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Multiplicity of solutions for semilinear variational inequalities via linking and ∇-theorems ☆

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Abstract

We prove the existence of three distinct nontrivial solutions for a class of semilinear elliptic variational inequalities involving a superlinear nonlinearity. The approach is variational and is based on linking and ∇-theorems. Some nonstandard adaptations are required to overcome the lack of the Palais–Smale condition.

Keywords: Semilinear elliptic variational inequalities; Linking theorems; ∇-theorems; Local Palais–Smale condition

1. Introduction

In this paper we will give a multiplicity result for solutions of the following class of semilinear elliptic variational inequalities:

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$$\begin{cases} u \in H_0^1(\Omega), \quad \psi_1 \leqslant u \leqslant \psi_2 \quad \text{in } \Omega, \\ \langle Au, v-u\rangle - \lambda \int_\Omega u(v-u)\,dx \geqslant \int_\Omega p(x,u)(v-u)\,dx, \quad \forall v \in H_0^1(\Omega) \\ \quad \text{with } \psi_1 \leqslant v \leqslant \psi_2 \quad \text{in } \Omega, \end{cases} \tag{$\mathcal{P}$}$$

where A is a uniformly elliptic operator, λ is a real parameter, Ω is a bounded open subset of $\mathbb{R}^N$, $N \geqslant 1$, ψ_1 and ψ_2 are the "obstacles" and $p : \Omega \times \mathbb{R} \to \mathbb{R}$ is superlinear at zero and infinity in the second variable (see Section 2 for the precise assumptions on ψ_1, ψ_2 and p).

In the case of equations, namely for $\psi_1 \equiv -\infty$ and $\psi_2 \equiv +\infty$, the existence of multiple solutions for $(\mathcal{P})$ has been extensively treated in the literature, starting from the pioneering paper of Ambrosetti and Rabinowitz [1] (see, e.g., [23] and references therein). More recently, new multiplicity results for semilinear elliptic equations have been obtained by means of the so-called "theorems of mixed type" or "∇-theorems" introduced by Marino and Saccon in [16] (see also [15,19,20]).

On the other hand, multiple solutions for the obstacle problem $(\mathcal{P})$ have been obtained by several authors (see [15,20–22,25–27]). In [9] the existence of two solution for a fully nonlinear variational inequality is achieved by means of the nonsmooth critical point theory introduced in [6,7]. Mainly this problem was studied in the case in which the nonlinearity p has linear growth at infinity in the second variable. Here we are interested in the case of superlinear nonlinearities, already considered in [8,14,18]. In particular, our results should be compared with those of [8,18], where the existence of a nontrivial solution to $(\mathcal{P})$ is proved by a mountain pass and a linking argument, respectively. Let us stress that, in order to prove the boundedness of Palais–Smale sequences, in [8,18] the obstacles ψ_1 and ψ_2 are assumed to belong to $H_0^1(\Omega)$.

The purpose of our paper is that of proving the existence of three solutions to $(\mathcal{P})$ (see Theorem 2.1) by combining the linking technique with that of ∇-theorems. An important feature is that, in our setting, it is not clear whether Palais–Smale sequences are bounded or not. For the application of the linking theorem, this is related to the fact that the obstacles ψ_1 and ψ_2 are here just Borel functions, while for the application of the ∇-theorems the problem would arise also for smooth obstacles. To overcome this difficulty, we adapt both the linking and the ∇-theorem to a situation in which the global Palais–Smale condition is substituted by a local Palais–Smale condition combined with a quantitative gradient estimate on a suitable bounded set. In Section 3 (see Theorems 3.10 and 3.11) we will prove such adaptations in the setting of the metric critical point theory developed in [6,7, 11,13].

The variational approach we use to treat problem $(\mathcal{P})$ is described in Section 4. In Section 5 we will get the existence of at least two solutions for problem $(\mathcal{P})$ by a suitable ∇-theorem (see Theorem 5.1), while in Section 6 we will find a third solution of higher energy, by a linking argument (see Theorem 6.2).

2. The main result

Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \geqslant 1$, and let $H_0^1(\Omega)$ be the usual Sobolev space, i.e. the closure of $C_0^\infty(\Omega)$ in $H^1(\Omega)$. We denote by $\langle \cdot, \cdot \rangle$ the pairing between $H_0^1(\Omega)$ and its dual space $H^{-1}(\Omega)$.

We are interested in the existence and multiplicity of solutions u of semilinear elliptic variational inequalities of the form

$$\begin{cases} u \in K_\psi, \\ \langle Au, v-u\rangle - \lambda \int_\Omega u(v-u)\,dx \geqslant \int_\Omega p(x,u)(v-u)\,dx, \quad \forall v \in K_\psi. \end{cases} \tag{$\mathcal{P}$}$$

Here A denotes a uniformly elliptic operator of the form

$$A = -\sum_{i,j=1}^{N} D_i\big(a_{ij}(x)D_j\big)$$

with $a_{ij} : \Omega \to \mathbb{R}$, $i, j = 1, \dots, N$, satisfying the following conditions:

(A1) $a_{ij} \in L^\infty(\Omega)$;
(A2) $a_{ij}(x) = a_{ji}(x)$ a.e. x in Ω, $\forall i, j = 1, \dots, N$;
(A3) $\exists c_1, c_2 > 0$ such that

$$c_1|\xi|^2 \leqslant \sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \leqslant c_2|\xi|^2, \quad \forall \xi \in \mathbb{R}^N, \text{ a.e. } x \text{ in } \Omega.$$

Moreover, λ is a real parameter, $\psi_1 : \Omega \to [-\infty, 0]$ and $\psi_2 : \Omega \to [0, +\infty]$ are Borel functions, $K_\psi = \{v \in H_0^1(\Omega)\colon\ \psi_1 \leqslant \tilde{v} \leqslant \psi_2$ cap. q.e. in $\Omega\}$, where $\tilde{v}$ is a quasicontinuous representative of v, and $p : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function such that:

(P1) $p(x,\xi)$ is measurable in x for every $\xi \in \mathbb{R}$ and continuous in ξ for a.e. $x \in \Omega$;
(P2) $|p(x,\xi)| \leqslant a_1|\xi| + a_2|\xi|^s$ for some $a_1, a_2 > 0$, $\forall \xi \in \mathbb{R}$, a.e. x in Ω, with

$$s > 1, \quad \text{if } N = 1, 2, \quad \text{and} \quad 1 < s < \frac{N+2}{N-2}, \quad \text{if } N \geqslant 3;$$

(P3) for a.e. $x \in \Omega$, we have $p(x,\xi) = o(|\xi|)$ as $\xi \to 0$;
(P4) there exists $\mu > 2$ such that

$$0 < \mu P(x,\xi) \leqslant \xi p(x,\xi), \quad \forall \xi \in \mathbb{R} \setminus \{0\}, \text{ a.e. } x \text{ in } \Omega,$$

$$\text{where} \quad P(x,\xi) := \int_0^\xi p(x,t)\,dt, \quad \forall \xi \in \mathbb{R}, \text{ a.e. } x \text{ in } \Omega.$$

Given a positive real number R and a compact subset C of Ω, we will also consider the condition:

($\psi_{R,C}$) $\psi_1(x) \leqslant -R < R \leqslant \psi_2(x)$ cap. q.e. in C.

Let $\lambda_1 < \lambda_2 \leqslant \cdots$ be the sequence of the eigenvalues of the problem

$$\begin{cases} Au = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

repeated according to their multiplicity, and let us set $\lambda_0 = -\infty$.

Let $J_\lambda : H_0^1(\Omega) \to \mathbb{R}$ be the functional defined as

$$J_\lambda(u) = \frac{1}{2}\langle Au, u\rangle - \frac{\lambda}{2}\int_\Omega u^2\,dx - \int_\Omega P(x,u)\,dx.$$

From the assumptions on p, ψ_1 and ψ_2 it follows that $u \equiv 0$ is a solution of $(\mathcal{P})$.

Our main result is the following:

Theorem 2.1. *Let $1 \leqslant k \leqslant j$ be such that $\lambda_{k-1} < \lambda_k = \cdots = \lambda_j < \lambda_{j+1}$. Then, there exist $\eta > 0$, $R > 0$ and a compact set $C \subset \Omega$ such that, if $\lambda_k - \eta \leqslant \lambda < \lambda_k$ and ψ_1, ψ_2 satisfy $(\psi_{R,C})$, problem $(\mathcal{P})$ admits at least three nontrivial solutions.*

The proof will be given in Section 6.

Remark 2.2. In general, the solutions of problem $(\mathcal{P})$ given by Theorem 2.1 do not solve the corresponding equation. Indeed, suppose that Ω has smooth boundary, that $A = -\Delta$ and that $p(x,\xi) = |\xi|^{s-1}\xi$. Let $\varphi : \mathbb{R}^N \to \mathbb{R}$ be a function of class C^∞ with $\varphi = 0$ in $\mathbb{R}^N \setminus \Omega$ and $\varphi > 0$ in Ω.

From Theorem 2.1 it follows that, if $\lambda_k - \eta \leqslant \lambda < \lambda_k$ with η small enough and $\psi_1 = -R\varphi$, $\psi_2 = R\varphi$ with R large enough, then problem $(\mathcal{P})$ has at least three solutions.

On the other hand, if u is a solution of $(\mathcal{P})$ which also solves $-\Delta u - \lambda u = |u|^{s-1}u$, then we have that u is identically zero. To see it, observe that $u \in C^1(\overline{\Omega})$ by regularity theory. Since $-R\varphi \leqslant u \leqslant R\varphi$, it follows that also the normal derivative of u vanishes on $\partial\Omega$. Therefore we have

$$\int_\Omega \nabla u \cdot \nabla v\,dx - \lambda\int_\Omega uv\,dx = \int_\Omega |u|^{s-1}uv\,dx, \quad \forall v \in H^1(\Omega).$$

Let B be an open ball such that $\overline{\Omega} \subset B$. If we extend u to B with value 0 outside Ω, then u is a solution in B of the equation $-\Delta u = Vu$, where $V = \lambda + |u|^{s-1}$. Since u is identically zero on the open set $B \setminus \overline{\Omega}$, from [12, Theorem 6.3] it follows that u is identically zero on B.

Remark 2.3. Let us observe that, since we can allow $\psi_1 \equiv -\infty$ and $\psi_2 \equiv +\infty$, Theorem 2.1 gives a multiplicity result for the associated equation too, as proved in [19].

3. Critical point theorems

Let X be a metric space endowed with the metric d and let $J : X \to \mathbb{R} \cup \{+\infty\}$ be a function. We set

$$J^c = \{u \in X:\ J(u) \leqslant c\}, \quad c \in \mathbb{R},$$
$$\operatorname{epi}(J) = \{(u,s) \in X \times \mathbb{R}:\ J(u) \leqslant s\},$$

and denote by $\mathrm{B}_r(u)$ the open ball of center u and radius r.

The next definition is taken from [2, Definition 2.1]. For an equivalent approach, see [6,7] and, when J is continuous, [13].

Definition 3.1. For every $u \in X$ with $J(u) < +\infty$, we denote by $|dJ|(u)$ the supremum of the σ's in $[0, +\infty[$ such that there exist a neighborhood W of $(u, J(u))$ in $\operatorname{epi}(J)$, $\delta > 0$ and a continuous map $\mathcal{H} : W \times [0, \delta] \to X$ satisfying

$$d\big(\mathcal{H}\big((z,s),t\big), z\big) \leqslant t, \qquad J\big(\mathcal{H}\big((z,s),t\big)\big) \leqslant s - \sigma t,$$

whenever $(z,s) \in W$ and $t \in [0, \delta]$.

The extended real number $|dJ|(u)$ is called the *weak slope* of J at u. If $u \in X$, $J(u) < +\infty$ and $|dJ|(u) = 0$, we say that u is a (*lower*) *critical point* of J. If $c \in \mathbb{R}$ and there exists a critical point u of J with $J(u) = c$, we say that c is a *critical value* of J.

Remark 3.2. Let $(u_n)_n$ be a sequence in X. If $u_n \to u$ with $J(u) < +\infty$ and $J(u_n) \to J(u)$ as $n \to \infty$, then $\liminf_{n\to\infty} |dJ|(u_n) \geqslant |dJ|(u)$.

According to [2, Proposition 2.2], the above definition can be simplified when J is real-valued and continuous. Indeed we have the following

Proposition 3.3. *Let $J : X \to \mathbb{R}$ be a continuous function. Then $|dJ|(u)$ is the supremum of the σ's in $[0, +\infty[$ such that there exist a neighborhood U of u in X, $\delta > 0$ and a continuous map $\mathcal{H} : U \times [0, \delta] \to X$ satisfying*

$$d\big(\mathcal{H}(z,t), z\big) \leqslant t, \qquad J\big(\mathcal{H}(z,t)\big) \leqslant J(z) - \sigma t,$$

whenever $z \in U$ and $t \in [0, \delta]$.

Coming back to the general setting, now consider the continuous function $\mathcal{G}_J$: $\operatorname{epi}(J) \to \mathbb{R}$ defined as $\mathcal{G}_J(u,s) = s$. The main feature of $\mathcal{G}_J$ is that it allows to reduce, at a certain extent, the study of general functions to that of continuous functions. According to [2, Proposition 2.3], the key connection is given by the next

Proposition 3.4. *Consider in* $\operatorname{epi}(J)$ *the metric*

$$d\big((u,s),(v,t)\big) = \sqrt{d(u,v)^2 + (s-t)^2}, \tag{3.1}$$

so that the function $\mathcal{G}_J$ is Lipschitz continuous of constant 1.

Then, for every $u \in X$ with $J(u) < +\infty$, we have

$$|dJ|(u) = \begin{cases} \dfrac{|d\mathcal{G}_J|(u, J(u))}{\sqrt{1 - |d\mathcal{G}_J|(u, J(u))^2}} & \text{if } |d\mathcal{G}_J|(u, J(u)) < 1, \\ +\infty & \text{if } |d\mathcal{G}_J|(u, J(u)) = 1. \end{cases}$$

Definition 3.5. Let $c \in \mathbb{R}$. A sequence $(u_n)_n$ in X is said to be a *Palais–Smale sequence at level c* (*$(\mathrm{PS})_c$-sequence*, for short) for J, if $J(u_n) \to c$ and $|dJ|(u_n) \to 0$.

Given a closed subset A of X, we say that J satisfies the *Palais–Smale condition on A at level c* ($(\mathrm{PS})_c$ on A, for short), if every $(\mathrm{PS})_c$-sequence $(u_n)_n$ for J, with $u_n \in A$ for any $n \in \mathbb{N}$, admits a convergent subsequence. In the case $A = X$, we omit the indication of A.

For the next critical point theorem we need the notion of relative category, which was introduced in [24] and then reconsidered, with minor variants, by several authors. Here we follow the approach of [3].

