cap X_+ = \{0\},$$
$$\text{Index}\,(X_-\setminus\{0\}) = \text{Index}\,(X\setminus X_+) < \infty.$$

Let also $e \in X\setminus X_-$, $0 < r_+ < r_-$,

$$S_+ = \{v \in X_+ : \ \|v\| = r_+\},$$
$$Q = \{te + u : \ t \geq 0, \ u \in X_-, \ \|te + u\| \leq r_-\},$$
$$P = \{u \in X_- : \ \|u\| \leq r_-\} \cup \{te + u : \ t \geq 0, \ u \in X_-, \ \|te + u\| = r_-\}$$

be such that

$$\sup_P f < \inf_{S_+} f, \quad \sup_Q f < +\infty.$$

Then f admits a $(PS)_c$-sequence with

$$\inf_{S_+} f \leq c \leq \sup_Q f.$$

In particular, if f satisfies $(PS)_c$, then c is a critical value of f.

Proof If $f : X \longrightarrow \mathbb{R}$ is of class C^1, by [12, Corollary 2.9] the assertion is a particular case of [12, Theorem 2.2]. If $f : X \longrightarrow \mathbb{R}$ is continuous, the proof is exactly the same, by the Deformation theorem of [11]. The case we are treating can be reduced to the continuous one arguing, as in [16], on the continuous function $\mathcal{G}_f : \mathrm{epi}\,(f) \to \mathbb{R}$ defined by $\mathcal{G}_f(u, s) = s$. □

We also need an information in the constrained case.

Theorem 2.5 *Assume that f and g are even with $g(0) \neq 0$ and that*

$$\text{Index}\,(\{u \in M : \ f(u) < +\infty\}) = \infty.$$

Suppose also that $f_{|M}$ is bounded from below, satisfies $(PS)_c$ for any $c \in \mathbb{R}$ and that, for every $u \in M$ with $f(u) < +\infty$, there exist $u_\pm \in X$ such that $f(u_\pm) < +\infty$ and

$$g^0\,(u; u_- - u) < 0, \quad g^0\,(u; u - u_+) < 0.$$

For every $m \geq 1$, let

$$c_m = \inf\left\{\sup_A f : \ A \subseteq M, \ A \ \textit{ is symmetric and } \text{Index}\,(A) \geq m\right\}.$$

Then $c_m \to +\infty$ and, for every $m \geq 1$ and c with $c_m \leq c < c_{m+1}$, we have

$$\text{Index}\left(\{u \in M : \; f(u) \leq c\}\right) = m.$$

Proof In the C^1 setting, the assertion follows from the Deformation theorem (see e.g. [12, Theorem 3.2]). For the extension to the nonsmooth case we are treating, we may argue as in the previous proof. □

Finally, let us recall from [16, Theorem 3.5] two results which connect the metric notion of weak slope with that of subdifferential.

Theorem 2.6 *Let $u \in X$ with $f(u) < +\infty$ and $|df|(u) < +\infty$. Then there exist $w \in X'$ with $\|w\| \leq |df|(u)$ and $\alpha \in \partial f_1(u)$ such that $-\alpha + w \in \partial f_0(u)$, i.e.*

$$f_0(v) \geq f_0(u) - \langle \alpha, v - u\rangle + \langle w, v - u\rangle, \quad \forall v \in X.$$

Theorem 2.7 *Let $u \in M$ with $f(u) < +\infty$ and $\left|d\left(f_{|M}\right)\right|(u) < +\infty$. Assume also that there exist $u_\pm \in X$ such that $f(u_\pm) < +\infty$ and*

$$g^0(u; u_- - u) < 0, \quad g^0(u; u - u_+) < 0.$$

Then there exist $w \in X'$ with $\|w\| \leq \left|d\left(f_{|M}\right)\right|(u)$ and $\alpha \in \partial f_1(u)$, $\beta \in \partial g(u)$, $\lambda \in \mathbb{R}$ such that $-\alpha + \lambda\beta + w \in \partial f_0(u)$, i.e.

$$f_0(v) \geq f_0(u) - \langle \alpha, v - u\rangle + \lambda\langle \beta, v - u\rangle + \langle w, v - u\rangle, \quad \forall v \in X.$$

3 The functional framework

Let Ω be a bounded open subset of $\mathbb{R}^n$, $n \geq 2$, with Lipschitz boundary and let $\lambda \in \mathbb{R}$. According to [28], let us define a convex, lower semicontinuous functional $f_0 : L^{1^*}(\Omega) \longrightarrow [0, +\infty]$ by

$$f_0(u) = \begin{cases} |Du|(\Omega) + \int_{\partial\Omega} |u|\, d\mathscr{H}^{n-1} & \text{if } u \in BV(\Omega), \\ \\ +\infty & \text{if } u \in L^{1^*}(\Omega)\backslash BV(\Omega), \end{cases}$$

and two locally Lipschitz continuous functionals $f_1, g : L^{1^*}(\Omega) \longrightarrow \mathbb{R}$ by

$$f_1(u) = -\lambda \int_\Omega |u|\, dx - \frac{1}{1^*} \int_\Omega |u|^{1^*} dx,$$

$$g(u) = \int_\Omega |u|\, dx - 1.$$

As usual, the dual of $L^{1^*}(\Omega)$ will be identified with $L^{(1^*)'}(\Omega) = L^n(\Omega)$. Moreover, f_0 is a norm on $BV(\Omega)$ equivalent to the canonical one. According to [17,28], we have

$$S = S(n, 1) = \min\left\{\frac{f_0(u)}{\|u\|_{1^*}} : \; u \in BV(\Omega)\backslash\{0\}\right\}, \tag{3.1}$$

$$\lambda_1 = \lambda_1(\Omega, 1) = \min\left\{\frac{f_0(u)}{\|u\|_1} : \; u \in BV(\Omega)\backslash\{0\}\right\}, \tag{3.2}$$

where S, λ_1 are defined in (1.3), (1.4). In particular, contrary to the case $p > 1$, the constant S is achieved in (3.1), for instance on characteristic functions of balls contained in Ω (see [4]).

