A stability result for Mountain Pass type solutions of semilinear elliptic variational inequalities

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Abstract The aim of the present paper is to establish a stability result for the so called Mountain Pass type solutions of the following class of semilinear elliptic variational inequalities

\[ \begin{align*}
\left\{ \begin{array}{ll}
\langle P_n \rangle & \quad u_n \in H^1_0(\Omega), \quad u_n \leq \psi_n \quad \text{in } \Omega \\
\langle A_n u_n, v - u_n \rangle - \lambda \int_{\Omega} u_n(x)(v - u_n)(x)dx & \geq \int_{\Omega} p_n(x, u_n(x))(v - u_n)(x)dx \\
\forall v \in H^1_0(\Omega), \quad v \leq \psi_n & \quad \text{in } \Omega,
\end{array} \right.
\end{align*} \]

where \( \Omega \) is an open bounded subset of \( \mathbb{R}^N \) \( (N \geq 1) \) with a sufficiently smooth boundary and \( \lambda \) is a real parameter. Moreover, for any \( n \in \mathbb{N}, \ A_n \) is a uniformly elliptic operator, \( \psi_n \) belongs to \( H^1(\Omega) \), \( (\psi_n)_{|_{\partial \Omega}} \geq 0 \) and \( p_n \) is a continuous real function which satisfies some general superlinear and subcritical growth conditions at zero and at infinity.

1 Introduction and main results

In the last years an extensive literature has been developed concerning stability results for semilinear elliptic equations and existence results for semilinear elliptic variational inequalities. The aim of our paper is to establish a stability result for the following class of semilinear elliptic variational inequalities

\[ \begin{align*}
\left\{ \begin{array}{ll}
\langle P_n \rangle & \quad u_n \in H^1_0(\Omega), \quad u_n \leq \psi_n \quad \text{in } \Omega \\
\langle A_n u_n, v - u_n \rangle - \lambda \int_{\Omega} u_n(x)(v - u_n)(x)dx & \geq \int_{\Omega} p_n(x, u_n(x))(v - u_n)(x)dx \\
\forall v \in H^1_0(\Omega), \quad v \leq \psi_n & \quad \text{in } \Omega,
\end{array} \right.
\end{align*} \]
\[\begin{array}{l}
\{ \begin{array}{l}
\forall v \in H_0^1(\Omega), \quad v \leq \psi \quad \text{in} \quad \Omega,
\langle A_n u_n, v - u_n \rangle - \lambda \int_{\Omega} u_n(x)(v - u_n)(x)dx \\
\quad \geq \int_{\Omega} p_n(x, u_n(x))(v - u_n)(x)dx
\end{array} \}
\end{array} \]

where \( \Omega \) is an open bounded subset of \( \mathbb{R}^N \) (\( N \geq 1 \)) with a sufficiently smooth boundary and \( \lambda \) is a real parameter. Moreover, for any \( n \in \mathbb{N} \), \( A_n \) is a uniformly elliptic operator, \( \psi_n \) belongs to \( H^1(\Omega) \), \( (\psi_n)|_{\partial \Omega} \geq 0 \) and \( p_n \) is a continuous real function which satisfies some general superlinear and subcritical growth conditions at zero and at infinity.

More precisely, some stability results in the framework of semilinear elliptic equations were obtained in [6, 7] for uniformly elliptic operators (Dall’Aglio and Tchou in [6] deal with the nonlinear case, while in [7] the linear case is considered) and in [2] in the more general framework of Dirichlet forms.

On the other hand, some existence results for problem \((P_n)\), for any fixed \( n \in \mathbb{N} \), were obtained in [8] when \( \lambda < \tilde{\lambda}_1 \) and in [9] when \( \lambda \geq \tilde{\lambda}_1 \), where \( \tilde{\lambda}_1 \) is the first eigenvalue of \(-\Delta\) in \( H_0^1(\Omega) \).

As far as the authors know, the only previous results about stability for variational inequalities are due to Boccardo and Capuzzo Dolcetta (see [3, 4]). In particular we extend the result obtained in [3], since Boccardo and Capuzzo Dolcetta consider a problem of the kind \((P_n)\) with \( \lambda = 0 \) and \( p_n(\cdot, u_n(\cdot)) = p_n(\cdot) \), that is the linear case.

As for our paper, we find a nontrivial non-negative solution \( u_n \) of problem \((P_n)\) by using the penalization method (see [1]) and the Mountain Pass Theorem (see [10]) as in [8]. By giving suitable convergence conditions for \((A_n)_n\), \((\psi_n)_n\) and \((p_n)_n\) as \( n \) goes to infinity (see \((H5)-(H7))\) and introducing an auxiliary problem \((A_n)\) (see section 4), we obtain a stability result for the solutions \((u_n)_n\) of problems \(((P_n))_n\), that is we get a nontrivial non-negative solution \( u \) of the following limit problem

\[\begin{array}{l}
\{ \begin{array}{l}
\forall v \in H_0^1(\Omega), \quad v \leq \psi \quad \text{in} \quad \Omega,
\langle Au, v - u \rangle - \lambda \int_{\Omega} u(x)(v - u)(x)dx \\
\quad \geq \int_{\Omega} p(x, u(x))(v - u)(x)dx
\end{array} \}
\end{array} \]

as weak limit in \( H_0^1(\Omega) \) of a subsequence of \((u_n)_n\).

## 2 Preliminaries

Let \( \Omega \) be an open bounded subset of \( \mathbb{R}^N \) (\( N \geq 1 \)) with a sufficiently smooth boundary. Let \( H_0^1(\Omega) \) be the usual Sobolev space with the norm

\[ \| v \| = \left( \int_{\Omega} | \nabla v(x) |^2 \, dx \right)^{\frac{1}{2}} \]
and let $\langle \cdot, \cdot \rangle$ be the pairing between $H^1_0(\Omega)$ and its dual space $H^{-1}(\Omega)$.

