Nonlinear Studies
Vol. 9, No. 4, pp. 387-405, 2002

Nonlinear Studies

A stability result for Mountain Pass type solutions of semilinear elliptic variational inequalities

Paola Magrone and Raffaella Servadei

Dipartimento di Matematica,
Università degli Studi di Roma 'Tor Vergata'
via della Ricerca Scientifica 00133 Roma, ITALIA
e-mail:magrone@mat.uniroma2.it
e-mail:servadei@mat.uniroma2.it

Abstract The aim of the present paper is to establish a stability result for the so called Mountain Pass type solutions of the following class of semilinear elliptic variational inequalities

$$(\mathcal{P}_n)\begin{cases} u_n \in H_0^1(\Omega), \quad u_n \le \psi_n \ \text{ in } \ \Omega \\ \langle A_n u_n, v - u_n\rangle - \lambda \displaystyle\int_\Omega u_n(x)(v-u_n)(x)dx \\ \qquad\qquad\qquad\qquad \ge \displaystyle\int_\Omega p_n(x, u_n(x))(v-u_n)(x)dx \\ \forall v \in H_0^1(\Omega), \quad v \le \psi_n \ \text{ in } \ \Omega, \end{cases}$$

where Ω is an open bounded subset of $\mathbb{R}^N$ $(N \ge 1)$ with a sufficiently smooth boundary and λ is a real parameter. Moreover, for any $n \in \mathbb{N}$, A_n is a uniformly elliptic operator, ψ_n belongs to $H^1(\Omega)$, $(\psi_n)_{|\partial\Omega} \ge 0$ and p_n is a continuous real function which satisfies some general superlinear and subcritical growth conditions at zero and at infinity.

1 Introduction and main results

In the last years an extensive literature has been developed concerning stability results for semilinear elliptic equations and existence results for semilinear elliptic variational inequalities. The aim of our paper is to establish a stability result for the following class of semilinear elliptic variational inequalities

2000 *Supported by MURST, project 'Variational Methods and Nonlinear Differential Equations'.*
Mathematics Subject Classification: Primary 49J40, 35J60, 58E05, 35B30.; Secondary: 34.D.45.
Keywords: semilinear elliptic variational inequalities, penalization method, nonlinearity of superlinear growth, Mountain Pass type critical points, stability of solutions, G-convergence..

$$(\mathcal{P}_n)\begin{cases} u_n \in H_0^1(\Omega), \quad u_n \le \psi_n \quad \text{in} \quad \Omega \\ \langle A_n u_n, v-u_n\rangle - \lambda \displaystyle\int_\Omega u_n(x)(v-u_n)(x)dx \\ \qquad\qquad\qquad\qquad \ge \displaystyle\int_\Omega p_n(x,u_n(x))(v-u_n)(x)dx \\ \forall v \in H_0^1(\Omega), \quad v \le \psi_n \quad \text{in} \quad \Omega, \end{cases}$$

where Ω is an open bounded subset of $\mathbb{R}^N$ $(N \ge 1)$ with a sufficiently smooth boundary and λ is a real parameter. Moreover, for any $n \in \mathbb{N}$, A_n is a uniformly elliptic operator, ψ_n belongs to $H^1(\Omega)$, $(\psi_n)_{|\partial\Omega} \ge 0$ and p_n is a continuous real function which satisfies some general superlinear and subcritical growth conditions at zero and at infinity.

More precisely, some stability results in the framework of semilinear elliptic equations were obtained in [6, 7] for uniformly elliptic operators (Dall'Aglio and Tchou in [6] deal with the nonlinear case, while in [7] the linear case is considered) and in [2] in the more general framework of Dirichlet forms.

On the other hand, some existence results for problem $(\mathcal{P}_n)$, for any fixed $n \in \mathbb{N}$, were obtained in [8] when $\lambda < \widetilde{\lambda}_1$ and in [9] when $\lambda \ge \widetilde{\lambda}_1$, where $\widetilde{\lambda}_1$ is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$.

As far as the authors know, the only previous results about stability for variational inequalities are due to Boccardo and Capuzzo Dolcetta (see [3, 4]). In particular we extend the result obtained in [3], since Boccardo and Capuzzo Dolcetta consider a problem of the kind $(\mathcal{P}_n)$ with $\lambda = 0$ and $p_n(\cdot, u_n(\cdot)) = p_n(\cdot)$, that is the linear case.

As for our paper, we find a nontrivial non-negative solution u_n of problem $(\mathcal{P}_n)$ by using the penalization method (see [1]) and the Mountain Pass Theorem (see [10]) as in [8].By giving suitable convergence conditions for $(A_n)_n$, $(\psi_n)_n$ and $(p_n)_n$ as n goes to infinity (see $(H5)-(H7)$) and introducing an auxiliary problem $(\mathcal{A}_n)$ (see section 4), we obtain a stability result for the solutions $(u_n)_n$ of problems $((\mathcal{P}_n))_n$, that is we get a nontrivial non-negative solution u of the following limit problem

$$(\mathcal{P})\begin{cases} u \in H_0^1(\Omega), \quad u \le \psi \quad \text{in} \quad \Omega \\ \langle Au, v-u\rangle - \lambda \displaystyle\int_\Omega u(x)(v-u)(x)dx \\ \qquad\qquad\qquad\qquad \ge \displaystyle\int_\Omega p(x,u(x))(v-u)(x)dx \\ \forall v \in H_0^1(\Omega), \quad v \le \psi \quad \text{in} \quad \Omega, \end{cases}$$

as weak limit in $H_0^1(\Omega)$ of a subsequence of $(u_n)_n$.

2 Preliminaries

Let Ω be an open bounded subset of $\mathbb{R}^N$ $(N \ge 1)$ with a sufficiently smooth boundary. Let $H_0^1(\Omega)$ be the usual Sobolev space with the norm

$$\| v \| = \left(\int_\Omega | \nabla v(x) |^2 \, dx\right)^{\frac{1}{2}}$$

and let $\langle \cdot, \cdot \rangle$ be the pairing between $H_0^1(\Omega)$ and its dual space $H^{-1}(\Omega)$.

Let us denote by $\mathcal{E}(c_1, c_2)$, where c_1 and c_2 are positive constants, the class of the operators of the kind

$$B = -\sum_{i,j=1}^{N} D_i\left(b_{ij}(x) D_j\right)$$

with $b_{ij} : \Omega \to \mathbb{R}$ verifying the following conditions

$(B1)$ b_{ij} is measurable in Ω, $\forall\, i, j = 1, \ldots, N$;

$(B2)$ $b_{ij}(x) = b_{ji}(x)$ a.e. x in $\Omega, \forall\, i, j = 1, \ldots, N, i \neq j$;

$(B3)$ $c_1|\xi|^2 \leq \displaystyle\sum_{i,j=1}^{N} b_{ij}(x)\xi_i\xi_j \leq c_2|\xi|^2 \quad \forall \xi \in \mathbb{R}^N$ a.e. x in Ω.