We are interested in the application of variational methods to $f = f_0 + f_1$ on the whole space $L^{1^*}(\Omega)$ and to f_0 constrained on

$$M = \left\{u \in L^{1^*}(\Omega) : \; g(u) = 0\right\}.$$

In order to apply the results of the previous section, let us first recall from [28] the next

Proposition 3.1 *Let $u \in BV(\Omega)$ and $w \in L^n(\Omega)$. Then the following facts are equivalent:*

(a) *we have $w \in \partial f_0(u)$;*
(b) *we have*

$$\int_\Omega uw\,dx = |Du|(\Omega) + \int_{\partial\Omega} |u|\,d\mathscr{H}^{n-1}$$

and there exists $z \in L^\infty(\Omega; \mathbb{R}^n)$ such that $\|z\|_\infty \le 1$ and $-\mathrm{div}\, z = w$;
(c) *there exists $z \in L^\infty(\Omega; \mathbb{R}^n)$ such that $\|z\|_\infty \le 1$, $-\mathrm{div}\, z = w$ and*

$$\int_\Omega uw\varphi\,dx - \int_\Omega uz \cdot \nabla\varphi\,dx = \sup\left\{\left|\int_\Omega u\,\mathrm{div}\,\psi\,dx\right| : \; \psi \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n), \; |\psi| \le \varphi\right\}$$

for every $\varphi \in C_c^\infty(\mathbb{R}^n)$ with $\varphi \ge 0$.

Proof It is enough to combine [28, Proposition 4.23] with [28, Proposition A.12] and recall that the function defined as

$$\begin{cases} u & \text{on } \Omega, \\ 0 & \text{on } \mathbb{R}^n \setminus \Omega, \end{cases}$$

belongs to $BV(\mathbb{R}^n)$. □

In general, the graph of the subdifferential of a convex, lower semicontinuous functional is strong-weak* closed. In our case, we have a better property which will be useful later.

Proposition 3.2 *Let (u_k) be a sequence in $BV(\Omega)$ and (w_k) a sequence in $L^n(\Omega)$ such that (u_k) is weakly convergent to u in $L^{1^*}(\Omega)$, (w_k) is weakly convergent to w in $L^n(\Omega)$ and $w_k \in \partial f_0(u_k)$ for every $k \in \mathbb{N}$.*

Then $u \in BV(\Omega)$ and $w \in \partial f_0(u)$.

Proof For every $h > 0$, define $T_h, R_h : \mathbb{R} \longrightarrow \mathbb{R}$ by $T_h(s) = \min\{\max\{s, -h\}, h\}$, $R_h(s) = s - T_h(s)$. By [2, Theorem 3.99] we have

$$|Du|(\Omega) = |D(T_h(u))|(\Omega) + |D(R_h(u))|(\Omega),$$

hence

$$f_0(u) = f_0(T_h(u)) + f_0(R_h(u)), \quad \forall u \in BV(\Omega). \tag{3.3}$$

First of all, from the inequality

$$0 = f_0(0) \ge f_0(u_k) - \int_\Omega w_k u_k\,dx$$

we see that (u_k) is bounded in $BV(\Omega)$. It follows that $u \in BV(\Omega)$ and that $(T_h(u_k))$ is strongly convergent to $T_h(u)$ in $L^{1^*}(\Omega)$, for every $h > 0$.

We also have

$$\begin{aligned} f_0(v) + f_0(R_h(u_k)) \geq f_0(v + R_h(u_k)) &\geq f_0(u_k) + \int_\Omega w_k(v + R_h(u_k) - u_k)\,dx \\ &= f_0(T_h(u_k)) + f_0(R_h(u_k)) + \int_\Omega w_k(v - T_h(u_k))\,dx, \end{aligned}$$

whence

$$f_0(v) \geq f_0(T_h(u_k)) + \int_\Omega w_k(v - T_h(u_k))\,dx.$$

Passing to the limit as $k \to \infty$ and taking into account the lower semicontinuity of f_0, we get

$$f_0(v) \geq f_0(T_h(u)) + \int_\Omega w(v - T_h(u))\,dx.$$

Passing to the limit as $h \to \infty$, the assertion follows. □

Let us also prove a simple regularity property. A related result is contained in [18, Proposition 7].

Proposition 3.3 *Let $u \in BV(\Omega)$ with $\partial f_0(u) \neq \emptyset$. Then $u \in L^\infty(\Omega)$.*

Proof Let $w \in L^n(\Omega)$ with $w \in \partial f_0(u)$. For every $h > 0$, we have

$$f_0(T_h(u)) \geq f_0(u) + \int_\Omega w(T_h(u) - u)\,dx.$$

By (3.1), (3.3) and Hölder's inequality, it follows

$$S\|R_h(u)\|_{1^*} \leq f_0(R_h(u)) \leq \int_\Omega wR_h(u)\,dx \leq \left(\int_{\{|u|>h\}} |w|^n\,dx\right)^{1/n} \|R_h(u)\|_{1^*}.$$

If h is large enough to guarantee that

$$\left(\int_{\{|u|>h\}} |w|^n\,dx\right)^{1/n} < S,$$

we infer that $\|R_h(u)\|_{1^*} = 0$ and the assertion follows. □

Finally, from [28] we have the

Proposition 3.4 *Let $u \in BV(\Omega)$ with $|df|(u) < +\infty$. Then $u \in L^\infty(\Omega)$ and there exist $\gamma \in L^\infty(\Omega)$ and $w \in L^n(\Omega)$ such that $\|\gamma\|_\infty \le 1$, $\gamma|u| = u$ a.e. in Ω, $\|w\|_n \le |df|(u)$ and*

$$f_0(v) \ge f_0(u) + \lambda \int_\Omega \gamma(v-u)\,dx + \int_\Omega |u|^{1^*-2}u(v-u)\,dx$$
$$+ \int_\Omega w(v-u)\,dx, \quad \forall v \in BV(\Omega).$$

Proof It is enough to combine Theorem 2.6 with Proposition 3.3 and [28, Proposition 4.23]. □

Corollary 3.5 *If $u \in L^{1^*}(\Omega)$ is a critical point of f, then $u \in BV(\Omega) \cap L^\infty(\Omega)$ and u is a solution of* (1.6).