Let us denote by $\mathcal{E}(c_1, c_2)$, where $c_1$ and $c_2$ are positive constants, the class of the operators of the kind

$$B = -\sum_{i,j=1}^{N} D_i \left( b_{ij}(x) D_j \right)$$

with $b_{ij} : \Omega \to \mathbb{R}$ verifying the following conditions

(B1) $b_{ij}$ is measurable in $\Omega$, $\forall$ $i, j = 1, \ldots, N$;

(B2) $b_{ij}(x) = b_{ji}(x)$ a.e. $x$ in $\Omega$, $\forall$ $i, j = 1, \ldots, N, i \neq j$;

(B3) $c_1 |\xi|^2 \leq \sum_{i,j=1}^{N} b_{ij}(x) \xi_i \xi_j \leq c_2 |\xi|^2$ a.e. $x$ in $\Omega$.

Let

$$A_n = -\sum_{i,j=1}^{N} D_i \left( a_{ij}^{(n)}(x) D_j \right)$$

and

$$A = -\sum_{i,j=1}^{N} D_i \left( a_{ij}(x) D_j \right)$$

in $\mathcal{E}(c_1, c_2)$, for any $n \in \mathbb{N}$.

One can easily check that

$$|a_{ij}^{(n)}(x)| \leq c_2 \text{ a.e. } x \text{ in } \Omega, \forall i, j = 1, \ldots, N, \forall n \in \mathbb{N}. \quad (1)$$

We recall the notion of $G$-convergence for the operators of the class $\mathcal{E}(c_1, c_2)$:

**Definition 1.** We will say that the sequence $(A_n)_{n \in \mathbb{N}}$ $G$-converges to $A$ and we will write $A_n \xrightarrow{G} A$ if, for any $T \in H^{-1}(\Omega)$, $(A_n^{-1}T)_{n \in \mathbb{N}}$ weakly converges to $A^{-1}T$ in $H^1_0(\Omega)$.

Let $\lambda_{1,n} > 0$ and $\lambda_1 > 0$ be respectively the first eigenvalue of the operator $A_n$ and $A$ in $H^1_0(\Omega)$ with Dirichlet boundary conditions, for any $n \in \mathbb{N}$. Finally let $\tilde{\lambda}_1 > 0$ be the first eigenvalue of the operator $-\Delta$ in $H^1_0(\Omega)$ with Dirichlet boundary conditions. By the uniform ellipticity of $A_n$ and $A$ and by the variational characterization of the eigenvalues, i.e.

$$\lambda_{1,n} = \inf_{v \in H^1_0(\Omega) \setminus \{0\}} \frac{\langle A_n v, v \rangle}{\int_{\Omega} v^2(x) dx}$$

and

$$\lambda_1 = \inf_{v \in H^1_0(\Omega) \setminus \{0\}} \frac{\langle Av, v \rangle}{\int_{\Omega} v^2(x) dx}$$
one has
\[ \frac{c_1}{c_2} \lambda_1 \leq \lambda_{1,n} \leq \frac{c_2}{c_1} \lambda_1, \quad \forall n \in \mathbb{N}. \tag{2} \]

For any operator \( B \in \mathcal{E}(c_1, c_2) \) and for any \( \lambda < \frac{c_1}{c_2} \lambda_1 \) the following inequalities hold:
\[ \tilde{c}_1 ||u||^2 \leq \langle Bu, u \rangle - \lambda \int_{\Omega} u^2(x)dx \leq \tilde{c}_2 ||u||^2, \tag{3} \]
where \( \tilde{c}_1 := \min \left\{ c_1, c_1 - \frac{\lambda}{\lambda_1} c_2 \right\} \) and \( \tilde{c}_2 := \max \left\{ c_2, c_2 \left( 1 - \frac{\lambda c_2}{\lambda_1 c_1} \right) \right\} \).

Furthermore it is well-known that the sequence \((\lambda_{1,n})_n\) converges to \( \lambda_1 \) as \( n \) goes to infinity, if the operators \( A_n \) G-converge to \( A \) in \( \mathcal{E}(c_1, c_2) \) (see [6]).

Let us denote by \( F(a_1, a_2, a_3, r) \), where \( a_1, a_2, a_3 \) and \( r \) are positive constants, the class of the functions \( f : \Omega \times \mathbb{R} \to \mathbb{R} \) which verify the following conditions

\begin{itemize}
  \item [(F1)] \( f(\cdot, \cdot) \) is continuous in \( \overline{\Omega} \times \mathbb{R} \);
  \item [(F2)] \( |f(x, \xi)| \leq a_1 + a_2 |\xi|^s, \quad \forall (x, \xi) \in \overline{\Omega} \times \mathbb{R}, \)
    
    with \( 1 < s < \frac{N+2}{N-2} \) if \( N \geq 3 \), \( 1 < s \) if \( N = 1, 2 \);
  \item [(F3)] \( f(x, \xi) = o(|\xi|) \) as \( \xi \to 0, \quad \forall x \in \overline{\Omega} \).
\end{itemize}

Moreover, putting
\[ F(x, \xi) := \int_0^\xi f(x, t)dt, \quad \forall (x, \xi) \in \overline{\Omega} \times \mathbb{R}, \]
we assume that

\begin{itemize}
  \item [(F4)] \( 0 < (s + 1) F(x, \xi) \leq \xi f(x, \xi), \quad \forall (x, \xi) \in \overline{\Omega} \times \mathbb{R}, \quad \xi \geq r; \)
  \item [(F5)] \( F(x, \xi) \geq a_3 |\xi|^{s+1}, \quad \forall (x, \xi) \in \overline{\Omega} \times \mathbb{R}, \quad \xi \geq r. \)
\end{itemize}

Choosing \( a_4 = a_1 r + 2^s a_2 r^s \), by \( (F1), (F2) \) and \( (F5) \) one has
\[ F(x, \xi) \geq a_3 |\xi|^{s+1} - a_4, \quad \forall (x, \xi) \in \overline{\Omega} \times \mathbb{R}^+. \]