Let

$$A_n = -\sum_{i,j=1}^{N} D_i\left(a_{ij}^{(n)}(x) D_j\right)$$

and

$$A = -\sum_{i,j=1}^{N} D_i\left(a_{ij}(x) D_j\right)$$

in $\mathcal{E}(c_1, c_2)$, for any $n \in \mathbb{N}$.
One can easily check that

$$|a_{ij}^{(n)}(x)| \leq c_2 \quad \text{a.e. } x \text{ in } \Omega, \quad \forall\, i, j = 1, \ldots, N, \ \forall\, n \in \mathbb{N}. \tag{1}$$

We recall the notion of G-convergence for the operators of the class $\mathcal{E}(c_1, c_2)$:

Definition 1. *We will say that the sequence $(A_n)_n$ G-converges to A and we will write $A_n \xrightarrow{G} A$ if, for any $T \in H^{-1}(\Omega)$, $(A_n^{-1}T)_n$ weakly converges to $A^{-1}T$ in $H_0^1(\Omega)$.*

Let $\lambda_{1,n} > 0$ and $\lambda_1 > 0$ be respectively the first eigenvalue of the operator A_n and A in $H_0^1(\Omega)$ with Dirichlet boundary conditions, for any $n \in \mathbb{N}$. Finally let $\widetilde{\lambda}_1 > 0$ be the first eigenvalue of the operator $-\Delta$ in $H_0^1(\Omega)$ with Dirichlet boundary conditions. By the uniform ellipticity of A_n and A and by the variational characterization of the eigenvalues, i.e.

$$\lambda_{1,n} = \inf_{v \in H_0^1(\Omega)\setminus\{0\}} \frac{\langle A_n v, v\rangle}{\displaystyle\int_\Omega v^2(x)dx}$$

and

$$\lambda_1 = \inf_{v \in H_0^1(\Omega)\setminus\{0\}} \frac{\langle A v, v\rangle}{\displaystyle\int_\Omega v^2(x)dx} \quad ,$$

one has

$$\frac{c_1}{c_2}\lambda_1 \le \lambda_{1,n} \le \frac{c_2}{c_1}\lambda_1, \quad \forall n \in \mathbb{N}. \tag{2}$$

For any operator $B \in \mathcal{E}(c_1, c_2)$ and for any $\lambda < \dfrac{c_1}{c_2}\lambda_1$ the following inequalities hold:

$$\tilde{c}_1 ||u||^2 \le \langle Bu, u\rangle - \lambda \int_\Omega u^2(x)dx \le \tilde{c}_2 ||u||^2, \tag{3}$$

where $\tilde{c}_1 := \min\left\{c_1,\ c_1 - \dfrac{\lambda}{\lambda_1}c_2\right\}$ and $\tilde{c}_2 := \max\left\{c_2,\ c_2\left(1 - \dfrac{\lambda c_2}{\lambda_1 c_1}\right)\right\}$.
Furthermore it is well-known that the sequence $(\lambda_{1,n})_n$ converges to λ_1 as n goes to infinity, if the operators A_n G-converge to A in $\mathcal{E}(c_1, c_2)$ (see [6]).

Let us denote by $\mathcal{F}(a_1, a_2, a_3, r)$, where a_1, a_2, a_3 and r are positive constants, the class of the functions $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ which verify the following conditions

$(F1)$ $f(\cdot,\cdot)$ is continuous in $\overline{\Omega} \times \mathbb{R}$;

$(F2)$ $| f(x,\xi) | \le a_1 + a_2 | \xi |^s, \quad \forall (x,\xi) \in \overline{\Omega} \times \mathbb{R},$

with $1 < s < \frac{N+2}{N-2}$ if $N \ge 3$, $1 < s$ if $N = 1, 2$;

$(F3)$ $f(x,\xi) = o(| \xi |)$ as $\xi \to 0, \quad \forall\, x \in \overline{\Omega}$.

Moreover, putting

$$F(x,\xi) := \int_0^\xi f(x,t)dt, \quad \forall (x,\xi) \in \overline{\Omega} \times \mathbb{R},$$

we assume that

$(F4)$ $0 < (s+1)F(x,\xi) \le \xi f(x,\xi), \quad \forall\ (x,\xi) \in \overline{\Omega} \times \mathbb{R}, \quad \xi \ge r;$

$(F5)$ $F(x,\xi) \ge a_3 | \xi |^{s+1}, \quad \forall\ (x,\xi) \in \overline{\Omega} \times \mathbb{R}, \quad \xi \ge r.$

Choosing $a_4 = a_1 r + 2^s a_2 r^s$, by $(F1), (F2)$ and $(F5)$ one has

$(F6)$ $F(x,\xi) \ge a_3 | \xi |^{s+1} - a_4, \quad \forall\ (x,\xi) \in \overline{\Omega} \times \mathbb{R}^+.$

Finally, for any function $g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ we define

$$\overline{g}(x,\xi) = \begin{cases} 0 & \text{if } \xi < 0 \\ g(x,\xi) & \text{if } \xi \ge 0, \end{cases}$$

for any $x \in \overline{\Omega}$. It is easy to check that, if g is a function belonging to the class $\mathcal{F}(a_1, a_2, a_3, r)$, then $\overline{g}$ is in $\mathcal{F}(a_1, a_2, a_3, r)$ too.

3 The existence result of a nontrivial non-negative solution for problem $(\mathcal{P}_n)$

Let us consider the following family of semilinear variational inequalities

$$(\mathcal{P}_n)\begin{cases} u_n \in H_0^1(\Omega), \quad u_n \leq \psi_n \ \text{ in } \ \Omega \\ \langle A_n u_n, v-u_n\rangle - \lambda \displaystyle\int_\Omega u_n(x)(v-u_n)(x)dx \\ \qquad\qquad\qquad\qquad \geq \displaystyle\int_\Omega p_n(x,u_n(x))(v-u_n)(x)dx \\ \forall v \in H_0^1(\Omega), \quad v \leq \psi_n \ \text{ in } \ \Omega, \end{cases}$$

where Ω is an open bounded subset of $\mathbb{R}^N$ $(N \geq 3)$ with a sufficiently smooth boundary and λ is a real parameter. Moreover, for any $n \in \mathbb{N}$, A_n is an operator of the class $\mathcal{E}(c_1, c_2)$, ψ_n belongs to $H^1(\Omega)$, with $(\psi_n)_{|\partial\Omega} \geq 0$ and $p_n : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ belongs to the class $\mathcal{F}(a_1, a_2, a_3, r)$.

In case that $\psi_n(x) \geq 0$ on $\overline{\Omega}$, it is obvious that $u_0 \equiv 0$ is a trivial solution of problem $(\mathcal{P}_n)$, for any $n \in \mathbb{N}$.

One can state the following

Theorem 1. *Let $\lambda < \dfrac{c_1}{c_2}\lambda_1$. Let $A_n \in \mathcal{E}(c_1, c_2)$ and $p_n \in \mathcal{F}(a_1, a_2, a_3, r)$, for any $n \in \mathbb{N}$. Moreover, let the following hypotheses hold*

$(H1)$ $\quad \psi_n \in H_0^1(\Omega),\ \psi_n \geq 0$ *in* Ω, $\forall\, n \in \mathbb{N}$;

$(H2)$ $\quad \exists\, \overline{v} \in H_0^1(\Omega),\ 0 \leq \overline{v} \leq \psi_n$ *in* Ω, $\forall\, n \in \mathbb{N}$ *such that*

$$\tilde{c}_2 \parallel \overline{v} \parallel^2 \leq 2\left(a_3 \int_\Omega |\overline{v}(x)|^{s+1}dx - a_4|\Omega|\right);$$

$(H3)$ $\quad s < 2$ *in* $(F2)$ *and* $(F4)$;

$(H4)$ $\quad \exists\, v_0 \in H_0^1(\Omega)$ *such that* $\psi_n \leq v_0$ *in* Ω, $\forall\, n \in \mathbb{N}$.

Then, for any $n \in \mathbb{N}$, there exists a nontrivial non-negative solution u_n of problem $(\mathcal{P}_n)$.

