Nonlinear differ. equ. appl. 15 (2008), 717–728

1021-9722/060717-12
published online 19 November 2008
DOI 10.1007/s00030-008-8006-z

Nonlinear Differential Equations and Applications NoDEA

An Existence Result for a Problem with Critical Growth and Lack of Strict Convexity

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Abstract. We prove the existence of a nontrivial solution for a quasilinear elliptic equation involving a nonlinearity having critical growth and a convex principal part, which is not required to be strictly convex.

Mathematics Subject Classification (2000). 35J65, 58E05.

Keywords. Critical growth, linking theorem, nontrivial solution.

1. Introduction and main results

Let us consider the problem

$$\begin{cases} -\mathrm{div}(\Psi'(\nabla u)) = \lambda u + |u|^{2^*-2}u & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{$\mathcal{P}$}$$

where λ is a real parameter, Ω is a bounded open subset of $\mathbb{R}^N$, $N \geq 4$, and $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent for the embedding of $H_0^1(\Omega)$ in $L^p(\Omega)$. Moreover, assume that $\Psi : \mathbb{R}^N \to \mathbb{R}$ is a convex function of class C^1 satisfying the following conditions:

$$\lim_{\xi \to 0} \frac{\Psi(\xi)}{|\xi|^2} = \frac{1}{2}\,; \tag{Ψ_1}$$

$$\lim_{|\xi| \to \infty} \frac{\Psi'(\xi)\cdot\xi}{|\xi|^2} = 1\,; \tag{Ψ_2}$$

$$\Psi(\xi) \leq \frac{1}{2}|\xi|^2 \quad \text{for every} \quad \xi \in \mathbb{R}^N\,. \tag{Ψ_3}$$

Let us also denote by (λ_k) the eigenvalues of $-\Delta$ with homogeneous Dirichlet boundary condition. It is easily seen that (Ψ_1) implies $\Psi'(0) = 0$. Therefore problem $(\mathcal{P})$ possesses the trivial solution $u = 0$.

Supported by MURST, Project "Variational Methods and Nonlinear Differential Equations".

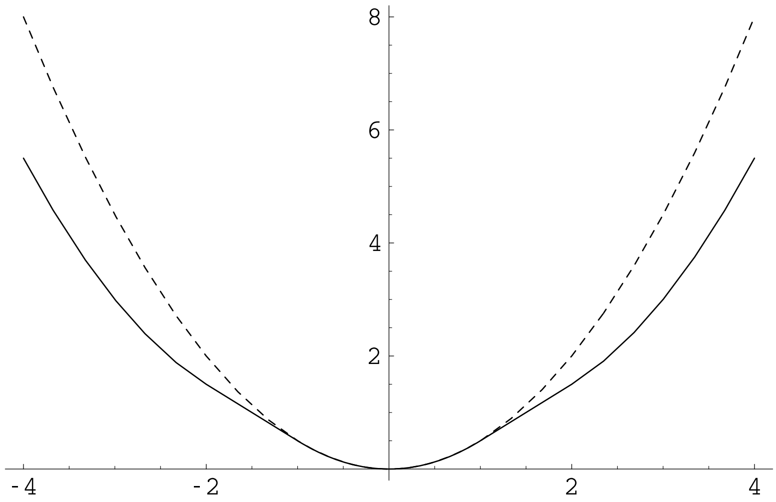

FIGURE 1

Problem $(\mathcal{P})$ can be treated by variational techniques. Indeed, weak solutions u of $(\mathcal{P})$ can be found as critical points of the C^1 functional $J : H_0^1(\Omega) \to \mathbb{R}$ defined as

$$J(u) = \int_\Omega \Psi(\nabla u)\, dx - \frac{\lambda}{2} \int_\Omega u^2\, dx - \frac{1}{2^*} \int_\Omega |u|^{2^*}\, dx\,. \tag{1.1}$$

In the typical case $\Psi(\xi) = \frac{1}{2}|\xi|^2$, there is now a wide literature on problem $(\mathcal{P})$, starting from [3]. The key point here is that, although Ψ shares some properties with this typical case, there is no assumption of strict convexity with respect to ξ.

