Linking type solutions for elliptic equations with indefinite nonlinearities up to the critical growth

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Abstract

In this paper we state some existence results for the semilinear elliptic equation $-\Delta u(x) - \lambda u(x) = W(x)f(u)$ where $W(x)$ is a function possibly changing sign , f has superlinear growth and λ is a positive real parameter. We discuss both the cases of subcritical and critical growth for f, and prove the existence of Linking type solutions.

1 Introduction.

Let us consider a semilinear Dirichlet problem of the kind

$$\begin{cases} -\Delta u(x) - \lambda u(x) = \; p(x, u(x)) \;\; \text{in } \Omega \subset \mathbb{R}^N, \\ \\ u(x) = 0 \text{ on } \partial\Omega. \end{cases} \tag{1.1}$$

where λ is a real parameter, p is a sufficiently regular function on $\overline{\Omega} \times \mathbb{R}$ and Ω is a bounded domain of $\mathbb{R}^N$, $(N \geq 2)$ with a smooth boundary $\partial\Omega$.

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It is well known that, under a suitable assumption of superlinear subcritical growth at infinity for p in the u variable, and in case that $P(x,\xi) = \int_0^\xi p(x,s)ds$ is positive, (1.1) has positive solutions if $\lambda < \lambda_1$, the first eigenvalue of the Laplacian in $H_0^1(\Omega)$. These solutions are found through the Mountain-Pass Theorem by Ambrosetti-Rabinowitz [4], while, for $\lambda \geq \lambda_1$, Rabinowitz proved (see [19]) the existence of Linking type solutions, which in general can change sign. Always in the case that P is positive, an extensive literature has been developed if the growth β of P is *critical*, that is the exponent of its superquadratic behaviour at zero and at infinity is given by the Sobolev critical exponent $\beta = 2^* = \frac{2N}{N-2}$ $(N \geq 3)$.

In this setting we only recall the pioneering paper by Brezis and Nirenberg (see [12]) where it was proved that there exists $\lambda^* \geq 0$ such that if $\lambda^* < \lambda < \lambda_1$ there exists a positive solution to (1.1). In this paper $\lambda^* = 0$ if $N \geq 4$. Some existence results, for $\lambda > \lambda_1$, were estabilished in [13],[14], where $P(x,\xi) = |\xi|^{2^*}$.

On the other hand, in the framework of the subcritical growth (i.e. $\beta < 2^*$), when the assumption on the sign of P falls, some important results were obtained in [7],[8], in case that $P(x,\xi) = W(x)F(\xi)$, where W changes sign, $F(\xi) = |\xi|^\beta$ as well as for more general elliptic operators including the Laplacian, and for more general choices than the power function $|\xi|^\beta$.

In the same period Alama and Tarantello stated in [2] some more general results (always for the pure Laplacian case) in order to find positive solutions of (1.1) in case that $\lambda_1 < \lambda < \Lambda^*$, with Λ^* suitably near to λ_1. In the following many other interesting papers were devoted to the existence or nonexistence of (possibly infinitely many) solutions of problem (1.1), either in the case that $\lambda \in \; [\lambda_1, \; \Lambda^*]$, or also for every λ, in case the nonlinear term satisfies some oddness assumption (see [3],[1], [6],[2],[5]).

A recent result concerning all the possible choices of λ *different* from any eigenvalue of the Laplacian, under some rather general assumptions, has been stated in [20].

The aim of this paper is twofold. From one side we extend the results by Rabinowitz for Linking type solutions of (1.1) in case that $P(x,\xi) = W(x)F(\xi)$ with W changing sign and F having superquadratic growth $\beta < 2^*$. On the other hand we also consider, for the same choice of W changing sign, the case of critical growth. In this case, the only results at our knowledge are due to Alama and Tarantello in the case $P(x,u) = W(x)|u|^{2^*}$ with W changing sign (see Theorem 4.1 of [2]).

Suitably reinforcing the assumptions given for the subcritical growth we obtain some existence results for any $\lambda \geq \lambda_1$. In this case we are able to prove, as well as in the subcritical case, the boundedness of the Palais–Smale sequences $\{u_n\}$, but it is well known that the lack of compactness of the embedding of H_0^1 into L^{2^*} does not allow to prove the compactness of $\{u_n\}$ in the critical case.

However, using some known techniques, it is possible to estimate the level c of the associated functional I where the Palais–Smale condition fails. Indeed we will prove that, below a suitable level, the PS condition is restored (see section 5). At this stage, using the same Linking structure of the subcritical case, we are able to deduce the existence of at least one solution if we show that its energy level is less than some suitable constant c^*. In order to prove this estimate we were inspired by the arguments carried out in [13]. We point out that the same geometrical conditions which provide a "Linking structure" for the associated functional I play a crucial role to show that I is actually controlled from above by the level c^*.

2 Statement of the Main Results.

We consider the following problem

$$\begin{cases} -\Delta u(x) - \lambda u(x) = \ W(x)f(u) \ \text{ in } \Omega \subset \mathbb{R}^N, \\ \\ u(x)|_{\partial\Omega} = 0. \end{cases} \tag{2.1}$$

where $W \in C(\overline{\Omega})$ is a changing sign function. Let $0 < \lambda_1 < \lambda_2 \leq ..... \leq \lambda_k \leq \lambda_{k+1} \leq ...$ be the sequence of eigenvalues of the operator $-\Delta$ with respect to the zero boundary conditions on Ω. As it is well-known, each eigenvalue λ_k has a finite multiplicity, which we choose to be coinciding with the number of its different indexes. So let us call X_k the (k-dimensional) subspace of the Sobolev space $H_0^1(\Omega)$ spanned by the eigenfunctions related to $\{\lambda_1 ... \lambda_k\}$ with $\lambda_k < \lambda_{k+1}$. We only consider here the case $\lambda \geq \lambda_1$ (for the case $\lambda < \lambda_1$ see [2],[18]). Finally $f \in C^0(\mathbb{R})$ and put

$$F(t) = \int_0^t f(\xi) d\xi \quad \forall t \in \mathbb{R}\,.$$

and, for $N \geq 3$, $2^* = \frac{2N}{N-2}$. We will prove the following theorems

Theorem 2.1 *Let us assume that the following conditions hold:*

$$f(t)t \geq \beta F(t) \quad \forall t \in \mathbb{R}\,. \tag{2.2}$$

$$|f(t)| \leq C\,|t|^{\beta-1} \quad \forall t \in \mathbb{R},\ \textit{for some}\ \beta \in (2,\, 2^*). \tag{2.3}$$

Moreover if k is the positive integer number such that $\lambda \in [\lambda_k, \lambda_{k+1})$, let W verify the following assumptions:

$$W^-(f(t)t - \beta F(t)) \leq \gamma\,|t|^2 \quad \forall\, t \geq R > 0 \ \textit{sufficiently large} \tag{2.4}$$
$$\textit{for some}\ \gamma \in \left(0, \left(\frac{\beta}{2} - 1\right)(\lambda_{k+1} - \lambda)\right)$$

where $W^- = max\{W^-(x) : x \in \overline{\Omega}\}$.