Finally, for any function \( g : \overline{\Omega} \times \mathbb{R} \to \mathbb{R} \) we define
\[ \overline{g}(x, \xi) = \begin{cases} 0 & \text{if } \xi < 0, \\ g(x, \xi) & \text{if } \xi \geq 0, \end{cases} \]
for any \( x \in \overline{\Omega} \). It is easy to check that, if \( g \) is a function belonging to the class \( \mathcal{F}(a_1, a_2, a_3, r) \), then \( \overline{g} \) is in \( \mathcal{F}(a_1, a_2, a_3, r) \) too.
3 The existence result of a nontrivial non-negative solution for problem \((P_n)\)

Let us consider the following family of semilinear variational inequalities

\[
(P_n) \left\{ \begin{array}{l}
u_n \in H^1_0(\Omega), \, u_n \leq \psi_n \quad \text{in} \quad \Omega \\
(A_n u_n, v - u_n) - \lambda \int_{\Omega} u_n(x)(v - u_n)(x)dx \geq \int_{\Omega} p_n(x, u_n(x))(v - u_n)(x)dx \\
\forall v \in H^1_0(\Omega), \, v \leq \psi_n \quad \text{in} \quad \Omega,
\end{array} \right.
\]

where \(\Omega\) is an open bounded subset of \(\mathbb{R}^N\) \((N \geq 3)\) with a sufficiently smooth boundary and \(\lambda\) is a real parameter. Moreover, for any \(n \in \mathbb{N}\), \(A_n\) is an operator of the class \(E(c_1, c_2)\), \(\psi_n\) belongs to \(H^1(\Omega)\), with \((\psi_n)|_{\partial \Omega} \geq 0\) and \(p_n : \Omega \times \mathbb{R} \rightarrow \mathbb{R}\) belongs to the class \(F(a_1, a_2, a_3, r)\).

In case that \(\psi_n(x) \geq 0\) on \(\Omega\), it is obvious that \(u_0 \equiv 0\) is a trivial solution of problem \((P_n)\), for any \(n \in \mathbb{N}\).

One can state the following

**Theorem 1.** Let \(\lambda < \frac{c_1}{c_2} \lambda_1\). Let \(A_n \in E(c_1, c_2)\) and \(p_n \in F(a_1, a_2, a_3, r)\), for any \(n \in \mathbb{N}\). Moreover, let the following hypotheses hold

\((H1)\) \(\psi_n \in H^1_0(\Omega), \, \psi_n \geq 0 \quad \text{in} \quad \Omega, \, \forall \, n \in \mathbb{N};\)

\((H2)\) \(\exists \, \psi \in H^1_0(\Omega), \, 0 \leq \psi \leq \psi_n \quad \text{in} \quad \Omega, \, \forall \, n \in \mathbb{N} \) such that

\[\tilde{c}_2 \| \psi \|^2 \leq 2 \left( a_3 \int_\Omega |\psi(x)|^{s+1}dx - a_4 |\Omega| \right);\]

\((H3)\) \(s < 2 \quad \text{in} \quad (F2) \) and \((F4);\)

\((H4)\) \(\exists \, v_0 \in H^1_0(\Omega) \) such that \(\psi_n \leq v_0 \quad \text{in} \quad \Omega, \, \forall \, n \in \mathbb{N}.\)

Then, for any \(n \in \mathbb{N}\), there exists a nontrivial non-negative solution \(u_n\) of problem \((P_n)\).

**Remark 1.** We need \(\lambda < \lambda_{1,n}\), for any \(n \in \mathbb{N}\), in order to obtain estimates independent of \(n\). By \((2)\), \(\lambda < \frac{c_1}{c_2} \lambda_1\) is a sufficient condition for our request. If the sequence of operators \((A_n)\) \(G\)-converges to an operator \(A\) in \(E(c_1, c_2)\) and \(\lambda_1\) is the first eigenvalue of \(A\), then \((\lambda_{1,n})\) \(G\)-converges to \(\lambda_1\) as \(n\) goes to \(\infty\). So we could substitute the condition \(\lambda < \frac{c_1}{c_2} \lambda_1\) with \(\lambda < \lambda_1\), which implies \(\lambda < \lambda_{1,n}\) for \(n\) large enough. In this case we obtain a result of existence of a nontrivial non-negative solution for problem \((P_n)\) for \(n\) large enough.

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We have decided to consider only the case \(N \geq 3\) as, for \(N = 1, 2\), the results are the same, even using easier arguments.
Remark 2. We need that the constant \( a_3 \) in (F5) is independent of \( n \). So we cannot deduce (F5) from (F4) as usual, but we have to assume it.

Remark 3. Hypotheses (H2) and (H4) are not empty. Indeed, taking a non-zero non-negative function \( \tilde{v} \) in \( H^1_0(\Omega) \) and a bounded sequence \( (\mu_n)_n \) in \( \mathbb{R}^+ \) such that \( \inf_{n \in \mathbb{N}} \mu_n > 1 \), one can choose \( \psi_n = \mu_n \tilde{v} \), for any \( n \in \mathbb{N} \). Then, putting \( v_0 = \sup_{n \in \mathbb{N}} \mu_n \tilde{v} \) and \( \bar{v} = \inf_{n \in \mathbb{N}} \mu_n \tilde{v} \), (H2) and (H4) hold, since \( s > 1 \).

The method of finding the solution \( u_n \) for problem \((P_n)\) relies on the consideration of a family of ‘penalized’ equations associated, in a standard way, with \((P_n)\) (see [1]). Indeed, one can prove that any penalized equation possesses a solution of ‘Mountain Pass type’ and that a sequence chosen in this family actually converges to a nontrivial non-negative solution \( u_n \) of \((P_n)\), suitably using some estimates from below and from above for the \( H^1_0(\Omega) \)-norm of the solutions of the penalized equations. As mentioned before, we apply the following Mountain Pass Theorem (see [10]):

**Mountain Pass Theorem:** Let \( E \) be a real Banach space and \( J \in C^1(E, \mathbb{R}) \). Suppose \( J \) satisfies (PS) condition, i.e.

\[
\forall (u_m)_m \in H^1_0(\Omega) \text{ such that } (J(u_m))_m \text{ is bounded and } J'(u_m) \to 0
\]

in \( H^{-1}(\Omega) \) as \( m \to \infty \), there exists a subsequence of \((u_m)_m\) strongly converging in \( H^1_0(\Omega) \).