Remark 1. We need $\lambda < \lambda_{1,n}$, for any $n \in \mathbb{N}$, in order to obtain estimates independent of n. By (2), $\lambda < \dfrac{c_1}{c_2}\lambda_1$ is a sufficient condition for our request. If the sequence of operators $(A_n)_n$ G-converges to an operator A in $\mathcal{E}(c_1, c_2)$ and λ_1 is the first eigenvalue of A, then $(\lambda_{1,n})_n$ converges to λ_1 as n goes to ∞. So we could substitute the condition $\lambda < \dfrac{c_1}{c_2}\lambda_1$ with $\lambda < \lambda_1$, which implies $\lambda < \lambda_{1,n}$ for n large enough. In this case we obtain a result of existence of a nontrivial non-negative solution for problem $(\mathcal{P}_n)$ for n large enough.

We have decided to consider only the case $N \geq 3$ as, for $N = 1, 2$, the results are the same, even using easier arguments.

Remark 2. We need that the constant a_3 in $(F5)$ is independent of n. So we cannot deduce $(F5)$ from $(F4)$ as usual, but we have to assume it.

Remark 3. Hypotheses $(H2)$ and $(H4)$ are not empty. Indeed, taking a non-zero non-negative function $\tilde{v}$ in $H_0^1(\Omega)$ and a bounded sequence $(\mu_n)_n$ in $\mathbb{R}^+$ such that $\inf_{n\in\mathbb{N}} \mu_n > 1$, one can choose $\psi_n = \mu_n \tilde{v}$, for any $n \in \mathbb{N}$. Then, putting $v_0 = \sup_{n\in\mathbb{N}} \mu_n \tilde{v}$ and $\overline{v} = \inf_{n\in\mathbb{N}} \mu_n \tilde{v}$, $(H2)$ and $(H4)$ hold, since $s > 1$.

The method of finding the solution u_n for problem $(\mathcal{P}_n)$ relies on the consideration of a family of 'penalized' equations associated, in a standard way, with $(\mathcal{P}_n)$ (see [1]). Indeed, one can prove that any penalized equation possesses a solution of 'Mountain Pass type' and that a sequence chosen in this family actually converges to a nontrivial non-negative solution u_n of $(\mathcal{P}_n)$, suitably using some estimates from below and from above for the $H_0^1(\Omega)$-norm of the solutions of the penalized equations. As mentioned before, we apply the following Mountain Pass Theorem (see [10]):

Mountain Pass Theorem: *Let E be a real Banach space and $J \in C^1(E, \mathbb{R})$. Suppose J satisfies (PS) condition, i.e.*

$$\forall \ (u_m)_m \in H_0^1(\Omega) \ \textit{ such that } \ (J(u_m))_m \ \textit{ is bounded and } J'(u_m) \to 0$$

$$\textit{in } \ H^{-1}(\Omega) \ \textit{ as } \ m \to \infty, \textit{ there exists a subsequence of } (u_m)_m \ \textit{ strongly}$$

$$\textit{converging in } \ H_0^1(\Omega).$$

Furthermore let J satisfy $J(0) = 0$ and the following assumptions

(J1) there exist constants $\rho, \alpha > 0$ such that $J|_{\partial B_\rho} \geq \alpha$;

(J2) there exists an $e \in E \setminus B_\rho$ such that $J(e) \leq 0$.

Then J possesses a critical value $c \geq \alpha$ which can be characterized as

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C([0,1], E) \colon \gamma(0) = 0, \ \gamma(1) = e\}.$$

First of all, for any $n \in \mathbb{N}$, let us introduce the 'penalized' problem associated with $(\mathcal{P}_n)$, that is, for any $\epsilon > 0$, the weak equation

$$(\mathcal{P}_n)_\epsilon \begin{cases} u_n^\epsilon \in H_0^1(\Omega) \ \text{ such that} \\ \langle A_n u_n^\epsilon, v\rangle \ - \ \lambda \displaystyle\int_\Omega u_n^\epsilon(x) v(x) dx \\ \qquad + \ \dfrac{1}{\epsilon} \displaystyle\int_\Omega (u_n^\epsilon - \psi_n)^+(x) v(x) dx \ = \ \displaystyle\int_\Omega p_n(x, u_n^\epsilon(x)) v(x) dx \\ \forall v \in H_0^1(\Omega), \end{cases}$$

where f^+ denotes the positive part of the function f. Let us note that the last integral is well defined for all $v \in H_0^1(\Omega)$ as a consequence of $(F2)$ and of the continuous embedding of $H_0^1(\Omega)$ into $L^{2^*}(\Omega)$.

In order to look for non-negative solutions of problem $(\mathcal{P}_n)_\epsilon$ it is convenient to modify it with the following one:

$$(\overline{\mathcal{P}}_n)_\epsilon \begin{cases} u_n^\epsilon \in H_0^1(\Omega) \quad \text{such that} \\ \langle A_n u_n^\epsilon, v\rangle \quad - \quad \lambda \displaystyle\int_\Omega u_n^\epsilon(x)v(x)dx \\ \quad + \quad \dfrac{1}{\epsilon}\displaystyle\int_\Omega (u_n^\epsilon - \psi_n)^+(x)v(x)dx \quad = \quad \displaystyle\int_\Omega \overline{p}_n(x, u_n^\epsilon(x))v(x)dx \\ \forall v \in H_0^1(\Omega). \end{cases}$$

Actually in order to look for solutions of $(\overline{\mathcal{P}}_n)_\epsilon$, we study the critical points of the real functional I_n^ϵ defined on $H_0^1(\Omega)$ in this way

$$I_n^\epsilon(v) = \frac{1}{2}\langle A_n v, v\rangle - \frac{\lambda}{2}\int_\Omega v^2(x)dx + \frac{1}{\epsilon}\int_\Omega\int_0^{v(x)} (t-\psi_n(x))^+ dt dx - \int_\Omega \overline{P}_n(x, v(x))dx,$$

where

$$\overline{P}_n(x,\xi) := \int_0^\xi \overline{p}_n(x,t)dt, \quad \forall (x,\xi) \in \overline{\Omega} \times \mathbb{R}.$$

One can easily check that I_n^ϵ belongs to $C^1(H_0^1(\Omega), \mathbb{R})$ and that the pairing $\langle (I_n^\epsilon)'(u_n^\epsilon), v\rangle$ coincides with the difference between the first and the second member in $(\overline{\mathcal{P}}_n)_\epsilon$.
At this point, to prove Theorem 1, let us verify that the functional I_n^ϵ satisfies all the hypotheses of the Mountain Pass Theorem.

Proof. (of Theorem 1) Let us proceed by steps.

Step 1. *The functional I_n^ϵ verifies, for any $n \in \mathbb{N}$ and for any $\epsilon > 0$, the conditions*

$$I_n^\epsilon(0) = 0, \tag{4}$$

$$I_n^\epsilon(v) \geq \alpha \quad \textit{for some} \quad \rho, \alpha > 0, \ \forall v \in H_0^1(\Omega), \ \| v \| = \rho. \tag{5}$$

Proof. Let us fix $n \in \mathbb{N}$ and $\epsilon > 0$.Property (4) is trivial. As for (5), let us note that the positivity of ψ_n on Ω yields

$$\int_\Omega \int_0^{v(x)} (t - \psi_n(x))^+ dt dx = \int_{\{x \in \Omega : v(x) \geq \psi_n(x)\}} \int_{\psi_n(x)}^{v(x)} (t - \psi_n(x)) dt dx \geq 0, \quad (6)$$

for all $v \in H_0^1(\Omega)$.

As a consequence of $(F2)$ and $(F3)$, one gets that

$$\forall \delta > 0 \ \ \exists \, c(\delta) > 0 \ \text{ such that } \ \overline{P}_n(x, \xi) \leq \frac{\delta}{2} \mid \xi \mid^2 + c(\delta) \mid \xi \mid^{s+1}, \qquad (7)$$

$\forall \ (x, \xi) \in \overline{\Omega} \times \mathbb{R}, \ \forall \ n \in \mathbb{N}$.