Proof It is enough to combine Proposition 3.1 with Proposition 3.4. □

4 Symmetric cones related to the 1-Laplace operator

In this section we show how to build, for the 1-Laplace operator, two cones X_-, X_+ with the properties required in Theorem 2.4. The construction is based on a sequence of eigenvalues for the 1-Laplace operator. We refer the reader to Milbers and Schuricht [30] for a slightly different construction of such a sequence.

Proposition 4.1 *The following facts hold:*

(a) *for every $u \in BV(\Omega) \cap M$, there exist $u_\pm \in BV(\Omega)$ such that*

$$g^0(u; u_- - u) < 0, \quad g^0(u; u - u_+) < 0;$$

(b) *for every $u \in BV(\Omega) \cap M$ with $\left|d\left(f_{0|M}\right)\right|(u) < +\infty$, we have $u \in L^\infty(\Omega)$ and there exist $\lambda \in \mathbb{R}$, $\gamma \in L^\infty(\Omega)$ and $w \in L^n(\Omega)$ such that $\|\gamma\|_\infty \le 1$, $\gamma|u| = u$ a.e. in Ω, $\|w\|_n \le \left|d\left(f_{0|M}\right)\right|(u)$ and*

$$f_0(v) \ge f_0(u) + \lambda \int_\Omega \gamma(v-u)\,dx + \int_\Omega w(v-u)\,dx, \quad \forall v \in BV(\Omega);$$

(c) *the functionals f_0 and g are even with $g(0) \ne 0$ and* Index $(BV(\Omega) \cap M) = \infty$ *with respect to the topology of $L^{1^*}(\Omega)$; moreover, $f_{0|M}$ is bounded from below and satisfies $(PS)_c$ for any $c \in \mathbb{R}$.*

Proof In the proof of [28, Theorem 4.6] it is shown that (a) holds. Then assertion (b) follows from Theorem 2.7, Proposition 3.3 and [28, Proposition 4.23]. Since $BV(\Omega)$ has infinite dimension, it is obvious that $\gamma^+(BV(\Omega) \cap M) = \infty$, also with respect to the topology of $L^{1^*}(\Omega)$. Therefore Index $(BV(\Omega) \cap M) = \infty$.

If (u_k) is a (PS)-sequence for $f_{0|M}$, by (b) we have

$$f_0(v) \ge f_0(u_k) + \lambda_k \int_\Omega \gamma_k(v-u_k)\,dx + \int_\Omega w_k(v-u_k)\,dx, \quad \forall v \in BV(\Omega)$$

with $\lambda_k \in \mathbb{R}$, $\gamma_k \in L^\infty(\Omega)$ and $w_k \in L^n(\Omega)$ satisfying $\|\gamma_k\|_\infty \le 1$, $\gamma_k|u_k| = u_k$ a.e. in Ω and $\|w_k\|_n \to 0$. Since f_0 is an equivalent norm in $BV(\Omega)$, up to a subsequence (u_k) is convergent to $u \in BV(\Omega)$ weakly in $L^{1^*}(\Omega)$ and strongly in $L^1(\Omega)$, while (γ_k) is convergent to γ in the weak* topology of $L^\infty(\Omega)$. Moreover, by Proposition 3.1 we have

$$\begin{aligned} f_0(u_k) &= \lambda_k \int_\Omega \gamma_k u_k\,dx + \int_\Omega w_k u_k\,dx \\ &= \lambda_k \int_\Omega |u_k|\,dx + \int_\Omega w_k u_k\,dx = \lambda_k + \int_\Omega w_k u_k\,dx. \end{aligned}$$

Therefore, also (λ_k) is bounded, hence convergent, up to a subsequence, to some λ. From Proposition 3.2 it follows that $\lambda\gamma \in \partial f_0(u)$, whence, by Proposition 3.1,

$$\lim_k f_0(u_k) = \lim_k \left(\lambda_k \int_\Omega \gamma_k u_k\,dx + \int_\Omega w_k u_k\,dx\right) = \lambda \int_\Omega \gamma u\,dx = f_0(u).$$

From [15, Theorem 4.10] we conclude that (u_k) is strongly convergent to u in $L^{1^*}(\Omega)$.

The other assertions contained in (c) are obvious. □

For every $m \geq 1$, let

$$\lambda_m = \inf\left\{\sup_A f_0 : \ A \subseteq M, A \text{ is symmetric and Index}\,(A) \geq m\right\}.$$

Since Index $(A) = 0$ only for $A = \emptyset$, the definition of λ_1 agrees with (3.2).

Theorem 4.2 *We have that $\lambda_m \to +\infty$. Moreover, for every $m \geq 1$ and μ with $\lambda_m \leq \mu < \lambda_{m+1}$, we have*

$$\text{Index}\left(\left\{u \in BV(\Omega)\setminus\{0\} : \ |Du|(\Omega) + \int_{\partial\Omega} |u|\,d\mathscr{H}^{n-1} \leq \mu \int_\Omega |u|\,dx\right\}\right) = m$$

with respect to the topology of $L^{1^*}(\Omega)$.