For instance, one could consider (see Figure 1)

$$\Psi(\xi) = \psi(\xi_1) + \frac{1}{2} \sum_{j=2}^{N} \xi_j^2\,, \tag{1.2}$$

where

$$\psi(t) = \begin{cases} \frac{1}{2}t^2 & \text{if } |t| < 1\,, \\ |t| - \frac{1}{2} & \text{if } 1 \le |t| \le 2\,, \\ \frac{1}{2}|t|^2 - |t| + \frac{3}{2} & \text{if } |t| > 2\,. \end{cases}$$

If we look at the principal part of J as the energy stored in the deformation u, this means that the material has a plastic behavior when $1 \le |D_1 u| \le 2$. We refer the reader to [10, Chapter 6] for a discussion of several models of plasticity.

From a variational point of view, the effect is a lack of compactess even stronger than in the usual case. For instance, in the case of (1.2), suppose that u is a critical point of J with $5/4 < D_1 u < 7/4$ on some open subset ω of Ω. There exists a sequence (v_n) in $H_0^1(\Omega)$ such that $\operatorname{supt} v_n \subset \omega$, $|D_1 v_n| \le 1/4$, $D_j v_n \to 0$ in

$L^2(\Omega)$ for $j \geq 2$, $v_n \to 0$ in $L^\infty(\Omega)$, but (v_n) is not strongly precompact in $H_0^1(\Omega)$. Then $(u + v_n)$ is a Palais–Smale sequence just at the critical level $J(u)$ which is not strongly precompact in $H_0^1(\Omega)$. On the other hand, there is no way to prevent an interaction between the area where Ψ fails to be strictly convex and the values of ∇u.

Let us mention that, in the subcritical case, (nonsmooth) variational methods for functionals with lack of strict convexity have been successfully applied in [6].

Our first purpose is to extend the main result of [3] to the setting of problem $(\mathcal{P})$.

Theorem 1.1. *Let $N \geq 4$ and let $\Psi : \mathbb{R}^N \to \mathbb{R}$ be a convex function of class C^1 satisfying (Ψ_1)–(Ψ_3). Then, for every $\lambda \in]0, \lambda_1[$, problem $(\mathcal{P})$ admits a nontrivial and nonnegative weak solution $u \in H_0^1(\Omega)$.*

Then we will also extend the result of [4].

Theorem 1.2. *Let $\Psi : \mathbb{R}^N \to \mathbb{R}$ be a convex function of class C^1 satisfying (Ψ_1)–(Ψ_3) and let $\lambda > 0$. Moreover, suppose that either:*

(a) $N \geq 5$;

or

(b) $N \geq 4$ *and* $\lambda \neq \lambda_k$ *for every* $k \geq 1$.

Then problem $(\mathcal{P})$ admits a nontrivial weak solution $u \in H_0^1(\Omega)$.

For proving both results, we will construct in a standard way a Palais–Smale sequence (u_n) for J. Then we will show that, up to a subsequence, (u_n) is weakly convergent in $H_0^1(\Omega)$ to a nontrivial solution u of $(\mathcal{P})$, even if there is no hope to ensure the strong convergence in $H_0^1(\Omega)$. In order to prove that the weak limit is a solution of $(\mathcal{P})$, we will show in Lemma 2.2 a variant of the main result of [2, 5] which can be of independent interest.

From now on, $\|\cdot\|_p$ will denote the usual norm in L^p and $\|\cdot\|$ the H_0^1-norm defined as $\|u\| = \|\nabla u\|_2$.

2. Some convergence properties for convex functions

This section deals with some results of convergence for convex functions.

We point out that, by a simple extension of Hôpital's theorem, (Ψ_2) implies that

$$\lim_{|\xi|\to\infty} \frac{2\Psi(\xi)}{|\xi|^2} = 1\,. \tag{2.1}$$

In turn, (Ψ_2) and (2.1) yield that

$$\lim_{|\xi|\to\infty} \frac{\frac{1}{2}\Psi'(\xi)\cdot\xi - \Psi(\xi)}{|\xi|^2} = 0\,. \tag{2.2}$$

On the other hand, (Ψ_1), (2.1) and the convexity of Ψ imply that $\Psi(0) = 0$, $\Psi'(0) = 0$ and that there exists $\mu > 0$ such that

$$\Psi(\xi) \geq \mu|\xi|^2 \quad \forall \xi \in \mathbb{R}^N \,. \tag{2.3}$$

Since

$$0 = \Psi(0) \geq \Psi(\xi) - \Psi'(\xi)\cdot\xi \,,$$

it easily follows that

$$\Psi'(\xi)\cdot\xi \geq \mu|\xi|^2 \quad \forall \xi \in \mathbb{R}^N \,, \tag{2.4}$$

$$|\Psi'(\xi)| \geq \mu|\xi| \quad \forall \xi \in \mathbb{R}^N \,. \tag{2.5}$$

We also have

$$\Psi(\xi + |\xi|\nu) \geq \Psi(\xi) + |\xi|\Psi'(\xi)\cdot\nu \quad \forall \xi, \nu \in \mathbb{R}^N \quad \text{with} \quad |\nu| = 1 \,.$$