Definition 3.6. Let X be a metric space and Y a closed subset of X. For every closed subset A of X, we define the *relative category* of A in (X, Y), denoted by $\mathrm{cat}_{X,Y} A$, as the least integer $n \geqslant 0$ such that A can be covered by $n + 1$ open subsets $U_0, \ldots, U_n$ of X with the following properties:

(a) there exists a deformation $\mathcal{K}: X \times [0, 1] \to X$ with $\mathcal{K}(Y \times [0, 1]) \subset Y$ and $\mathcal{K}(U_0 \times \{1\}) \subset Y$ (if $Y = \emptyset$, we mean that U_0 must be empty);
(b) for $1 \leqslant k \leqslant n$, each U_k is contractible in X.

If no such integer n exists, we set $\mathrm{cat}_{X,Y} A = \infty$.

Finally, the *category* of A in X, denoted by $\mathrm{cat}_X A$, is defined as the category of A in $(X, \emptyset)$.

The next result is an adaptation to our setting of one of the "∇-theorems" proved in [17].

Theorem 3.7. *Let E be a Banach space such that $E = E_1 \oplus E_2 \oplus E_3$, with $\dim(E_1) < \infty$, $1 \leqslant \dim(E_2) < \infty$ and E_3 closed in E, let X be a closed subset of E and let $J: X \to \mathbb{R}$ be a continuous function satisfying the following assumptions*:

(a) *there exist $\alpha, \beta \in \mathbb{R}$, $0 < r_1 < r_2 < r_3$ and a bounded, convex, open neighborhood B of 0 in E_2 such that*

$$\Delta := \{u \in E_1 \oplus E_2\colon \|P_1 u\| \leqslant r_3,\ r_1 \leqslant \|P_2 u\| \leqslant r_3\} \subset X,$$

$$\sup\{J(u)\colon u \in \partial_{E_1 \oplus E_2} \Delta\} < \alpha < \inf\{J(u)\colon u \in X \cap (E_2 \oplus E_3),\ \|u\| = r_2\},$$

$$\sup\{J(u)\colon u \in \Delta\} < \beta,$$

$$\Delta \subset M := \{u \in E\colon P_2 u \notin B\};$$

(b) *J satisfies* $(\mathrm{PS})_c$ *for every* $c \in [\alpha, \beta]$ *and we have*

$$\inf\big\{\big|d(J|_{X\cap M})\big|(u):\ u \in X,\ P_2u \in \partial B,\ \alpha \leqslant J(u) \leqslant \beta\big\} > 0.$$

Then, either

(i) *there exist two distinct critical values* $c_1, c_2 \in [\alpha, \beta]$ *of* J

or

(ii) *there exists a critical value* $c \in [\alpha, \beta]$ *of* J *and a compact subset of*

$$\big\{u \in X:\ |dJ|(u) = 0,\ J(u) = c,\ P_2u \notin \overline{B}\big\}$$

having at least category 2 *in* $\{u \in X:\ P_2u \notin \overline{B}\}$.

Proof. Let us consider the restricted function $\widehat{J} := J|_{X\cap M}$ defined on the closed subset $X \cap M$ of E. Moreover, let α', β' be such that

$$\alpha < \alpha' < \inf\big\{J(u):\ u \in X \cap (E_2 \oplus E_3),\ \|u\| = r_2\big\},$$
$$\sup\big\{J(u):\ u \in \Delta\big\} < \beta' < \beta.$$

First of all, let us prove that

$$\operatorname{cat}_{X\cap M, \widehat{J}^{\alpha'}} \widehat{J}^{\beta'} \geqslant 2. \tag{3.2}$$

Indeed, by assumption (a) we can consider the inclusion map

$$i : (\Delta, \partial_{E_1\oplus E_2}\Delta) \to (X \cap M, \widehat{J}^{\alpha'}).$$

If we define $\pi_1 : M \to \Delta$ by

$$\pi_1(u) = \min\left\{\frac{r_3}{\|P_1u\|}, 1\right\} P_1u + \min\big\{\max\big\{\|P_2u + P_3u\|, r_1\big\}, r_3\big\}\frac{P_2u}{\|P_2u\|},$$

we have that π_1 is a retraction satisfying

$$\pi_1\big(M \setminus \big\{u \in E_2 \oplus E_3:\ \|u\| = r_2\big\}\big) \subset \Delta \setminus \big\{u \in E_2:\ \|u\| = r_2\big\}.$$

On the other hand, it is easy to find a retraction $\pi_2 : \Delta \setminus \{u \in E_2:\ \|u\| = r_2\} \to \partial_{E_1\oplus E_2}\Delta$ such that $(1 - t)u + t\pi_2(u) \in \Delta$ for every $t \in [0, 1]$. By the choice of α', we have that $J^{\alpha'}$ is disjoint from $\{u \in E_2 \oplus E_3:\ \|u\| = r_2\}$. Let $\vartheta : E \to [0, 1]$ be a continuous function with

$\vartheta = 1$ on $J^{\alpha'}$ and $\vartheta = 0$ in a neighborhood of $\{u \in E_2 \oplus E_3\colon \|u\| = r_2\}$. Then the map $g : X \cap M \to \Delta$ defined as

$$g(u) = \begin{cases} \pi_1(u) & \text{if } u \in E_2 \oplus E_3 \text{ and } \|u\| = r_2, \\ (1 - \vartheta(u))\pi_1(u) + \vartheta(u)\pi_2(\pi_1(u)) & \text{otherwise}, \end{cases}$$

is continuous and satisfies $g(\widehat{J}^{\alpha'}) \subset \partial_{E_1 \oplus E_2} \Delta$. Moreover, $\mathcal{H} : \Delta \times [0, 1] \to \Delta$ defined as $\mathcal{H}(u, t) = (1 - t)u + tg(u)$ is a homotopy between the identity map of $(\Delta, \partial_{E_1 \oplus E_2} \Delta)$ and $g \circ i$.

Since $\Delta \subset \widehat{J}^{\beta'}$, we have $\Delta = i^{-1}(\widehat{J}^{\beta'})$, hence by [3, Theorem 1.4.5]

$$\mathrm{cat}_{\Delta, \partial_{E_1 \oplus E_2} \Delta} \Delta \leqslant \mathrm{cat}_{X \cap M, \widehat{J}^{\alpha'}} \widehat{J}^{\beta'}.$$

On the other hand, it is well known (see, e.g., [17, Lemma 2.3]) that

$$\mathrm{cat}_{\Delta, \partial_{E_1 \oplus E_2} \Delta} \Delta = 2.$$

Then (3.2) follows.

Now let us observe that, thanks to (b), the function $\widehat{J}$ satisfies $(\mathrm{PS})_c$ for every $c \in [\alpha', \beta']$. Indeed, if $(u_n)_n$ is a $(\mathrm{PS})_c$-sequence for $\widehat{J}$, then $J(u_n) \in [\alpha, \beta]$ and $P_2 u_n \notin \overline{B}$ eventually as $n \to \infty$. It follows that $|d\widehat{J}|(u_n) = |dJ|(u_n)$, whence the fact that $(u_n)_n$ admits a convergent subsequence by assumption (b).

By [3, Theorem 1.4.11] we can define $\alpha' \leqslant c_1 \leqslant c_2 \leqslant \beta'$ such that each c_i, $i = 1, 2$, is a critical value of $\widehat{J}$. Moreover, as in the proof of the Palais–Smale condition, we have that each critical point u of $\widehat{J}$ at level c_i satisfies $P_2 u \notin \overline{B}$ and is critical also for J.

If $c_1 < c_2$, then assertion (i) immediately follows. Otherwise, set $c := c_1 = c_2$ and consider $C = \{u \in X \cap M\colon |d\widehat{J}|(u) = 0,\ \widehat{J}(u) = c\}$. Since $\widehat{J}$ satisfies $(\mathrm{PS})_c$, it is clear that C is compact. Moreover, according to [3, Theorem 1.4.11], we have $\mathrm{cat}_{X \cap M} C \geqslant 2$. On the other hand, we have that $C \subset \{u \in X\colon P_2 u \notin \overline{B}\}$ and each $u \in C$ is a critical point of J. Since $\{u \in X\colon P_2 u \notin \overline{B}\}$ is open in $X \cap M$, it is easy to see that C has at least category 2 also in $\{u \in X\colon P_2 u \notin \overline{B}\}$ and assertion (ii) follows. □

Definition 3.8. A metric space X is said to be *locally contractible* if, for every $u \in X$ and every neighborhood U of u, there exists a neighborhood V of u with $V \subset U$ and V is contractible in U.

Lemma 3.9. *Let X be a metric space, $J : X \to \mathbb{R}$ a continuous function, $\beta \in \mathbb{R}$, $\sigma, \varrho > 0$ and U an open subset of X. Assume that*

$$\forall u \in \mathrm{B}_\varrho(U) \setminus U, \quad \beta - \sigma\varrho < J(u) \leqslant \beta \quad \Rightarrow \quad |dJ|(u) > \sigma. \tag{3.3}$$

Then, if we set

$$Y = \bigl\{u \in X\colon J(u) \leqslant \beta - \sigma d\bigl(u, J^\beta \cap U\bigr)\bigr\},$$

we have

$$\big|d(J|_Y)\big|(u) \geqslant |dJ|(u), \quad \forall u \in Y \cap U, \tag{3.4}$$

$$\forall u \in Y \setminus U \quad J(u) > \beta - \sigma\varrho \quad \Rightarrow \quad \big|d(J|_Y)\big|(u) > \sigma. \tag{3.5}$$

Proof. To prove (3.4), consider $u \in Y \cap U$. If $|dJ|(u) = 0$, then the assertion is trivial. So let us suppose $|dJ|(u) > 0$ and take any $\sigma' > 0$ with $\sigma' < |dJ|(u)$. By Proposition 3.3, there exist δ and a continuous map $\mathcal{H} : \mathrm{B}_\delta(u) \times [0, \delta] \to X$ such that

$$\forall v \in \mathrm{B}_\delta(u),\ \forall t \in [0, \delta], \quad d\big(\mathcal{H}(v, t), v\big) \leqslant t, \qquad J\big(\mathcal{H}(v, t)\big) \leqslant J(v) - \sigma' t. \tag{3.6}$$

The first inequality easily yields $\mathcal{H}(u, 0) = u$, so, by the continuity of $\mathcal{H}$ and by the fact that U is open, by possibly reducing δ, we may assume that $\mathcal{H}(\mathrm{B}_\delta(u) \times [0, \delta]) \subset U$. Now, let us consider $v \in \mathrm{B}_\delta(u) \cap Y$. One has $J(\mathcal{H}(v, t)) \leqslant J(v) \leqslant \beta$. So $\mathcal{H}(v, t) \in J^\beta \cap U$, which yields $d(\mathcal{H}(v, t), J^\beta \cap U) = 0$. It follows

$$J\big(\mathcal{H}(v, t)\big) \leqslant \beta - \sigma' d\big(\mathcal{H}(v, t), J^\beta \cap U\big),$$

which yields $\mathcal{H}((\mathrm{B}_\delta(u) \cap Y) \times [0, \delta]) \subset Y$. By Proposition 3.3 we get $|d(J|_Y)|(u) \geqslant \sigma'$ and the assertion follows by the arbitrariness of $\sigma' < |dJ|(u)$.

In order to prove (3.5), consider $u \in Y \setminus U$ with $J(u) > \beta - \sigma\varrho$. First of all, we have

$$\beta - \sigma\varrho < J(u) \leqslant \beta - \sigma d\big(u, J^\beta \cap U\big),$$

whence $\varrho > d(u, J^\beta \cap U) \geqslant d(u, U)$. It follows that $u \in \mathrm{B}_\varrho(U) \setminus U$ which yields $|dJ|(u) > \sigma$ by (3.3). Let $\sigma < \sigma' < |dJ(u)|$, $\delta > 0$ and $\mathcal{H} : \mathrm{B}_\delta(u) \times [0, \delta] \to X$ be a continuous map satisfying (3.6). If we consider $v \in \mathrm{B}_\delta(u) \cap Y$, we have

$$\begin{aligned}
J\big(\mathcal{H}(v, t)\big) &\leqslant J(v) - \sigma' t \leqslant \beta - \sigma d(v, J^\beta \cap U) - \sigma' t \\
&\leqslant \beta - \sigma d(v, J^\beta \cap U) - \sigma t \leqslant \beta - \sigma d(v, J^\beta \cap U) - \sigma d\big(\mathcal{H}\big((v, t), v\big)\big) \\
&= \beta - \sigma\big[d\big(\mathcal{H}(v, t), v\big) + d(v, J^\beta \cap U)\big] \\
&\leqslant \beta - \sigma\big[d\big(\mathcal{H}(v, t), J^\beta \cap U\big)\big],
\end{aligned}$$

whence $\mathcal{H}(v, t) \in Y$. It follows that $\mathcal{H}((\mathrm{B}_\delta(u) \cap Y) \times [0, \delta]) \subset Y$. Again, by Proposition 3.3 we get $|d(J|_Y)|(u) \geqslant \sigma' > \sigma$ and the assertion follows. □

In the next result we prove the "∇-theorem" we will actually use. With respect to Theorem 3.7, the main difference is that the Palais–Smale condition is required to hold only on the closure of a suitable open subset.

Theorem 3.10. *Let E be a Banach space such that $E = E_1 \oplus E_2 \oplus E_3$, with* $\dim(E_1) < \infty$, $1 \leqslant \dim(E_2) < \infty$ *and E_3 closed in E, let X be a closed subset of E and let $J : X \to \mathbb{R}$ be a continuous function satisfying the following assumptions*:

(a) *there exist* $\alpha, \beta \in \mathbb{R}$, $0 < r_1 < r_2 < r_3$ *and a bounded, convex, open neighborhood* B *of* 0 *in* E_2 *such that*

$$\Delta := \bigl\{u \in E_1 \oplus E_2\colon \ \|P_1 u\| \leqslant r_3,\ r_1 \leqslant \|P_2 u\| \leqslant r_3\bigr\} \subset X,$$
$$\sup\bigl\{J(u)\colon\ u \in \partial_{E_1\oplus E_2}\Delta\bigr\} < \alpha < \inf\bigl\{J(u)\colon\ u \in X \cap (E_2 \oplus E_3),\ \|u\| = r_2\bigr\},$$
$$\sup\bigl\{J(u)\colon\ u \in \Delta\bigr\} < \beta,$$
$$\Delta \subset M := \{u \in E\colon\ P_2 u \notin B\};$$

(b′) *there exist an open subset* U *of* X *and* $\varrho > 0$ *such that* $\Delta \subset U$ *and*

$$J \text{ satisfies } (\mathrm{PS})_c \text{ on } \overline{U} \text{ for every } c \in [\alpha, \beta],$$
$$\inf\bigl\{\bigl|d(J|_{X\cap M})\bigr|(u)\colon\ u \in \overline{U},\ P_2 u \in \partial B,\ \alpha \leqslant J(u) \leqslant \beta\bigr\} > 0,$$
$$\forall u \in \mathrm{B}_\varrho(U) \setminus U \quad \alpha \leqslant J(u) \leqslant \beta \quad \Rightarrow \quad |dJ|(u) > \frac{\beta - \alpha}{\varrho},$$
$$\forall u \in \mathrm{B}_\varrho(U) \setminus U \quad \alpha \leqslant J(u) \leqslant \beta,\ P_2 u \in \partial B \quad \Rightarrow \quad \bigl|d(J|_{X\cap M})\bigr|(u) > \frac{\beta - \alpha}{\varrho};$$

(c) X *is locally contractible.*

Then J *admits at least two distinct critical points* $u_1, u_2 \in U$ *with* $J(u_i) \in [\alpha, \beta]$, $i = 1, 2$.