Furthermore let us assume

$$meas\{x \in \Omega \colon W(x) = 0\} = 0, \tag{2.5}$$

$$W^+(x) \not\equiv 0, \tag{2.6}$$

$$\int_\Omega W(x)F(v(x))dx \geq 0 \;\; \forall v \in X_k, \tag{2.7}$$

$$\exists \overline{v} \in X_k^\perp \setminus \{0\} : \int_\Omega W(x)F(v(x))dx \geq C_0 \int_\Omega |v(x)|^\beta \, dx, \forall v \in X_k \oplus span\{\overline{v}\}, \\ with \; \|v\| \geq R. \tag{2.8}$$

Then problem (2.1) admits a nontrivial solution u.

Remark 1 A condition similar to (2.4) was introduced in [15] in the context of periodic solutions of Hamiltonian Systems and plays a crucial role in the proof of Palais Smale condition. Indeed in case that W^- and $(F'(u)u - \beta F(u))$ are both strictly positive (otherwise (2.4) is trivially verified) the inequality gives a relation between the negative part of W and the difference in homogeneity at infinity of the function F. For example one can choose $F(u) = a_1|u|^\beta + a_2|u|^\theta$, where here $a_1 > 0$ can be choosen as an arbitrary positive number, while $a_2 > 0$ still can be taken as an arbitrary positive number if $\theta < 2$ or as γ in (2.4) in case $\theta = 2$.

Theorem 2.2 *Let all the assumptions of Theorem 2.1 be satisfied with $\beta = 2^*$, and let $N \geq 5$. Let F be convex and let us require that*

$$W \in C^3(\overline{\Omega}), \tag{2.9}$$

$$\lim_{\varepsilon \to 0} \varepsilon^{\frac{N+2}{2}} f\left(s\varepsilon^{\frac{2-N}{2}}\right) = |s|^{2^*-2}s, \quad \textit{uniformly w.r. to } s \in \mathbb{R}, \tag{2.10}$$

$$\lim_{\mu \to 0} \mu^{\frac{N-2}{4}} \int_0^{\frac{1}{\sqrt{\mu}}} \frac{\rho^2 - 1}{(1+\rho^2)^{\frac{N}{2}}} \left[f\left(\frac{\mu^{\frac{2-N}{4}}}{(1+\rho^2)^{\frac{N-2}{2}}} \right) - \frac{\mu^{-\frac{N+2}{4}}}{(1+\rho^2)^{\frac{N+2}{2}}} \right] \rho^{N-1} d\rho \; = \; 0. \tag{2.11}$$

Then problem (2.1) admits a nontrivial solution u.

Remark 2 Note that (2.2) implies , for $\beta \in (2,\, 2^*]$

$$F(t) \geq C|t|^\beta \quad \text{for } |t| > R \text{ sufficiently large.} \tag{2.12}$$

Remark 3 Let us point out that, in general, one cannot exclude that the solution u in Theorems 2.1, 2.2 can change its sign, differently from the case of Mountain Pass solutions.

Remark 4 Let us note that if one chooses $f(t) = t|t|^{2^*-2}$ then the "technical" conditions (2.10), (2.11) as well as, obviously, all the other conditions (2.2)-(2.4) (even in case that 2^* is replaced by $\beta \in (2,\, 2^*)$), are indeed satisfied.

Remark 5 A condition of the same kind as (2.11) appeared in [12], section 2 (2.44).

3 The Boundedness of Palais-Smale sequences.

We are going to find the solution to problem (2.1) by looking for (nontrivial) critical points of the functional

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \frac{\lambda}{2}\int_\Omega u^2 dx - \int_\Omega W(x)F(u)dx \qquad u \in H_0^1(\Omega). \tag{3.1}$$

In order to prove Theorem 2.1, we will first state that I satisfies the Palais–Smale condition. On the other hand, in the case $\beta = 2^*$, it can be shown that there exist some levels of I where the Palais–Smale condition does not hold. Nevertheless, for *all* $\beta \in (2, 2^*]$ the boundedness of Palais–Smale sequences is guaranteed. We recall the following

Definition 3.1 *We call $\{u_n\}_n \subset H_0^1(\Omega)$ a Palais–Smale sequence for the functional I if*

$$\{I(u_n)\} \ \textit{is bounded and} \{I'(u_n)\} \to 0 \ \textit{in} \ H^{-1}(\Omega). \tag{3.2}$$

Now one can state the following

Proposition 3.2 *Under assumptions (2.2),(2.4),(2.5), any Palais–Smale sequence for the functional I is bounded.*

Proof
Actually the properties of a Palais-Smale sequence $\{u_n\}$ can be esplicitely written as

$$C_1 \leq \frac{1}{2}\left(\|\nabla u_n\|_2^2 - \lambda\|u_n\|_2^2\right) - \int_\Omega W(x)F(u_n)dx \leq C_2 \text{ for some } C_1, C_2 > 0 \tag{3.3}$$

and

$$\sup_{\{\phi \in H_0^1, \int_\Omega |\nabla\phi|^2 dx = 1\}} \left\{ \int_\Omega \nabla u_n \nabla\phi \, dx - \lambda \int_\Omega u_n\phi \, dx - \int_\Omega W(x)f(u_n)\phi \, dx \right\} \to 0, \tag{3.4}$$

as $n \to \infty$.

Then, by (2.2),(2.4),(3.3),(3.4), one gets, for some $\{\varepsilon_n\} \to 0^+$ and some C_R (depending on the number R in (2.4) and (2.12)),

$$\int_\Omega |\nabla u_n|^2 dx = \lambda\int_\Omega u_n^2 dx + \int_\Omega W(x)f(u_n)u_n dx + \varepsilon_n\left(\int_\Omega |\nabla u_n|^2 dx\right)^{\frac{1}{2}} \geq \tag{3.5}$$

$$\geq \lambda\int_\Omega u_n^2 dx + \beta\left(\frac{1}{2}\int_\Omega |\nabla u_n|^2 dx - \frac{\lambda}{2}\int_\Omega u_n^2 dx - C_2\right) -$$

$$-\gamma\int_\Omega u_n^2 dx + C_R + \varepsilon_n\left(\int_\Omega |\nabla u_n|^2 dx\right)^{\frac{1}{2}}.$$

Let us split now u_n into the sum

$$u_n = v_n + w_n \quad \text{with } v_n \in X_k,\ w_n \in X_k^\perp. \tag{3.6}$$

Thus, if $\{e_1, \dots, e_k\}$ is an (orthogonal) basis for X_k, one has

$$v_n = \sum_{i=1}^{k} t_n^i e_i \quad \text{for some } t_n^i \in \mathbb{R},\ i = 1, \dots, k. \tag{3.7}$$

Then (3.5),(3.6),(3.7),and the variational characterization of λ_{k+1}, that is

$$\lambda_{k+1} = \min_{v \in X_k^{\perp} \setminus \{0\}} \frac{\int_\Omega |\nabla v|^2 dx}{\int_\Omega v^2 dx} \tag{3.8}$$

yield, using the Poincare' inequality,

$$\left[\varepsilon_n + (\beta/2 - 1)\left(1 - \frac{\lambda}{\lambda_{k+1}}\right) - \frac{\gamma}{\lambda_{k+1}}\right] \int_\Omega |\nabla w_n|^2 dx \leq \tag{3.9}$$

$$\leq (\beta/2 - 1) \sum_{i=1}^{k} (\lambda - \lambda_i)(t_n^i)^2 \int_\Omega e_i^2 dx + \gamma \sum_{i=1}^{n} (t_n^i)^2 \int_\Omega e_i^2 dx + C.$$