Combining this fact with (2.3), (Ψ_1) and (2.1), we deduce that there exists $M > 0$ such that

$$|\Psi'(\xi)| \leq M|\xi| \quad \forall \xi \in \mathbb{R}^N \,. \tag{2.6}$$

Lemma 2.1. *Let $\Psi : \mathbb{R}^N \to \mathbb{R}$ be a convex function of class C^1, let (ξ_k) be a sequence in $\mathbb{R}^N$ and let $\xi \in \mathbb{R}^N$ be such that*

$$\lim_{k\to\infty} \big(\Psi'(\xi_k) - \Psi'(\xi)\big)\cdot(\xi_k - \xi) = 0 \,.$$

Then $(\Psi'(\xi_k))$ is convergent to $\Psi'(\xi)$.

Proof. By substituting Ψ with $\widetilde{\Psi}(\zeta) = \Psi(\zeta) - \Psi'(\xi)\cdot\zeta$, we may suppose, without loss of generality, that ξ is a minimum of Ψ. By contradiction, assume that, up to a subsequence, there exists $\delta > 0$ such that $|\Psi'(\xi_k)| > \delta$ for every $k \in \mathbb{N}$. Let $t_k \in]0,1[$ be such that $|\Psi'((1-t_k)\xi + t_k\xi_k)| = \delta$ and let $\zeta_k = (1-t_k)\xi + t_k\xi_k$. Up to a subsequence, $(\Psi'(\zeta_k))$ is convergent to some $\alpha \in \mathbb{R}^N$ with $|\alpha| = \delta$. As $\Psi'(\xi) = 0$, we have

$$\begin{aligned} 0 \leq \Psi'(\zeta_k)\cdot(\zeta_k - \xi) &= t_k\, \Psi'(\zeta_k)\cdot(\xi_k - \xi) \\ &= \frac{t_k}{1-t_k}\, \Psi'(\zeta_k)\cdot(\xi_k - \zeta_k) \leq \frac{t_k}{1-t_k}\, \Psi'(\xi_k)\cdot(\xi_k - \zeta_k) \\ &= t_k\, \Psi'(\xi_k)\cdot(\xi_k - \xi) \,, \end{aligned}$$

whence

$$\lim_{k\to\infty} \Psi'(\zeta_k)\cdot(\zeta_k - \xi) = 0 \,. \tag{2.7}$$

On the other hand, the convexity of Ψ also implies that

$$\Psi(\xi) \geq \Psi(\zeta_k) + \Psi'(\zeta_k)\cdot(\xi - \zeta_k) \,.$$

Combining this fact with (2.7) and the minimality of ξ, we infer that

$$\lim_{k\to\infty} \Psi(\zeta_k) = \Psi(\xi) \,.$$

For every $\eta \in \mathbb{R}^N$, we also have

$$\Psi(\eta) \geq \Psi(\zeta_k) + \Psi'(\zeta_k)\cdot(\eta - \zeta_k) = \Psi(\zeta_k) + \Psi'(\zeta_k)\cdot(\eta - \xi) + \Psi'(\zeta_k)\cdot(\xi - \zeta_k) \,.$$

Passing to the limit as $k \to \infty$, we get

$$\Psi(\eta) \geq \Psi(\xi) + \alpha \cdot (\eta - \xi) \quad \forall \eta \in \mathbb{R}^N .$$

Since $\alpha \neq 0$, this contradicts the fact that ξ is a minimum of Ψ and Ψ is of class C^1. □

In the next result we adapt to our setting the main theorem of [2, 5].