Proof. Let $\sigma = \frac{\beta-\alpha}{\varrho}$ and let Y be as in Lemma 3.9. Moreover, let α', β' be such that

$$\alpha < \alpha' < \inf\bigl\{J(u)\colon\ u \in X \cap (E_2 \oplus E_3),\ \|u\| = r_2\bigr\},$$
$$\sup\bigl\{J(u)\colon\ u \in \Delta\bigr\} < \beta' < \beta.$$

We will show that the function $J|_Y$ satisfies the assumptions of Theorem 3.7 with α, β replaced by α', β'. First of all, thanks to the continuity of J, we have that Y is closed in X, hence in E. Moreover, we have $\Delta \subset J^\beta \cap U$, whence $d(u, J^\beta \cap U) = 0$ for any $u \in \Delta$. This implies that $\Delta \subset Y$. Then it is easy to see that $J|_Y$ satisfies (a) of Theorem 3.7 with α, β replaced by α', β'.

Now let us look at condition (b). Let $c \in [\alpha', \beta']$ and let $(u_n)_n$ be a $(\mathrm{PS})_c$-sequence for $J|_Y$. Knowing that $J(u_n) > \alpha = \beta - \sigma\varrho$ for n sufficiently large, we deduce by (3.5) that $u_n \in U$ eventually as $n \to \infty$, and by (3.4) that $|dJ|(u_n) \to 0$. Therefore $(u_n)_n$ admits a converging subsequence, and this implies that $J|_Y$ satisfies $(\mathrm{PS})_c$ for every $c \in [\alpha', \beta']$.

As for the second condition of (b) in Theorem 3.7, let us apply again Lemma 3.9 with $X \cap M$ in place of X and $J|_{X\cap M}$ in place of J. Let $u \in \mathrm{B}_\varrho(U) \setminus U$ with $\alpha < J(u) \leqslant \beta$. If $P_2 u \in \partial B$, from the last condition in (b′) we directly see that $|d(J|_{X\cap M})|(u) > \sigma$. If $P_2 u \notin \overline{B}$, we have $|d(J|_{X\cap M})|(u) = |dJ|(u)$, hence again $|d(J|_{X\cap M})|(u) > \sigma$ by the third condition in (b′). From Lemma 3.9 we deduce that

$$\forall u \in Y \cap M \cap U, \quad \big|d(J|_{Y\cap M})\big|(u) \geqslant \big|d(J|_{X\cap M})\big|(u), \tag{3.7}$$

$$\forall u \in (Y \cap M) \setminus U, \quad J(u) > \alpha \quad \Rightarrow \quad \big|d(J|_{Y\cap M})\big|(u) > \sigma. \tag{3.8}$$

Now, arguing by contradiction, assume that

$$\inf\Big\{\big|d(J|_{Y\cap M})\big|(u)\colon\ u \in Y,\ P_2u \in \partial B,\ \alpha' \leqslant J(u) \leqslant \beta'\Big\} = 0$$

and consider a sequence $(u_n)_n$ in Y such that $P_2u_n \in \partial B$, $\alpha' \leqslant J(u_n) \leqslant \beta'$ and $|d(J|_{Y\cap M})|(u_n) \to 0$. From (3.8) we see that $u_n \in U$ eventually as $n \to \infty$, and from (3.7) that $|d(J|_{X\cap M})|(u_n) \to 0$. This fact contradicts the second condition in (b′). Therefore $J|_Y$ satisfies also (b) of Theorem 3.7.

If c_1, c_2 are distinct critical values of $J|_Y$ in $[\alpha', \beta']$ and $u_1, u_2 \in Y$ are corresponding critical points, from (3.5) we see that $u_1, u_2 \in U$ and from (3.4) we conclude that $|dJ|(u_i) = 0$, $i = 1, 2$.

Otherwise, there exists $c \in [\alpha', \beta']$ and a compact subset C of

$$\Big\{u \in Y\colon\ \big|d(J|_Y)\big|(u) = 0,\ J(u) = c,\ P_2u \notin \overline{B}\Big\}$$

having at least category 2 in $\{u \in Y\colon\ P_2u \notin \overline{B}\}$. Again by (3.5), we have $C \subset U$. Therefore C is actually contained in the set

$$V := \Big\{u \in X\colon\ P_2u \notin \overline{B},\ J(u) < \beta - \sigma d\big(u, J^\beta \cap U\big)\Big\},$$

which is clearly open in X. *A fortiori* we have that C has at least category 2 in V. On the other hand, being an open subset of a locally contractible space, V itself is locally contractible (see, e.g., [10]). From [3, Theorem 1.4.11] it follows that C contains at least two elements, which are free critical points of J by (3.4). □

Finally, for our purposes we need a variant of the linking theorem (see [23]) adapted to a nonsmooth setting as in [6,7]. As in Theorem 3.10, the main feature is that the Palais–Smale condition is required only on the closure of a suitable open set.

Theorem 3.11. *Let E be a Banach space such that $E = E_- \oplus E_+$, with* $\dim(E_-) < \infty$ *and E_+ closed in E, and let $J : E \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous function satisfying the following assumptions*:

(a) *there exist $\alpha, \beta \in \mathbb{R}$, $0 < r < R$ and $e \in E_+ \setminus \{0\}$ such that*

$$\sup\Big\{J(u)\colon\ u \in \partial_{E_-\oplus\mathbb{R}e} Q\Big\} < \alpha,$$
$$\inf\Big\{J(u)\colon\ u \in E_+,\ \|u\| = r\Big\} > \alpha,$$
$$\sup\Big\{J(u)\colon\ u \in Q\Big\} < \beta,$$

where $Q = \{v + te\colon\ v \in E_-,\ t \geqslant 0,\ \|v + te\| \leqslant R\}$;

(b) *there exist an open subset U of E and $\varrho > 0$ such that $Q \subset U$ and*

$$J \text{ satisfies } (\mathrm{PS})_c \quad \text{on } \overline{U} \text{ for every } c \in [\alpha, \beta], \tag{3.9}$$

$$|d\mathcal{G}_J|(u,s) = 1 \text{ with respect to the metric (3.1)}, \quad \text{whenever } J(u) < s \leqslant \beta, \tag{3.10}$$

$$\forall u \in \mathrm{B}_\varrho(U) \setminus U, \quad \alpha \leqslant J(u) \leqslant \beta \quad \Rightarrow \quad |dJ|(u) > \frac{\beta - \alpha}{\varrho}. \tag{3.11}$$

Then J admits a critical point $u \in U$ with $J(u) \in [\alpha, \beta]$.

Proof. It is easily seen that $\alpha < \beta$. Let $\sigma = \frac{\beta-\alpha}{\varrho}$ and let

$$\alpha' = \inf\big\{J(u):\ u \in E_+,\ \|u\| = r\big\}.$$

Let us first treat the particular case in which

$$\beta - \frac{\sigma}{\sqrt{1+\sigma^2}}\varrho < \alpha'. \tag{3.12}$$

Let us consider the set $\mathrm{epi}(J)$ endowed with the metric (3.1) and put

$$\widehat{U} := \big\{(u,s) \in \mathrm{epi}(J):\ u \in U\big\},$$
$$Y := \left\{(u,s) \in \mathrm{epi}(J):\ s \leqslant \beta - \frac{\sigma}{\sqrt{1+\sigma^2}} d\big((u,s), \mathcal{G}_J^\beta \cap \widehat{U}\big)\right\}.$$

We are going to apply [7, Theorem 3.12] to the continuous function $\mathcal{G}_J|_Y : Y \to \mathbb{R}$. Then we will prove that J possesses a critical point too.

Since J is lower semi-continuous, $\mathrm{epi}(J)$ is closed in $E \times \mathbb{R}$, hence complete. In turn, also Y is complete.

Let us consider the compact pair (D, S) given by

$$D = \big(\partial_{E_- \oplus \mathbb{R}e} Q \times [\alpha, \beta]\big) \cup \big(Q \times \{\beta\}\big), \qquad S = \partial_{E_- \oplus \mathbb{R}e} Q \times \{\alpha\}.$$

From assumption (a) it follows that $D \subset \mathrm{epi}(J)$. Then it is easy to see that $D \subset \mathcal{G}_J^\beta \cap \widehat{U}$, whence $D \subset Y$. Let $\psi : S \to Y$ be the inclusion map and

$$\Phi = \big\{\varphi \in C(D, Y):\ \varphi|_S = \psi\big\}.$$

If $\overline{\varphi} : D \to Y$ is the inclusion map, we have $\overline{\varphi} \in \Phi$, whence $\Phi \neq \emptyset$. Let

$$c = \inf_{\varphi \in \Phi} \sup_D (\mathcal{G}_J \circ \varphi).$$

Since $\overline{\varphi} \in \Phi$, we immediately find that $c \leqslant \beta$. On the other hand, it is easily seen that the pair (D, S) is homeomorphic to the pair $(Q, \partial_{E_- \oplus Re} Q)$. Then, standard considerations on the linking theorem show that

$$\varphi_1(D) \cap \{u \in E_+ \colon \|u\| = r\} \neq \emptyset \quad \text{for every } \varphi \equiv (\varphi_1, \varphi_2) \in \Phi.$$

It follows

$$\begin{aligned}\sup_D (\mathcal{G}_J \circ \varphi) = \sup_D \varphi_2 \geqslant \sup_D (J \circ \varphi_1) \geqslant \inf\{J(u)\colon\ u \in E_+, \|u\| = r\} \\ = \alpha' > \sup_S (\mathcal{G}_J \circ \psi)\end{aligned}$$

for every $\varphi \equiv (\varphi_1, \varphi_2) \in \Phi$. In particular, we have $c \geqslant \alpha'$.

Consider now the $(\mathrm{PS})_c$ condition for $\mathcal{G}_J|_Y$. If

$$(u, s) \in \mathrm{B}_\varrho(\widehat{U}) \setminus \widehat{U} \quad \text{and} \quad \beta - \frac{\sigma}{\sqrt{1+\sigma^2}}\varrho < s \leqslant \beta,$$

we have $u \in \mathrm{B}_\varrho(U) \setminus U$, hence

$$|d\mathcal{G}_J|(u, s) > \frac{\sigma}{\sqrt{1+\sigma^2}}$$

by assumption (3.11) and Proposition 3.4. From Lemma 3.9 we deduce that

$$\forall (u, s) \in Y \cap \widehat{U}, \quad \left|d(\mathcal{G}_J|_Y)\right|(u, s) \geqslant |d\mathcal{G}_J|(u, s), \tag{3.13}$$

$$\forall (u, s) \in Y \setminus \widehat{U}, \quad s > \beta - \frac{\sigma}{\sqrt{1+\sigma^2}}\varrho \quad \Rightarrow \quad \left|d(\mathcal{G}_J|_Y)\right|(u, s) > \frac{\sigma}{\sqrt{1+\sigma^2}}. \tag{3.14}$$

If (u_h, s_h) is a $(\mathrm{PS})_c$-sequence for $\mathcal{G}_J|_Y$, by (3.12) we have

$$s_h = \mathcal{G}_J(u_h, s_h) > \beta - \frac{\sigma}{\sqrt{1+\sigma^2}}\varrho$$

eventually as $h \to \infty$, hence $(u_h, s_h) \in \widehat{U}$ by (3.14). From (3.13) it follows that (u_h, s_h) is a $(\mathrm{PS})_c$-sequence also for $\mathcal{G}_J$. Again by assumption (3.10) and Proposition 3.4 we have that $J(u_h) = s_h$ eventually as $h \to \infty$ and that (u_h) is a $(\mathrm{PS})_c$-sequence for J with $u_h \in U$. By assumption (3.9), up to a subsequence (u_h) is convergent to some u in $\overline{U}$. Then (u_h, s_h) is convergent to (u, c) in Y. Therefore $\mathcal{G}_J|_Y$ satisfies condition $(\mathrm{PS})_c$.

By [7, Theorem 3.12] there exists a critical point $(u, s) \in Y$ of $\mathcal{G}_J|_Y$ with $s = \mathcal{G}_J(u, s) = c$. By (3.14) we have $(u, c) \in \widehat{U}$, hence $u \in U$. From (3.13) it follows that (u, c) is a critical point also for $\mathcal{G}_J$. By assumption (3.10) and Proposition 3.4 we conclude that $J(u) = c$ and that u is a critical point of J.

In order to remove the extra assumption (3.12), take $\lambda > 0$ and consider on E the norm

$$\|u\|_\lambda := \lambda \|u\|. \tag{3.15}$$

Of course, the norm (3.15) is equivalent to the original one of E.

It is easy to see that assumption (a) is still valid after the change of norm with the same α, β and $r_\lambda := \lambda r$, $R_\lambda := \lambda R$ and $e_\lambda := e/\lambda$.

From Definition 3.1 it is also easy to see that $|d_\lambda J|(u) = \frac{1}{\lambda}|dJ|(u)$, where d_λ denotes the weak slope induced by the norm (3.15). If we set $\varrho_\lambda := \lambda\varrho$ and keep the same U, we have that (3.9) and (3.11) still hold.