Proof Since f_0 and $\| \ \|_1$ are both positively homogeneous of degree 1, it is enough to combine Theorem 2.5 with Proposition 4.1. □

In view of the application of Theorem 2.4, let us see a first possible choice of X_-, X_+.

Theorem 4.3 *Let $m \geq 1$ and let $\lambda_m < \mu < \lambda_{m+1}$. Then there exist a symmetric cone X_- in $BV(\Omega)$ and a symmetric cone X_+ in $L^{1^*}(\Omega)$ such that X_- is closed in $L^1(\Omega)$, X_+ is closed in $L^{1^*}(\Omega)$ and:*

(a) *we have*

$$X_- \subseteq \left\{u \in BV(\Omega) : \ |Du|(\Omega) + \int_{\partial\Omega} |u|\,d\mathscr{H}^{n-1} \leq \lambda_m \int_\Omega |u|\,dx\right\} \cap L^\infty(\Omega);$$

(b) *$X_- \cap M$ is bounded in $L^\infty(\Omega)$ and strongly compact in $L^1(\Omega)$;*

(c) *we have*

$$X_+ \cap BV(\Omega) \subseteq \left\{u \in BV(\Omega) : \ |Du|(\Omega) + \int_{\partial\Omega} |u|\,d\mathscr{H}^{n-1} \geq \mu \int_\Omega |u|\,dx\right\};$$

(d) *we have* Index $(X_-\setminus\{0\}) = $ Index $\left(L^{1^*}(\Omega)\setminus X_+\right) = m$ *with respect to the topology of* $L^{1^*}(\Omega)$.

Proof Let

$$\widetilde{X}_- = \left\{ u \in BV(\Omega) : \ |Du|(\Omega) + \int_{\partial\Omega} |u| \, d\mathscr{H}^{n-1} \le \lambda_m \int_{\Omega} |u| \, dx \right\}.$$

Since $\widetilde{X}_- \cap M$ is an odd deformation retract of $\widetilde{X}_-\setminus\{0\}$, by Theorem 4.2 we have that Index $\left(\widetilde{X}_- \cap M\right) = m$. Moreover, $\widetilde{X}_- \cap M$ is strongly compact in $L^1(\Omega)$.

Let T_h, R_h be defined as before. First of all, we claim that there exists $h > 0$ such that

$$f_0(T_h(u)) \le \lambda_m \int_{\Omega} |T_h(u)| \, dx, \quad \forall u \in \widetilde{X}_- \cap M; \tag{4.1}$$

$$\int_{\Omega} |T_h(u)| \, dx \ge \frac{1}{2}, \quad \forall u \in \widetilde{X}_- \cap M. \tag{4.2}$$

Actually, for every $u \in BV(\Omega)$ Hölder's inequality and (3.1) yield

$$\int_{\Omega} |u| \, dx \le \mathscr{L}^n(\{u \ne 0\})^{\frac{1}{n}} \left(\int_{\Omega} |u|^{1^*} \, dx \right)^{\frac{1}{1^*}} \le \frac{1}{S} \mathscr{L}^n(\{u \ne 0\})^{\frac{1}{n}} \ f_0(u).$$

Since for every $u \in BV(\Omega) \cap M$ we have $R_h(u) \in BV(\Omega)$ and

$$1 = \int_{\Omega} |u| \, dx \ge \int_{\{R_h(u)\ne 0\}} |u| \, dx \ge h\mathscr{L}^n(\{R_h(u) \ne 0\}),$$

it follows

$$S\, h^{\frac{1}{n}} \int_{\Omega} |R_h(u)| \, dx \le f_0(R_h(u)) \quad \forall u \in BV(\Omega) \cap M.$$

Then, if h is large enough, we have

$$\lambda_m \int_{\Omega} |R_h(u)| \, dx \le f_0(R_h(u)) \quad \forall u \in BV(\Omega) \cap M$$

and (4.1) follows from (3.3). Moreover, if $u \in \widetilde{X}_- \cap M$, we also have

$$S\, h^{\frac{1}{n}} \int_{\Omega} |R_h(u)| \, dx \le f_0(R_h(u)) \le f_0(u) \le \lambda_m.$$

Then (4.2) also follows, provided that h is large enough.

With this choice of h, let

$$X_- = \left\{ t\, T_h(u) : \ t \ge 0, \ u \in \widetilde{X}_- \cap M \right\}.$$

Then X_- is a symmetric cone in $BV(\Omega) \cap L^\infty(\Omega)$. From (4.1) it follows that $X_- \subseteq \widetilde{X}_-$, while (4.2) implies that

$$\|v\|_\infty \le 2h \, \|v\|_1, \quad \forall v \in X_-.$$

In particular, $X_- \cap M$ is bounded in $L^\infty(\Omega)$. Since the surjective map

$$\begin{aligned} \widetilde{X}_- \cap M &\longrightarrow X_- \cap M \\ u &\mapsto \frac{T_h(u)}{\|T_h(u)\|_1} \end{aligned}$$

is odd and continuous with respect to the topology of $L^{1^*}(\Omega)$, we have

$$\text{Index}\,(X_-\backslash\{0\}) \geq \text{Index}\,(X_- \cap M) \geq \text{Index}\left(\widetilde{X}_- \cap M\right) = m.$$

Actually, equality holds, as $X_- \subseteq \widetilde{X}_-$. Finally, the above map is also continuous with respect to the topology of $L^1(\Omega)$. Therefore $X_- \cap M$ is strongly compact in $L^1(\Omega)$ and X_- is closed in $L^1(\Omega)$.