Lemma 2.2. *Let $\Psi : \mathbb{R}^N \to \mathbb{R}$ be a convex function of class C^1 satisfying (2.6). Let (u_k) be a sequence weakly convergent to u in $H_0^1(\Omega)$ such that*

$$-\mathrm{div}\big(\Psi'(\nabla u_k)\big) = \mu_k + w_k \quad \text{in} \quad H^{-1}(\Omega) ,$$

where (w_k) is a sequence strongly convergent in $H^{-1}(\Omega)$ and (μ_k) is a sequence in $H^{-1}(\Omega)$ such that

$$\sup \big\{ |\langle \mu_k, v\rangle| : \ k \in \mathbb{N},\, v \in C_c^\infty(\Omega),\, \|v\|_\infty \leq 1,\, \mathrm{supt}\, v \subseteq K \big\} < +\infty \quad \forall K \subset\subset \Omega . \tag{2.8}$$

Then there exists a subsequence (u_{k_n}) such that

$$\lim_{n\to\infty} \Psi'\big(\nabla u_{k_n}(x)\big) = \Psi'\big(\nabla u(x)\big) \quad \text{a.e in} \quad \Omega .$$

Proof. Following the proof of [5, Theorem 5] with $f_k = w_k$ and $b_k(x, \xi) = b(x, \xi) = \Psi'(\xi)$, it turns out that there exists a subsequence (u_{k_n}) such that

$$\lim_{n\to\infty} \Big(\Psi'\big(\nabla u_{k_n}(x)\big) - \Psi'\big(\nabla u(x)\big)\Big) \cdot \big(\nabla u_{k_n}(x) - \nabla u(x)\big) = 0 \quad \text{a.e. in} \quad \Omega .$$

Actually, for this conclusion the assumption that $b(x, \cdot\,)$ is strictly monotone is not used in [5].

By Lemma 2.1 the assertion follows. □

Lemma 2.3. *Let $\Psi : \mathbb{R}^N \to \mathbb{R}$ be a convex function of class C^1 and let (u_k) be a sequence weakly convergent to u in $H_0^1(\Omega)$.*

(a) *If $(\Psi'(\nabla u_k(x)))$ is convergent to $\Psi'(\nabla u(x))$ a.e in Ω and we have*

$$\limsup_{|\xi|\to\infty} \frac{\frac{1}{2}\Psi'(\xi)\cdot\xi - \Psi(\xi)}{|\xi|^2} \leq 0 , \tag{2.9}$$

then

$$\limsup_{k\to\infty} \int_\Omega \left[\frac{1}{2}\,\Psi'(\nabla u_k)\cdot\nabla u_k - \Psi(\nabla u_k)\right] dx \leq \int_\Omega \left[\frac{1}{2}\,\Psi'(\nabla u)\cdot\nabla u - \Psi(\nabla u)\right] dx .$$

(b) *If $(\nabla u_k(x))$ is convergent to $\nabla u(x)$ a.e in Ω and we have*

$$\liminf_{|\xi|\to\infty} \frac{\Psi'(\xi)\cdot\xi}{|\xi|^2} \geq 1 , \tag{2.10}$$

then

$$\liminf_{k\to\infty} \int_\Omega \big[\Psi'(\nabla u_k)\cdot\nabla u_k - |\nabla u_k|^2\big]\, dx \geq \int_\Omega \big[\Psi'(\nabla u)\cdot\nabla u - |\nabla u|^2\big]\, dx .$$

Proof. Let us prove assertion (a). By (2.9) there exists $C > 0$ such that

$$\frac{1}{2}\Psi'(\xi)\cdot\xi - \Psi(\xi) \le |\xi|^2 + C \quad \forall \xi \in \mathbb{R}^N . \tag{2.11}$$

Moreover, given $\varepsilon > 0$, there exists $K_\varepsilon > 0$ such that

$$\frac{1}{2}\Psi'(\xi)\cdot\xi - \Psi(\xi) \le \varepsilon|\xi|^2 \quad \text{whenever} \quad |\Psi'(\xi)| \ge K_\varepsilon . \tag{2.12}$$

Consider $K \ge K_\varepsilon$ such that

$$\text{the set} \quad \Big\{x \in \Omega : \ \big|\Psi'\big(\nabla u(x)\big)\big| = K\Big\} \quad \text{is negligible} . \tag{2.13}$$

Apart from a countable set, each $K \ge K_\varepsilon$ satisfies (2.13). If we denote by χ_A the characteristic function of the set A, it follows that the sequence $(\chi_{\{|\Psi'(\nabla u_k)|<K\}}\Psi'(\nabla u_k))$ is convergent to $\chi_{\{|\Psi'(\nabla u)|<K\}}\Psi'(\nabla u)$ a.e. in Ω, hence strongly in $L^2(\Omega;\mathbb{R}^N)$. Therefore we have

$$\lim_{k\to\infty}\int_\Omega \chi_{\{|\Psi'(\nabla u_k)|<K\}}\Psi'(\nabla u_k)\cdot\nabla u_k\,dx = \int_\Omega \chi_{\{|\Psi'(\nabla u)|<K\}}\Psi'(\nabla u)\cdot\nabla u\,dx . \tag{2.14}$$