Then, by using (6),(7), (3) and by choosing $\rho > 0$ such that $\tilde{c}_1 - \dfrac{\delta}{\widetilde{\lambda}_1} > \ 2c(\delta) c_s \rho^{s-1}$ (c_s denoting the embedding Sobolev constant of $H_0^1(\Omega)$ into $L^{s+1}(\Omega)$), for all $v \in H_0^1(\Omega)$ such that $\| \, v \, \| = \rho$, one has

$$\begin{aligned} I_n^\epsilon(v) \quad &\geq \quad \frac{1}{2} \langle A_n v, v \rangle - \frac{\lambda}{2} \int_\Omega v^2(x) dx - \int_\Omega \overline{P}_n(x, v(x)) dx \\ &\geq \quad \frac{\tilde{c}_1}{2} \| \, v \, \|^2 - \frac{\delta}{2\widetilde{\lambda}_1} \| \, v \, \|^2 - c(\delta) c_s \| \, v \, \|^{s+1} \\ &= \quad \left(\frac{1}{2} \left(\tilde{c}_1 - \frac{\delta}{\widetilde{\lambda}_1} \right) - \ c(\delta) c_s \rho^{s-1} \right) \rho^2. \end{aligned}$$

So Step 1 is proved.

Remark 4. Note that the positive constant α is independent of ϵ and of n and this fact will be used in the proof of Theorem 1.

Step 2. *There exists an element $e \in H_0^1(\Omega) \setminus \{0\}$ such that $I_n^\epsilon(e) \leq 0$, for any $n \in \mathbb{N}$ and for any $\epsilon > 0$.*

Proof. Let us fix $n \in \mathbb{N}$ and $\epsilon > 0$. Let $e = \overline{v}$ as in $(H2)$. Observe that $e \not\equiv 0$. Moreover, $0 \leq e \leq \psi_n$ in Ω implies

$$\int_\Omega \int_0^{e(x)} (t - \psi_n(x))^+ dt dx = \int_{\{x \in \Omega : e(x) \geq \psi_n(x)\}} \int_{\psi_n(x)}^{e(x)} (t - \psi_n(x)) dt dx = 0.$$

So, by (3) and $(F6)$, one has

$$I_n^\epsilon(e) \leq \frac{\tilde{c}_2}{2} \| \, e \, \|^2 - a_3 \int_\Omega |e(x)|^{s+1} dx + a_4 |\Omega|.$$

Then Step 2 follows from $(H2)$.

Step 3. *For any $n \in \mathbb{N}$ and for any $\epsilon > 0$, I_n^ϵ satisfies the Palais-Smale condition, i.e.*

$$
(PS)\quad \begin{array}{l} \textit{for any}\quad (u_m)_m \in H_0^1(\Omega)\ \textit{ such that }\ \left(I_n^\epsilon(u_m)\right)_m\ \textit{ is bounded and} \\ (I_n^\epsilon)'(u_m) \to 0 \ \text{ in }\ H^{-1}(\Omega)\ \textit{ as }\ m \to \infty, \textit{there exists a subsequence} \\ \textit{of}\, (u_m)_m\ \textit{ strongly converging in }\ H_0^1(\Omega). \end{array}
$$

Proof. Let us fix $n \in \mathbb{N}$, $\epsilon > 0$ and $\beta \in \left(\frac{1}{s+1}, \frac{1}{2}\right)$. By the properties of $(u_m)_m$ one deduces

$$
I_n^\epsilon(u_m) - \beta\langle (I_n^\epsilon)'(u_m), u_m\rangle \leq K_{n,\epsilon} + \beta M_{n,\epsilon} \parallel u_m \parallel, \tag{8}
$$

where $K_{n,\epsilon}, M_{n,\epsilon}$ are positive constants independent of m.
By definition of I_n^ϵ, $(I_n^\epsilon)'$ and by (3), $(F1)$, $(F4)$ one gets

$$
\begin{aligned}
& I_n^\epsilon(u_m) - \beta\langle (I_n^\epsilon)'(u_m), u_m\rangle \\
& \geq \tilde{c}_1\left(\frac{1}{2} - \beta\right) \parallel u_m \parallel^2 - \frac{1}{\epsilon}\int_{\{x\in\Omega: u_m(x)\geq\psi_n(x)\}} (1-\beta)\psi_n(x)u_m(x)dx \\
& + (s+1)\left(\beta - \frac{1}{s+1}\right)\int_{\{x\in\Omega: u_m(x)\geq r\}} \overline{P}_n(x, u_m(x))dx - K_{n,r} \\
& \geq \tilde{c}_1\left(\frac{1}{2} - \beta\right) \parallel u_m \parallel^2 - \frac{1-\beta}{\epsilon\sqrt{\widetilde{\lambda_1}}} \parallel \psi_n \parallel_{L^2(\Omega)} \parallel u_m \parallel - K_{n,r},
\end{aligned} \tag{9}
$$

for any $m \in \mathbb{N}$, where $K_{n,r}$ is a positive constant independent of m.

Finally, combining (8) and (9), one gets

$$
\parallel u_m \parallel^2 \leq C_{n,\epsilon} \parallel u_m \parallel + D_{n,\epsilon},
$$

for any $m \in \mathbb{N}$, for suitable positive constants $C_{n,\epsilon}$ and $D_{n,\epsilon}$ independent of m. Thus $(u_m)_m$ is bounded in $H_0^1(\Omega)$. At this point, Step 3 easily follows from a standard argument based on the compact embedding of $H_0^1(\Omega)$ into $L^p(\Omega)$ for $p \in [2,\ 2^*)$.

Step 4. *For any $n \in \mathbb{N}$ and for any $\epsilon > 0$, there exists a solution u_n^ϵ of problem $(\overline{\mathcal{P}}_n)_\epsilon$ such that*

$$
I_n^\epsilon(u_n^\epsilon) = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_n^\epsilon(\gamma(t)),
$$

where $\Gamma = \{\gamma \in C([0,1], H_0^1(\Omega)) : \gamma(0) = 0,\ \gamma(1) = e\}$.
Moreover

$$
I_n^\epsilon(u_n^\epsilon) \geq \alpha.
$$

Proof. It is a consequence of Steps $1, 2, 3$ and of the Mountain Pass Theorem.

Step 5. *For any $n \in \mathbb{N}$ and for any $\epsilon > 0$, u_n^ϵ is a nontrivial non-negative solution of problem $(\mathcal{P}_n)_\epsilon$.*

Proof. Let us fix $n \in \mathbb{N}$ and $\epsilon > 0$. $u_n^\epsilon \not\equiv 0$ as it is a Mountain Pass type critical point.
In order to prove that $u_n^\epsilon \geq 0$, let us take $v = (u_n^\epsilon)^-$ in $(\overline{\mathcal{P}}_n)_\epsilon$. It follows that

$$\begin{aligned} \langle A_n u_n^\epsilon, (u_n^\epsilon)^- \rangle \quad &- \quad \lambda \int_\Omega u_n^\epsilon(x)(u_n^\epsilon)^-(x)dx \\ &+ \quad \frac{1}{\epsilon}\int_\Omega (u_n^\epsilon - \psi_n)^+(x)(u_n^\epsilon)^-(x)dx = \int_\Omega \overline{p}_n(x, u_n^\epsilon(x))(u_n^\epsilon)^-(x)dx. \end{aligned} \tag{10}$$

By the definition of $(u_n^\epsilon)^-$ and of $\overline{p}_n(\cdot,\cdot)$ and using (3), (10) yields

$$0 = \langle A_n u_n^\epsilon, (u_n^\epsilon)^- \rangle - \lambda \int_\Omega u_n^\epsilon(x)(u_n^\epsilon)^-(x)dx \geq \quad \tilde{c}_1 ||(u_n^\epsilon)^-||^2,$$

from which we obtain $u_n^\epsilon \geq 0$. The non-negativity of u_n^ϵ and the definition of $\overline{p}_n(\cdot,\cdot)$ yield Step 5.