At this point, taking into account the choice of γ in (2.4), one easily deduces from (3.9) the relation

$$\int_\Omega |\nabla w_n|^2 dx \leq K_1 \sum_{i=1}^{k} (t_n^i)^2 (\lambda - \lambda_i) + K_2 \sum_{i=1}^{k} (t_n^i)^2 + K_3, \tag{3.10}$$

for some positive constant numbers K_1, K_2, K_3.
Let us prove now that $\{t_n^i\}_n$ is a bounded sequence for $i = 1, \dots, k$. By contradiction, let us suppose that, putting

$$T_n = \max_{\{i=1,\dots,k\}} |t_n^i| \quad \forall n \in N,$$

the sequence T_n is unbounded, so , at least for a subsequence, $\{T_n\} \to +\infty$ as $n \to \infty$.
Therefore the sequence $\{w_n/T_n\}$ is bounded in $H_0^1(\Omega)$, so a subsequence, also named $\{w_n/T_n\}$, weakly converges in $H_0^1(\Omega)$. Let us put

$$h(x) = \lim_{n \to \infty} (T_n)^{-1} w_n \quad \text{weakly in } H_0^1(\Omega). \tag{3.11}$$

On the other side there exists an index $\bar{l} \in \{1, \dots, k\}$ such that $|t_n^{\bar{l}}| = T_n$ for infinite indexes $n \in \mathbb{N}$. So, up to a subsequence,

$$\sum_{i=1}^{k} (t_n^i)(T_n)^{-1} e_i \ \to\ \sum_{i=1}^{k} \tau_i e_i \text{ in } H_0^1(\Omega)$$

with $|\tau_{\bar{l}}| = 1$. We claim that

$$\left| h(x) + \sum_{i=1}^{k} \tau_i e_i(x) \right|^\beta \neq 0 \text{ on a subset of } \Omega \text{ with positive measure.} \tag{3.12}$$

Indeed, if (3.12) did not hold, taking into account that

$$\int_\Omega \nabla w_n \nabla e_i dx = 0 \qquad \forall\, n \in \mathbb{N}\ , \quad \forall\, i = 1, \ldots, k \tag{3.13}$$

one would get from (3.11),(3.13), the relation

$$\sum_{i=1}^{k} (\tau_i)^2 \int_\Omega |\nabla e_i|^2 dx = -\int_\Omega |\nabla h|^2 dx = 0,$$

which would imply $\tau_i = 0$ for $i = 1, \ldots, k$, which is a contradiction with the fact that $|\tau_{\bar{l}}| = 1$. Thus (3.12) is proved.

Let us choose now a sequence $\{\psi_n\}$ in $H_0^1(\Omega)$ defined as

$$\psi_n = (T_n)^{-1} \sum_{i=1}^{k} (t_n^i e_i \,+\, w_n)\psi \tag{3.14}$$

where ψ is a suitable non-zero regular function with a compact support in Ω which will be better specified in the following. Since $\{\psi_n\}$ is bounded in $H_0^1(\Omega)$, from $\{I'(u_n)\} \to 0$ and (3.13), one gets in $H^{-1}(\Omega)$,

$$(T_n)^{-1} \int_\Omega \nabla w_n \nabla \psi_n dx - \lambda (T_n)^{-1} \int_\Omega w_n \psi_n dx \,= \tag{3.15}$$

$$= \frac{1}{T_n} \int_\Omega W(x) f\left(\sum_{i=1}^{k} t_n^i e_i \,+\, w_n \right) \psi_n dx \,+\, \eta_n \quad \text{with } \eta_n \to 0 \text{ in } \mathbb{R}\,.$$

Let us note that (2.5) implies that at least one of the relations

$$\left| h(x) + \sum_{i=1}^{k} \tau_i e_i(x) \right|^\beta W^+(x) > 0 \text{ on a subset of supp } W^+ \text{ with positive measure} \tag{3.16}$$

$$\left| h(x) + \sum_{i=1}^{k} \tau_i e_i(x) \right|^\beta W^-(x) > 0 \text{ on a subset of supp } W^- \text{ with positive measure} \tag{3.17}$$

must hold.

If, for example, (3.16) holds, one chooses $\psi \not\equiv 0$ in (3.14) as a regular non negative function, with supp $\psi \subset$ supp W^+ such that one has

$$\int_{\text{supp } W^+} W^+(x)\psi(x) \left| h(x) + \sum_{i=1}^{k} \tau_i e_i(x) \right|^\beta dx > 0. \tag{3.18}$$

Then (2.12) and the very definition of ψ_n yield, for some positive constant numbers K_3, K_4,

$$(T_n)^{-1} \int_\Omega W(x) f\left(\sum_{i=1}^{k} t_n^i e_i + w_n \right) \psi_n dx = \tag{3.19}$$

$$= (T_n)^{-2} \int_{\text{supp } W^+} W^+(x) f\left(\sum t_n^i e_i + w_n\right)\left(\sum t_n^i e_i + w_n\right)\psi(x)\, dx \geq$$

$$\geq K_3 (T_n)^{-2} \int_{\text{supp } W^+} W^+(x)\psi(x) \left|\sum_{i=1}^k t_n^i e_i + w_n\right|^\beta dx - K_4 =$$

$$= K_3 (T_n)^{\beta-2} \int_{\text{supp } W^+} W^+(x)\psi(x) \left|\sum_{i=1}^k \frac{t_n^i}{T_n} e_i + \frac{w_n}{T_n}\right|^\beta dx - K_4 ,$$

where, by (3.18)

$$b_n = \int_{\text{supp } W^+} W^+(x)\psi(x) \left|\sum_{i=1}^k \frac{t_n^i}{T_n} e_i + \frac{w_n}{T_n}\right|^\beta dx \ \to K_5 > 0 \text{ as } n \to +\infty \tag{3.20}$$

On the other hand, all the terms of the first member of (3.15) are bounded as $n \to \infty$, which is a contradiction with (3.20) since $\beta > 2$ in (3.19).
In case that (3.17) holds, the argument is quite similar: only one has to replace W^+ with W^- and ψ as in (3.18) with a non–zero regular *non positive* function $\tilde{\psi}$ with supp $\tilde{\psi} \subset$ supp W^- and such that

$$\int_{\text{supp } W^-} W^-(x)\tilde{\psi}(x) \left|h(x) + \sum_{i=1}^k \tau_i e_i(x)\right|^\beta dx < 0.$$

Therefore $\{t_n^i\}_n$ is a bounded sequence for $i = 1, \ldots, k$, hence $\{w_n\}$ is bounded in $H_0^1(\Omega)$ by (3.10), then $\{u_n\}$ given by (3.6),(3.7) is a bounded sequence in $H_0^1(\Omega)$. □

4 Proof of Theorem 2.1.

The proof of Theorem 2.1 relies on the following Linking Theorem by Rabinowitz (see [19])

Proposition 4.1 *Let E be a real Banach space with $E = E_1 \oplus E_2$, where E_2 is finite dimensional. Suppose $J \in C^1(E; \mathbb{R})$ satisfies the Palais Smale condition and the further assumptions*

(J1) $\exists \rho, \alpha > 0$ such that $J(v) \geq \alpha\ \forall v \in E_1 \colon \|v\| = \rho$,

(J2) $J(v) \leq 0 \quad \forall v \in E_2$,

(J3) $\exists \tilde{v} \in E_1$ and $\overline{R} > \rho$ such that $J(v) \leq 0\, \forall\, v \in E_2 \oplus span\{\tilde{v}\}$ with $\|v\| \geq \overline{R}$.