Furthermore let \( J \) satisfy \( J(0) = 0 \) and the following assumptions

(J1) there exist constants \( \rho, \alpha > 0 \) such that \( J|_{\partial B_\rho} \geq \alpha \);

(J2) there exists an \( e \in E \setminus B_\rho \) such that \( J(e) \leq 0 \).

Then \( J \) possesses a critical value \( c \geq \alpha \) which can be characterized as

\[
c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t)),
\]

where

\[
\Gamma = \{ \gamma \in C([0,1], E) : \gamma(0) = 0, \gamma(1) = e \}.
\]

First of all, for any \( n \in \mathbb{N} \), let us introduce the ‘penalized’ problem associated with \((P_n)\), that is, for any \( \epsilon > 0 \), the weak equation

\[
(P_n)_\epsilon
\]

\[
\begin{align*}
&\left\{ \begin{array}{ll}
&u_n^\epsilon \in H^1_0(\Omega) \text{ such that }
&
\langle A_n u_n^\epsilon, v \rangle - \lambda \int_\Omega u_n^\epsilon(x)v(x)dx
\nonumber \\
&\quad + \frac{1}{\epsilon} \int_\Omega (u_n^\epsilon - \psi_n)^+(x)v(x)dx = \int_\Omega p_n(x, u_n^\epsilon(x))v(x)dx
\end{array} \right. \\
&\forall v \in H^1_0(\Omega),
\end{align*}
\]
where \( f^+ \) denotes the positive part of the function \( f \). Let us note that the last integral is well defined for all \( v \in H_0^1(\Omega) \) as a consequence of \((F2)\) and of the continuous embedding of \( H_0^1(\Omega) \) into \( L^2(\Omega) \).

In order to look for non-negative solutions of problem \((P_n)\), it is convenient to modify it with the following one:

\[
(P_n)_\epsilon \begin{cases}
  u_{n,\epsilon}^+ \in H_0^1(\Omega) \\
  (A_n u_{n,\epsilon}^+, v) - \lambda \int_\Omega u_{n,\epsilon}^+(x)v(x)dx \\
  + \frac{1}{\epsilon} \int_\Omega (u_{n,\epsilon}^+ - \psi_n)^+(x)v(x)dx = \int_\Omega p_n(x,u_{n,\epsilon}^+(x))v(x)dx
\end{cases}
\forall v \in H_0^1(\Omega).

Actually in order to look for solutions of \((P_n)_\epsilon\), we study the critical points of the real functional \( I_{n,\epsilon} \) defined on \( H_0^1(\Omega) \) in this way

\[
I_{n,\epsilon}(v) = \frac{1}{2} \langle A_n v, v \rangle - \lambda \int_\Omega v^2(x)dx + \frac{1}{\epsilon} \int_\Omega \int_0^{v(x)} (t-\psi_n(x))^+ dt dx - \int_\Omega \bar{p}_n(x,v(x))dx,
\]

where

\[
\bar{p}_n(x,\xi) := \int_0^\xi p_n(x,t)dt, \quad \forall (x,\xi) \in \Omega \times \mathbb{R}.
\]

One can easily check that \( I_{n,\epsilon} \) belongs to \( C^1(H_0^1(\Omega), \mathbb{R}) \) and that the pairing \( \langle (I_{n,\epsilon})'(u_{n,\epsilon}^+), v \rangle \) coincides with the difference between the first and the second member in \((P_n)_\epsilon\).

At this point, to prove Theorem 1, let us verify that the functional \( I_{n,\epsilon} \) satisfies all the hypotheses of the Mountain Pass Theorem.

**Proof.** (of Theorem 1) Let us proceed by steps.

**Step 1.** The functional \( I_{n,\epsilon}^+ \) verifies, for any \( n \in \mathbb{N} \) and for any \( \epsilon > 0 \), the conditions

\[
I_{n,\epsilon}^+(0) = 0,
\]

\[
I_{n,\epsilon}^+(v) \geq \alpha \quad \text{for some} \quad \rho, \alpha > 0, \quad \forall v \in H_0^1(\Omega), \quad \|v\| = \rho.
\]

**Proof.** Let us fix \( n \in \mathbb{N} \) and \( \epsilon > 0 \). Property (4) is trivial. As for (5), let us note that the positivity of \( \psi_n \) on \( \Omega \) yields
for all $v \in H_0^1(\Omega)$.

As a consequence of (F2) and (F3), one gets that

$$\forall \delta > 0 \exists c(\delta) > 0 \text{ such that } P_n(x,\xi) \leq \frac{\delta}{2} |\xi|^2 + c(\delta) |\xi|^{s+1},$$

for all $v \in H_1^0(\Omega)$.

Then, by using (6),(7), (3) and by choosing $\rho > 0$ such that

$$\frac{\tilde{c}_1}{2} \rho - \frac{\delta}{2\tilde{\lambda}_1} \rho - c(\delta) c_s \rho^{s-1},$$

one has

So Step 1 is proved.

Remark 4. Note that the positive constant $\alpha$ is independent of $\epsilon$ and of $n$ and this fact will be used in the proof of Theorem 1.

Step 2. There exists an element $e \in H_0^1(\Omega) \setminus \{0\}$ such that $I_n^\epsilon(e) \leq 0$, for any $n \in \mathbb{N}$ and for any $\epsilon > 0$.

Proof. Let us fix $n \in \mathbb{N}$ and $\epsilon > 0$. Let $e = v$ as in (H2). Observe that $e \not\equiv 0$. Moreover, $0 \leq e \leq \psi_n$ in $\Omega$ implies

$$I_n^\epsilon(e) \leq \frac{\tilde{c}_2}{2} \epsilon^2 - a_3 \int_{\Omega} |e(x)|^{s+1} dx + a_4 |\Omega|.$$
for any \((u_m)_m \in H^1_0(\Omega)\) such that \((I^\epsilon_n(u_m))_m\) is bounded and
\((PS)\ (I^\epsilon')(u_m) \to 0\ in \ H^{-1}(\Omega)\ as \ m \to \infty,\ there\ exists\ a\ subsequence\ of\ (u_m)_m\ strongly\ converging\ in\ \ H^1_0(\Omega)\).