Concerning (3.10), let $(u,s) \in \mathrm{epi}(J)$ with $J(u) < s \leqslant \beta$ and let $\varepsilon \in \,]0,1[$ be such that $\lambda^2(2\varepsilon - \varepsilon^2) < 1$. Since $|d\mathcal{G}_J|(u,s) = 1$, by Proposition 3.3 there exist $\delta > 0$ and a continuous map $\mathcal{H} \equiv (\mathcal{H}_1, \mathcal{H}_2): \mathrm{B}_\delta(u,s) \times [0,\delta] \to \mathrm{epi}(J)$ such that

$$\big\|\mathcal{H}_1\big((v,\tau),t\big) - v\big\|^2 + \big(\mathcal{H}_2\big((v,\tau),t\big) - \tau\big)^2 \leqslant t^2, \tag{3.16}$$

$$\mathcal{H}_2\big((v,\tau),t\big) = \mathcal{G}_J\big(\mathcal{H}\big((v,\tau),t\big)\big) \leqslant \mathcal{G}_J(v,\tau) - (1-\varepsilon)t = \tau - (1-\varepsilon)t, \tag{3.17}$$

whence

$$\big\|\mathcal{H}_1\big((v,\tau),t\big) - v\big\|^2 \leqslant \big(1 - (1-\varepsilon)^2\big)t^2 = (2\varepsilon - \varepsilon^2)t^2. \tag{3.18}$$

If we set $\widehat{\mathcal{H}}_2((v,\tau),t) := \tau - t\sqrt{1 - \lambda^2(2\varepsilon - \varepsilon^2)}$, from (3.16)–(3.18) it follows that

$$\begin{aligned}
&\big\|\mathcal{H}_1\big((v,\tau),t\big) - v\big\|_\lambda^2 + \big(\widehat{\mathcal{H}}_2\big((v,\tau),t\big) - \tau\big)^2 \leqslant t^2,\\
&\widehat{\mathcal{H}}_2\big((v,\tau),t\big) = \tau - t\sqrt{1 - \lambda^2(2\varepsilon - \varepsilon^2)} \geqslant \tau - (1-\varepsilon)t \geqslant \mathcal{H}_2\big((v,\tau),t\big)\\
&\qquad \geqslant J\big(\mathcal{H}_1\big((v,\tau),t\big)\big).
\end{aligned}$$

In particular, $(\mathcal{H}_1, \widehat{\mathcal{H}}_2)$ also takes its values in $\mathrm{epi}(J)$. This yields that $|d_\lambda\mathcal{G}_J|(u,s) \geqslant \sqrt{1 - \lambda^2(2\varepsilon - \varepsilon^2)}$, hence $|d_\lambda\mathcal{G}_J|(u,s) = 1$ by the arbitrariness of ε.

Therefore, all the assumptions of the theorem are satisfied, also after the change of norm. On the other hand, if we set $\sigma_\lambda := (\beta - \alpha)/\varrho_\lambda = \sigma/\lambda$, the extra assumption (3.12) reads

$$\beta - \frac{\sigma_\lambda}{\sqrt{1 + \sigma_\lambda^2}}\varrho_\lambda = \beta - \frac{\sigma}{\sqrt{1 + (\sigma/\lambda)^2}}\varrho < \alpha'. \tag{3.19}$$

Since $\alpha < \alpha'$, if λ is large enough also (3.19) is satisfied and the assertion follows. □

4. Some preliminary lemmas

Now we come back to the setting of Theorem 2.1. We observe that $\langle A\cdot,\cdot\rangle^{\frac{1}{2}}$ is a norm in $H_0^1(\Omega)$ equivalent to that induced by $H^1(\Omega)$. From now on, we will consider the space $H_0^1(\Omega)$ endowed with the norm associated with the operator A, i.e. we will put

$$\|u\| = \langle Au, u\rangle^{\frac{1}{2}}, \quad \forall u \in H_0^1(\Omega).$$

We will also denote by $\|\cdot\|_q$ the usual norm of $L^q(\Omega)$, $q \geqslant 1$.

In what follows, let $1 \leqslant k \leqslant j$ be such that $\lambda_{k-1} < \lambda_k = \cdots = \lambda_j < \lambda_{j+1}$ and let $e_1, \ldots, e_{j+1}$ be eigenfunctions related to $\lambda_1, \ldots, \lambda_{j+1}$ such that $\{e_1, \ldots, e_{j+1}\}$ is an $L^2(\Omega)$-orthonormal system of functions. For any $1 \leqslant i \leqslant j+1$, we also denote by V_i the i-dimensional space generated by $\{e_1, \ldots, e_i\}$ and we set

$$V_i^{\perp} = \left\{ v \in H_0^1(\Omega) \colon \int_\Omega v e_h \, dx = 0, \ \forall h = 1, \ldots, i \right\}.$$

For $i = 0$ we set $V_0 = \{0\}$. Finally, for $h = 1, \ldots, k-1$ we choose $\tilde{e}_h \in C_0^\infty(\Omega)$ close enough to e_h in order to have that

$$H_0^1(\Omega) = \widetilde{V}_{k-1} \oplus \operatorname{span}\{e_k, \ldots, e_j\} \oplus V_j^{\perp}, \tag{4.1}$$

$$\langle Av, v \rangle \leqslant \frac{\lambda_{k-1} + \lambda_k}{2} \|v\|_2^2, \quad \forall v \in \widetilde{V}_{k-1}, \tag{4.2}$$

where $\widetilde{V}_{k-1}$ is the $(k-1)$-dimensional space generated by $\{\tilde{e}_1, \ldots, \tilde{e}_{k-1}\}$. We denote by $P_1 : H_0^1(\Omega) \to \widetilde{V}_{k-1}$, $P_2 : H_0^1(\Omega) \to \operatorname{span}\{e_k, \ldots, e_j\}$ and $P_3 : H_0^1(\Omega) \to V_j^{\perp}$ the projections associated with the (nonorthogonal) direct sum (4.1) and we observe that

$$\langle Av, v \rangle \leqslant \lambda_k \|v\|_2^2, \quad \forall v \in \widetilde{V}_{k-1} \oplus \operatorname{span}\{e_k, \ldots, e_j\}. \tag{4.3}$$

About the nonlinearity p, we observe that condition (P4) easily yields

$$\lim_{|\xi| \to \infty} \frac{P(x, \xi)}{\xi^2} = +\infty, \quad \text{for a.e. } x \in \Omega. \tag{4.4}$$

Finally, we denote by I_{K_ψ} the indicator function of the closed convex set K_ψ, namely

$$I_{K_\psi}(u) = \begin{cases} 0 & \text{if } u \in K_\psi, \\ +\infty & \text{if } u \in H_0^1(\Omega) \setminus K_\psi, \end{cases}$$

and by $J_{\lambda,\psi} : K_\psi \to \mathbb{R}$ the restriction of J_λ to K_ψ.

For any $\varrho > 0$ we also set

$$M_\varrho = \left\{ u \in H_0^1(\Omega) \colon \int_\Omega |P_2 u|^2 \, dx \geqslant \varrho^2 \right\}$$

and for any $u \in M_\varrho$

$$\mathrm{H}_{M_\varrho}(u) = \begin{cases} \{v \in H_0^1(\Omega) \colon \int_\Omega (P_2 u)(P_2 v)\, dx > 0\} & \text{if } \int_\Omega |P_2 u|^2 \, dx = \varrho^2, \\ H_0^1(\Omega) & \text{if } \int_\Omega |P_2 u|^2 \, dx > \varrho^2. \end{cases}$$

The set $\mathrm{H}_{M_\varrho}(u)$ is called the *hypertangent cone* to M_ϱ at u (see [4]).

Now, we are able to prove some preliminary lemmas we will need in order to get the existence of two solutions for the variational inequality $(\mathcal{P})$.

Lemma 4.1. *The following statements hold true*:

(1) *for every $u \in K_\psi$, there exists $\varphi \in H^{-1}(\Omega)$ with $\|\varphi\| \leqslant |dJ_{\lambda,\psi}|(u)$ such that*

$$\langle Au, v-u\rangle - \lambda \int_\Omega u(v-u)\,dx - \int_\Omega p(x,u)(v-u)\,dx \geqslant \langle \varphi, v-u\rangle, \quad \forall v \in K_\psi;$$

(2) *for every $u \in K_\psi \cap M_\varrho$, there exists $\varphi \in H^{-1}(\Omega)$ with $\|\varphi\| \leqslant |d(J_{\lambda,\psi}|_{M_\varrho})|(u)$ such that*

$$\langle Au, v-u\rangle - \lambda \int_\Omega u(v-u)\,dx - \int_\Omega p(x,u)(v-u)\,dx \geqslant \langle \varphi, v-u\rangle,$$

$\forall v \in K_\psi \cap (u + \mathrm{H}_{M_\varrho}(u))$.

Proof. (1) Since the functional $J_{\lambda,\psi} : K_\psi \to \mathbb{R}$ is locally Lipschitz continuous, it is easily seen that $|dJ_{\lambda,\psi}|(u) < +\infty$. If we consider the functional $J_\lambda + I_{K_\psi} : H_0^1(\Omega) \to \mathbb{R} \cup \{+\infty\}$, by Definition 3.1 we have that

$$\left|d(J_\lambda + I_{K_\psi})\right|(u) = \left|d\left((J_\lambda + I_{K_\psi})|_{K_\psi}\right)\right|(u) = |dJ_{\lambda,\psi}|(u).$$

On the other hand, the function I_{K_ψ} is convex and lower semi-continuous, while J_λ is of class C^1. From [7, Proposition 2.10 and Theorem 2.11] we deduce that there exists $\varphi \in H^{-1}(\Omega)$ with $\|\varphi\| \leqslant |dJ_{\lambda,\psi}|(u)$ and $\varphi - J'_\lambda(u) \in \partial I_{K_\psi}(u)$, where ∂ stands for the usual subdifferential of convex analysis. Then assertion (1) easily follows.

(2) Consider first the case in which $\int_\Omega |P_2 u|^2\,dx > \varrho^2$. Since the notion of weak slope is local, we have $|d(J_{\lambda,\psi}|_{M_\varrho})|(u) = |dJ_{\lambda,\psi}|(u)$ and the assertion follows from (1).

Then, let $\int_\Omega |P_2 u|^2\,dx = \varrho^2$. We claim that

$$\left\langle J'_\lambda(u), v-u\right\rangle \geqslant -\left|d(J_{\lambda,\psi}|_{M_\varrho})\right|(u)\|v-u\|, \quad \forall v \in K_\psi \cap \left(u + \mathrm{H}_{M_\varrho}(u)\right). \tag{4.5}$$

By contradiction, let $v \in K_\psi \cap (u + \mathrm{H}_{M_\varrho}(u))$ and let $\sigma > |d(J_{\lambda,\psi}|_{M_\varrho})|(u)$ be such that

$$\left\langle J'_\lambda(u), v-u\right\rangle < -\sigma\|v-u\|. \tag{4.6}$$

Let $\delta > 0$ and let $\mathcal{H} : (K_\psi \cap M_\varrho \cap \mathrm{B}_\delta(u)) \times [0,\delta] \to K_\psi \cap M_\varrho$ be defined by

$$\mathcal{H}(z,t) = z + \frac{t}{\|v-z\|}(v-z).$$

Since $v \neq u$, it is readily seen that $\mathcal{H}$ is well defined with values in K_ψ, provided that δ is small enough. Moreover, we have:

$$\begin{aligned}\int_\Omega \left|P_2\mathcal{H}(z,t)\right|^2 dx &= \int_\Omega |P_2 z|^2\,dx + \frac{2t}{\|v-z\|}\int_\Omega P_2 z P_2(v-z)\,dx \\ &\quad + \frac{t^2}{\|v-z\|^2}\int_\Omega \left|P_2(v-z)^2\right|dx \\ &\geqslant \varrho^2 + \frac{2t}{\|v-z\|}\int_\Omega P_2 z P_2(v-z)\,dx \geqslant \varrho^2,\end{aligned}$$

provided, again, that δ is small enough, as $v-u \in \mathrm{H}_{M_\varrho}(u)$. Therefore $\mathcal{H}$ takes its values in $K_\psi \cap M_\varrho$.

It is clear that $\|\mathcal{H}(z,t)-z\| = t$. Since J_λ is of class C^1, from (4.6) we deduce that $J_\lambda(\mathcal{H}(z,t)) \leqslant J_\lambda(z) - \sigma t$, provided again that δ is small enough.

It follows that $|d(J_{\lambda,\psi}|_{M_\varrho})|(u) \geqslant \sigma$, whence a contradiction. Therefore (4.5) is proved.

If we define $g : H_0^1(\Omega) \to \mathbb{R}\cup\{+\infty\}$ by

$$g(v) = \begin{cases} \langle J_\lambda'(u), v-u\rangle & \text{if } v \in K_\psi \cap (u + (\{0\}\cup \mathrm{H}_{M_\varrho}(u))), \\ +\infty & \text{otherwise,}\end{cases}$$

we have that g is a convex function with $g(u) = 0$ and

$$g(v) \geqslant -\left|d(J_{\lambda,\psi}|_{M_\varrho})\right|(u)\|v-u\|$$

for every $v \in H_0^1(\Omega)$. From [5, Lemma] we deduce that there exists $\varphi \in H^{-1}(\Omega)$ with $\|\varphi\| \leqslant |d(J_{\lambda,\psi}|_{M_\varrho})|(u)$ and $g(v) \geqslant \langle \varphi, v-u\rangle$ for every $v \in H_0^1(\Omega)$. Then assertion (2) easily follows. □

Lemma 4.2. *The following statements hold true*:

(1) *we have*

$$\lim_{\|u\|_{s+1}\to 0} \frac{\|p(x,u)\|_{(s+1)/s}}{\|u\|_{s+1}} = 0, \tag{4.7}$$

$$\lim_{\|u\|_{s+1}\to 0} \frac{\|P(x,u)\|_1}{\|u\|_{s+1}^2} = 0; \tag{4.8}$$

(2) *for every finite-dimensional subspace V of $H_0^1(\Omega)$, we have*

$$\lim_{\|v\|\to\infty,\ v\in V} \frac{\int_\Omega P(x,u)\,dx}{\|u\|^2} = +\infty.$$

Proof. (1) If we set

$$\pi(x,\xi)=\begin{cases}\dfrac{p(x,\xi)}{\xi} & \text{if } \xi\neq 0,\\ 0 & \text{if } \xi=0,\end{cases}$$

from (P2) and (P3) it follows that $\pi:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function satisfying

$$\left|\pi(x,\xi)\right|\leqslant a_1+a_2|\xi|^{s-1},\quad \forall\xi\in\mathbb{R},\ \text{a.e. } x\in\Omega. \tag{4.9}$$

It follows that

$$\lim_{\|u\|_{s+1}\to 0}\pi(x,u)=0\quad \text{in } L^{\frac{s+1}{s-1}}(\Omega).$$

On the other hand, Hölder inequality implies that

$$\left\|p(x,u)\right\|_{\frac{s+1}{s}}=\left\|\pi(x,u)u\right\|_{\frac{s+1}{s}}\leqslant\left\|\pi(x,u)\right\|_{\frac{s+1}{s-1}}\|u\|_{s+1}.$$

Then (4.7) follows.

The proof of (4.8) is similar.