Since also the sequence $(\chi_{\{|\Psi'(\nabla u_k)|<K\}})$ is convergent to $\chi_{\{|\Psi'(\nabla u)|<K\}}$ strongly in $L^2(\Omega)$, we can apply the result of [8] to the integrand $f(x,s,\xi) = \min\{|s|,1\}\,\Psi(\xi)$, obtaining

$$\begin{aligned}\liminf_{k\to\infty}\int_\Omega \chi_{\{|\Psi'(\nabla u_k)|<K\}}\Psi(\nabla u_k)\,dx &= \liminf_{k\to\infty}\int_\Omega f(x,\chi_{\{|\Psi'(\nabla u_k)|<K\}},\nabla u_k)\,dx \\ &\ge \int_\Omega f(x,\chi_{\{|\Psi'(\nabla u)|<K\}},\nabla u)\,dx \\ &= \int_\Omega \chi_{\{|\Psi'(\nabla u)|<K\}}\Psi(\nabla u)\,dx . \end{aligned} \tag{2.15}$$

Combining (2.14) and (2.15), we infer that

$$\limsup_{k\to\infty}\int_{\{|\Psi'(\nabla u_k)|<K\}}\left[\frac{1}{2}\,\Psi'(\nabla u_k)\cdot\nabla u_k - \Psi(\nabla u_k)\right]dx \le \int_{\{|\Psi'(\nabla u)|<K\}}\left[\frac{1}{2}\,\Psi'(\nabla u)\cdot\nabla u - \Psi(\nabla u)\right]dx .$$

On the other hand, by (2.12) we have

$$\int_{\{|\Psi'(\nabla u_k)|\ge K\}}\left[\frac{1}{2}\,\Psi'(\nabla u_k)\cdot\nabla u_k - \Psi(\nabla u_k)\right]dx \le \varepsilon\|\nabla u_k\|_2^2 ,$$

hence

$$\limsup_{k\to\infty} \int_\Omega \left[\frac{1}{2}\,\Psi'(\nabla u_k)\cdot\nabla u_k - \Psi(\nabla u_k)\right] dx \\ \leq \varepsilon \sup_{k\in\mathbb{N}} \|\nabla u_k\|_2^2 + \int_{\{|\Psi'(\nabla u)|<K\}} \left[\frac{1}{2}\,\Psi'(\nabla u)\cdot\nabla u - \Psi(\nabla u)\right] dx \quad (2.16)$$

for every $K \geq K_\varepsilon$ satisfying (2.13). By (2.11) and the monotone convergence theorem, we have

$$\lim_{K\to+\infty} \int_{\{|\Psi'(\nabla u)|<K\}} \left[\frac{1}{2}\,\Psi'(\nabla u)\cdot\nabla u - \Psi(\nabla u)\right] dx \\ = \int_\Omega \left[\frac{1}{2}\,\Psi'(\nabla u)\cdot\nabla u - \Psi(\nabla u)\right] dx\,.$$

Combining this fact with (2.16), we get

$$\limsup_{k\to\infty} \int_\Omega \left[\frac{1}{2}\,\Psi'(\nabla u_k)\cdot\nabla u_k - \Psi(\nabla u_k)\right] dx \\ \leq \varepsilon \sup_{k\in\mathbb{N}} \|\nabla u_k\|_2^2 + \int_\Omega \left[\frac{1}{2}\,\Psi'(\nabla u)\cdot\nabla u - \Psi(\nabla u)\right] dx$$

and assertion (a) follows by the arbitrariness of ε.

The proof of assertion (b) is similar and even simpler, as

$$\lim_{k\to\infty} \int_{\{|\nabla u_k|<K\}} \left[\Psi'(\nabla u_k)\cdot\nabla u_k - |\nabla u_k|^2\right] dx = \int_{\{|\nabla u|<K\}} \left[\Psi'(\nabla u)\cdot\nabla u - |\nabla u|^2\right] dx$$

whenever the set $\{x \in \Omega : \ |\nabla u(x)| = K\}$ is negligible. □

3. Existence of a nonnegantive, nontrivial solution

In this section we prove Theorems 1.1 and 1.2.

The functional J defined in (1.1) is of class C^1 on $H_0^1(\Omega)$ by (2.6).