Step 6. *There exists a constant $K_1 > 0$ such that $||u_n^\epsilon|| \geq K_1$, for any $n \in \mathbb{N}$ and for any $\epsilon > 0$.*

Proof. Let us fix $n \in \mathbb{N}$ and $\epsilon > 0$. Let us take $v = u_n^\epsilon$ in $(\mathcal{P}_n)_\epsilon$. It follows that

$$\begin{aligned} \langle A_n u_n^\epsilon, u_n^\epsilon \rangle \quad &- \quad \lambda \int_\Omega (u_n^\epsilon)^2(x)dx \\ &+ \quad \frac{1}{\epsilon}\int_\Omega (u_n^\epsilon - \psi_n)^+(x)u_n^\epsilon(x)dx = \int_\Omega p_n(x, u_n^\epsilon(x))u_n^\epsilon(x)dx. \end{aligned} \tag{11}$$

Arguing as in Step 1 we obtain

$$\forall \delta > 0 \;\; \exists\, c(\delta) > 0 \;\; \text{such that} \;\; p_n(x,\xi) \leq \delta \mid \xi \mid + c(\delta) \mid \xi \mid^s \tag{12}$$

for any $(x,\xi) \in \overline{\Omega} \times \mathbb{R}$.
By using the non-negativity of u_n^ϵ, (3), the variational characterization of $\widetilde{\lambda}_1$ and (12), one has

$$\begin{aligned} &\tilde{c}_1 ||u_n^\epsilon||^2 \leq \int_\Omega p_n(x, u_n^\epsilon(x))u_n^\epsilon(x)dx \\ &\leq \delta ||u_n^\epsilon||^2_{L^2(\Omega)} + c(\delta)||u_n^\epsilon||^{s+1}_{L^{s+1}(\Omega)} \\ &\leq \frac{\delta}{\widetilde{\lambda}_1}||u_n^\epsilon||^2 + c(\delta)c_s||u_n^\epsilon||^{s+1}. \end{aligned}$$

By choosing δ such that $\delta < \tilde{c}_1\widetilde{\lambda}_1$, we can conclude that $||u_n^\epsilon||$ is bounded from below as $s+1>2$.

Step 7. *There exists a constant $K_2 > 0$ such that $I_n^\epsilon(u_n^\epsilon) \le K_2$, for any $n \in \mathbb{N}$ and for any $\epsilon > 0$.*

Proof. Let us fix $n \in \mathbb{N}$ and $\epsilon > 0$.By Step 4 with $\gamma(t) = t\overline{v}$, where $\overline{v}$ is as in $(H2)$, we obtain

$$I_n^\epsilon(u_n^\epsilon) \le \max_{t\in[0,1]} I_n^\epsilon(t\overline{v}).$$

From the fact that $0 \le t\overline{v} \le \psi_n$ in Ω, from the definition of $\overline{P}_n$ and by (3) one has

$$I_n^\epsilon(u_n^\epsilon) \le \max_{t\in[0,1]} \left\{ \frac{1}{2}\langle A_n t\overline{v}, t\overline{v}\rangle - \frac{\lambda}{2}\int_\Omega (t\overline{v})^2(x)dx \right\} \le \max_{t\in[0,1]} \frac{\tilde{c}_2}{2}||t\overline{v}||^2 \le \tilde{c}_2||\overline{v}||^2.$$

So Step 7 is proved as the right member of the previous relation is independent of n and ϵ.

Step 8. *There exists a constant $K_3 > 0$ such that $||u_n^\epsilon|| \le K_3$, for any $n \in \mathbb{N}$ and for any $\epsilon > 0$.*

Proof. Let us fix $n \in \mathbb{N}$ and $\epsilon > 0$. By Step 7 one has

$$\begin{aligned} &\frac{1}{2}\,\langle A_n u_n^\epsilon, u_n^\epsilon\rangle - \frac{\lambda}{2}\int_\Omega (u_n^\epsilon)^2(x)dx + \frac{1}{\epsilon}\int_\Omega\int_0^{u_n^\epsilon(x)} (t-\psi_n(x))^+ dtdx \le \\ &\le K_2 + \int_\Omega \overline{P}_n(x, u_n^\epsilon(x))dx. \end{aligned} \tag{13}$$

By the non-negativity of u_n^ϵ, by (13), $(F2)$, $(F4)$, using the fact that u_n^ϵ is a solution of $(\mathcal{P}_n)_\epsilon$ in particular with $v = u_n^\epsilon$ and that $s+1>2$, we obtain

$$\begin{aligned} &\frac{1}{2}\langle A_n u_n^\epsilon, u_n^\epsilon\rangle - \frac{\lambda}{2}\int_\Omega (u_n^\epsilon)^2(x)dx + \frac{1}{2\epsilon}\int_{\{x\in\Omega: u_n^\epsilon(x)\ge\psi_n(x)\}} (u_n^\epsilon - \psi_n)^2(x)dx \\ &\le K_2 + C_r|\Omega| + \frac{1}{s+1}\int_\Omega p_n(x, u_n^\epsilon(x))u_n^\epsilon(x)dx \\ &= K_2 + C_r|\Omega| + \frac{1}{s+1}\left[\langle A_n u_n^\epsilon, u_n^\epsilon\rangle - \lambda\int_\Omega (u_n^\epsilon)^2(x)dx\right] \\ &+\frac{1}{s+1}\frac{1}{\epsilon}\int_\Omega (u_n^\epsilon - \psi_n)^+(x)u_n^\epsilon(x)dx \\ &\le K_2 + C_r|\Omega| + \frac{1}{s+1}\left[\langle A_n u_n^\epsilon, u_n^\epsilon\rangle - \lambda\int_\Omega (u_n^\epsilon)^2(x)dx\right] \\ &+\frac{1}{2\epsilon}\int_\Omega (u_n^\epsilon - \psi_n)^+(x)u_n^\epsilon(x)dx, \end{aligned} \tag{14}$$

where C_r is a positive constant independent of n and ϵ.
By (14), $(H4)$, the fact that u_n^ϵ is a solution of $(\mathcal{P}_n)_\epsilon$ with $v = v_0$, $(F2)$ and (1), we obtain

$$
\begin{aligned}
&\left(\frac{1}{2}-\frac{1}{s+1}\right)\left[\langle A_n u_n^\epsilon, u_n^\epsilon\rangle - \lambda\int_\Omega (u_n^\epsilon)^2(x)dx\right] \\
&\le K_2 + C_r|\Omega| + \frac{1}{2\epsilon}\int_{\{x\in\Omega: u_n^\epsilon(x)\ge\psi_n(x)\}} (u_n^\epsilon-\psi_n)(x)\psi_n(x)dx \\
&\le K_2 + C_r|\Omega| + \frac{1}{2\epsilon}\int_{\{x\in\Omega: u_n^\epsilon(x)\ge\psi_n(x)\}} (u_n^\epsilon-\psi_n)(x)v_0(x)dx \\
&= K_2 + C_r|\Omega| - \frac{1}{2}\langle A_n u_n^\epsilon, v_0\rangle + \frac{\lambda}{2}\int_\Omega u_n^\epsilon(x)v_0(x)dx + \frac{1}{2}\int_\Omega p_n(x,u_n^\epsilon(x))v_0(x)dx \\
&\quad \le K_2 + C_r|\Omega| + \frac{c_2}{2}||v_0||\ ||u_n^\epsilon|| + \frac{\lambda}{2}\int_\Omega u_n^\epsilon(x)v_0(x)dx + \frac{a_1}{2}||v_0||_{L^1} \\
&\quad + \frac{a_2}{2}||v_0||_{L^{\left(\frac{2^*}{s}\right)'}(\Omega)}||(u_n^\epsilon)^s||_{L^{\frac{2^*}{s}}(\Omega)}.
\end{aligned}
\tag{15}
$$