Proof. Let us fix \(n \in \mathbb{N}, \epsilon > 0\) and \(\beta \in \left(\frac{1}{s+1}, \frac{1}{2}\right)\). By the properties of \((u_m)_m\) one deduces
\[
I^\epsilon_n(u_m) - \beta\langle (I^\epsilon_n)'(u_m), u_m \rangle \leq K_{n,\epsilon} + \beta M_{n,\epsilon} \left\| u_m \right\|, \tag{8}
\]
where \(K_{n,\epsilon}, M_{n,\epsilon}\) are positive constants independent of \(m\).

By definition of \(I^\epsilon_n, (I^\epsilon_n)'\) and by (3), (F1), (F4) one gets
\[
\begin{align*}
I^\epsilon_n(u_m) - \beta\langle (I^\epsilon_n)'(u_m), u_m \rangle &\geq \tilde{c}_1 \left(\frac{1}{2} - \beta\right) \left\| u_m \right\|^2 - \frac{1}{\epsilon} \int_{\{x \in \Omega: u_m(x) \geq \psi_n(x)\}} (1 - \beta)\psi_n(x)u_m(x)dx \\
&\quad + (s + 1) \left(\beta - \frac{1}{s + 1}\right) \int_{\{x \in \Omega: u_m(x) \geq r\}} P_n(x, u_m(x))dx - K_{n,r} \tag{9}
\end{align*}
\]
for any \(m \in \mathbb{N}\), where \(K_{n,r}\) is a positive constant independent of \(m\).

Finally, combining (8) and (9), one gets
\[
\left\| u_m \right\|^2 \leq C_{n,\epsilon} \left\| u_m \right\| + D_{n,\epsilon},
\]
for any \(m \in \mathbb{N}\), for suitable positive constants \(C_{n,\epsilon}\) and \(D_{n,\epsilon}\) independent of \(m\).
Thus \((u_m)_m\) is bounded in \(H^1_0(\Omega)\). At this point, Step 3 easily follows from a standard argument based on the compact embedding of \(H^1_0(\Omega)\) into \(L^p(\Omega)\) for \(p \in [2, 2^*\).

Step 4. For any \(n \in \mathbb{N}\) and for any \(\epsilon > 0\), there exists a solution \(u^*_n\) of problem \((\mathcal{P}_n)_\epsilon\) such that
\[
I^\epsilon_n(u^*_n) = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I^\epsilon_n(\gamma(t)),
\]
where \(\Gamma = \{\gamma \in C([0,1], H^1_0(\Omega)) : \gamma(0) = 0, \gamma(1) = \epsilon\}\).

Moreover
\[
I^\epsilon_n(u^*_n) \geq \alpha.
\]
Proof. It is a consequence of Steps 1, 2, 3 and of the Mountain Pass Theorem.

**Step 5.** For any \( n \in \mathbb{N} \) and for any \( \epsilon > 0 \), \( u_n^\epsilon \) is a nontrivial non-negative solution of problem \((P_n)_\epsilon\).

**Proof.** Let us fix \( n \in \mathbb{N} \) and \( \epsilon > 0 \). \( u_n^\epsilon \not\equiv 0 \) as it is a Mountain Pass type critical point. In order to prove that \( u_n^\epsilon \geq 0 \), let us take \( v = (u_n^\epsilon)^- \) in \((P_n)_\epsilon\). It follows that

\[
\langle A_n u_n^\epsilon, (u_n^\epsilon)^- \rangle - \lambda \int_\Omega u_n^\epsilon(x)(u_n^\epsilon)^-(x)dx \\
+ \frac{1}{\epsilon} \int_\Omega (u_n^\epsilon - \psi_n)^+(x)(u_n^\epsilon)^-(x)dx = \int_\Omega p_n(x, u_n^\epsilon(x))(u_n^\epsilon)^-(x)dx.
\]

By the definition of \((u_n^\epsilon)^-\) and of \( p_n(\cdot, \cdot) \) and using (3), (10) yields

\[
0 = \langle A_n u_n^\epsilon, (u_n^\epsilon)^- \rangle - \lambda \int_\Omega u_n^\epsilon(x)(u_n^\epsilon)^-(x)dx \geq \tilde{c}_1 \|(u_n^\epsilon)^-\|^2,
\]

from which we obtain \( u_n^\epsilon \geq 0 \). The non-negativity of \( u_n^\epsilon \) and the definition of \( p_n(\cdot, \cdot) \) yield Step 5.

**Step 6.** There exists a constant \( K_1 > 0 \) such that \( \|u_n^\epsilon\| \geq K_1 \), for any \( n \in \mathbb{N} \) and for any \( \epsilon > 0 \).

**Proof.** Let us fix \( n \in \mathbb{N} \) and \( \epsilon > 0 \). Let us take \( v = u_n^\epsilon \) in \((P_n)_\epsilon\). It follows that

\[
\langle A_n u_n^\epsilon, u_n^\epsilon \rangle - \lambda \int_\Omega u_n^\epsilon(x)^2dx \\
+ \frac{1}{\epsilon} \int_\Omega (u_n^\epsilon - \psi_n)^+(x)u_n^\epsilon(x)dx = \int_\Omega p_n(x, u_n^\epsilon(x))u_n^\epsilon(x)dx.
\]

Arguing as in Step 1 we obtain

\[
\forall \delta > 0 \; \exists c(\delta) > 0 \; \text{such that} \; p_n(x, \xi) \leq \delta \| \xi \| + c(\delta) \| \xi \|^s
\]

for any \((x, \xi) \in \Omega \times \mathbb{R}\).