Then J possesses a critical point $\overline{u} \not\equiv 0$ such that

$$\bar{c} = J(\overline{u}) = \inf_{h \in \Gamma} \max_{v \in \overline{Q}} J(h(v)) \tag{4.1}$$

where

$$Q \equiv (B_{\overline{R}} \cap E_2) \oplus \left\{ r\tilde{v}: 0 < r < \overline{R} \right\}$$

and

$$\Gamma = \left\{ h \in C(\overline{Q}, E): h = id \text{ on } \partial Q \right\}.$$

Remark 6 Actually the precise characterization of c given by (4.1) is not relevant when dealing with the subcritical case (except for the fact that it guarantees the nontriviality of the solution).
On the contrary, in the critical case (see Proposition 6.6), this characterization plays a crucial role. Indeed one can prove that Proposition 4.1 still holds if the PS condition is verified only in a neighborhood of the critical level c. This fact will allow us to apply Proposition 4.1 after estimating level c.

At this point one can state the

Proof of Theorem 2.1
First of all one has to prove that the functional I verifies the Palais-Smale condition on $H_0^1(\Omega)$. As $\beta < 2^*$, this is a consequence of Proposition 3.2 and the strong convergence of a subsequence of a Palais-Smale sequence. At this point one applies Proposition 4.1. Precisely let us take $J = I$, $E = H_0^1(\Omega)$, $E_1 = (X_k)^\perp$, $E_2 = X_k$. As for $(J1)$, it is easy to conclude that (2.2),(2.3),(3.8) imply

$$I(v) \geq \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_{k+1}}\right)\rho^2 - C\rho^\beta \quad \forall v \in E_1, \|v\| = \rho,$$

then, as $\beta > 2$ one gets

$$I(v) \geq \alpha > 0 \quad \forall v \in E_1, \|v\| = \rho > 0 \text{ sufficiently small.}$$

As for the proof of $(J2)$, it is an obvious consequence of (2.7). Finally, let us verify $(J3)$. Let us choose $\tilde{v}$ as the function $\overline{v}$ as in (2.8). From the equivalence of all the norms in the finite-dimensional space $E_2 \oplus \{\tilde{v}\}$, one obtains, for some $C > 0$,

$$I(v) \leq C\,(\|v\|^2 - \|v\|^\beta) \qquad \forall v \in E_2 \oplus \{\tilde{v}\}, \|v\| \geq R$$

which yields $(J3)$ for $\overline{R} > R$ and $\beta > 2$. □

5 The behaviour of Palais–smale sequences in the critical case.

In this section we consider the case $\beta = 2^* = \frac{2N}{N-2}$ and $N \geq 5$.
It is well known that, since the embedding of $H_0^1(\Omega)$ in $L^{2^*}(\Omega)$ is not compact, the Palais–Smale condition for the functional I does not hold in general. In the next remark we state this more precisely.

Remark 7 Let us consider the sequence

$$u_n(x) \;=\; \phi(x)\frac{[N(N-2)]^{\frac{N-2}{4}}}{(\frac{1}{n^2}+|x-x_0|^2)^{\frac{N-2}{2}}}\cdot\frac{1}{n^{\frac{N-2}{2}}}\cdot\frac{1}{W(x_0)^{\frac{N-2}{4}}} \tag{5.1}$$

with $x_0 \in \mathit{supp}(W^+)$, $\phi \in C_0^\infty(\Omega)$ and $\phi \equiv 1$ in $B(x_0, r)$ where $0 < r < \mathit{dist}\,(x_0, \partial\Omega)$. It is easy to verify that $I(u_n) \to \frac{1}{N}S^{\frac{N}{2}}\frac{1}{W(x_0)^{\frac{N-2}{2}}}$ (here S denotes the best Sobolev constant), $I'(u_n) \to 0$ in H^{-1} and u_n does not admit any strongly converging subsequence in $H_0^1(\Omega)$. This shows that the Palais–Smale condition does not hold for the functional I at the level $c = \frac{1}{N}S^{\frac{N}{2}}\frac{1}{W(x_0)^{\frac{N-2}{2}}}$ for any $x_0 \in \mathit{supp}(W^+)$.

In the next proposition, which is an easy generalization of some known results in literature (see e.g, [9], [16],[17],[21]), we prove that below the levels c considered in Remark 7, the Palais–Smale condition holds for I.

Proposition 5.1 *Let us assume the same assumptions of Proposition 3.2. Suppose that F is convex and (2.10) holds. If*

$$c < \frac{1}{N}S^{\frac{N}{2}}\frac{1}{\|W^+\|_\infty^{\frac{N-2}{2}}} + I(u) = c^* + I(u), \tag{5.2}$$

for any solution u of (2.1), then the Palais–Smale condition holds at level c, i.e. any PS sequence $\{u_n\}$ s.t. $I(u_n) \to c$ admits a strongly converging subsequence in $H_0^1(\Omega)$.

Proof

We will prove that, if $\{u_n\}$ is a Palais–Smale sequence with $I(u_n) \to c$, which does not admit any subsequence strongly converging in $H_0^1(\Omega)$, then

$$c \geq I(u_0) + \frac{1}{N}S^{\frac{N}{2}}\frac{1}{\|W^+\|_\infty^{\frac{N-2}{2}}}, \tag{5.3}$$

for some u_0 solution of (2.1). We follow the same argument of [9], section 1. By Proposition 3.2 we have that

$$\|u_n\|_{H_0^1(\Omega)} \leq C \quad \text{and} \quad \left|\int_\Omega W(x)F(u_n)dx\right| \leq C,\ \forall n \in \mathbf{N}. \tag{5.4}$$

So there exists $u_0 \in H_0^1(\Omega)$ such that, up to subsequences,

$$\begin{cases} u_n \rightharpoonup u_0 & \text{weakly in } H_0^1(\Omega) \\ u_n \to u_0 & \text{strongly in } L^q(\Omega), \text{ for any } q \in [2, 2^*[\\ u_n \to u_0 & \text{almost everywhere in } \Omega. \end{cases} \tag{5.5}$$

Therefore u_0 is a solution of (2.1). Let $v_n = u_n - u_0$. Since, by contradiction,

$\{u_n\}$ does not admit any subsequence which converges strongly to u_0, we have that $\int_\Omega |\nabla v_n|^2 dx \geq \alpha > 0$. At this point, proceeding as in [9], we can find two sequences $a_n \in \Omega$ and $\varepsilon_n \searrow 0$ such that

$$\lim_{n\to\infty} \frac{dist(a_n, \partial\Omega)}{\varepsilon_n} = +\infty \tag{5.6}$$

and

$$\int_{a_n+\varepsilon_n\Omega} |\nabla v_n|^2 dx \geq \alpha > 0 \tag{5.7}$$

Set $\tilde{v}_n(x) = \varepsilon_n^{\frac{N-2}{2}} v_n(\varepsilon_n x)$ and $\Omega_n = \frac{\Omega - a_n}{\varepsilon_n}$. From (5.6) we obtain that, for any compact set $K \subset \mathbb{R}^N$, there exists $n_0 \in \mathbf{N}$ such that for any $n \geq n_0$ we have $K \subset \frac{\Omega - a_n}{\varepsilon_n}$.
By standard computations we get, for any compact $K \subset \mathbb{R}^N$,

$$-\Delta\tilde{v}_n = W(\varepsilon_n x + a_n)\varepsilon_n^{\frac{N+2}{2}} f\left(\frac{\tilde{v}_n}{\varepsilon_n^{\frac{N-2}{2}}}\right) + g_n, \text{ with } g_n \to 0 \text{ in } H^{-1}(K) \tag{5.8}$$