(2) Let $(v_n)_n$ be a sequence in V with $\|v_n\|\to\infty$. If we set $t_n=\|v_n\|$ and $w_n=v_n/t_n$, we may assume, since V is finite-dimensional, that $(w_n)_n$ is convergent a.e. to some $w\in V\setminus\{0\}$. From (4.4) it follows that

$$\lim_{n\to\infty}\frac{P(x,t_nw_n(x))}{t_n^2}=+\infty\quad \text{on a set of positive measure.}$$

Therefore, from (P4) and Fatou's lemma we deduce that

$$\lim_{n\to\infty}\frac{\int_\Omega P(x,t_nw_n)\,dx}{t_n^2}=+\infty$$

and the assertion follows. Lemma 4.2 is thus completely proved. □

Lemma 4.3. *There exist $R^*, R^{**}>0$ and $\eta^*,\varepsilon^*>0$ such that, for every $\lambda\in[\lambda_k-\eta^*,\lambda_k+\eta^*]$, the following statements hold true*:

(1) *for any $u\in\widetilde{V}_{k-1}\oplus\operatorname{span}\{e_k,\dots,e_j\}$ with $\max\{\|P_1u\|,\|P_2u\|\}=R^*$, we have $\|u\|<R^{**}$ and $J_\lambda(u)<0$;*
(2) *if u is a critical point of the functional J_λ with $\|u\|\leqslant R^{**}+1$ and critical value $J_\lambda(u)\leqslant\varepsilon^*$, then $\|u\|<R^{**}$;*
(3) *if $u\in\widetilde{V}_{k-1}\oplus V_j^\perp$ is a constrained critical point of the functional $J_\lambda|_{\widetilde{V}_{k-1}\oplus V_j^\perp}$ with $\|u\|\leqslant R^{**}+1$ and critical value $J_\lambda(u)\leqslant\varepsilon^*$, then u is identically zero.*

Proof. (1) By (4.3) for any $\lambda \in [\lambda_k - 1, \lambda_k + 1]$ we have that

$$J_\lambda(u) \leqslant \frac{1}{2}\int_\Omega u^2\,dx - \int_\Omega P(x,u)\,dx, \quad \forall u \in \widetilde{V}_{k-1} \oplus \operatorname{span}\{e_k, \dots, e_j\}. \tag{4.10}$$

Combining (4.10) with (2) of Lemma 4.2, we deduce that there exists $R^* > 0$ such that, for any $\lambda \in [\lambda_k - 1, \lambda_k + 1]$ and any $u \in \widetilde{V}_{k-1} \oplus \operatorname{span}\{e_k, \dots, e_j\}$ with $\max\{\|P_1 u\|, \|P_2 u\|\} = R^*$, we have $J_\lambda(u) < 0$. Since $\max\{\|P_1 u\|, \|P_2 u\|\}$ is a norm equivalent to the original one, we also find R^{**} with the required property.

(2) Let R^{**} be as in assertion (1). By contradiction, let us suppose that there exist a sequence $(\lambda^{(n)})_n$ such that $\lambda^{(n)} \to \lambda_k$ and $(u_n)_n$ such that u_n is a critical point for $J_{\lambda^{(n)}}$ with $R^* \leqslant \|u_n\| \leqslant R^{**} + 1$, for any $n \in \mathbb{N}$ and $\limsup_{n\to\infty} J_{\lambda^{(n)}}(u_n) \leqslant 0$. Up to a subsequence, we may also assume that $(u_n)_n$ is convergent to some $u \in H_0^1(\Omega)$ weakly in $H_0^1(\Omega)$ and a.e. in Ω.

We have that

$$\langle Au_n, v\rangle - \lambda^{(n)} \int_\Omega u_n v\,dx - \int_\Omega p(x,u_n)v\,dx = 0, \tag{4.11}$$

for any $v \in H_0^1(\Omega)$ and for any $n \in \mathbb{N}$. Taking $v = u_n$ as a test function in (4.11) and using (P4) we get

$$0 \leqslant (\mu - 2)\int_\Omega P(x,u_n)\,dx \leqslant 2J_{\lambda^{(n)}}(u_n), \tag{4.12}$$

for any $n \in \mathbb{N}$. Passing to the limit in (4.12) as n goes to infinity, we deduce that $P(x,u) = 0$, hence that $u = 0$ a.e. in Ω. So,

$$u_n \overset{w}{\to} 0 \quad \text{in } H_0^1(\Omega) \tag{4.13}$$

as n goes to infinity.

Coming back to (4.11) with u_n as a test function and using (4.13), we obtain

$$\|u_n\| \to 0 \tag{4.14}$$

as n goes to infinity. This is in contradiction with $\|u_n\| \geqslant R^*$ for any $n \in \mathbb{N}$.

(3) Let R^{**} be as in assertion (2). Let us suppose by contradiction that there exist $(\lambda^{(n)})_n$ such that $\lambda^{(n)} \to \lambda_k$ and $(u_n)_n$ in $(\widetilde{V}_{k-1} \oplus V_j^\perp) \setminus \{0\}$ such that u_n is a constrained critical point for $J_{\lambda^{(n)}}|_{\widetilde{V}_{k-1}\oplus V_j^\perp}$ with $\|u_n\| \leqslant R^{**} + 1$ and $\limsup_{n\to\infty} J_{\lambda^{(n)}}(u_n) \leqslant 0$. We have that

$$\langle Au_n, v\rangle - \lambda^{(n)} \int_\Omega u_n v\,dx - \int_\Omega p(x,u_n)v\,dx = 0, \tag{4.15}$$

for any $v \in \widetilde{V}_{k-1} \oplus V_j^{\perp}$ and for any $n \in \mathbb{N}$. Choosing $v = u_n$ as a test function and arguing as in part (2) of this lemma, we get that

$$u_n \to 0 \quad \text{in } H_0^1(\Omega) \tag{4.16}$$

as n goes to infinity. We can split u_n as $u_n = v_n + z_n$ with $v_n \in \widetilde{V}_{k-1}$ and $z_n \in V_j^{\perp}$. Taking $v = -v_n + z_n$ as a test function in (4.15) we obtain

$$\langle Av_n, v_n\rangle - \lambda^{(n)} \int_\Omega v_n^2\,dx - \langle Az_n, z_n\rangle + \lambda^{(n)} \int_\Omega z_n^2\,dx = \int_\Omega p(x, u_n)(v_n - z_n)\,dx \tag{4.17}$$

for any $n \in \mathbb{N}$. Taking into account (4.2), we have that

$$\langle Av_n, v_n\rangle - \lambda^{(n)} \int_\Omega v_n^2\,dx \leqslant -\left(\frac{2\lambda^{(n)}}{\lambda_{k-1} + \lambda_k} - 1\right)\langle Av_n, v_n\rangle \leqslant -K_1\langle Av_n, v_n\rangle \tag{4.18}$$

and

$$-\langle Az_n, z_n\rangle + \lambda^{(n)} \int_\Omega z_n^2\,dx \leqslant -\left(1 - \frac{\lambda^{(n)}}{\lambda_{j+1}}\right)\langle Az_n, z_n\rangle \leqslant -K_2\langle Az_n, z_n\rangle, \tag{4.19}$$

for any $n \in \mathbb{N}$, where K_1 and K_2 are positive constants independent of n. Then, combining (4.17)–(4.19), it easily follows that there exist positive constants K', K'', independent of n, such that

$$\int_\Omega p(x, u_n)(v_n - z_n)\,dx \leqslant -K'\big(\|v_n\|^2 + \|z_n\|^2\big) \leqslant -K''\|u_n\|^2, \tag{4.20}$$

for n sufficiently large. On the other hand, by Hölder inequality and the continuous embedding of $H_0^1(\Omega)$ into $L^{s+1}(\Omega)$, we have that

$$\left|\int_\Omega p(x, u_n)(v_n - z_n)\,dx\right| \leqslant \big\|p(x, u_n)\big\|_{\frac{s+1}{s}}\|v_n - z_n\|_{s+1} \leqslant K'''\big\|p(x, u_n)\big\|_{\frac{s+1}{s}}\|u_n\|, \tag{4.21}$$

where K''' is a positive constant independent of n.

Taking into account (4.20), (4.21) and the fact that $u_n \to 0$ in $H_0^1(\Omega)$, we obtain that

$$K'' \leqslant K''' \frac{\|p(x, u_n)\|_{(s+1)/s}}{\|u_n\|},$$

for n sufficiently large. This fact contradicts (4.7). Lemma 4.3 is now proved. □

Lemma 4.4. *Let $R^{**} > 0$ and $\eta^*, \varepsilon^* > 0$ be as in Lemma 4.3. Then the following statements hold true*:

(1) *there exist $\varrho', \sigma > 0$, $R' > 0$ and a compact set $C' \subset \Omega$ such that, if $\lambda \in [\lambda_k - \eta^*, \lambda_k + \eta^*]$, if ψ_1 and ψ_2 satisfy $(\psi_{R',C'})$ and if $\varrho \in]0, \varrho']$, we have*

$$\forall u \in K_\psi, \quad R^{**} \leqslant \|u\| \leqslant R^{**} + 1, \qquad J_{\lambda,\psi}(u) \leqslant \varepsilon^* \quad \Rightarrow \quad |dJ_{\lambda,\psi}|(u) > \sigma,$$
$$\forall u \in K_\psi \cap \partial M_\varrho, \quad R^{**} \leqslant \|u\| \leqslant R^{**} + 1,$$
$$J_{\lambda,\psi}(u) \leqslant \varepsilon^* \quad \Rightarrow \quad \left|d(J_{\lambda,\psi}|_{K_\psi \cap M_\varrho})\right|(u) > \sigma;$$

(2) *for every $\alpha > 0$, there exist $\varrho_\alpha, \sigma_\alpha > 0$, $R_\alpha > 0$ and a compact set $C_\alpha \subset \Omega$ such that, if $\lambda \in [\lambda_k - \eta^*, \lambda_k + \eta^*]$, if ψ_1 and ψ_2 satisfy $(\psi_{R_\alpha,C_\alpha})$ and if $\varrho \in]0, \varrho_\alpha]$, then $|d(J_{\lambda,\psi}|_{K_\psi \cap M_\varrho})|(u) > \sigma_\alpha$, whenever $u \in K_\psi \cap \partial M_\varrho$, $\|u\| \leqslant R^{**} + 1$ and $\alpha \leqslant J_{\lambda,\psi}(u) \leqslant \varepsilon^*$.*

Proof. (1) We proceed by contradiction and we suppose that there exist a sequence $(\lambda^{(n)})_n$ in $\mathbb{R}$ with

$$\lambda^{(n)} \to \bar{\lambda} \in [\lambda_k - \eta^*, \lambda_k + \eta^*], \tag{4.22}$$

two sequences of Borel functions $\psi_{1,n} : \Omega \to [-\infty, 0]$ and $\psi_{2,n} : \Omega \to [0, +\infty]$ such that

$$\psi_{1,n}(x) \leqslant -n < n \leqslant \psi_{2,n} \quad \text{cap. q.e.} \quad \text{on } C_n := \left\{x \in \mathbb{R}^N \colon d(x, \mathbb{R}^N \setminus \Omega) \geqslant \frac{1}{n}\right\}, \tag{4.23}$$

and a sequence $(u_n)_n$ in K_{ψ_n} such that

$$R^{**} \leqslant \|u_n\| \leqslant R^{**} + 1, \tag{4.24}$$
$$J_{\lambda^{(n)},\psi_n}(u_n) \leqslant \varepsilon^*, \tag{4.25}$$

and $|d(J_{\lambda^{(n)},\psi_n})|(u_n) \to 0$. By (2) of Lemma 4.1, we deduce that there exists $\varphi_n \in H^{-1}(\Omega)$ such that

$$\|\varphi_n\|_{H^{-1}(\Omega)} \to 0, \tag{4.26}$$
$$\langle Au_n, v - u_n\rangle - \lambda^{(n)} \int_\Omega u_n(v - u_n)\,dx - \int_\Omega p(x, u_n)(v - u_n)\,dx$$
$$\geqslant \langle \varphi_n, v - u_n\rangle, \quad \forall v \in K_{\psi_n}. \tag{4.27}$$

Since $(u_n)_n$ is bounded, up to a subsequence there exists $u \in H_0^1(\Omega)$ such that

$$u_n \overset{w}{\to} u \quad \text{in } H_0^1(\Omega) \tag{4.28}$$

as n goes to infinity. Let $\bar{v} \in C_0^\infty(\Omega)$. By (4.23) we have that $\psi_{1,n} \leqslant \bar{v} \leqslant \psi_{2,n}$ cap. q.e. in Ω for n sufficiently large. So, $\bar{v}$ can be taken as a test function in (4.27) for n sufficiently large, whence

$$\begin{aligned}&\langle Au_n, u_n\rangle\\ &\quad\leqslant \langle Au_n, \bar{v}\rangle - \lambda^{(n)} \int_\Omega u_n(\bar{v} - u_n)\,dx - \int_\Omega p(x, u_n)(\bar{v} - u_n)\,dx - \langle \varphi_n, \bar{v} - u_n\rangle.\end{aligned} \tag{4.29}$$

By (4.22), (4.26) and (4.28), passing to the upper limit, we get

$$\begin{aligned}&\limsup_{n\to\infty}\langle Au_n, u_n\rangle\\ &\quad\leqslant \langle Au, \bar{v}\rangle - \lambda \int_\Omega u(\bar{v} - u)\,dx - \int_\Omega p(x, u)(\bar{v} - u)\,dx, \quad \forall \bar{v} \in C_0^\infty(\Omega).\end{aligned} \tag{4.30}$$

By density we get that (4.30) holds for any $\bar{v} \in H_0^1(\Omega)$. In particular, we can take $\bar{v} = u$ in (4.30). So, by the weak lower semi-continuity of the norm, we obtain that $u_n \to u$ in $H_0^1(\Omega)$ as n goes to infinity. Moreover, by (4.24), (4.25) and (4.29), we get

$$R^{**} \leqslant \|u\| \leqslant R^{**} + 1, \tag{4.31}$$

$$J_{\bar{\lambda}}(u) \leqslant \varepsilon^* \tag{4.32}$$

and

$$\langle Au, v - u\rangle - \bar{\lambda} \int_\Omega u(v - u)\,dx - \int_\Omega p(x, u)(v - u)\,dx \geqslant 0, \quad \forall v \in H_0^1(\Omega). \tag{4.33}$$

Then, it is easy to see that

$$\langle Au, v\rangle - \bar{\lambda} \int_\Omega uv\,dx - \int_\Omega p(x, u)v\,dx = 0, \quad \forall v \in H_0^1(\Omega), \tag{4.34}$$

i.e. u is a critical point for the functional $J_{\bar{\lambda}}$. By (2) of Lemma 4.3 we have that $\|u\| < R^{**}$, which is in contradiction with (4.31). Then, the first assertion in (1) is proved.