Therefore, by (15), (3), the variational characterization of $\widetilde{\lambda}_1$ and the continuous embedding of $H_0^1(\Omega)$ into $L^{2^*}(\Omega)$, one has

$$
\begin{aligned}
&\tilde{c}_1\left(\frac{1}{2}-\frac{1}{s+1}\right)||u_n^\epsilon||^2 \\
&\le K_2 + C_r|\Omega| + \frac{c_2}{2}||v_0||\ ||u_n^\epsilon|| + \frac{\lambda}{2}\int_\Omega u_n^\epsilon(x)v_0(x)dx + \frac{a_1}{2}||v_0||_{L^1} \\
&+ \frac{a_2}{2}||v_0||_{L^{\left(\frac{2^*}{s}\right)'}(\Omega)}||(u_n^\epsilon)^s||_{L^{\frac{2^*}{s}}(\Omega)} \\
&\le K_2 + C_r|\Omega| + \frac{c_2}{2}||v_0||\ ||u_n^\epsilon|| + \frac{\lambda}{2\widetilde{\lambda}_1}||v_0||\ ||u_n^\epsilon|| + \frac{a_1}{2}||v_0||_{L^1} \\
&+ \frac{\widetilde{a}_2}{2}||v_0||_{L^{\left(\frac{2^*}{s}\right)'}(\Omega)}||u_n^\epsilon||^s,
\end{aligned}
\tag{16}
$$

where $\widetilde{a}_2$ is a positive constant independent of n and ϵ.
Finally, Step 8 follows from $(H3)$.

Step 9. *There exists a constant $K_4 > 0$ such that $\parallel (u_n^\epsilon - \psi_n)^+ \parallel_{L^2(\Omega)} \le K_4\sqrt{\epsilon}$, for any $n \in \mathbb{N}$ and for any $\epsilon > 0$.*

Proof. Let us fix $n \in \mathbb{N}$ and $\epsilon > 0$. Taking $v = u_n^\epsilon$ in $(\mathcal{P}_n)_\epsilon$, we obtain

$$\begin{aligned} &\frac{1}{\epsilon}\int_\Omega (u_n^\epsilon - \psi_n)^+(x) u_n^\epsilon(x)dx \\ &= -\langle A_n u_n^\epsilon, u_n^\epsilon\rangle + \lambda \int_\Omega (u_n^\epsilon)^2(x)dx + \int_\Omega p_n(x, u_n^\epsilon(x)) u_n^\epsilon(x)dx. \end{aligned} \tag{17}$$

Using the non-negativity of ψ_n, (17), (3), $(F2)$ and the compact embedding of $H_0^1(\Omega)$ into $L^p(\Omega)$ for $p \in [1, 2^*)$, we deduce

$$\begin{aligned} &\frac{1}{\epsilon}\int_\Omega ((u_n^\epsilon - \psi_n)^+)^2(x)dx \\ &\leq \frac{1}{\epsilon}\int_\Omega (u_n^\epsilon - \psi_n)^+(x) u_n^\epsilon(x)dx \\ &= -\langle A_n u_n^\epsilon, u_n^\epsilon\rangle + \lambda \int_\Omega (u_n^\epsilon)^2(x)dx + \int_\Omega p_n(x, u_n^\epsilon(x)) u_n^\epsilon(x)dx \\ &\leq -\tilde{c}_1 \parallel u_n^\epsilon \parallel^2 + a_1 ||u_n^\epsilon||_{L^1(\Omega)} + a_2 ||u_n^\epsilon||^{s+1}_{L^{s+1}(\Omega)} \\ &\leq \tilde{a}_1 ||u_n^\epsilon|| + \tilde{a}_2 ||u_n^\epsilon||^{s+1}, \end{aligned}$$

where $\tilde{a}_1$ and $\tilde{a}_2$ are positive constants independent of n and ϵ.
So Step 9 follows from Step 8.

Step 10. *There exists a sequence $(\epsilon_k)_k$ converging to 0 as k goes to ∞ such that $(u_n^{\epsilon_k})_k$ weakly converges in $H_0^1(\Omega)$ to some $u_n \not\equiv 0$ and $u_n \geq 0$, for any $n \in \mathbb{N}$.*

Proof. Let us fix $n \in \mathbb{N}$. First of all, by Step 8, there exists a sequence $(u_n^{\epsilon_k})_k$ weakly converging in $H_0^1(\Omega)$ to some u_n as ϵ_k goes to 0. Therefore, as ϵ_k goes to 0,

$$\begin{aligned} &u_n^{\epsilon_k} \to u_n \quad \text{in } L^2(\Omega) \\ \text{and} \quad &u_n^{\epsilon_k} \to u_n \quad \text{a.e. in } \Omega. \end{aligned}$$

Moreover, by Step 5, $u_n^{\epsilon_k} \geq 0$ for any $\epsilon_k > 0$ and therefore $u_n \geq 0$.
We claim that u_n is not identically zero. Indeed, from the fact that $u_n^{\epsilon_k}$ is a non-negative solution of problem $(\mathcal{P}_n)_\epsilon$ with $\epsilon = \epsilon_k$, using (3) and Step 6, we obtain

$$\begin{aligned} &\tilde{c}_1 K_1^2 \leq \tilde{c}_1 \parallel u_n^{\epsilon_k} \parallel^2 \\ &\leq \langle A_n u_n^{\epsilon_k}, u_n^{\epsilon_k}\rangle - \lambda \int_\Omega (u_n^{\epsilon_k})^2(x)dx \leq \int_\Omega p_n(x, u_n^{\epsilon_k}(x)) u_n^{\epsilon_k}(x)dx. \end{aligned} \tag{18}$$

By $(F1)$, $(F2)$ and the weak convergence in $H_0^1(\Omega)$ of $(u_n^{\epsilon_k})_k$ to u_n, one has

$$\int_\Omega p_n(x, u_n^{\epsilon_k}(x))u_n^{\epsilon_k}(x)dx \to \int_\Omega p_n(x, u_n(x))u_n(x)dx$$

as $\epsilon_k \to 0$. If u_n was identically zero, then, going to the limit as $\epsilon_k \to 0$ in (18), we get a contradiction. So $u_n \not\equiv 0$.