Since $\Psi'(0) = 0$, of course 0 is a solution of $(\mathcal{P})$. Therefore we are interested in *nontrivial* solutions. In order to find nonnegative solutions of $(\mathcal{P})$, we consider the modified functional $\overline{J} : H_0^1(\Omega) \to \mathbb{R}$ defined as

$$\overline{J}(u) = \int_\Omega \Psi(\nabla u)\, dx - \frac{\lambda}{2}\int_\Omega (u^+)^2\, dx - \frac{1}{2^*}\int_\Omega (u^+)^{2^*}\, dx\,.$$

Of course, $\overline{J}$ also is of class C^1.

Proposition 3.1. *Let $\Psi : \mathbb{R}^N \to \mathbb{R}$ be a convex function of class C^1 satisfying* (2.4), *with $\mu > 0$, and* (2.6). *Then each critical point $u \in H_0^1(\Omega)$ of $\overline{J}$ is a nonnegative solution of* $(\mathcal{P})$.

Proof. We have

$$\mu \int_\Omega |\nabla u^-|^2\,dx \le \int_\Omega \Psi'(\nabla u)\cdot(-\nabla u^-)\,dx$$
$$= \lambda \int_\Omega u^+(-u^-)\,dx + \int_\Omega (u^+)^{2^*-1}(-u^-)\,dx = 0\,,$$

whence the assertion. □

Proof of Theorem 1.1. We aim to apply to $\overline{J}$ the mountain pass theorem [1, 9]. First of all, let us observe that, by (Ψ_1), we have

$$\frac{\int_\Omega \Psi(\nabla u)\,dx}{\int_\Omega |\nabla u|^2\,dx} \to \frac{1}{2} \quad \text{as} \quad u \to 0 \quad \text{in} \quad H^1_0(\Omega)\,.$$

Then, as in the case $\Psi(\xi) = \frac{1}{2}|\xi|^2$ treated in [3], we deduce that there exist $\varrho > 0$ and $\alpha > 0$ such that $\overline{J}(u) \ge \alpha$ whenever $\|u\| = \varrho$. On the other hand, there exists $e \in H^1_0(\Omega)$ with $e \ge 0$ a.e. in Ω such that

$$\lim_{t\to+\infty} \overline{J}(te) = -\infty\,,$$
$$\sup\left\{\overline{J}(te) : t \ge 0\right\} < \frac{1}{N} S^{\frac{N}{2}}\,,$$

where

$$S = \inf\left\{\int_\Omega |\nabla u|^2\,dx :\ u \in H^1_0(\Omega)\,,\ \|u\|_{2^*} = 1\right\}.$$

Again, this is proved in [3] in the case $\Psi(\xi) = \frac{1}{2}|\xi|^2$, but by (Ψ_3) the assertion is true also in our case.

By the mountain pass theorrem, there exist a sequence (u_k) in $H^1_0(\Omega)$ and a sequence (w_k) in $H^{-1}(\Omega)$ strongly convergent to 0 such that

$$\int_\Omega \Psi'(\nabla u_k)\cdot\nabla v\,dx - \lambda\int_\Omega u_k^+ v\,dx - \int_\Omega (u_k^+)^{2^*-1} v\,dx = \langle w_k, v\rangle \quad \forall v \in H^1_0(\Omega)\,, \tag{3.1}$$

$$\lim_{k\to\infty}\left(\int_\Omega \Psi(\nabla u_k)\,dx - \frac{\lambda}{2}\int_\Omega (u_k^+)^2\,dx - \frac{1}{2^*}\int_\Omega (u_k^+)^{2^*}\,dx\right) = c \in \left[\alpha, \frac{1}{N}S^{\frac{N}{2}}\right[. \tag{3.2}$$

From (3.1), we get

$$\int_\Omega \Psi'(\nabla u_k)\cdot\nabla u_k\,dx - \lambda\int_\Omega (u_k^+)^2\,dx - \int_\Omega (u_k^+)^{2^*}\,dx = \langle w_k, u_k\rangle\,. \tag{3.3}$$

We claim that (u_k) is bounded in $H^1_0(\Omega)$. By contradiction, up to a subsequence we may assume that $\|u_k\| \to \infty$. From (3.2) it follows that

$$\lim_{k\to\infty}\left[\frac{\int_\Omega \Psi(\nabla u_k)\,dx}{\|u_k\|^{2^*}} - \frac{\lambda\int_\Omega (u_k^+)^2\,dx}{2\|u_k\|^{2^*}} - \frac{1}{2^*}\int_\Omega \left(\frac{u_k^+}{\|u_k\|}\right)^{2^*} dx\right] = 0\,.$$