Now, we consider the second statement of (1). We proceed again by contradiction and we suppose that there exist a sequence $(\varrho_n)_n$ in $]0, +\infty[$ such that

$$\varrho_n \to 0, \tag{4.35}$$

a sequence $(\lambda^{(n)})_n$ in $\mathbb{R}$ satisfying (4.22), two sequences of Borel functions $\psi_{1,n}:\Omega \to [-\infty, 0]$ and $\psi_{2,n}:\Omega \to [0, +\infty]$ satisfying (4.23) and a sequence $(u_n)_n$ in $K_{\psi_n} \cap \partial M_{\varrho n}$ satisfying (4.24), (4.25) and

$$\left|d(J_{\lambda^{(n)},\psi_n}|_{K_{\psi_n}\cap M_{\varrho n}})\right|(u_n) \to 0. \tag{4.36}$$

By (2) of Lemma 4.1, we deduce that there exists $\varphi_n \in H^{-1}(\Omega)$ such that

$$\|\varphi_n\|_{H^{-1}(\Omega)} \to 0, \tag{4.37}$$

$$\begin{aligned}&\langle Au_n, v-u_n\rangle - \lambda^{(n)} \int_\Omega u_n(v-u_n)\,dx - \int_\Omega p(x,u_n)(v-u_n)\,dx \\ &\quad \geqslant \langle \varphi_n, v-u_n\rangle, \quad \forall v \in K_{\psi_n} \cap \big(u_n + \mathrm{H}_{M_{\varrho n}}(u_n)\big).\end{aligned} \tag{4.38}$$

Since $(u_n)_n$ is bounded, there exists $u \in H_0^1(\Omega)$ such that, up to a subsequence,

$$u_n \stackrel{w}{\to} u \quad \text{in } H_0^1(\Omega). \tag{4.39}$$

Moreover, $u_n \in \partial M_{\varrho n}$ and (4.35) yield

$$\int_\Omega |P_2 u|^2\,dx = 0, \tag{4.40}$$

i.e. $u \in \widetilde{V}_{k-1} \oplus V_j^\perp$. Let $z_n = \frac{P_2 u_n}{\varrho_n}$. It easily follows that $\int_\Omega |z_n|^2\,dx = 1$ and $z_n \in \operatorname{span}\{e_k, \ldots, e_j\}$. Then, up to a subsequence,

$$z_n \to z \in \operatorname{span}\{e_k, \ldots, e_j\} \tag{4.41}$$

with $\int_\Omega |z|^2\,dx = 1$.

Let $\bar{v} \in C_0^\infty(\Omega)$ be such that $\int_\Omega z P_2 v\,dx > 0$. By (4.40) we have that $\int_\Omega z P_2(v-u)\,dx > 0$. It follows $\int_\Omega z_n P_2(v-u_n)\,dx > 0$ for n sufficiently large, namely, $v - u_n \in \mathrm{H}_{M_{\varrho n}}(u_n)$. On the other hand, (4.23) yields $\psi_{1,n} \leqslant \bar{v} \leqslant \psi_{2,n}$ cap. q.e. in Ω for n sufficiently large. So, $\bar{v}$ can be taken as a test function in (4.38), and passing to the upper limit we get

$$\begin{aligned}&\limsup_{n\to+\infty}\langle Au_n, u_n\rangle \leqslant \langle Au, \bar{v}\rangle - \bar{\lambda}\int_\Omega u(\bar{v}-u)\,dx - \int_\Omega p(x,u)(\bar{v}-u)\,dx, \\ &\quad \text{for any } \bar{v} \in C_0^\infty(\Omega) \quad \text{with} \int_\Omega z P_2 \bar{v}\,dx > 0.\end{aligned} \tag{4.42}$$

By density (4.42) holds for any $v \in H_0^1(\Omega)$ such that $\int_\Omega z P_2 v\,dx \geqslant 0$. So, we can take $v = u$ as a test function in (4.42). By the weak lower semi-continuity of the norm we obtain that $u_n \to u$ in $H_0^1(\Omega)$ as n goes to infinity. Then, by (4.24), (4.25) and (4.38), we get

$$R^{**} \leqslant \|u\| \leqslant R^{**} + 1, \tag{4.43}$$

$$J_{\bar\lambda}(u) \leqslant \varepsilon^* \tag{4.44}$$

and

$$\langle Au, v - u\rangle - \bar\lambda \int_\Omega u(v-u)\,dx - \int_\Omega p(x,u)(v-u)\,dx \geqslant 0$$
$$\text{for any } v \in H_0^1(\Omega) \quad \text{with} \int_\Omega z P_2 v\,dx \geqslant 0. \tag{4.45}$$

In particular, (4.45) holds for any $v \in \widetilde{V}_{k-1} \oplus V_j^{\perp}$, whence

$$\langle Au, v\rangle - \bar\lambda \int_\Omega uv\,dx - \int_\Omega p(x,u)v\,dx = 0, \quad \forall v \in \widetilde{V}_{k-1} \oplus V_j^{\perp}. \tag{4.46}$$

It follows that u is a constrained critical point for the functional $J_{\bar\lambda}|_{\widetilde{V}_{k-1}\oplus V_j^{\perp}}$. By (3) of Lemma 4.3 we have that $u \equiv 0$, which contradicts (4.43).

Assertion (1) is therefore proved.

(2) We proceed by contradiction and we suppose that there exist $\bar\alpha > 0$, a sequence $(\varrho_n)_n$ in $]0,+\infty[$ satisfying (4.35), a sequence $(\lambda^{(n)})_n$ in $\mathbb{R}$ satisfying (4.22), two sequences of Borel functions $\psi_{1,n} : \Omega \to [-\infty, 0]$ and $\psi_{2,n} : \Omega \to [0,+\infty]$ satisfying (4.23) and a sequence $(u_n)_n$ in $K_{\psi_n} \cap \partial M_{\varrho_n}$ satisfying

$$\|u_n\| \leqslant R^{**} + 1,$$

(4.25) and (4.36).

Arguing as in the second part of (1), we can prove that $u_n \to u$ in $H_0^1(\Omega)$, where u is a constrained critical point for the functional $J_{\bar\lambda}|_{\widetilde{V}_{k-1}\oplus V_j^{\perp}}$ with $\|u\| \leqslant R^{**} + 1$ and $\bar\alpha \leqslant J_{\bar\lambda}(u) \leqslant \varepsilon^*$. By (3) of Lemma 4.3 it follows that $u \equiv 0$, which is in contradiction with $J_{\bar\lambda}(u) \geqslant \bar\alpha > 0$.

Assertion (2) follows. Lemma 4.4 is completely proved. □

5. Existence of two solutions for $(\mathcal{P})$ with low energy

In this section we will prove the existence of two distinct solutions u_1 and u_2 for the variational inequality $(\mathcal{P})$ characterized by the fact that the energy $J_\lambda(u_i)$ is small for $i = 1, 2$. For this purpose, we will apply Theorem 3.10 to the continuous functional $J_{\lambda,\psi}$.

Theorem 5.1. *Let $1 \leqslant k \leqslant j$ be such that $\lambda_{k-1} < \lambda_k = \cdots = \lambda_j < \lambda_{j+1}$. Then, for every $\varepsilon > 0$, there exist $\eta > 0$, $R > 0$ and a compact set $C \subset \Omega$ such that, if $\lambda_k - \eta \leqslant \lambda < \lambda_k$ and ψ_1 and ψ_2 satisfy $(\psi_{R,C})$, problem $(\mathcal{P})$ has at least two solutions u_1 and u_2 such that $0 < J_\lambda(u_i) \leqslant \varepsilon$, $i = 1, 2$.*

Proof. Let $\varepsilon > 0$. Let also R^*, R^{**}, η^*, ε^* be as in Lemma 4.3 and ϱ', σ, R' and C' be as in Lemma 4.4.

We will proceed by steps.

Step 1. There exist $\beta > 0$ and $\eta > 0$ such that, for every $\lambda \in [\lambda_k - \eta, \lambda_k + \eta]$, we have

$$\beta \leqslant \min\{\varepsilon, \varepsilon^*, \sigma\}, \tag{5.1}$$

$$\eta \leqslant \eta^*, \tag{5.2}$$

$$\sup\bigl\{J_\lambda(u)\colon u \in \widetilde{V}_{k-1}\bigr\} \leqslant 0, \tag{5.3}$$

$$\sup\bigl\{J_\lambda(u)\colon u \in \widetilde{V}_{k-1} \oplus \operatorname{span}\{e_k, \ldots, e_j\},\ \max\bigl\{\|P_1 u\|, \|P_2 u\|\bigr\} = R^*\bigr\} < 0, \tag{5.4}$$

$$\sup\bigl\{J_\lambda(u)\colon u \in \widetilde{V}_{k-1} \oplus \operatorname{span}\{e_k, \ldots, e_j\},\ \max\bigl\{\|P_1 u\|, \|P_2 u\|\bigr\} \leqslant R^*\bigr\} < \beta. \tag{5.5}$$

Proof. From (4.2) and (P4) we see that we can satisfy (5.3), while (5.4) follows from (1) of Lemma 4.3. By (4.3) and (P4) we have that

$$J_\lambda(u) \leqslant \frac{\lambda_k - \lambda}{2} \|u\|_2^2 \leqslant \frac{\eta}{2\lambda_1} \|u\|^2$$

for any $u \in \widetilde{V}_{k-1} \oplus \operatorname{span}\{e_k, \ldots, e_j\}$. Then assertions (5.1) and (5.5) easily follow. Of course, we can also satisfy (5.2). □

At this point, fix $\lambda \in [\lambda_k - \eta, \lambda_k[$.

Step 2. There exist $r, r' > 0$ and $\alpha > 0$ such that $r' < r < R^*$ and

$$\inf\bigl\{J_\lambda(u)\colon u \in \operatorname{span}\{e_k, \ldots, e_j\} \oplus V_j^\perp,\ \|u\| = r\bigr\} > \alpha, \tag{5.6}$$

$$\sup\bigl\{J_\lambda(u)\colon u \in \widetilde{V}_{k-1} \oplus \operatorname{span}\{e_k, \ldots, e_j\},\ \|P_1 u\| \leqslant R^*, \|P_2 u\| = r'\bigr\} < \alpha. \tag{5.7}$$

Proof. Since

$$\langle Au, u\rangle - \lambda \int_\Omega u^2\, dx \geqslant \left(1 - \frac{\lambda}{\lambda_k}\right)\langle Au, u\rangle, \quad \forall u \in \operatorname{span}\{e_k, \ldots, e_j\} \oplus V_j^\perp,$$

by (4.8) there exist $r \in]0, R^*[$ and $\alpha > 0$ satisfying (5.6). By (5.3) there exists $r' \in]0, r[$ satisfying (5.7). By (2) of Lemma 4.4 we have that there exist $\varrho_\alpha, \sigma_\alpha > 0$, $R_\alpha > 0$ and a compact set $C_\alpha \subset \Omega$ such that, if ψ_1 and ψ_2 satisfy $(\psi_{R_\alpha, C_\alpha})$ and $\varrho \in]0, \varrho_\alpha]$, we have $|d(J_{\lambda,\psi}|_{M_\varrho})|(u) > \sigma_\alpha$, for any $u \in K_\psi \cap \partial M_\varrho$ with $\|u\| \leqslant R^{**} + 1$ and $\alpha \leqslant J_{\lambda,\psi}(u) \leqslant \varepsilon^*$. □

Now, fix $\varrho \in (0, \min\{\varrho', \varrho_\alpha, r'\})$.

Step 3. There exist $\tilde{e}_k, \dots, \tilde{e}_j \in C_0^\infty(\Omega)$ such that

$$H_0^1(\Omega) = \widetilde{V}_{k-1} \oplus \operatorname{span}\{\tilde{e}_k, \dots, \tilde{e}_j\} \oplus V_j^\perp, \tag{5.8}$$

$$\sup\left\{J_\lambda(u)\colon u \in \partial_{\widetilde{V}_{k-1} \oplus \operatorname{span}\{\tilde{e}_k, \dots, \tilde{e}_j\}} \Delta\right\} < \alpha, \tag{5.9}$$

$$\sup\{J_\lambda(u)\colon u \in \Delta\} < \beta, \tag{5.10}$$

$$\inf\left\{J_\lambda(u)\colon u \in \operatorname{span}\{\tilde{e}_k, \dots, \tilde{e}_j\} \oplus V_j^\perp,\ \|u\| = r\right\} > \alpha, \tag{5.11}$$

$$\Delta \subset M_\varrho \cap \mathrm{B}_{R^{**}}(0), \tag{5.12}$$

where $\Delta = \{u \in \widetilde{V}_k \oplus \operatorname{span}\{\tilde{e}_k, \dots, \tilde{e}_j\}\colon \|\widetilde{P}_1 u\| \leqslant R^*,\ r' \leqslant \|\widetilde{P}_2 u\| \leqslant R^*\}$ and $\widetilde{P}_1\colon H_0^1(\Omega) \to \widetilde{V}_{k-1}$, $\widetilde{P}_2\colon H_0^1(\Omega) \to \operatorname{span}\{\tilde{e}_k, \dots, \tilde{e}_j\}$ and $\widetilde{P}_3\colon H_0^1(\Omega) \to V_j^\perp$ are the projections associated with the (nonorthogonal) direct sum (5.8).

Proof. If we choose $\tilde{e}_k, \dots, \tilde{e}_j$ close enough to $e_k, \dots, e_j$, respectively, and take into account the continuity of J_λ, then assertion (5.9) follows from (5.4) and (5.7), assertion (5.10) from (5.5), while (5.8) is clear. Since J_λ is Lipschitz continuous on bounded sets, assertion (5.11) follows from (5.6), while assertion (5.12) follows from (1) of Lemma 4.3 and the fact that $\varrho < r'$. □

At this point, let $E_1 = \widetilde{V}_{k-1}$, $E_2 = \operatorname{span}\{\tilde{e}_k, \dots, \tilde{e}_j\}$ and $E_3 = V_j^\perp$. Let also $B = (H_0^1(\Omega) \setminus M_\varrho) \cap E_2$ and observe that B is a bounded, convex, open neighborhood of 0 in E_2 such that

$$M_\varrho = \left\{u \in H_0^1(\Omega)\colon\ \widetilde{P}_2 u \notin B\right\}.$$

Let $C = C' \cup C_\alpha \cup \bigcup_{i=1}^j \operatorname{supt}(\tilde{e}_i)$ and let $R > \max\{R', R_\alpha\}$ be such that $\|u\|_\infty \leqslant R$ whenever $u \in \Delta$. If $\psi_1 : \Omega \to [-\infty, 0]$ and $\psi_2 : \Omega \to [0, +\infty]$ are Borel functions satisfying condition $(\psi_{R,C})$, then we clearly have

$$\Delta \subset K_\psi. \tag{5.13}$$

Now let $X = K_\psi$ and consider the continuous functional $J_{\lambda,\psi} : K_\psi \to \mathbb{R}$. By step 3 and (5.13), we have that assumption (a) of Theorem 3.10 is satisfied.