Again from Theorem 4.2 we know that

$$\text{Index}\left(\left\{u \in BV(\Omega) \cap M :\ |Du|(\Omega) + \int_{\partial\Omega} |u|\, d\mathscr{H}^{n-1} \leq \mu\right\}\right) = m.$$

Let U be a symmetric open neighborhood of such a set satisfying Index $(U) = m$. Then

$$X_+ = L^{1^*}(\Omega)\backslash \{tu :\ t > 0,\ u \in U\}$$

has the required properties. □

5 The Palais–Smale condition

Lemma 5.1 *Let (u_k) be a (PS) sequence for f and let $u \in BV(\Omega)$. Assume that (u_k) is bounded in $BV(\Omega)$ and weakly convergent to u in $L^{1^*}(\Omega)$.*

Then we have

$$\lim_k \left(f_0(u_k) - \|u_k\|_{1^*}^{1^*}\right) = f_0(u) - \|u\|_{1^*}^{1^*},$$

$$\limsup_k \left(f_0(R_h(u_k)) - \|R_h(u_k)\|_{1^*}^{1^*}\right) \leq f_0(R_h(u)) - \|R_h(u)\|_{1^*}^{1^*}, \quad \forall h > 0.$$

Proof By Proposition 3.4, there exist (γ_k) in $L^\infty(\Omega)$ and (w_k) in $L^n(\Omega)$ such that $\|\gamma_k\|_\infty \leq 1$, $\gamma_k|u_k| = u_k$ a.e. in Ω, $\|w_k\|_n \to 0$ and $\lambda\gamma_k + |u_k|^{1^*-2}u_k + w_k \in \partial f_0(u_k)$. Moreover, (u_k) is also strongly convergent to u in $L^1(\Omega)$ and, up to a subsequence, (γ_k) is convergent to some γ in the weak* topology of $L^\infty(\Omega)$. By Proposition 3.2 it follows $\lambda\gamma + |u|^{1^*-2}u \in \partial f_0(u)$. Then by Proposition 3.1 we have

$$\begin{aligned} f_0(u_k) &= \lambda \int_\Omega \gamma_k u_k\, dx + \int_\Omega |u_k|^{1^*}\, dx + \int_\Omega w_k u_k\, dx \\ &= \lambda \int_\Omega |u_k|\, dx + \int_\Omega |u_k|^{1^*}\, dx + \int_\Omega w_k u_k\, dx, \end{aligned} \tag{5.1}$$

$$f_0(u) = \lambda \int_\Omega \gamma u\, dx + \int_\Omega |u|^{1^*}\, dx,$$

whence

$$\lim_k \left(f_0(u_k) - \int_\Omega |u_k|^{1^*} dx \right) = \lim_k \left(\lambda \int_\Omega \gamma_k u_k \, dx + \int_\Omega w_k u_k \, dx \right)$$
$$= \lambda \int_\Omega \gamma u \, dx = f_0(u) - \int_\Omega |u|^{1^*} dx.$$

By (3.3) we also have

$$f_0(R_h(u_k)) - \|R_h(u_k)\|_{1^*}^{1^*}$$
$$= \left(f_0(u_k) - \|u_k\|_{1^*}^{1^*} \right) - f_0(T_h(u_k)) + \left(\|u_k\|_{1^*}^{1^*} - \|R_h(u_k)\|_{1^*}^{1^*} \right).$$

On the other hand, $(T_h(u_k))$ is convergent to $T_h(u)$ in $L^{1^*}(\Omega)$ and we have that

$$0 \le |s|^{1^*} - |R_h(s)|^{1^*} \le \varepsilon |s|^{1^*} + C_{h,\varepsilon}, \quad \forall \varepsilon > 0.$$

From [12, Lemma 4.2] it follows that

$$\lim_k \left(\|u_k\|_{1^*}^{1^*} - \|R_h(u_k)\|_{1^*}^{1^*} \right) = \left(\|u\|_{1^*}^{1^*} - \|R_h(u)\|_{1^*}^{1^*} \right).$$

By the lower semicontinuity of f_0, the second assertion also follows. □

Lemma 5.2 *Each (PS) sequence for f is bounded in $BV(\Omega)$.*

Proof Let (u_k) be a (PS) sequence for f. Assume, for a contradiction, that $f_0(u_k) \to +\infty$. If we set

$$v_k = \frac{u_k}{f_0(u_k)},$$

up to a subsequence (v_k) is strongly convergent in $L^1(\Omega)$ to some $v \in BV(\Omega)$. Since

$$\frac{f(u_k)}{f_0(u_k)} = 1 - \lambda \|v_k\|_1 - \frac{1}{1^*} (f_0(u_k))^{1^*-1} \|v_k\|_{1^*}^{1^*},$$

from the boundedness of $(f(u_k))$ we deduce that (v_k) is strongly convergent to 0 in $L^{1^*}(\Omega)$. On the other hand, as before it holds (5.1) with $\|w_k\|_n \to 0$. It follows

$$f(u_k) = \frac{1}{n} \left[f_0(u_k) - \lambda \|u_k\|_1 \right] + \frac{1}{1^*} \int_\Omega w_k u_k \, dx,$$

namely

$$\frac{f(u_k)}{f_0(u_k)} = \frac{1}{n} \left[1 - \lambda \|v_k\|_1 \right] + \frac{1}{1^*} \int_\Omega w_k v_k \, dx.$$

Passing to the limit as $k \to \infty$, we get $0 = 1/n$ and a contradiction follows. □

Theorem 5.3 *For any $\lambda \in \mathbb{R}$, the functional f satisfies $(PS)_c$ whenever $c < (1/n)S^n$.*

Proof Let (u_k) be a $(PS)_c$ sequence with $c < (1/n)S^n$. We already know that (u_k) is bounded in $BV(\Omega)$, hence convergent, up to a subsequence, to some $u \in BV(\Omega)$ weakly in $L^{1^*}(\Omega)$ and strongly in $L^1(\Omega)$. From (5.1) it also follows that

$$f(u_k) = \frac{1}{n}\,\|u_k\|_{1^*}^{1^*} + \int_\Omega w_k u_k\,dx,$$

with $\|w_k\|_n \to 0$, whence

$$\lim_k \|u_k\|_{1^*}^{1^*-1} = (nc)^{1/n} < S.$$

Given $\varepsilon > 0$, let $h > 0$ be such that

$$f_0(R_h(u)) - \|R_h(u)\|_{1^*}^{1^*} < \varepsilon\left(S - (nc)^{1/n}\right).$$