Since $\int_{\Omega_n} |\nabla\tilde{v}_n|^2 dx = \int_\Omega |\nabla v_n|^2 dx \leq \int_\Omega |\nabla u_n|^2 dx + \int_\Omega |\nabla u_0|^2 dx \leq C$, we deduce that

$$\tilde{v}_n \rightharpoonup U \text{ weakly in } D^{1,2}(\mathbb{R}^N) = \left\{ u \in L^{2^*}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |\nabla u|^2 dx < +\infty \right\}. \tag{5.9}$$

Finally, from (2.10) we can pass to the limit in (5.8) and so U solves the problem

$$-\Delta U = W(a_0)|U|^{2^*-2}U \quad \text{in } \mathbb{R}^N,\ U \in D^{1,2}(\mathbb{R}^N)$$

with $a_0 = \lim_{n\to\infty} a_n,\ a_0 \in \overline{\Omega}$.
Arguing as in [9] it is possible to deduce that $\tilde{v}_n \to U$ strongly in $H^1_{Loc}(\mathbb{R}^N)$.
From this and (5.7) we get that $U \not\equiv 0$.
Now we claim that $W(a_0) > 0$. Actually, since $u \in D^{1,2}(\mathbb{R}^N)$ we have that $W(a_0) \geq 0$. If $W(a_0) = 0$, U is a harmonic function in $\mathbb{R}^N$ thus it does not admit any maximum or minimum in $\mathbb{R}^N$. But this yields a contradiction with $U \in D^{1,2}(\mathbb{R}^N)$. Let us set

$$\overline{U} = W(a_0)^{\frac{N-2}{4}} U \tag{5.10}$$

which verifies

$$\begin{cases} -\Delta u = |u|^{\frac{4}{N-2}} u \text{ in } \mathbb{R}^N \\ u \in D^{1,2}(\mathbb{R}^N) \end{cases} \tag{5.11}$$

Since $\tilde{v}_n \rightharpoonup \frac{\overline{U}}{W(a_0)^{\frac{N-2}{4}}}$ weakly in $D^{1,2}(\mathbb{R}^N)$, we have

$$\liminf_{n\to\infty} \int_{\Omega_n} |\nabla\tilde{v}_n|^2 dx \geq \frac{1}{W(a_0)^{\frac{N-2}{2}}} \int_{\mathbb{R}^N} |\nabla\overline{U}|^2 dx, \tag{5.12}$$

then

$$\liminf_{n\to\infty}\left[\int \left(|\nabla u_n|^2 - \lambda u_n^2\right) dx\right] \ \geq\ \liminf_{n\to\infty}\int |\nabla v_n|^2 dx + \int |\nabla u_0|^2 dx - \lambda\int u_0^2 dx \geq \tag{5.13}$$

$$\geq\ \int |\nabla u_0|^2 dx - \lambda \int u_0^2\ dx + \frac{1}{W(a_0)^{\frac{N-2}{2}}}\int |\nabla \overline{U}|^2 dx.$$

Moreover, by Brezis-Lieb Lemma (see [11]), by the convexity of F, one has

$$\int_\Omega W(x)F(u_n)dx = \int_\Omega W(x)F(v_n)dx + \int_\Omega W(x)F(u_0)dx\ + o(1). \tag{5.14}$$

Finally, using again (2.4) and by (3.4)

$$\int_\Omega W(x)F(v_n)dx = \frac{N-2}{2N}\int_\Omega W(x)f(v_n)v_n dx + o(1) = \frac{N-2}{2N}\int_\Omega |\nabla v_n|^2 dx + o(1). \tag{5.15}$$

Hence, by (5.12)-(5.15) we get

$$c = \frac{1}{2}\int_\Omega |\nabla u_n|^2 dx\ -\ \frac{\lambda}{2}\int_\Omega u_n^2 dx\ -\ \int_\Omega W(x)F(u_n)dx\ \geq \tag{5.16}$$

$$\geq\ I(u_0) + \frac{1}{2}\int_\Omega |\nabla v_n|^2 dx\ -\ \frac{N-2}{2N}\int_\Omega |\nabla v_n|^2 dx + o(1)\ \geq$$

$$\geq\ I(u_0)\ +\ \frac{1}{\|W^+\|^{\frac{N-2}{2}}}\frac{1}{N}\int_{\mathbb{R}^N} |\nabla \overline{U}|^2 dx\ \geq\ I(u_0) + \frac{1}{\|W^+\|^{\frac{N-2}{2}}}\frac{1}{N}\int_{\mathbb{R}^N} |\nabla U_0|^2 dx$$

where U_0 is the unique positive solution of (5.11). The last inequality follows by the known fact that

$\inf_{u\in\mathcal{C}}\left(\frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{1}{2^*}\int_{\mathbb{R}^N} |u|^{2^*} dx\right)$

where $\mathcal{C} = \{u \text{ is a solution of } (5.11)\}$, is achieved at $u\ =\ U_0$.

Since $\int_{\mathbb{R}^N} |\nabla U_0|^2 dx\ =\ S^{\frac{N}{2}}$, then (5.3) follows. □

6 Proof of Theorem 2.2

In this section, we start by estimating the critical level of the functional

$$I(u)\ =\ \frac{1}{2}\int_\Omega (|\nabla u|^2 - \lambda u^2)dx\ -\ \int_\Omega W(x)F(u)dx \tag{6.1}$$

obtained via the Linking theorem.
Let us assume that $\|W\|_\infty\ =\ W(0)$ and $B(0,2) \subset supp\ W^+ \cap\ \Omega$. Let us denote by

$$U_\mu(x)\ =\ \frac{|N(N-2)\mu|^{\frac{N-2}{4}}}{W(0)^{\frac{1}{2^*-1}}|\mu + |x|^2|^{\frac{N-2}{2}}}\quad \mu > 0,$$

and $\tilde{\Psi}_\mu(x) = \phi(x)U_\mu(x)$ where $\phi \in C_0^\infty(B(0,2))$, $\phi \equiv 1$ in $B(0,1)$.
In this section we assume $N \geq 5$. Let X_k be the k-dimensional vector space considered in Theorem 2.1 and P_k the projector on $X_k^\perp$ and set

$$\Psi_\mu = P_k \tilde{\Psi}_\mu. \tag{6.2}$$

One can get the following

Lemma 6.1 *The following estimates hold*

$$\int_\Omega |\nabla \Psi_\mu|^2 dx = \frac{S^{\frac{N}{2}}}{W(0)^{\frac{N-2}{2}}} + O(\mu^{\frac{N-2}{2}}), \tag{6.3}$$