Step 4. If we set $U = X \cap \mathrm{B}_{R^{**}}(0)$, we have that

$$J_{\lambda,\psi} \text{ satisfies } (\mathrm{PS})_c \quad \text{on } \overline{U} \text{ for every } c \in \mathbb{R}, \tag{5.14}$$

$$\inf\left\{\left|d(J_{\lambda,\psi}|_{K_\psi \cap M})\right|(u)\colon u \in \overline{U},\ \widetilde{P}_2 u \in \partial B,\ \alpha \leqslant J_{\lambda,\psi}(u) \leqslant \beta\right\} > 0, \tag{5.15}$$

$$\forall u \in \mathrm{B}_1(U) \setminus U, \quad \alpha \leqslant J_{\lambda,\psi}(u) \leqslant \beta \quad \Rightarrow \quad |dJ_{\lambda,\psi}|(u) > \beta - \alpha, \tag{5.16}$$

$$\begin{aligned} &\forall u \in \mathrm{B}_1(U) \setminus U, \quad \alpha \leqslant J_{\lambda,\psi}(u) \leqslant \beta, \\ &\quad \widetilde{P}_2 u \in \partial B \quad \Rightarrow \quad \left|d(J_{\lambda,\psi}|_{K_\psi \cap M})\right|(u) > \beta - \alpha. \end{aligned} \tag{5.17}$$

Proof. Let (u_n) be a $(\mathrm{PS})_c$-sequence for $J_{\lambda,\psi}$ with $\|u_n\| \leqslant R^{**}$. By (1) of Lemma 4.1, there exists a sequence $(\varphi_n)_n$ in $H^{-1}(\Omega)$ such that $\|\varphi_n\|_{H^{-1}(\Omega)} \to 0$ as $n \to \infty$ and

$$\langle Au_n, v-u_n\rangle - \lambda\int_\Omega u_n(v-u_n)\,dx - \int_\Omega p(x,u_n)(v-u_n)\,dx \geqslant \langle\varphi_n, v-u_n\rangle, \quad \forall v \in K_\psi. \tag{5.18}$$

Since $(u_n)_n$ is bounded, it is weakly convergent, up to a subsequence, to some $u \in K_\psi$. Taking u as a test function in (5.18), we get by (P2)

$$\limsup_{n\to\infty}\langle Au_n, u_n\rangle \leqslant \langle Au, u\rangle.$$

By the weak lower semi-continuity of the norm, (5.14) follows.

Assertion (5.15) follows by (2) of Lemma 4.4. Finally, (5.1) and (1) of Lemma 4.4 easily yield (5.16) and (5.17).

By (5.12) and step 4, we have that also assumption (b′) of Theorem 3.10 is satisfied. □

Step 5. The metric space X is locally contractible.

Proof. Being a convex subset of a normed space, X is an ANR, hence locally contractible (see, e.g., [10]). □

Since also assumption (c) of Theorem 3.10 is satisfied, there exist two distinct critical points u_1 and u_2 of $J_{\lambda,\psi}$ such that $J_{\lambda,\psi}(u_i) \in [\alpha,\beta]$, $i = 1, 2$, hence with $0 < J_{\lambda,\psi}(u_i) \leqslant \varepsilon$. From (1) of Lemma 4.1 we conclude that u_1 and u_2 are solutions of $(\mathcal{P})$.

The proof of Theorem 5.1 is complete. □

6. The linking solution

In the first part of the section we will get the existence of a solution u for the problem $(\mathcal{P})$ with energy $J_\lambda(u) > 0$ in a uniform way with respect to λ. In the second part we will prove Theorem 2.1.

Let s be given by assumption (P2) and let $\bar{J}_\lambda, \bar{I}_{K_\psi} : L^{s+1}(\Omega) \to \mathbb{R} \cup \{+\infty\}$ be the functionals defined as

$$\bar{J}_\lambda(u) = \begin{cases} J_\lambda(u) & \text{if } u \in H_0^1(\Omega), \\ +\infty & \text{if } u \in L^{s+1}(\Omega) \setminus H_0^1(\Omega), \end{cases}$$

$$\bar{I}_{K_\psi}(u) = \begin{cases} 0 & \text{if } u \in K_\psi, \\ +\infty & \text{if } u \in L^{s+1}(\Omega) \setminus K_\psi. \end{cases}$$

We also set $\bar{J}_{\lambda,\psi} = \bar{J}_\lambda + \bar{I}_{K_\psi}$. First of all, we prove the following:

Lemma 6.1. *The following facts hold*:

(1) *for every $u \in H_0^1(\Omega)$, we have $|d\overline{J}_\lambda|(u) < +\infty$ if and only if $Au \in L^{(s+1)'}(\Omega)$ and in this case*

$$|d\overline{J}_\lambda|(u) = \left\| Au - \lambda u - p(x,u) \right\|_{(s+1)'};$$

(2) *for every $u \in K_\psi$ with $|d\overline{J}_{\lambda+\psi}|(u) < +\infty$, there exists $\varphi \in L^{(s+1)'}(\Omega)$ with*

$$\|\varphi\|_{(s+1)'} \leqslant |d\overline{J}_{\lambda,\psi}|(u)$$

such that

$$\langle Au, v-u\rangle - \lambda \int_\Omega u(v-u)\,dx - \int_\Omega p(x,u)(v-u)\,dx \geqslant \int_\Omega \varphi(v-u)\,dx, \quad \forall v \in K_\psi.$$

Proof. We only prove assertion (2). The proof of (1) is similar and simpler. First of all, observe that the functional $\mathcal{A}: L^{s+1}(\Omega) \to \mathbb{R} \cup \{+\infty\}$ defined as

$$\mathcal{A}(u) = \begin{cases} \frac{1}{2}\langle Au, u\rangle & \text{if } u \in K_\psi, \\ +\infty & \text{if } u \in L^{s+1}(\Omega) \setminus K_\psi, \end{cases}$$

is convex and lower semi-continuous, while the functional

$$u \mapsto \frac{\lambda}{2}\int_\Omega u^2\,dx + \int_\Omega P(x,u)\,dx$$

is of class C^1 on $L^{s+1}(\Omega)$. From [7, Proposition 2.10 and Theorem 2.11] we deduce that there exists $\varphi \in L^{(s+1)'}(\Omega)$ with $\|\varphi\|_{(s+1)'} \leqslant |d\overline{J}_{\lambda,\psi}|(u)$ and

$$\varphi + \lambda u + p(x,u) \in \partial\mathcal{A}(u),$$

where ∂ stands for the usual subdifferential of convex analysis. Then assertion (2) easily follows. □

Now, we are able to prove the main result of this section.

Theorem 6.2. *Let $1 \leqslant k \leqslant j$ be such that $\lambda_{k-1} < \lambda_k = \cdots = \lambda_j < \lambda_{j+1}$. Then, there exist $\alpha > 0$, $\eta > 0$, $R > 0$ and a compact set $C \subset \Omega$ such that, if $\lambda_k - \eta \leqslant \lambda \leqslant \lambda_k + \eta$, and ψ_1 and ψ_2 satisfy $(\psi_{R,C})$, we have that problem $(\mathcal{P})$ has at least one solution u such that $J_\lambda(u) \geqslant \alpha$.*

Proof. We will apply Theorem 3.11. First of all, let $e_1, \dots, e_{j+1}$ be the eigenfunctions related to $\lambda_1, \dots, \lambda_{j+1}$ and let $V_j^\perp$ be as in Section 2. Let also

$$W = \left\{ u \in L^{s+1}(\Omega)\colon \int_\Omega u e_h \, dx = 0, \ \forall h = 1, \dots, j+1 \right\},$$

so that

$$L^{s+1}(\Omega) = \operatorname{span}\{e_1, \dots, e_{j+1}\} \oplus W.$$

Now we will proceed by steps.

Step 1. There exist $0 < r < R^*$ and $\alpha > 0$ such that

$$\inf\left\{ \overline{J}_{\lambda_k}(u)\colon u \in \operatorname{span}\{e_{j+1}\} \oplus W, \ \|u\|_{s+1} = r \right\} > \alpha, \tag{6.1}$$

$$\sup\left\{ \overline{J}_{\lambda_k}(u)\colon u \in \operatorname{span}\{e_1, \dots, e_{j+1}\}, \ \|u\|_{s+1} = R^* \right\} < 0. \tag{6.2}$$

Proof. Since

$$\langle Au, u\rangle - \lambda_k \int_\Omega u^2 \, dx \geqslant \left(1 - \frac{\lambda_k}{\lambda_{j+1}}\right) \langle Au, u\rangle, \quad \forall u \in V_j^\perp,$$

by (4.8) and the continuous embedding of $H_0^1(\Omega)$ into $L^{s+1}(\Omega)$, there exist $r > 0$ and $\alpha > 0$ satisfying (6.1). Then, by (2) of Lemma 4.2 there exists $R^* > r$ satisfying (6.2). □

Step 2. There exist $\tilde{e}_1, \dots, \tilde{e}_{j+1} \in C_0^\infty(\Omega)$, $\eta > 0$ and $\beta > 0$ such that, for every $\lambda \in [\lambda_k - \eta, \lambda_k + \eta]$, we have

$$L^{s+1}(\Omega) = \widetilde{V}_j \oplus \operatorname{span}\{\tilde{e}_{j+1}\} \oplus W, \tag{6.3}$$

$$\inf\left\{ \overline{J}_\lambda(u)\colon u \in \operatorname{span}\{\tilde{e}_{j+1}\} \oplus W, \ \|u\|_{s+1} = r \right\} > \alpha, \tag{6.4}$$

$$\sup\left\{ \overline{J}_\lambda(u)\colon u \in \widetilde{V}_j, \ \|u\|_{s+1} \leqslant R^* \right\} < \alpha, \tag{6.5}$$

$$\sup\left\{ \overline{J}_\lambda(u)\colon u \in \widetilde{V}_j \oplus \operatorname{span}\{\tilde{e}_{j+1}\}, \ \|u\|_{s+1} = R^* \right\} < 0, \tag{6.6}$$

$$\sup\left\{ \overline{J}_\lambda(u)\colon u \in \widetilde{V}_j \oplus \operatorname{span}\{\tilde{e}_{j+1}\}, \ \|u\|_{s+1} \leqslant R^* \right\} < \beta, \tag{6.7}$$

where $\widetilde{V}_j$ is the j-dimensional space generated by $\{\tilde{e}_1, \dots, \tilde{e}_j\}$.

Proof. If $\tilde{e}_h$ is close enough to e_h, $1 \leqslant h \leqslant j+1$, and η is small enough, we have that (6.3) holds, while (6.5) follows from (P4) and (6.4), (6.6) from (6.1), (6.2), respectively. Of course, there exists $\beta > 0$ satisfying (6.7). So, step 2 is proved. □

Now let us fix $\lambda \in [\lambda_k - \eta, \lambda_k + \eta]$.

Step 3. For every $c \in \mathbb{R}$, the functional $\overline{J}_\lambda$ satisfies $(\mathrm{PS})_c$.

Proof. Let $(u_n)_n$ be a $(\mathrm{PS})_c$-sequence for $\overline{J}_\lambda$. We have that $(\overline{J}_\lambda(u_n))_n$ is bounded. Moreover, by Lemma 6.1 there exists $(\varphi_n)_n$ in $L^{(s+1)'}(\Omega)$ such that $\|\varphi_n\|_{(s+1)'} \to 0$ as n goes to infinity and

$$\langle Au_n, v\rangle - \lambda \int_\Omega u_n v\,dx - \int_\Omega p(x,u_n)v\,dx = \int_\Omega \varphi_n v\,dx, \quad \forall v \in H_0^1(\Omega). \tag{6.8}$$

Taking $v = u_n$ as a test function in (6.8) and using (P4), by standard arguments we obtain that $(u_n)_n$ is bounded in $H_0^1(\Omega)$. Then, up to subsequences, $(u_n)_n$ strongly converges to some u in $L^{s+1}(\Omega)$. □

Step 4. Let $Q = \{v + t\tilde{e}_{j+1}\colon v \in \widetilde{V}_j,\ t \geqslant 0,\ \|v + t\tilde{e}_{j+1}\|_{s+1} \leqslant R^*\}$. Then there exist $R^{**} > 0$ and $\varrho > 0$ such that

$$Q \subset \bigl\{u \in L^{s+1}(\Omega)\colon \|u\|_{s+1} < R^{**}\bigr\}, \tag{6.9}$$

$$|d\overline{J}_\lambda|(u) > \frac{\beta - \alpha}{\varrho} \quad \text{whenever } u \in H_0^1(\Omega),\ \alpha \leqslant \overline{J}_\lambda(u) \leqslant \beta \text{ and } \|u\|_{s+1} \geqslant R^{**}. \tag{6.10}$$

Proof. By step 3, the set $\{u \in L^{s+1}(\Omega)\colon \alpha \leqslant \overline{J}_\lambda(u) \leqslant \beta,\ |d\overline{J}_\lambda|(u) = 0\}$ is compact. Since Q also is compact, there exists $R^{**} > 0$ satisfying (6.9) and

$$\bigl\{u \in L^{s+1}(\Omega)\colon \alpha \leqslant \overline{J}_\lambda(u) \leqslant \beta,\ |d\overline{J}_\lambda|(u) = 0\bigr\} \subset \bigl\{u \in L^{s+1}(\Omega)\colon \|u\|_{s+1} < R^{**}\bigr\}. \tag{6.11}$$

To find ϱ satisfying (6.10), assume for a contradiction that there exists a sequence $(u_n)_n$ in $H_0^1(\Omega)$ such that $\|u_n\|_{s+1} \geqslant R^{**}$,

$$\alpha \leqslant \overline{J}_\lambda(u_n) \leqslant \beta \tag{6.12}$$

for any $n \in \mathbb{N}$ and

$$\bigl|d\overline{J}_\lambda\bigr|(u_n) \to 0 \tag{6.13}$$

as n goes to infinity. Then, $(u_n)_n$ is a (PS)-sequence for $\overline{J}_\lambda$. So, up to a subsequence, we have that

$$\overline{J}_\lambda(u_n) \to c \in [\alpha, \beta] \tag{6.14}$$

and

$$u_n \to u \quad \text{with } \|u\|_{s+1} \geqslant R^{**} \tag{6.15}$$

as n goes to infinity. By (6.13) and Proposition 3.4 we deduce that $|dG_{\overline{J}_\lambda}|(u_n, \overline{J}_\lambda(u_n)) \to 0$ as n goes to infinity. Remark 3.2 and (6.14) easily yield that

$$|dG_{\overline{J}_\lambda}|(u, c) = 0. \tag{6.16}$$

On the other hand, by [7, Theorem 3.13] we have $|dG_{\overline{J}_\lambda}|(u, t) = 1$ whenever $\overline{J}_\lambda(u) < t$. It follows that $\overline{J}_\lambda(u) = c$. Using again Proposition 3.4 and (6.16) we obtain that $|d\overline{J}_\lambda|(u) = 0$. Then, u is a critical point for $\overline{J}_\lambda$ such that $\|u\|_{s+1} \geqslant R^{**}$ and $\alpha \leqslant \overline{J}_\lambda(u) \leqslant \beta$. This contradicts (6.11). Then, step 4 is proved. □

Now, let $U = \{u \in L^{s+1}(\Omega)\colon \|u\|_{s+1} < R^{**}\}$. We apply Theorem 3.11 to the lower semicontinuous functional $\overline{J}_{\lambda,\psi} : L^{s+1}(\Omega) \to \mathbb{R} \cup \{+\infty\}$.