Then we have

$$\limsup_k \|R_h(u_k)\|_{1^*}^{1^*-1} \le (nc)^{1/n}$$

and, by (3.1),

$$\left(S - \|R_h(u_k)\|_{1^*}^{1^*-1}\right)\|R_h(u_k)\|_{1^*} \le f_0(R_h(u_k)) - \|R_h(u_k)\|_{1^*}^{1^*}.$$

From Lemma 5.1 it follows

$$\limsup_k \|R_h(u_k)\|_{1^*} < \varepsilon,$$

whence $\|R_h(u)\|_{1^*} < \varepsilon$. Since $(T_h(u_k))$ is strongly convergent to $T_h(u)$ in $L^{1^*}(\Omega)$, we have

$$\begin{aligned}\limsup_k \|u_k - u\|_{1^*} &\le \limsup_k \|T_h(u_k) - T_h(u)\|_{1^*}\\ &\quad + \limsup_k \|R_h(u_k)\|_{1^*} + \|R_h(u)\|_{1^*} \le 2\varepsilon\end{aligned}$$

and the assertion follows by the arbitrariness of ε. □

6 Proof of the main result

Let $x_0 \in \Omega$ and let

$$e_\rho = n^{n-1}\,\rho^{1-n}\,\chi_{\mathrm{B}_\rho(x_0)}.$$

Then it is well known (see [4]) that $e_\rho \in BV(\mathbb{R}^n)$ and

$$|De_\rho|(\mathbb{R}^n) = \int_{\mathbb{R}^n} \left|e_\rho\right|^{1^*} dx = S^n, \tag{6.1}$$

$$\int_{\mathbb{R}^n} \left|u_\rho\right| dx = n^{n-1}\,\mathscr{L}^n\left(\mathrm{B}_1(0)\right)\rho. \tag{6.2}$$

Let $\lambda \ge \lambda_1$, let $m \ge 1$ be such that $\lambda_m \le \lambda < \lambda_{m+1}$ and let $\lambda < \mu < \lambda_{m+1}$. Let X_-, X_+ be as in Theorem 4.3. Let also

$$v_\rho = \chi_{\mathbb{R}^n \setminus \mathrm{B}_{2\rho}(x_0)}\,v, \quad \forall v \in X_-;$$
$$X_-^\rho = \left\{v_\rho :\; v \in X_-\right\}.$$

Lemma 6.1 *There exist $C, \overline{\rho} > 0$ such that $\overline{\mathrm{B}_{2\overline{\rho}}(x_0)} \subseteq \Omega$ and*

$$f_0(v_\rho) \leq f_0(v) + C\rho^{n-1} \left(\int_\Omega |v|^{1^*} \, dx \right)^{1/1^*}, \tag{6.3}$$

$$\int_\Omega |v_\rho|^{1^*} \, dx \geq \int_\Omega |v|^{1^*} \, dx - C\rho^n \int_\Omega |v|^{1^*} \, dx, \tag{6.4}$$

$$\int_\Omega |v_\rho| \, dx \geq \int_\Omega |v| \, dx - C\rho^n \left(\int_\Omega |v|^{1^*} \, dx \right)^{1/1^*}, \tag{6.5}$$

$$e_\rho \notin X_-^\rho \text{ and } X_-^\rho \text{ is closed in } L^1(\Omega), \tag{6.6}$$

$$X_-^\rho \cap X_+ = \{0\}, \quad \mathrm{Index}\left(X_-^\rho \setminus \{0\}\right) = \mathrm{Index}\left(L^{1^*}(\Omega) \setminus X_+\right) = m, \tag{6.7}$$

for every $v \in X_-$ and $\rho \in]0, \overline{\rho}]$.

Proof Let first $\overline{\rho} > 0$ be such that $\overline{\mathrm{B}_{2\overline{\rho}}(x_0)} \subseteq \Omega$ and let $0 < \rho \leq \overline{\rho}$. According to [2] and Theorem 4.3, we have

$$f_0(v_\rho) \leq f_0(v) + \|v\|_\infty \, |D\chi_{\mathrm{B}_{2\rho}(x_0)}|(\Omega) \leq f_0(v) + C\,\rho^{n-1}\,\|v\|_{1^*},$$

whence (6.3). The proof of (6.4) and (6.5) is similar and even simpler.

It is clear that $e_\rho \notin X_-^\rho$. From (6.3), (6.5) and Theorem 4.3 it also follows that

$$f_0(v_\rho) \leq \frac{1}{2}(\lambda_m + \mu) \int_\Omega |v_\rho| \, dx, \quad \forall v \in X_-,$$

provided that ρ is small enough. Therefore $X_-^\rho \cap X_+ = \{0\}$. Moreover, for every $v \in X_-$ we have

$$\begin{aligned} \int_\Omega |v| \, dx &\leq \mathscr{L}^n\left(\mathrm{B}_{2\rho}(x_0)\right)^{\frac{1}{n}} \left(\int_\Omega |v|^{1^*} \, dx \right)^{\frac{1}{1^*}} + \int_{\Omega \setminus \mathrm{B}_{2\rho}(x_0)} |v| \, dx \\ &\leq S^{-1}\,\mathscr{L}^n\left(\mathrm{B}_{2\rho}(x_0)\right)^{\frac{1}{n}} f_0(v) + \int_{\Omega \setminus \mathrm{B}_{2\rho}(x_0)} |v| \, dx \\ &\leq S^{-1}\lambda_m\,\mathscr{L}^n\left(\mathrm{B}_{2\rho}(x_0)\right)^{\frac{1}{n}} \int_\Omega |v| \, dx + \int_{\Omega \setminus \mathrm{B}_{2\rho}(x_0)} |v| \, dx. \end{aligned}$$