By using the non-negativity of \( u_n^\epsilon \), (3), the variational characterization of \( \tilde{\lambda}_1 \) and (12), one has

\[
\tilde{c}_1 \|u_n^\epsilon\|^2 \leq \int_\Omega p_n(x, u_n^\epsilon(x))u_n^\epsilon(x)dx \\
\leq \delta \|u_n^\epsilon\|_{L^2(\Omega)}^2 + c(\delta) \|u_n^\epsilon\|_{L^{s+1}(\Omega)}^{s+1} \\
\leq \frac{\delta}{\tilde{\lambda}_1} \|u_n^\epsilon\|^2 + c(\delta) c_s \|u_n^\epsilon\|^{s+1}.
\]
By choosing $\delta$ such that $\delta < \tilde{c}_1 \lambda_1$, we can conclude that $\|u^*_n\|$ is bounded from below as $s + 1 > 2$.

**Step 7.** There exists a constant $K_2 > 0$ such that $I'_n(u^*_n) \leq K_2$, for any $n \in \mathbb{N}$ and for any $\epsilon > 0$.

**Proof.** Let us fix $n \in \mathbb{N}$ and $\epsilon > 0$. By Step 4 with $\gamma(t) = t\overline{v}$, where $\overline{v}$ is as in $(H2)$, we obtain

$$I'_n(u^*_n) \leq \max_{t \in [0,1]} I'_n(t\overline{v}).$$

From the fact that $0 \leq t\overline{v} \leq \psi_n$ in $\Omega$, from the definition of $\mathcal{P}_n$ and by (3) one has

$$I'_n(u^*_n) \leq \max_{t \in [0,1]} \left\{ \frac{1}{2} \langle A_n t\overline{v}, t\overline{v} \rangle - \frac{\lambda}{2} \int\! (t\overline{v})^2(x)dx \right\} \leq \max_{t \in [0,1]} \frac{\tilde{c}_2}{2} \|t\overline{v}\|^2 \leq \tilde{c}_2 \|\overline{v}\|^2.$$

So Step 7 is proved as the right member of the previous relation is independent of $n$ and $\epsilon$.

**Step 8.** There exists a constant $K_3 > 0$ such that $\|u^*_n\| \leq K_3$, for any $n \in \mathbb{N}$ and for any $\epsilon > 0$.

**Proof.** Let us fix $n \in \mathbb{N}$ and $\epsilon > 0$. By Step 7 one has

$$\frac{1}{2} \langle A_n u^*_n, u^*_n \rangle - \frac{\lambda}{2} \int\! (u^*_n)^2(x)dx + \frac{1}{\epsilon} \int\! \int_t^{u^*_n(x)} (t - \psi_n(x))^+ dt dx \leq K_2 + \int\! \mathcal{P}_n(x, u^*_n(x)) dx.$$  \hfill (13)

By the non-negativity of $u^*_n$, by (13), $(F2)$, $(F4)$, using the fact that $u^*_n$ is a solution of $(P_n)$, in particular with $v = u^*_n$ and that $s + 1 > 2$, we obtain

$$\frac{1}{2} \langle A_n u^*_n, u^*_n \rangle - \frac{\lambda}{2} \int\! (u^*_n)^2(x)dx + \frac{1}{2\epsilon} \int\! \int_{\{x \in \Omega: u^*_n(x) \geq \psi_n(x)\}} (u^*_n - \psi_n)^2(x)dx \leq K_2 + C_r |\Omega| + \frac{1}{s+1} \int\! p_n(x, u^*_n(x)) u^*_n(x) dx$$

$$= K_2 + C_r |\Omega| + \frac{1}{s+1} \left[ \langle A_n u^*_n, u^*_n \rangle - \lambda \int\! (u^*_n)^2(x)dx \right]$$

$$+ \frac{1}{s+1} \epsilon \int\! \int_{\{x \in \Omega: u^*_n(x) \geq \psi_n(x)\}} (u^*_n - \psi_n)^+ (x) u^*_n(x) dx$$

$$\leq K_2 + C_r |\Omega| + \frac{1}{s+1} \left[ \langle A_n u^*_n, u^*_n \rangle - \lambda \int\! (u^*_n)^2(x)dx \right]$$

$$+ \frac{1}{2\epsilon} \int\! (u^*_n - \psi_n)^+ (x) u^*_n(x) dx,$$  \hfill (14)
where $C_r$ is a positive constant independent of $n$ and $\epsilon$.
By (14), $(H4)$, the fact that $u_n^\epsilon$ is a solution of $(P_n)_\epsilon$ with $v = v_0$, $(F2)$ and (1), we obtain

$$
\left( \frac{1}{2} - \frac{1}{s+1} \right) \left[ (A_n u_n^\epsilon, u_n^\epsilon) - \lambda \int_\Omega (u_n^\epsilon)^2(x)dx \right]
$$

$$
\leq K_2 + C_r |\Omega| + \frac{1}{2\epsilon} \int_{\{x \in \Omega : u_n^\epsilon(x) \geq \psi_n(x)\}} (u_n^\epsilon - \psi_n)(x) \psi_n(x)dx
$$

$$
\leq K_2 + C_r |\Omega| + \frac{1}{2\epsilon} \int_{\{x \in \Omega : u_n^\epsilon(x) \geq \psi_n(x)\}} (u_n^\epsilon - \psi_n)(x)v_0(x)dx
$$

$$
= K_2 + C_r |\Omega| - \frac{1}{2} \langle A_n u_n^\epsilon, v_0 \rangle + \frac{\lambda}{2} \int_\Omega u_n^\epsilon(x)v_0(x)dx + \frac{1}{2} \int_\Omega p_n(x, u_n^\epsilon(x))v_0(x)dx
$$

(15)

$$
\leq K_2 + C_r |\Omega| + \frac{c_2}{2} ||v_0|| ||u_n^\epsilon|| + \frac{\lambda}{2} \int_\Omega u_n^\epsilon(x)v_0(x)dx + \frac{a_1}{2} ||v_0||_{L^1}
$$

$$
+ \frac{a_2}{2} ||v_0||_{L^2}(\Omega) \frac{1}{(\omega)} ||(u_n^\epsilon)'||_{L^2}(\Omega).
$$