By (Ψ_3) we deduce that $(u_k^+/\|u_k\|)$ is convergent to 0 strongly in $L^{2^*}(\Omega)$ and weakly in $H_0^1(\Omega)$. On the other hand, by (3.2) and (3.3) we have

$$\int_\Omega \Big[(2^* - 2)\,\Psi(\nabla u_k) + \big(2\Psi(\nabla u_k) - \Psi'(\nabla u_k)\cdot \nabla u_k\big) \Big]\, dx - \lambda \left(\frac{2^*}{2} - 1\right) \int_\Omega (u_k^+)^2\, dx = 2^* c - \langle w_k, u_k\rangle + o(1)\,.$$

By (2.2) and (2.3) it follows that there exist $\tilde\mu > 0$ and $C \in \mathbb{R}$ such that

$$\tilde\mu \int_\Omega |\nabla u_k|^2\, dx \le \lambda \left(\frac{2^*}{2} - 1\right) \int_\Omega (u_k^+)^2\, dx + C\,,$$

whence

$$\tilde\mu \le \lambda \left(\frac{2^*}{2} - 1\right) \int_\Omega \left(\frac{u_k^+}{\|u_k\|}\right)^2 dx + o(1)\,.$$

Since $(u_k^+/\|u_k\|)$ is strongly convergent to 0 in $L^2(\Omega)$, a contradiction follows. Therefore (u_k) is bounded in $H_0^1(\Omega)$, hence convergent, up to a subsequence, to some u weakly in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$.

If we set

$$\mu_k = \lambda u_k^+ + (u_k^+)^{2^*-1}\,,$$

we have that (μ_k) is bounded in $L^{\frac{2N}{N+2}}(\Omega)$, in particular (2.8) is satisfied. By Lemma 2.2 we have that, up to a further subsequence,

$$\lim_{k\to\infty} \Psi'\big(\nabla u_k(x)\big) = \Psi'\big(\nabla u(x)\big) \quad \text{a.e in} \quad \Omega\,.$$

In particular, $(\Psi'(\nabla u_k))$ is convergent to $\Psi'(\nabla u)$ weakly in $L^2(\Omega; \mathbb{R}^N)$.

Passing to the limit as $k \to \infty$ in (3.1), we get that

$$\int_\Omega \Psi'(\nabla u)\cdot \nabla v\, dx - \lambda \int_\Omega u^+ v\, dx - \int_\Omega (u^+)^{2^*-1} v\, dx = 0 \quad \forall v \in H_0^1(\Omega)\,,$$

namely that u is a critical point of $\overline{J}$, hence a nonnegative weak solution of $(\mathcal{P})$ by Proposition 3.1.

It is left to prove that u is not trivial. Arguing by contradiction, let us assume that $u = 0$ a.e. in Ω. Since (2.2) implies (2.9), from Lemma 2.3 we deduce that

$$\limsup_{k\to\infty} \int_\Omega \left[\frac{1}{2}\Psi'(\nabla u_k)\cdot \nabla u_k - \Psi(\nabla u_k)\right] dx \le 0\,.$$

Then from

$$\left(\frac{1}{2} - \frac{1}{2^*}\right) \int_\Omega (u_k^+)^{2^*}\, dx = \overline{J}(u_k) + \int_\Omega \left[\frac{1}{2}\Psi'(\nabla u_k)\cdot \nabla u_k - \Psi(\nabla u_k)\right] dx - \frac{1}{2}\langle w_k, u_k\rangle\,,$$

we get

$$\frac{1}{N} \int_\Omega (u_k^+)^{2^*}\, dx \le \overline{J}(u_k) + o(1)\,.$$

On the other hand, by (2.5) we have that $(\nabla u_k(x))$ is convergent to 0 a.e. in Ω. Since (Ψ_2) implies (2.10), from Lemma 2.3 we deduce that

$$\liminf_{k\to\infty}\int_\Omega \left[\Psi'(\nabla u_k)\cdot\nabla u_k - |\nabla u_k|^2\right] dx \geq 0\,.$$