$$\int_\Omega W(x)\Psi_\mu^{2^*} dx = \int_\Omega W(x)\tilde{\Psi}_\mu^{2^*} dx + O(\mu^{\frac{N-2}{2}}), \tag{6.4}$$

$$\int_\Omega |\Psi_\mu|^{2^*-1} dx = O(\mu^{\frac{N-2}{4}}), \tag{6.5}$$

$$\int_\Omega |\Psi_\mu| dx = O(\mu^{\frac{N-2}{4}}), \tag{6.6}$$

$$\int_\Omega \Psi_\mu^2 dx = k_1\mu + O(\mu^{\frac{N-2}{2}}). \tag{6.7}$$

Proof
See [13], Remark 2.4. □

Lemma 6.2 *Let us assume (2.10), (2.11) and let $W \in C^3(\overline{\Omega})$. Then the following relation holds*

$$\int_\Omega W(x)F(t\Psi_\mu)dx = \frac{t^\beta}{\beta W(0)^{\frac{N-2}{2}}} S^{\frac{N}{2}} + c(N)\Delta W(0)\mu + o(\mu). \tag{6.8}$$

Proof
Let us look at the identity

$$\int_\Omega W(x)F(t\tilde{\Psi}_\mu)dx = \frac{t^\beta}{\beta}\int_\Omega W(x)\tilde{\Psi}_\mu^\beta dx + \int_\Omega W(x)\left(F(t\tilde{\Psi}_\mu) - \frac{t^\beta}{\beta}\Psi_\mu^\beta\right)dx. \tag{6.9}$$

Set $C_N = (N(N-2))^{\frac{N}{2}}$. Then, by Taylor's formula (recalling that 0 is the maximum point for W), one has

$$W(0)^{\frac{N}{2}}\int_\Omega W(x)\Psi_\mu^\beta dx = \mu^{\frac{N}{2}} C_N \int_{B(0,1)} \frac{W(x)}{(\mu+|x|^2)^N} dx = \tag{6.10}$$

$$W(0)\mu^{\frac{N}{2}} C_N \int_{B(0,1)} \frac{1}{(\mu+|x|^2)^N} dx + \frac{\mu^{\frac{N}{2}}}{2} C_N \sum_{i,j=1}^N \frac{\partial^2 W(0)}{\partial x_i \partial x_j}\int_{B(0,1)} \frac{x_i x_j}{(\mu+|x|^2)^N} dx +$$

$$+\ \frac{\mu^{\frac{N}{2}}}{3!}\sum_{i,j,k=1}^{N}\int_{B(0,1)}\frac{\partial^3 W(\xi)}{\partial x_i\partial x_j\partial x_k}\frac{x_ix_jx_k}{(\mu+|x|^2)^N}dx$$

for a suitable ξ belonging to the segment joining 0 and x. We have the following asymptotic estimates

$$\mu^{\frac{N}{2}}\int_{B(0,1)}\frac{1}{(\mu+|x|^2)^N}dx \ = \ C_N\int_{\mathbb{R}^N}\frac{1}{(\mu+|x|^2)^N}dx \ +O(\mu^{\frac{N}{2}}), \tag{6.11}$$

$$\frac{\mu^{\frac{N}{2}}}{2}\sum_{i,j=1}^{N}\frac{\partial^2 W(0)}{\partial x_i\partial x_j}\int_{B(0,1)}\frac{x_ix_j}{(\mu+|x|^2)^N}dx \ = \ \frac{N}{2}\Delta W(0)\omega_N\int_{\mathbb{R}^N}\frac{x_1^2}{(\mu+|x|^2)^N}\mu dx+o(\mu), \tag{6.12}$$

where ω_N is the $(N-1)$-dimensional measure of S^{N-1}, and

$$\frac{\mu^{\frac{N}{2}}}{3!}\sum_{i,j,k=1}^{N}\int_{B(0,1)}\frac{\partial^3 W(\xi)}{\partial x_i\partial x_j\partial x_k}\frac{x_ix_jx_k}{(\mu+|x|^2)^N}dx \ = \ O(\mu^{\frac{3}{2}}). \tag{6.13}$$

So (6.10) becomes

$$\begin{aligned}\int_\Omega W(x)\tilde\Psi_\mu^\beta dx \ &= \ \int_{\mathbb{R}^N}\frac{C_N}{(1+|x|^2)^N}\frac{1}{W(0)^{\frac{N-2}{2}}}dx+C(N)\frac{\Delta W(0)}{W(0)^{\frac{N}{2}}}\mu+o(\mu)= \\ &= \frac{1}{W(0)^{\frac{N-2}{2}}}S^{\frac{N}{2}}+C(N)\frac{\Delta W(0)}{W(0)^{\frac{N}{2}}}\mu+o(\mu).\end{aligned} \tag{6.14}$$

Now, by (2.3)

$$\int_\Omega W(x)\left(F(\tilde\Psi_\mu)-\frac{t^\beta}{\beta}\Psi_\mu^\beta\right)dx \ = \ \int_{B(0,1)}W(x)\left(F(\tilde\Psi_\mu)-\frac{t^\beta}{\beta}\Psi_\mu^\beta\right)dx. \tag{6.15}$$

Let us prove now that

$$\lim_{\mu\to 0}\frac{\int_{B(0,1)}W(x)\left(F(\tilde\Psi_\mu)-\frac{t^\beta}{\beta}\Psi_\mu^\beta\right)dx}{\mu} \ = \ 0. \tag{6.16}$$

By (2.10) we have that $\int_{B(0,1)}W(x)\left(F(\tilde\Psi_\mu)-\frac{t^\beta}{\beta}\Psi_\mu^\beta\right)dx \ \to 0$ as $\mu\to 0$. Hence, by De l'Hopital rule

$$\lim_{\mu\to 0}\frac{\int_{B(0,1)}W(x)\left(F(\tilde\Psi_\mu)-\frac{t^\beta}{\beta}\Psi_\mu^\beta\right)dx}{\mu} \ =$$

$$= \ \lim_{\mu\to 0}\int_{B(0,1)}W(x)\left[f\left(t\left(\frac{C_N}{W(0)}\right)^{\frac{N-2}{4}}\frac{\mu^{\frac{N-2}{2}}}{(\mu+|x|^2)^{\frac{N-2}{2}}}\right)t\left(\frac{C_N}{W(0)}\right)^{\frac{N-2}{4}}\right.$$

$$\left.\frac{N-2}{4}\mu^{\frac{N-6}{4}}\frac{|x|^2-\mu}{(\mu+|x|^2)^{\frac{N}{2}}} \ - \ \frac{t^\beta}{\beta}\left(\frac{C_N}{W(0)}\right)^{\frac{N}{2}}\frac{N}{2}\mu^{\frac{N-2}{2}}\frac{|x|^2-\mu}{(\mu+|x|^2)^{N+1}}\right]dx \ =$$

$$= \; t \left(\frac{C_N}{W(0)} \right)^{\frac{N-2}{4}} \lim_{\mu \to 0} \mu^{\frac{N-2}{4}} \int_0^{\frac{1}{\sqrt{\mu}}} W(\sqrt{\mu}\rho, \sqrt{\mu}\theta) \frac{\rho^2 - 1}{(1+\rho^2)^{\frac{N}{2}}}$$

$$\left[f\left(t \left(\frac{C_N}{W(0)} \right)^{\frac{N-2}{4}} \frac{\mu^{\frac{2-N}{4}}}{(1+\rho^2)^{\frac{N-2}{2}}} \right) \; - \; t^{\frac{N+2}{N-2}} \left(\frac{C_N}{W(0)} \right)^{\frac{N+2}{4}} \cdot \frac{\mu^{-\frac{N+2}{4}}}{(1+\rho^2)^{\frac{N+2}{2}}} \right] \rho^{N-1} \, dx.$$

Setting $b_0 \; = \; t \left(\frac{C_N}{W(0)} \right)^{\frac{N-2}{4}}$, by (2.11) we obtain (6.16).