Step 5. There exist $R' > 0$ and a compact set $C' \subset \Omega$ such that, if ψ_1 and ψ_2 satisfy $(\psi_{R',C'})$, we have that

$$\forall u \in B_\varrho(U) \setminus U, \quad \alpha \leqslant \overline{J}_{\lambda,\psi}(u) \leqslant \beta \quad \Rightarrow \quad |d\overline{J}_{\lambda,\psi}|(u) > \frac{\beta - \alpha}{\varrho}.$$

Proof. We argue by contradiction and we suppose that there exist two sequences of Borel functions $\psi_{1,n} : \Omega \to [-\infty, 0]$ and $\psi_{2,n} : \Omega \to [0, +\infty]$ such that

$$\psi_{1,n}(x) \leqslant -n < n \leqslant \psi_{2,n}(x) \quad \text{cap. q.e.} \quad \text{on } C_n := \left\{x \in \mathbb{R}^N\colon\ d(x, \mathbb{R}^N \setminus \Omega) \geqslant \frac{1}{n}\right\} \tag{6.17}$$

and a sequence $(u_n)_n$ in K_{ψ_n} such that

$$R^{**} \leqslant \|u_n\|_{s+1} < R^{**} + \varrho, \tag{6.18}$$

$$\alpha \leqslant \overline{J}_{\lambda,\psi_n}(u_n) \leqslant \beta, \tag{6.19}$$

and $|d\overline{J}_{\lambda,\psi_n}|(u_n) \leqslant \frac{\beta-\alpha}{\varrho}$. By (2) of Lemma 6.1, we deduce that there exists $\varphi_n \in L^{(s+1)'}(\Omega)$ such that

$$\|\varphi_n\|_{L^{(s+1)'}(\Omega)} \leqslant \frac{\beta - \alpha}{\varrho} \tag{6.20}$$

and

$$\langle Au_n, v - u_n\rangle - \lambda \int_\Omega u_n(v - u_n)\,dx - \int_\Omega p(x, u_n)(v - u_n)\,dx$$
$$\geqslant \int_\Omega \varphi_n(v - u_n)\,dx, \quad \forall v \in K_{\psi_n}. \tag{6.21}$$

By (6.18) we have that $(u_n)_n$ is bounded in $L^{s+1}(\Omega)$. By (P2) and (6.19) we deduce that $(u_n)_n$ is bounded in $H_0^1(\Omega)$. Then, up to subsequences, there exists u in $H_0^1(\Omega)$ such that $u_n \to u$ in $L^{s+1}(\Omega)$ and $u_n \overset{w}{\to} u$ in $H_0^1(\Omega)$ as n tends to infinity.

Let $\bar{v} \in C_0^\infty(\Omega)$. By (6.17) we have that $\psi_{1,n} \leqslant \bar{v} \leqslant \psi_{2,n}$ cap. q.e. in Ω for n sufficiently large. Then we can take $\bar{v}$ as a test function in (6.21), obtaining

$$\langle Au_n, u_n\rangle \leqslant \langle Au_n, \bar{v}\rangle - \lambda \int_\Omega u_n(\bar{v} - u_n)\,dx - \int_\Omega p(x, u_n)(v - u_n)\,dx - \int_\Omega \varphi_n(\bar{v} - u_n)\,dx, \tag{6.22}$$

for any $\bar{v} \in C_0^\infty(\Omega)$ and n sufficiently large. Moreover, there exists φ in $L^{(s+1)'}(\Omega)$ such that $\|\varphi\|_{(s+1)'} \leqslant \frac{\beta-\alpha}{\varrho}$ and $\varphi_n \overset{w}{\to} \varphi$ in $L^{(s+1)'}(\Omega)$ as n goes to infinity. Passing to the upper limit in (6.22), we get

$$\limsup_{n\to\infty}\langle Au_n, u_n\rangle \leqslant \langle Au, \bar{v}\rangle - \lambda \int_\Omega u(\bar{v} - u)\,dx - \int_\Omega p(x, u)(\bar{v} - u)\,dx + \langle \varphi, \bar{v} - u\rangle, \tag{6.23}$$

for any $\bar{v} \in C_0^\infty(\Omega)$. By density we get that (6.23) holds for any $\bar{v} \in H_0^1(\Omega)$. In particular, we can take $\bar{v} = u$ in (6.23). So, by the weak lower semi-continuity of the norm we obtain that $u_n \to u$ in $H_0^1(\Omega)$ as n goes to infinity. Moreover, by (6.18) and (6.19) have that $R^{**} \leqslant \|u\| \leqslant R^{**} + \varrho$ and $\alpha \leqslant \overline{J}_\lambda(u) \leqslant \beta$. Finally, (6.23) yields

$$\langle Au, u\rangle \leqslant \langle Au, v\rangle - \lambda \int_\Omega u(v - u)\,dx - \int_\Omega p(x, u)(v - u)\,dx + \langle \varphi, \bar{v} - u\rangle, \quad \forall v \in H_0^1(\Omega),$$

whence $Au - \lambda u - p(x, u) = \varphi$. By (1) of Lemma 6.1 it follows that $|d\overline{J}_\lambda|(u) \leqslant \frac{\beta-\alpha}{\varrho}$ and this fact contradicts step 4. Then, step 5 is proved. □

At this point, let $C = C' \cup \bigcup_{i=1}^{j+1} \operatorname{supt}(\tilde{e}_i)$ where C' and C_α are those of Lemma 4.4, and let $R > R'$ be such that $\|u\|_\infty \leqslant R$ whenever $u \in Q$. If ψ_1 and ψ_2 satisfy condition $(\psi_{R,C})$, then we clearly have $Q \subset K_\psi$. So, by (6.5)–(6.7) it follows

$$\sup\bigl\{\overline{J}_{\lambda,\psi}(u)\colon u \in \partial_{\widetilde{V}_j \oplus \operatorname{span}\{\tilde{e}_{j+1}\}} Q\bigr\} < \alpha,$$
$$\sup\bigl\{\overline{J}_{\lambda,\psi}(u)\colon u \in Q\bigr\} < \beta.$$

Taking into account also (6.4), we can infer that the functional $\overline{J}_{\lambda,\psi}$ satisfies assumption (a) of Theorem 3.11 with $E_- = \widetilde{V}_j$ and $E_+ = \operatorname{span}\{\tilde{e}_{j+1}\} \oplus W$.

Step 6. For every $c \in \mathbb{R}$, $\overline{J}_{\lambda,\psi}$ satisfies $(\mathrm{PS})_c$ on $\overline{U}$. Moreover, we have that

$$|d\mathcal{G}_{\overline{J}_{\lambda,\psi}}|(u,t) = 1 \quad \text{whenever } \overline{J}_{\lambda,\psi}(u) < t. \tag{6.24}$$

Proof. Let $(u_n)_n$ be a $(\mathrm{PS})_c$-sequence for $\overline{J}_{\lambda,\psi}$ on $\overline{U}$. We have that

$$\|u_n\|_{s+1} \leqslant R^{**} \tag{6.25}$$

for any $n \in \mathbb{N}$, and

$$\overline{J}_{\lambda,\psi}(u_n) \to c \tag{6.26}$$

and

$$|d\overline{J}_{\lambda,\psi}|(u_n) \to 0 \tag{6.27}$$

as n goes to infinity. By (6.25), (P2) and (6.26) we deduce that $(u_n)_n$ is bounded in $H_0^1(\Omega)$, hence convergent, up to a subsequence, in $L^{s+1}(\Omega)$. On the other hand, property (6.24) directly follows from [7, Theorem 3.13]. Then, step 6 is proved. □

Since also assumption (b) of Theorem 3.11 is satisfied, there exists a critical point u of $\overline{J}_{\lambda,\psi}$ such that $\overline{J}_{\lambda,\psi}(u) \in [\alpha, \beta]$. Therefore, $u \in K_\psi$ and by (2) of Lemma 6.1 we conclude that u is a solution of problem $(\mathcal{P})$. □

Let us observe that Theorem 2.1 now easily follows from Theorems 5.1 and 6.2.

Remark 6.3. While in Section 5 we have considered the continuous functional $J_{\lambda,\psi}$, taking as reference the $H_0^1(\Omega)$-metric, here the same approach would have caused a further compactness difficulty. More precisely, in the proof of step 5 of Theorem 6.2, it is not clear how to manage the term $\langle \varphi_n, u_n\rangle$ if one only knows that $(u_n)_n$ is weakly convergent to u in $H_0^1(\Omega)$ and $(\varphi_n)_n$ is weakly convergent to φ in $H^{-1}(\Omega)$. On the contrary, the use of the $L^{s+1}(\Omega)$-metric forces to consider the lower semicontinuous functional $\overline{J}_{\lambda,\psi}$, but now $(\varphi_n)_n$ is weakly convergent to φ in $L^{(s+1)'}(\Omega)$, while $(u_n)_n$ is still bounded in $H_0^1(\Omega)$, hence strongly convergent, up to a subsequence, in $L^{s+1}(\Omega)$. In these conditions the term $\langle \varphi_n, u_n\rangle$ can be easily handled.

Remark 6.4. Actually, following the same proof of Theorem 6.2, one can state a more precise assertion. Namely, that, for every $\varepsilon > 0$, there exist $\alpha > 0$, $\eta > 0$, $R > 0$ and a compact set $C \subset \Omega$ such that, if $\lambda_k - \eta \leqslant \lambda \leqslant \lambda_{j+1} - \varepsilon$ and ψ_1 and ψ_2 satisfy $(\psi_{R,C})$, we have that problem $(\mathcal{P})$ has at least one solution u such that $J_\lambda(u) \geqslant \alpha$.

This is an improvement of a result of [18], concerning the existence of a nontrivial solution for problem $(\mathcal{P})$.

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References

[1] A. Ambrosetti, P.H. Rabinowitz, Dual variational methods in critical point theory and applications, J. Funct. Anal. 14 (1973) 349–381.
[2] I. Campa, M. Degiovanni, Subdifferential calculus and nonsmooth critical point theory, SIAM J. Optim. 10 (4) (2000) 1020–1048.
[3] A. Canino, M. Degiovanni, Nonsmooth critical point theory and quasilinear elliptic equations, in: A. Granas, M. Frigon, G. Sabidussi (Eds.), Topological Methods in Differential Equations and Inclusions, Montreal, 1994, in: NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 472, Kluwer Academic, Dordrecht, 1995, pp. 1–50.
[4] F.H. Clarke, Optimization and Nonsmooth Analysis, Canad. Math. Soc. Series of Monographs and Advanced Texts, Wiley–Interscience, New York, 1983.
[5] J.N. Corvellec, A note on coercivity of lower semi-continuous functions and nonsmooth critical point theory, Serdica Math. J. 22 (1) (1996) 57–68.
[6] J.N. Corvellec, M. Degiovanni, M. Marzocchi, Deformation properties for continuous functionals and critical point theory, Topol. Methods Nonlinear Anal. 1 (1) (1993) 151–171.
[7] M. Degiovanni, M. Marzocchi, A critical point theory for nonsmooth functionals, Ann. Mat. Pura Appl. (4) 167 (1994) 73–100.
[8] M. Girardi, L. Mastroeni, M. Matzeu, Existence and regularity results for non-negative solutions of some semilinear elliptic variational inequalities via mountain pass techniques, Z. Anal. Anwendungen 20 (4) (2001) 845–857.
[9] A. Groli, M. Squassina, Jumping problems for fully nonlinear elliptic variational inequalities, J. Convex Anal. 8 (2) (2001) 471–488.
[10] S. Hu, Theory of Retracts, Wayne State Univ. Press, Detroit, 1965.
[11] A. Ioffe, E. Schwartzman, Metric critical point theory. I. Morse regularity and homotopic stability of a minimum, J. Math. Pures Appl. (9) 75 (2) (1996) 125–153.
[12] D. Jerison, C.E. Kenig, Unique continuation and absence of positive eigenvalues for Schrödinger operators. With an appendix by E.M. Stein, Ann. of Math. (2) 121 (3) (1985) 463–494.
[13] G. Katriel, Mountain pass theorems and global homeomorphism theorems, Ann. Inst. H. Poincaré Anal. Non Linéaire 11 (2) (1994) 189–209.
[14] P. Magrone, R. Servadei, A stability result for mountain pass type solutions of semilinear elliptic variational inequalities, Nonlinear Stud. 9 (4) (2002) 387–405.
[15] A. Marino, D. Mugnai, Asymptotical multiplicity and some reversed variational inequalities, Topol. Methods Nonlinear Anal. 20 (1) (2002) 43–62.
[16] A. Marino, C. Saccon, Some variational theorems of mixed type and elliptic problems with jumping nonlinearities, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 25 (3/4) (1997) 631–665.
[17] A. Marino, C. Saccon, Nabla theorems and multiple solutions for some noncooperative elliptic systems, Topol. Methods Nonlinear Anal. 17 (2) (2001) 213–237.
[18] M. Matzeu, R. Servadei, A linking type method to solve a class of semilinear elliptic variational inequalities, Adv. Nonlinear Stud. 2 (1) (2002) 1–17.
[19] D. Mugnai, Multiplicity of critical points in presence of a linking: application to a superlinear boundary value problem, NoDEA Nonlinear Differential Equations Appl. 11 (3) (2004) 379–391.
[20] D. Mugnai, On a reversed variational inequality, Topol. Methods Nonlinear Anal. 17 (2) (2001) 321–358.
[21] D. Passaseo, Molteplicità di soluzioni per disequazioni variazionali di tipo ellittico, Boll. UMI 7 (3-B) (1989) 639–667.
[22] D. Passaseo, Molteplicità di soluzioni per disequazioni variazionali non lineari di tipo ellittico, Rend. Acc. Naz. Sc. detta dei XL, Memorie di Mat. 109 (1991) 19–56, XV, fasc. 2.

[23] P.H. Rabinowitz, Minimax methods in critical point theory with applications to differential equations, in: CBMS Reg. Conf. Ser. Math., vol. 65, Amer. Math. Soc., Providence, RI, 1986.
[24] M. Reeken, Stability of critical points under small perturbations. I. Topological theory, Manuscripta Math. 7 (1972) 387–411.
[25] R. Servadei, E. Valdinoci, A multiplicity result for a class of nonlinear variational inequalities, Nonlinear Stud. 12 (1) (2005) 37–48.
[26] A. Szulkin, On a class of variational inequalities involving gradient operators, J. Math. Anal. Appl. 100 (1984) 486–499.
[27] A. Szulkin, Minimax principles for lower semi-continuous functions and application to nonlinear boundary value problems, Ann. Inst. H. Poincaré Anal. Non Linéaire 3 (1986) 77–109.