If ρ is small enough, we get

$$\int_\Omega |v| \, dx \leq C \int_{\Omega \setminus \mathrm{B}_{2\rho}(x_0)} |v| \, dx \quad \text{for every } v \in X_-.$$

First of all, it follows that we have $v_\rho = 0$ only for $v = 0$. Since $\{v \mapsto v_\rho\}$ is continuous and odd with respect to the topology of $L^{1^*}(\Omega)$ from $X_- \setminus \{0\}$ to $X_-^\rho \setminus \{0\}$, we get

$$\mathrm{Index}\left(X_-^\rho \setminus \{0\}\right) \geq \mathrm{Index}\left(X_- \setminus \{0\}\right) = \mathrm{Index}\left(L^{1^*}(\Omega) \setminus X_+\right) = m.$$

Actually, equality holds, as $X_-^\rho \setminus \{0\} \subseteq L^{1^*}(\Omega)\setminus X_+$. Finally, let $(v^{(k)})$ be a sequence in X_- with $(v_\rho^{(k)})$ convergent to some u in $L^1(\Omega)$. Then $(v^{(k)})$ is bounded in $L^1(\Omega\setminus \mathrm{B}_{2\rho}(x_0))$, hence in $L^1(\Omega)$, hence in $BV(\Omega)$. Up to a subsequence, $(v^{(k)})$ is $L^1(\Omega)$-convergent to some element of X_-, whence $u \in X_-^\rho$. Therefore, X_-^ρ is closed in $L^1(\Omega)$. □

Lemma 6.2 *There exist* $\overline{\rho}, \delta > 0$ *such that*

$$\sup\left\{f(te_\rho + u) : t \geq 0,\ u \in X_-^\rho\right\} \leq \frac{1}{n} S^n (1 - \delta\rho)^n, \quad \forall \rho \in]0, \overline{\rho}]. \tag{6.8}$$

Proof Let $\overline{\rho} > 0$ be first such that the assertion of Lemma 6.1 holds and let $0 < \rho \leq \overline{\rho}$. Since X_-^ρ is a cone, it is easily seen that

$$\begin{aligned}
&\sup\left\{f\left(te_\rho + u\right) : t \geq 0,\ u \in X_-^\rho\right\}\\
&= \frac{1}{n}\left[\sup\left\{\frac{f_0(e_\rho + u) - \lambda\|e_\rho + u\|_1}{\|e_\rho + u\|_{1^*}} : u \in X_-^\rho\right\}\right]^n\\
&= \frac{1}{n}\left[\sup\left\{\frac{\left(f_0(e_\rho) - \lambda\|e_\rho\|_1\right) + \left(f_0(u) - \lambda\|u\|_1\right)}{\left(\|e_\rho\|_{1^*}^{1^*} + \|u\|_{1^*}^{1^*}\right)^{1/1^*}} : u \in X_-^\rho\right\}\right]^n,
\end{aligned}$$

as e_ρ and u have disjoint supports. Writing $u = v_\rho$ with $v \in X_-$, the assertion we need to prove takes the form

$$\sup\left\{\frac{\left(f_0(e_\rho) - \lambda\|e_\rho\|_1\right) + \left(f_0(v_\rho) - \lambda\|v_\rho\|_1\right)}{\left(\|e_\rho\|_{1^*}^{1^*} + \|v_\rho\|_{1^*}^{1^*}\right)^{1/1^*}} : v \in X_-\right\} \leq S(1 - \delta\rho).$$

If we set $\sigma = n^{n-1}\mathscr{L}^n(\mathrm{B}_1(0))$, by (6.1), (6.2), Lemma 6.1 and the fact that $\lambda_m \leq \lambda$, we have

$$\begin{aligned}
&\frac{\left(f_0(e_\rho) - \lambda\|e_\rho\|_1\right) + \left(f_0(v_\rho) - \lambda\|v_\rho\|_1\right)}{\left(\|e_\rho\|_{1^*}^{1^*} + \|v_\rho\|_{1^*}^{1^*}\right)^{1/1^*}}\\
&\quad \leq \frac{(S^n - \sigma\rho) + \left(C\rho^{n-1}\|v\|_{1^*} + \lambda C\rho^n\|v\|_{1^*}\right)}{\left(S^n + \|v\|_{1^*}^{1^*} - C\rho^n\|v\|_{1^*}^{1^*}\right)^{1/1^*}}.
\end{aligned}$$

Now, arguing by contradiction, let $\delta = 1/k$, let $\rho_k \to 0^+$ and let $v^{(k)} \in X_-$ be such that

$$\frac{\left(f_0(e_{\rho_k}) - \lambda\|e_{\rho_k}\|_1\right) + \left(f_0(v_{\rho_k}^{(k)}) - \lambda\|v_{\rho_k}^{(k)}\|_1\right)}{\left(\|e_{\rho_k}\|_{1^*}^{1^*} + \|v_{\rho_k}^{(k)}\|_{1^*}^{1^*}\right)^{1/1^*}} > S\left(1 - \frac{\rho_k}{k}\right).$$

It follows

$$\frac{(S^n - \sigma\rho_k) + \left(C\rho_k^{n-1}\|v_k\|_{1^*} + \lambda C\rho_k^n\|v_k\|_{1^*}\right)}{\left(S^n + \|v_k\|_{1^*}^{1^*} - C\rho_k^n\|v_k\|_{1^*}^{1^*}\right)^{1/1^*}} > S\left(1 - \frac{\rho_k}{k}\right).$$

Up to subsequences, it is enough to consider the three cases:

(i) $\|v_k\|_{1^*} \to +\infty$,
(ii) $\|v_k\|_{1^*} \to \ell \in]0, +\infty[$,
(iii) $\|v_k\|_{1^*} \to 0$.