Step 11. *The element u_n given by Step 10 is a nontrivial non-negative solution of problem$(\mathcal{P}_n)$, for any $n \in \mathbb{N}$.*

Proof. Let us fix $n \in \mathbb{N}$. As $u_n^{\epsilon_k}$ is a solution of $(\mathcal{P}_n)_{\epsilon_k}$, taking $v - u_n^{\epsilon_k}$ as test function, one gets

$$\begin{aligned} \langle A_n u_n^{\epsilon_k}, v - u_n^{\epsilon_k}\rangle \quad &- \quad \lambda \int_\Omega u_n^{\epsilon_k}(x)(v - u_n^{\epsilon_k})(x)dx \\ + \quad \frac{1}{\epsilon_k} &\int_\Omega (u_n^{\epsilon_k} - \psi_n)^+(x)(v - u_n^{\epsilon_k})(x)dx \\ = &\int_\Omega p_n(x, u_n^{\epsilon_k}(x))(v - u_n^{\epsilon_k})(x)dx. \end{aligned} \tag{19}$$

Arguing as in Step 10, we have

$$\int_\Omega p_n(x, u_n^{\epsilon_k}(x))(v - u_n^{\epsilon_k})(x)dx \to \int_\Omega p_n(x, u_n(x))(v - u_n)(x)dx \tag{20}$$

and

$$\lambda \int_\Omega u_n^{\epsilon_k}(x)(v - u_n^{\epsilon_k})(x)dx \to \lambda \int_\Omega u_n(x)(v - u_n)(x)dx \tag{21}$$

as ϵ_k goes to 0.
The weak lower semicontinuity of $\langle A_n u, u\rangle$ yields

$$\liminf_{\epsilon_k \to 0} \langle A_n u_n^{\epsilon_k}, v - u_n^{\epsilon_k}\rangle \le \langle A_n u_n, v - u_n\rangle. \tag{22}$$

Finally taking $v \le \psi_n$ in Ω, we have

$$\frac{1}{\epsilon_k} \int_\Omega (u_n^{\epsilon_k} - \psi_n)^+(x)(v - u_n^{\epsilon_k})(x)dx \le 0. \tag{23}$$

Taking the liminf in (19) as ϵ_k goes to 0, by (20), (21), (22) and (23) we get

$$\begin{aligned} &\langle A_n u_n, v - u_n\rangle - \lambda \int_\Omega u_n(x)(v - u_n)(x)dx \\ &\ge \int_\Omega p_n(x, u_n(x))(v - u_n)(x)dx, \end{aligned}$$

for any $v \in H_0^1(\Omega)$, $v \le \psi_n$ in Ω. Now let us prove that $u_n \le \psi_n$ in Ω: this fact follows by Step 9 and by the strong convergence in $L^2(\Omega)$ of $(u_n^{\epsilon_k})_k$ to u_n. So u_n is a solution of the variational inequality $(\mathcal{P}_n)$. □

4 The stability result

In this section we will prove the main result of this paper, i.e. the stability result for the Mountain Pass type solutions $(u_n)_n$ of the problems $((\mathcal{P}_n))_n$. In other words, we consider the following problem

$$(\mathcal{P})\begin{cases} u \in H_0^1(\Omega), \quad u \leq \psi \ \text{ in } \ \Omega \\ \langle Au, v-u\rangle - \lambda \displaystyle\int_\Omega u(x)(v-u)(x)dx \\ \qquad\qquad\qquad\qquad \geq \displaystyle\int_\Omega p(x,u(x))(v-u)(x)dx \\ \forall v \in H_0^1(\Omega), \quad v \leq \psi \ \text{ in } \ \Omega, \end{cases}$$

where $A \in \mathcal{E}(c_1, c_2)$, $p \in \mathcal{F}(a_1, a_2, a_3, r)$ and $\psi \in H^1(\Omega)$ with $\psi_{|\partial\Omega} \geq 0$, and we prove the following

Theorem 2. *Let $\lambda < \frac{c_1}{c_2}\lambda_1$. Let A_n, p_n be as in Theorem 1, u_n the solution of problem $(\mathcal{P}_n)$ given by Theorem 1, for any $n \in \mathbb{N}$, A an operator belonging to $\mathcal{E}(c_1, c_2)$ and p a function in the class $\mathcal{F}(a_1, a_2, a_3, r)$. Moreover, let us assume (H1), (H2), (H3), (H4) and the following hypotheses*

$(H5)$ $\psi_n \to \psi$ *weakly in* $H^1(\Omega)$, *as* $n \to \infty$;

$(H6)$ $A_n \xrightarrow{G} A$, *as* $n \to \infty$;

$(H7)$ $p_n(x, v(x)) \to p(x, v(x))$ *as* $n \to \infty$ *a.e. in* Ω *and uniformly on bounded set of* $H_0^1(\Omega)$;

$(H8)$ $\exists\, M > 0$ *such that* $||A_n\psi_n||_{L^{\frac{2^*}{s}}(\Omega)} \leq M, \quad \forall\, n \in \mathbb{N}$.

Then there exists a subsequence of $(u_n)_n$ weakly converging in $H_0^1(\Omega)$ to a function u which is a nontrivial non-negative solution of problem $(\mathcal{P})$.

Remark 5. It is easy to check that problem $(\mathcal{P})$ has a Mountain Pass type solution in a quite analogous way as $(\mathcal{P}_n)$. Then, from Theorem 2, we can deduce that either the Mountain Pass type solutions $(u_n)_n$ of problems $((\mathcal{P}_n))_n$ converge, up to subsequence, to a solution of $(\mathcal{P})$ which still is of Mountain Pass type, or problem $(\mathcal{P})$ has at least two different solutions.

Remark 6. Hypotheses $(H2)$ and $(H5)$ assure that ψ is not identically zero.

Remark 7. In $(H7)$ we require that

there exists a set N *of zero measure such that* $\forall x \in \Omega \setminus N$,

$\forall\, B$ *bounded set of* $H_0^1(\Omega)$ *and* $\forall \eta > 0$ *there exists* $\overline{n} = \overline{n}(\eta, x, B)$

such that $\forall n \geq \overline{n}$ *we have* $|p_n(x, v(x)) - p(x, v(x))| < \eta, \quad \forall v \in B$.

This assumption holds, in particular, if

$$p_n(x,\xi) \to p(x,\xi)$$

as $n \to \infty$, a.e. in Ω and uniformly on bounded intervals of $\mathbb{R}$ (see [2], Remark 4.2).

Remark 8. Hypothesis $(H8)$ is not empty. Indeed, it is enough to take $\tilde{v}$ in Remark 3 sufficiently regular. In order to have $(H5)$, we can assume that the sequence $(\mu_n)_n$ of Remark 3 is converging.

First of all, we will prove that the sequence $(u_n)_n$ is bounded from above and from below in $H_0^1(\Omega)$. Then, denoting by u the weak limit of a subsequence of $(u_n)_n$ in $H_0^1(\Omega)$, we will introduce the following auxiliary problem

$$(\mathcal{A}_n)\begin{cases} \overline{u}_n \in H_0^1(\Omega), \quad \overline{u}_n \le \psi_n \quad \text{in} \quad \Omega \\ \langle A_n\overline{u}_n, v-\overline{u}_n\rangle - \lambda \displaystyle\int_\Omega u(x)(v-\overline{u}_n)(x)dx \\ \qquad\qquad\qquad\qquad \ge \displaystyle\int_\Omega p(x,u(x))(v-\overline{u}_n)(x)dx \\ \forall v \in H_0^1(\Omega), \quad v \le \psi_n \quad \text{in} \quad \Omega. \end{cases}$$

After proving that the sequence $(\overline{u}_n)_n$ of solutions of problems $((\mathcal{A}_n))_n$ weakly converges in $H_0^1(\Omega)$ to a solution $\overline{u}$ of the following problem

$$(\mathcal{A})\begin{cases} \overline{u} \in H_0^1(\Omega), \quad \overline{u} \le \psi \quad \text{in} \quad \Omega \\ \langle A\overline{u}, v-\overline{u}\rangle - \lambda \displaystyle\int_\Omega u(x)(v-\overline{u})(x)dx \\ \qquad\qquad\qquad\qquad \ge \displaystyle\int_\Omega p(x,u(x))(v-\overline{u})(x)dx \\ \forall v \in H_0^1(\Omega), \quad v \le \psi \quad \text{in} \quad \Omega, \end{cases}$$

we will conclude the proof of Theorem 2 by showing that $u = \overline{u}$.

Proof. (of Theorem 2) Let us proceed by steps.

Step 1. *There exists* $K_5 > 0$ *such that* $||u_n|| \ge K_5$, *for any* $n \in \mathbb{N}$.