Therefore, by (15), (3), the variational characterization of $\bar{\lambda}_1$ and the continuous embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$, one has

$$
\tilde{c}_1 \left( \frac{1}{2} - \frac{1}{s+1} \right) ||u_n^\epsilon||^2
$$

$$
\leq K_2 + C_r |\Omega| + \frac{c_2}{2} ||v_0|| ||u_n^\epsilon|| + \frac{\lambda}{2} \int_\Omega u_n^\epsilon(x)v_0(x)dx + \frac{a_1}{2} ||v_0||_{L^1}
$$

$$
+ \frac{a_2}{2} ||v_0||_{L^2}(\Omega) \frac{1}{(\omega)} ||(u_n^\epsilon)'||_{L^2}(\Omega)
$$

(16)

$$
\leq K_2 + C_r |\Omega| + \frac{c_2}{2} ||v_0|| ||u_n^\epsilon|| + \frac{\lambda}{2\lambda_1} ||v_0|| ||u_n^\epsilon|| + \frac{a_1}{2} ||v_0||_{L^1}
$$

$$
+ \frac{\tilde{a}_2}{2} ||v_0||_{L^2}(\Omega) \frac{1}{(\omega)} ||u_n^\epsilon||^s
$$

where $\tilde{a}_2$ is a positive constant independent of $n$ and $\epsilon$.
Finally, Step 8 follows from $(H3)$.

**Step 9.** There exists a constant $K_4 > 0$ such that $\| (u_n^\epsilon - \psi_n)^+ \|_{L^2(\Omega)} \leq K_4 \sqrt{\epsilon}$, for any $n \in \mathbb{N}$ and for any $\epsilon > 0$. 


Proof. Let us fix \( n \in \mathbb{N} \) and \( \epsilon > 0 \). Taking \( v = u^\epsilon_n \) in \((P_\epsilon)\), we obtain
\[
\frac{1}{\epsilon} \int_\Omega (u^\epsilon_n - \psi_n)^+(x) u^\epsilon_n(x) dx
= - \langle A_n u^\epsilon_n, u^\epsilon_n \rangle + \lambda \int_\Omega (u^\epsilon_n)^2(x) dx + \int_\Omega p_n(x, u^\epsilon_n(x)) u^\epsilon_n(x) dx.
\]

(17)

Using the non-negativity of \( \psi_n \), (17), (3), (F2) and the compact embedding of \( H^1_0(\Omega) \) into \( L^p(\Omega) \) for \( p \in [1, 2^*) \), we deduce
\[
\frac{1}{\epsilon} \int_\Omega ((u^\epsilon_n - \psi_n)^+)^2(x) dx
\leq \frac{1}{\epsilon} \int_\Omega (u^\epsilon_n - \psi_n)^+(x) u^\epsilon_n(x) dx

= - \langle A_n u^\epsilon_n, u^\epsilon_n \rangle + \lambda \int_\Omega (u^\epsilon_n)^2(x) dx + \int_\Omega p_n(x, u^\epsilon_n(x)) u^\epsilon_n(x) dx

\leq - \tilde{c}_1 \| u^\epsilon_n \|_2^2 + a_1 \| u^\epsilon_n \|_{L^1(\Omega)} + a_2 \| u^\epsilon_n \|_{L^{s+1}(\Omega)}^{s+1}

\leq \tilde{a}_1 \| u^\epsilon_n \| + \tilde{a}_2 \| u^\epsilon_n \|^{s+1},
\]
where \( \tilde{a}_1 \) and \( \tilde{a}_2 \) are positive constants independent of \( n \) and \( \epsilon \).

So Step 9 follows from Step 8.

Step 10. There exists a sequence \((\epsilon_k)_k\) converging to 0 as \( k \) goes to \( \infty \) such that \((u^\epsilon_n)_k\) weakly converges in \( H^1_0(\Omega) \) to some \( u_n \neq 0 \) and \( u_n \geq 0 \), for any \( n \in \mathbb{N} \).

Proof. Let us fix \( n \in \mathbb{N} \). First of all, by Step 8, there exists a sequence \((u^\epsilon_n)_k\) weakly converging in \( H^1_0(\Omega) \) to some \( u_n \) as \( \epsilon_k \) goes to 0. Therefore, as \( \epsilon_k \) goes to 0,
\[
u^\epsilon_n \to u_n \quad \text{in } L^2(\Omega)
\]
and
\[u^\epsilon_n \to u_n \quad \text{a.e. in } \Omega.
\]

Moreover, by Step 5, \( u^\epsilon_n \geq 0 \) for any \( \epsilon_k > 0 \) and therefore \( u_n \geq 0 \).

We claim that \( u_n \) is not identically zero. Indeed, from the fact that \( u^\epsilon_n \) is a non-negative solution of problem \((P_\epsilon)\) with \( \epsilon = \epsilon_k \), using (3) and Step 6, we obtain
\[
\tilde{c}_1 K^2 \leq \tilde{c}_1 \| u^\epsilon_n \|_2^2

\leq \langle A_n u^\epsilon_n, u^\epsilon_n \rangle - \lambda \int_\Omega (u^\epsilon_n)^2(x) dx \leq \int_\Omega p_n(x, u^\epsilon_n(x)) u^\epsilon_n(x) dx.
\]

(18)

By (F1), (F2) and the weak convergence in \( H^1_0(\Omega) \) of \((u^\epsilon_n)_k\) to \( u_n \), one has
\[
\int_{\Omega} p_n(x, u^{\epsilon_k}_n(x)) u^{\epsilon_k}_n(x) dx \to \int_{\Omega} p_n(x, u_n(x)) u_n(x) dx
\]
as \(\epsilon_k \to 0\). If \(u_n\) was identically zero, then, going to the limit as \(\epsilon_k \to 0\) in (18), we get a contradiction. So \(u_n \not\equiv 0\).

**Step 11.** The element \(u_n\) given by Step 10 is a nontrivial non-negative solution of problem \((P_n)\), for any \(n \in \mathbb{N}\).