Then, by (6.9)–(6.14) and (6.16), we deduce

$$\int_\Omega W(x) F(\tilde{\Psi}_\mu) dx \; = \; \frac{t^\beta}{\beta W(0)^{\frac{N-2}{2}}} S^{\frac{N}{2}} \; + \; C(N) \Delta W(0) \mu \; + \; o(\mu). \tag{6.17}$$

In order to get (6.8) we have to evaluate

$$\left| \int_\Omega \left(F(\Psi_\mu) - F(\tilde{\Psi}_\mu) \right) dx \right| \leq \int_\Omega \left(\int_0^1 f(\Psi_\mu - \tau P_k \Psi_\mu)) P_k \Psi_\mu d\tau \right) dx \; \leq$$

$$\leq \; c \int_0^1 d\tau \int_\Omega (|\Psi_\mu|^{2^*-1} + \tau^{2^*-1} |P_k \Psi_\mu|^{2^*-1}) |P_k \Psi_\mu| dx$$

$$\leq c \int_\Omega |\Psi_\mu|^{2^*-1} \|P_k \Psi_\mu\|_\infty dx \; + \; \int_\Omega |P_k \Psi_\mu|^{2^*} dx.$$

Recalling that

$$\|P_k \Psi_\mu\|_\infty \leq C \mu^{\frac{N-2}{4}}$$

(see [13] formulae (2.14), and (6.5) of the present paper) one gets

$$\left| \int_\Omega F(\Psi_\mu) - F(\tilde{\Psi}_\mu) dx \right| \; \leq \; c \mu^{\frac{N-2}{2}}. \tag{6.18}$$

From (6.17) and (6.18) we deduce (6.8). □

Lemma 6.3 *If μ is sufficiently small, then, for $t \leq const$ and $N \geq 5$, one has*

$$I(t\Psi_\mu) = \frac{1}{2} \int_\Omega (|\nabla(t\Psi_\mu)|^2 - \lambda |t\Psi_\mu|^2) dx \; - \; \int_\Omega W(x) F(\Psi_\mu) dx \; = \tag{6.19}$$

$$\frac{1}{\|W^+\|_\infty^{\frac{N-2}{2}}} \frac{1}{N} S^{\frac{N}{2}} + (C(N)\Delta W(0) - \lambda k_1)\mu + o(\mu).$$

Proof We recall that $W(0) = \|W^+\|_\infty$. From (6.3),(6.7) and (6.8) we get for $N \geq 4$

$$I(t\Psi_\mu) = \frac{t^2}{2} \frac{S^{\frac{N}{2}}}{W(0)^{\frac{N-2}{2}}} - \frac{t^\beta}{\beta} \frac{S^{\frac{N}{2}}}{W(0)^{\frac{N-2}{2}}} + (C(N)\Delta W(0) - \lambda k_1)\mu + o(\mu).$$

Since $\frac{t^2}{2} - \frac{t^\beta}{\beta} \leq \frac{1}{N}$ and $\Delta W(0) \leq 0$, we get (6.19). □

Lemma 6.4 *Let* $u \in X_k \oplus t\Psi_\mu$ *and* $\mu \leq const.$ *If* $u = u_k + t\Psi_\mu$, $u_k \in X_k$, *then*

$$\lim_{t\to\infty} I(u_k + t\Psi_\mu) = -\infty \text{ uniformly w.r. to } \mu. \tag{6.20}$$

Proof Let $u = \sum_{i=1}^k a_i e_i + t\Psi_\mu$ with $a_i \in \mathbb{R}$. Arguing as in the proof of Theorem 2.1, by (2.8) we get

$$I(u) = \frac{1}{2}\int_\Omega \left(|\nabla(\sum a_i e_i + t\Psi_\mu)|^2 - \lambda|\sum a_i e_i + t\Psi_\mu|^2\right) dx- \tag{6.21}$$

$$-\int_\Omega W(x)F(\sum a_i e_i + t\Psi_\mu)dx \leq$$

$$\leq c_\lambda \left(\sum a_i^2 + t^2\right) - c\int_\Omega |\sum a_i e_i + t\Psi_\mu|^{2^*} dx \leq$$

$$\leq c_\lambda \left(\sum a_i^2 + t^2\right) - c\left(\sum a_i^2 + t^2\right)^{\frac{2^*}{2}} \to -\infty \text{ as } t \to \infty.$$

□

Lemma 6.5 *Let* $u = u_k + t\Psi_\mu \in X_k \oplus t\Psi_\mu$. *Then, for* μ *small and for any* $t \in \mathbb{R}$

$$\int_\Omega W(x)F(u)dx \geq \int_\Omega W(x)F(t\Psi_\mu)dx + \frac{C_0}{2}\int_\Omega W(x)F(u_k)dx - c_1 t^\beta \mu^{\frac{N(N-2)}{2N+4}}. \tag{6.22}$$

Proof We closely follow the argument of the proof of Lemma 2.2 in [13]. For some $\theta \in [0,1]$

$$\left|\int_\Omega W(x)F(u)dx - \int_\Omega W(x)F(t\Psi_\mu)dx - \int_\Omega W(x)F(u_k)dx\right| \leq \tag{6.23}$$

$$\leq \|W\|_\infty \int_0^1 d\tau \int_\Omega |\tau u_k + \theta t\Psi\mu|^{2^*-2} t\Psi\mu |u_k| dx \leq$$

$$\leq c\left(\|u_k\|_\infty \int_\Omega |t\Psi\mu|^{2^*-1}dx + \|u_k\|_\infty^{2^*-1}\int_\Omega |t\Psi\mu| dx\right) \leq$$

$$\leq c\left(\left(\int_\Omega u_k^2 dx\right)^{\frac{1}{2}} \int_\Omega |t\Psi\mu|^{2^*-1}dx + \left(\int_\Omega u_k^{2^*} dx\right)^{\frac{2^*-1}{2^*}} \int_\Omega |t\Psi\mu| dx\right) \leq \text{ (by (6.5) and (6.6))}$$

$$\leq c\left(\left(\int_\Omega u_k^2 dx\right)^{\frac{1}{2}} t^{2^*-1}\mu^{\frac{N-2}{4}} + \left(\int_\Omega u_k^{2^*} dx\right)^{\frac{2^*-1}{2^*}} t\mu^{\frac{N-2}{4}}\right) \leq$$

$$(*) \quad \leq \frac{C_0}{2}\left(\int_\Omega u_k^2 dx\right)^{\frac{1}{2}} t^{2^*-1}\mu^{\frac{N-2}{4}} + \frac{C_0}{4}\int_\Omega u_k^{2^*} dx + \tilde{C}_0 t^{2^*}\mu^{\frac{N}{2}} \leq$$

$$\leq \frac{C_0}{2}\int_\Omega u_k^{2^*}\,dx + c_1 t^{2^*}\mu^{\frac{N(N-2)}{2N+4}}.$$