Proof. Let us fix $n \in \mathbb{N}$. As u_n is a solution of $(\mathcal{P}_n)$, taking $v \equiv 0$ as test function, one has

$$\langle A_n u_n, u_n\rangle - \lambda \int_\Omega u_n^2(x)dx \le \int_\Omega p_n(x,u_n(x))u_n(x)dx. \tag{24}$$

By (24), (3), $(F2)$ and the compact embedding of $H_0^1(\Omega)$ into $L^{s+1}(\Omega)$ one gets

$$\begin{aligned} &\widetilde{c}_1||u_n||^2 \le \int_\Omega p_n(x,u_n(x))u_n(x)dx \\ &\le a_1||u_n||_{L^1(\Omega)} + a_2||u_n||^{s+1}_{L^{s+1}(\Omega)} \le \tilde{a}_1||u_n|| + \tilde{a}_2||u_n||^{s+1}, \end{aligned}$$

where $\tilde{a}_1,\ \tilde{a}_2$ are positive constants independent of n.
Step 1 follows from the fact that $s > 1$.

Step 2. *There exists* $K_6 > 0$ *such that* $||u_n|| \leq K_6$, *for any* $n \in \mathbb{N}$.

Proof. Let us fix $n \in \mathbb{N}$. By Step 10 of Theorem 1, we have that there exists $(u_n^{\epsilon_k})_k$ such that $u_n^{\epsilon_k} \to u_n$ weakly in $H_0^1(\Omega)$ as $\epsilon_k \to 0$. The weakly lower semicontinuity of $||\cdot||$ and Step 8 of Theorem 1 yield

$$K_3 \geq \liminf_{\epsilon_k \to 0} ||u_n^{\epsilon_k}|| \geq ||u_n||,$$

where K_3 is independent of n.

Step 3. *The sequence* $(\overline{u}_n)_n$ *of solutions of the auxiliary problems* $((\mathcal{A}_n))_n$ *weakly converges in* $H_0^1(\Omega)$ *to a solution* $\overline{u}$ *of problem* $(\mathcal{A})$.

Proof. By Lions-Stampacchia Theorem (see [5]), for any fixed $n \in \mathbb{N}$ there exists a unique solution $\overline{u}_n$ of problem $(\mathcal{A}_n)$. Arguing as in Theorem 1 of [3] with $g_n(\cdot) = \lambda u(\cdot) + p(\cdot, u(\cdot))$, we obtain that

$$\overline{u}_n \to \overline{u} \text{ weakly in } H_0^1(\Omega),$$

as n goes to ∞, where $\overline{u}$ is a solution of problem $(\mathcal{A})$.

Remark 9. Let us observe that Boccardo and Capuzzo Dolcetta in [3] require that $g_n \in L^2(\Omega)$ and $||A_n\psi_n||_{L^2(\Omega)}$ is bounded for any $n \in \mathbb{N}$, while we assume the same kind of hypotheses in $L^{\frac{2^*}{s}}(\Omega)$. We remark that one can use the same arguments of [3] in case that $L^2(\Omega)$ is replaced by $L^p(\Omega)$, where $p \in [1, 2^*)$.

Step 4. *There exists a subsequence of* $(u_n)_n$ *weakly converging in* $H_0^1(\Omega)$ *to a function* u *which is a nontrivial non-negative solution of problem* $(\mathcal{P})$.

Proof. First of all, let us prove that

$$p_n(\cdot, u_n(\cdot)) \to p(\cdot, u(\cdot)) \text{ in } L^{\frac{2^*}{2^*-1}}(\Omega). \tag{25}$$

By Step 2, up to subsequences, $u_n \to u$ in $L^{\frac{2^* s}{2^*-1}}(\Omega)$, $u_n \to u$ a.e. in Ω as n goes to ∞ and there exists $h \in L^{\frac{2^* s}{2^*-1}}(\Omega)$ such that

$$|u_n(x)| \leq h(x) \text{ a.e. } x \text{ in } \Omega, \text{ for any } n \in \mathbb{N}. \tag{26}$$

Therefore, by $(F1)$ and $(H7)$, one has

$$p_n(x, u_n(x)) \to p(x, u(x)) \tag{27}$$

a.e. in Ω, as n goes to ∞. Hence (25) follows by (27), (26) and $(F2)$.
Now let us prove that

$$||u_n - \overline{u}_n|| \to 0 \text{ as } n \text{ goes to } \infty. \tag{28}$$

By $(B3)$ and using the fact that u_n is solution of $(\mathcal{P}_n)$ with $v = \overline{u}_n$ and $\overline{u}_n$ is solution of $(\mathcal{A}_n)$ with $v = u_n$, we obtain:

$$
\begin{aligned}
&||u_n - \overline{u}_n||^2 \leq \frac{1}{c_1}\langle A_n(u_n - \overline{u}_n), u_n - \overline{u}_n\rangle \\
&\leq \frac{\lambda}{c_1}\int_\Omega (u_n - u)(x)(u_n - \overline{u}_n)(x)dx + \qquad (29)\\
&+\frac{1}{c_1}\int_\Omega \left[p_n(x, u_n(x)) - p(x, u(x))\right](u_n - \overline{u}_n)(x)dx.
\end{aligned}
$$

By (25), Steps $2, 3$ and the continuous embedding of $H_0^1(\Omega)$ into $L^{2^*}(\Omega)$, taking the limit in (29) as n goes to ∞, one gets (28).
On the other hand (28) and Step 3 yield

$$u_n \rightarrow \overline{u} \text{ weakly in } H_0^1(\Omega).$$

Then $u \equiv \overline{u}$, so u is a solution of problem $(\mathcal{P})$.
Finally, arguing as in Step 10 of Theorem 1, one can prove that $u \geq 0$ and $u \not\equiv 0$. □

References

[1] Bensoussan A.-Lions J.L.: *'Applications des inéquations variationelles en contrôle stochastique'*, Dunod, Paris (1978).

[2] Biroli M.- Mataloni S.-Matzeu M.: *'Stability results for Mountain Pass and Linking type solutions of semilinear problems involving Dirichlet forms'*, preprint (2002).

[3] Boccardo L.- Capuzzo Dolcetta I.: *'G-convergenza e problema di Dirichlet unilaterale'*, Boll. UMI (4), 12 (1975), pp.115-123.

[4] Boccardo L.- Capuzzo Dolcetta I.: *'Stabilitá delle soluzioni di disequazioni variazionali ellittiche e paraboliche quasilineari'*, Ann. Univ. Ferrara Sez. 7, Sc. Mat. 24 (1978), pp.99-111.

[5] Brezis H.: *'Analyse fonctionnelle. Théorie et applications'*, Masson, Paris (1983).

[6] Dall'Aglio A.-Tchou N.A.: *'G-convergence and semilinear elliptic equations'*, Asympt. Anal. (4) (1991), pp.367-380.

[7] De Giorgi E.- Spagnolo S.: *'Sulla convergenza degli integrali dell'energia per operatori ellittici del secondo ordine'*, Boll. UMI (4), 8 (1973), pp.391-411.

[8] Girardi M.-Matzeu M.-Mastroeni L.: *'Existence and regularity results for non-negative solutions of some semilinear elliptic variational inequalities via Mountain Pass techniques'*, ZAA, J. Anal. Appl. vol.20, no.4 (2001), pp.845-857.

[9] Matzeu M.-Servadei R.: *'A Linking type method to solve a class of semilinear elliptic variational inequalities'*, Adv. Nonl. St. Vol.2, No.1 (2002), pp.1-17.

[10] Rabinowitz P.: *'Minimax methods in critical point theory with applications to differential equations'*, CBMS Regional Conf. Series Math. 65 Amer. Math. Soc. Providence R.I. (1986).