In case (i) we get

$$\frac{(S^n - \sigma\rho_k) + \left(C\rho_k^{n-1}\|v_k\|_{1^*} + \lambda C\rho_k^n\|v_k\|_{1^*}\right)}{\left(S^n + \|v_k\|_{1^*}^{1^*} - C\rho_k^n\|v_k\|_{1^*}^{1^*}\right)^{1/1^*}} \to 0$$

while in case (ii) we obtain

$$\frac{(S^n - \sigma\rho_k) + \left(C\rho_k^{n-1}\|v_k\|_{1^*} + \lambda C\rho_k^n\|v_k\|_{1^*}\right)}{\left(S^n + \|v_k\|_{1^*}^{1^*} - C\rho_k^n\|v_k\|_{1^*}^{1^*}\right)^{1/1^*}} \to \frac{S^n}{\left(S^n + \ell^{1^*}\right)^{1/1^*}} < S.$$

In both cases, a contradiction follows. In case (iii) we have, eventually as $k \to \infty$,

$$\begin{aligned}
&\frac{(S^n - \sigma\rho_k) + \left(C\rho_k^{n-1}\|v_k\|_{1^*} + \lambda C\rho_k^n\|v_k\|_{1^*}\right)}{\left(S^n + \|v_k\|_{1^*}^{1^*} - C\rho_k^n\|v_k\|_{1^*}^{1^*}\right)^{1/1^*}} \\
&\quad \le \frac{(S^n - \sigma\rho_k) + \left(C\rho_k^{n-1}\|v_k\|_{1^*} + \lambda C\rho_k^n\|v_k\|_{1^*}\right)}{S^{n-1}} \\
&\quad = S - S^{1-n}\rho_k\left(\sigma - C\rho_k^{n-2}\|v_k\|_{1^*} - \lambda C\rho_k^{n-1}\|v_k\|_{1^*}\right).
\end{aligned}$$

Then a contradiction follows also in this case. □

Proof of Theorem 1.1 Let $\lambda \ge \lambda_1$, let $m \ge 1$ be such that $\lambda_m \le \lambda < \lambda_{m+1}$ and let $\lambda < \mu < \lambda_{m+1}$. Let X_-, X_+ be as in Theorem 4.3 and let $\overline{\rho} > 0$ be small enough to guarantee that the assertions of Lemmata 6.1 and 6.2 hold.

Since $\lambda < \mu$, for every $u \in X_+$ we have

$$f(u) \ge \left(1 - \frac{\lambda}{\mu}\right) S\,\|u\|_{1^*} - \frac{1}{1^*}\,\|u\|_{1^*}^{1^*}.$$

Therefore, there exist $r_+, \alpha > 0$ such that $f(u) \ge \alpha$ for every $u \in X_+$ with $\|u\|_{1^*} = r_+$. On the other hand, since $\lambda \ge \lambda_m$, by Lemma 6.1 we also have, for every $v \in X_-$,

$$f(v_\rho) \le C\rho^{n-1}\|v\|_{1^*} + \lambda C\rho^n\|v\|_{1^*} - \frac{1}{1^*}\,\|v\|_{1^*}^{1^*} + \frac{C}{1^*}\,\rho^n\|v\|_{1^*}^{1^*} \le \frac{\alpha}{2} - \frac{1}{2\cdot 1^*}\,\|v\|_{1^*}^{1^*},$$

provided that $\rho > 0$ is small enough. Combining this fact with Lemmata 6.1 and 6.2, we see that there exists $\rho > 0$ such that $e_\rho \notin X_-^\rho$, X_-^ρ is closed in $L^1(\Omega)$ and

$$\begin{aligned}
&X_-^\rho \cap X_+ = \{0\}, \qquad \text{Index}\left(X_-^\rho \setminus \{0\}\right) = \text{Index}\left(L^{1^*}(\Omega)\setminus X_+\right) = m, \\
&\sup\left\{f\left(te_\rho + u\right) :\ t \ge 0,\ u \in X_-^\rho\right\} < \frac{1}{n}\,S^n, \\
&\sup\left\{f(u) :\ u \in X_-^\rho\right\} \le \frac{\alpha}{2}.
\end{aligned}$$

Since X_-^ρ is closed in $L^1(\Omega)$, hence in $L^{1^*}(\Omega)$, there exists $b > 0$ such that

$$\|te_\rho\|_{1^*} + \|u\|_{1^*} \le b\|te_\rho + u\|_{1^*} \qquad \text{for every } t \in \mathbb{R} \text{ and } u \in X_-^\rho$$

(see also [12]). Consequently, there exists $b' > 0$ such that

$$f_0(u) \le b'\|u\|_{1^*} \quad \text{for every } u \in \mathbb{R}e_\rho + X_-^\rho,$$

whence

$$f(u) \to -\infty \qquad \text{whenever } \|u\|_{1^*} \to \infty \text{ with } u \in \mathbb{R}e_\rho + X_-^\rho.$$

In particular, there exists $r_- > r_+$ such that $f(u) \leq 0$ whenever $u \in \mathbb{R}e_\rho + X_-^\rho$ with $\|u\|_{1^*} = r_-$.

From Theorems 2.4 and Theorem 5.3 we deduce that f admits a critical value c with $0 < c < \frac{1}{n} S^n$. By Corollary 3.5, there exists a solution $u \in BV(\Omega) \cap L^\infty(\Omega)$ of (1.6) with

$$0 < f(u) < \frac{1}{n} S^n.$$

Of course, u is a nontrivial solution. □

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