Then we have

$$\begin{aligned}\int_\Omega |\nabla u_k|^2\,dx &\leq \int_\Omega \Psi'(\nabla u_k)\cdot\nabla u_k\,dx + o(1)\\ &= \int_\Omega (u_k^+)^{2^*}\,dx + o(1)\\ &\leq \left(\int_\Omega (u_k^+)^{2^*}\,dx\right)^{\frac{2^*-2}{2^*}}\left(\int_\Omega |u_k|^{2^*}\,dx\right)^{\frac{2}{2^*}} + o(1)\\ &\leq \frac{1}{S}\big(N\overline{J}(u_k)\big)^{\frac{2}{N}}\int_\Omega |\nabla u_k|^2\,dx + o(1)\,.\end{aligned}$$

Combining this fact with (3.2), we deduce that (u_k) is convergent to 0 strongly in $H^1_0(\Omega)$. In turn, this implies that $c = 0$, while (3.2) asserts that $c > 0$. Therefore u is not trivial and the proof is complete. □

4. Existence of a nontrivial solution

In this section we are concerned with the existence of (possibly sign-changing) nontrivial solutions u of $(\mathcal{P})$. Let (λ_k) denote the sequence of the eigenvalues of $-\Delta$ with homogeneous Dirichlet condition, repeated according to multiplicity. We will prove the second result stated in the introduction.

Proof of Theorem 1.2. Since the case $0 < \lambda < \lambda_1$ is already contained in Theorem 1.1, we may assume that $\lambda \geq \lambda_1$. Let $k \geq 1$ be such that $\lambda_k \leq \lambda < \lambda_{k+1}$ and let $e_1, \ldots, e_k$ be eigenfunctions of $-\Delta$ associated to $\lambda_1, \ldots, \lambda_k$, respectively. Finally, let $E_- = \mathrm{span}\{e_1, ..., e_k\}$ and $E_+ = E_-^\perp$.

Consider the functional J defined in (1.1). We aim to apply the linking theorem [9]. Since

$$\frac{\int_\Omega \Psi(\nabla u)\,dx}{\int_\Omega |\nabla u|^2\,dx} \to \frac{1}{2} \quad \text{as} \quad u \to 0 \quad \text{in} \quad H^1_0(\Omega)\,,$$

as in the case $\Psi(\xi) = \frac{1}{2}|\xi|^2$ treated in [4], we deduce that there exist $\varrho > 0$ and $\alpha > 0$ such that $J(u) \geq \alpha$ whenever $u \in E_+$ with $\|u\| = \varrho$. On the other hand, there exists $e \in H^1_0(\Omega) \setminus E_-$ such that

$$\lim_{\substack{\|u\|\to\infty\\ u\in\mathbb{R}e\oplus E_-}} J(u) = -\infty\,,$$

$$\sup\left\{J(te+v):\; t \geq 0\,,\; v \in E_-\right\} < \frac{1}{N}S^{\frac{N}{2}}\,.$$

Again, this is proved in [4] in both cases (a) and (b) (in case (a), the condition $N \geq 5$ needs to be required, see also [7, Corollary 1]) when $\Psi(\xi) = \frac{1}{2}|\xi|^2$, but by (Ψ_3) the assertion is true also in our case. Finally, it is clear that $J(u) \leq 0$ for every $u \in E_-$.

By the linking theorem, there exist a sequence (u_k) in $H_0^1(\Omega)$ and a sequence (w_k) in $H^{-1}(\Omega)$ strongly convergent to 0 such that

$$\int_\Omega \Psi'(\nabla u_k) \cdot \nabla v \, dx - \lambda \int_\Omega u_k v \, dx - \int_\Omega |u_k|^{2^*-2} u_k v \, dx = \langle w_k, v \rangle \quad \forall v \in H_0^1(\Omega) \, ,$$

$$\lim_{k \to \infty} \left(\int_\Omega \Psi(\nabla u_k) \, dx - \frac{\lambda}{2} \int_\Omega u_k^2 \, dx - \frac{1}{2^*} \int_\Omega |u_k|^{2^*} \, dx \right) = c \in \left[\alpha, \frac{1}{N} S^{\frac{N}{2}} \right[\, .$$

At this point, we can continue, up to minor changes, as in the proof of Theorem 1.1. In particular, (u_k) is bounded in $H_0^1(\Omega)$, hence weakly convergent, up to a subsequence, to some weak solution u of $(\mathcal{P})$. Moreover u turns out to be nontrivial. □

Acknowledgements

The author thanks Prof. Marco Degiovanni for very helpful hints and stimulating discussions.

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Received: 13 February 2008.
Accepted: 29 July 2008.