Here C_0 is the same constant appearing in (2.8). Then we have

$$\int_\Omega W(x)F(u)dx \geq \int_\Omega W(x)F(t\Psi_\mu)dx + \int_\Omega W(x)F(u_k)dx - \tag{6.24}$$

$$-\frac{C_0}{2}\int_\Omega u_k^{2^*}\,dx - c_1 t^{2^*}\mu^{\frac{N(N-2)}{2N+4}}$$

and (6.22) follows by (2.7). $\square$

Now we can prove the above mentioned estimate for I

Proposition 6.6 *For μ small enough we have, for $N \geq 4$*

$$\sup_{v\in X_k\oplus\ [\Psi_\mu]} I(v) \ < \ \frac{1}{\|W^+\|_\infty^{\frac{N-2}{2}}}\frac{1}{N}S^{\frac{N}{2}}. \tag{6.25}$$

Proof As in Lemma 2.5 of [13] let us first consider the case $\lambda \neq \lambda_k$. Let us split $u = u_k + t\Psi_\mu$ with $u_k \in X_k$ and let $\overline{\lambda} \ = \ \max\{\lambda_k$ such that $\lambda_k < \lambda\}$. By Lemma 6.4 we can suppose that t is bounded. Using (2.21) of [13] we have

$$I(u) \leq \frac{\overline{\lambda}-\lambda}{2}\int_\Omega u_k^2 dx + \frac{1}{2}t^2\int_\Omega \left(|\nabla\Psi_\mu|^2 - \lambda\Psi_\mu^2\right)dx \ - \tag{6.26}$$

$$\int_\Omega W(x)F(u_k+t\Psi_\mu)dx + c_2(\left(\int_\Omega u_k^2 dx\right)^{\frac{1}{2}}\mu^{\frac{N-2}{4}}.$$

Let us set $A(u_k,\ \mu,\ c_2) = \frac{(\overline{\lambda}-\lambda)}{2}\int_\Omega u_k^2 dx + c_2((\int_\Omega u_k^2 dx)^{\frac{1}{2}}\mu^{\frac{N-2}{4}}$ and point out that

$$A(u_k,\ \mu,\ c_2) \leq \frac{c_2^2}{2(\overline{\lambda}-\lambda)}\mu^{\frac{N-2}{2}}.$$

If $\frac{C_0}{2}\int_\Omega W(x)F(u_k)dx > c_1 t^{2^*}\mu^{\frac{N(N-2)}{2N+4}}$, by Lemma 6.5 we get

$$\int_\Omega W(x)F(u_k+t\Psi_\mu)dx > \int_\Omega W(x)F(t\Psi_\mu)dx$$

and then, by (6.19), (6.26) becomes

$$I(u) \leq \frac{1}{2}t^2\int_\Omega \left(|\nabla\Psi_\mu|^2 - \lambda\Psi_\mu^2\right)dx - \int_\Omega W(x)F(t\Psi_\mu)dx + \frac{c_2^2}{2(\overline{\lambda}-\lambda)}\mu^{\frac{N-2}{2}} = \tag{6.27}$$

$$= \frac{1}{\|W^+\|_\infty^{\frac{N-2}{2}}}\frac{1}{N}S^{\frac{N}{2}} + (C(N)\Delta W(0)-\lambda)\mu + o(\mu) < \frac{1}{\|W^+\|_\infty^{\frac{N-2}{2}}}\frac{1}{N}S^{\frac{N}{2}}.$$

On the other hand if $\frac{C_0}{2}\int_\Omega W(x)F(u_k)dx \leq c_1 t^{2^*}\mu^{\frac{N(N-2)}{2N+4}}$ we have, by (*) of (6.23)

$$\int_\Omega W(x)F(t\Psi_\mu)dx \leq \int_\Omega W(x)F(u)dx - \int_\Omega W(x)F(u_k)dx + C_0\left(\int_\Omega u_k^2 dx\right)^{\frac{1}{2}} t^{2^*-1}\mu^{\frac{N-2}{4}} + \tag{6.28}$$

$$\frac{C_0}{4}\int_\Omega u_k^{2^*}dx+\tilde{C}_0 t^{2^*}\mu^{\frac{N}{2}}\leq$$

$$\leq\int_\Omega W(x)F(u)dx-\frac{3}{4}\int_\Omega W(x)F(u_k)dx+C_0\left(\int_\Omega u_k^2dx\right)^{\frac{1}{2}}t^{2^*-1}\mu^{\frac{N-2}{4}}+\tilde{C}_0 t^{2^*}\mu^{\frac{N}{2}}\leq$$

$$\leq\int_\Omega W(x)F(u)dx+C_0\left(\int_\Omega u_k^2dx\right)^{\frac{1}{2}}\mu^{\frac{N-2}{4}}+c\mu^{\frac{N}{2}}\leq\left(\int_\Omega W(x)F(u)dx+c\mu^{\frac{N^2-2N}{2N+4}}+\mu^{\frac{N}{2}}\right)=$$

$$=\int_\Omega W(x)F(u)dx+o(\mu).$$

Hence (6.26) becomes

$$I(u)\leq\frac{1}{2}t^2\int_\Omega\left(|\nabla\Psi_\mu|^2-\lambda\Psi_\mu^2dx\right)-\int_\Omega W(x)F(u_k+t\Psi_\mu)dx+A(u_k,\ \mu,\ c_2)\leq$$
$$\leq I(t\Psi_\mu)+o(\mu)\ <\ \frac{1}{\|W^+\|_\infty^{\frac{N-2}{2}}}\frac{1}{N}S^{\frac{N}{2}}. \tag{6.29}$$

If $\lambda=\lambda_k$ the claim follows in an analogous way (see [13]). □

Now we are in the position to give the

Proof of Thorem 2.2 We will apply Proposition 4.1 with the choices $E_1=X_k\oplus[\Psi_\mu]$ and $E_2=E_1^\perp$. Actually one has two possibilities.
The first one is that $\overline{c}$ given by (4.1) is a critical value of the functional I, i.e. there exists $\overline{u}$ such that $I(\overline{u})=\overline{c}$ and $\overline{u}$ is a nontrivial solution of (2.1), thus Theorem 2.2 is proved.
Otherwise, if $\overline{c}$ is not a critical value one would have the following statement

$$\exists u_0\not\equiv 0,\ \text{with } I(u_0)\neq\overline{c}\text{ s.t. } u_0\text{ solves (2.1)}, \tag{6.30}$$

thus still Theorem 2.2 would be proved.
Actually, if (6.30) was false, then by Proposition 5.1 (where u in (5.2) would be given by the only trivial solution $u=0$, so $I(u)=0$) one would get $\overline{c}$ given by (4.1) as a level where any PS sequence admits strongly converging subsequences, since, by definition,

$$\overline{c}\leq\sup\{I(v):\ v\in X_k\oplus[\Psi_\mu]\}$$

and (6.25) holds. Therefore, as a consequence of Proposition 5.1 (see Remark 6), $\overline{u}$ would be a solution of (2.1) which contradicts the hypothesis we started with. □

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