**Proof.** Let us fix \(n \in \mathbb{N}\). As \(u^{\epsilon_k}_n\) is a solution of \((P_n)_{\epsilon_k}\), taking \(v - u^{\epsilon_k}_n\) as test function, one gets

\[
\langle A_n u^{\epsilon_k}_n, v - u^{\epsilon_k}_n \rangle - \lambda \int_{\Omega} u^{\epsilon_k}_n(x)(v - u^{\epsilon_k}_n)(x) dx + \frac{1}{\epsilon_k} \int_{\Omega} (u^{\epsilon_k}_n - \psi_n)^+(x)(v - u^{\epsilon_k}_n)(x) dx = \int_{\Omega} p_n(x, u^{\epsilon_k}_n(x))(v - u^{\epsilon_k}_n)(x) dx.
\]

(19)

Arguing as in Step 10, we have

\[
\int_{\Omega} p_n(x, u^{\epsilon_k}_n(x))(v - u^{\epsilon_k}_n)(x) dx \to \int_{\Omega} p_n(x, u_n(x))(v - u_n)(x) dx
\]

(20)

and

\[
\lambda \int_{\Omega} u^{\epsilon_k}_n(x)(v - u^{\epsilon_k}_n)(x) dx \to \lambda \int_{\Omega} u_n(x)(v - u_n)(x) dx
\]

(21)
as \(\epsilon_k\) goes to 0.
The weak lower semicontinuity of \(\langle A_n u, v \rangle\) yields

\[
\liminf_{\epsilon_k \to 0} \langle A_n u^{\epsilon_k}_n, v - u^{\epsilon_k}_n \rangle \leq \langle A_n u_n, v - u_n \rangle.
\]

(22)

Finally taking \(v \leq \psi_n\) in \(\Omega\), we have

\[
\frac{1}{\epsilon_k} \int_{\Omega} (u^{\epsilon_k}_n - \psi_n)^+(x)(v - u^{\epsilon_k}_n)(x) dx \leq 0.
\]

(23)

Taking the liminf in (19) as \(\epsilon_k\) goes to 0, by (20), (21), (22) and (23) we get

\[
\langle A_n u_n, v - u_n \rangle - \lambda \int_{\Omega} u_n(x)(v - u_n)(x) dx \geq \int_{\Omega} p_n(x, u_n(x))(v - u_n)(x) dx,
\]

for any \(v \in H^1_0(\Omega), v \leq \psi_n\) in \(\Omega\). Now let us prove that \(u_n \leq \psi_n\) in \(\Omega\): this fact follows by Step 9 and by the strong convergence in \(L^2(\Omega)\) of \((u^{\epsilon_k}_n)_k\) to \(u_n\). So \(u_n\) is a solution of the variational inequality \((P_n)\). \(\square\)
4 The stability result

In this section we will prove the main result of this paper, i.e. the stability result for the Mountain Pass type solutions \((u_n)_n\) of the problems \(((\mathcal{P}_n))_n\). In other words, we consider the following problem

\[
(\mathcal{P}) \quad \left\{ \begin{array}{ll}
u \in H^1_0(\Omega), & u \leq \psi \text{ in } \Omega \\
(Au, v-u) - \lambda \int_\Omega u(x)(v-u)(x)dx & \geq \int_\Omega p(x,u(x))(v-u)(x)dx \\
& \forall v \in H^1_0(\Omega), \; v \leq \psi \text{ in } \Omega,
\end{array} \right.
\]

where \(A \in \mathcal{E}(c_1,c_2), \; p \in \mathcal{F}(a_1,a_2,a_3,r)\) and \(\psi \in H^1(\Omega)\) with \(\psi|_{\partial \Omega} \geq 0\), and we prove the following

**Theorem 2.** Let \(\lambda < \frac{c_1}{c_2}\lambda_1\). Let \(A_n, p_n\) be as in Theorem 1, \(u_n\) the solution of problem \(((\mathcal{P}_n))\) given by Theorem 1, for any \(n \in \mathbb{N}\), \(A\) an operator belonging to \(\mathcal{E}(c_1,c_2)\) and \(p\) a function in the class \(\mathcal{F}(a_1,a_2,a_3,r)\). Moreover, let us assume \((H1), (H2), (H3), (H4)\) and the following hypotheses

\begin{enumerate}
\item[(H5)] \(\psi_n \rightharpoonup \psi\) weakly in \(H^1(\Omega)\), as \(n \to \infty\);
\item[(H6)] \(A_n \rightharpoonup A\), as \(n \to \infty\);
\item[(H7)] \(p_n(x,v(x)) \to p(x,v(x))\) as \(n \to \infty\) a.e. in \(\Omega\) and uniformly on bounded set of \(H^1_0(\Omega)\);
\item[(H8)] \(\exists M > 0\) such that \(||A_n \psi_n||_{L^r(\Omega)} \leq M, \; \forall n \in \mathbb{N}\).
\end{enumerate}

Then there exists a subsequence of \((u_n)_n\) weakly converging in \(H^1_0(\Omega)\) to a function \(u\) which is a nontrivial non-negative solution of problem \((\mathcal{P})\).

**Remark 5.** It is easy to check that problem \((\mathcal{P})\) has a Mountain Pass type solution in a quite analogous way as \(((\mathcal{P}_n))\). Then, from Theorem 2, we can deduce that either the Mountain Pass type solutions \((u_n)_n\) of problems \(((\mathcal{P}_n))_n\) converge, up to subsequence, to a solution of \((\mathcal{P})\) which still is of Mountain Pass type, or problem \((\mathcal{P})\) has at least two different solutions.

**Remark 6.** Hypotheses \((H2)\) and \((H5)\) assure that \(\psi\) is not identically zero.

**Remark 7.** In \((H7)\) we require that

there exists a set \(N\) of zero measure such that \(\forall x \in \Omega \setminus N,\)

\(\forall B\) bounded set of \(H^1_0(\Omega)\) and \(\forall \eta > 0\) there exists \(\bar{\pi} = \bar{\pi}(\eta,x,B)\)

such that \(\forall n \geq \bar{\pi}\) we have \(|p_n(x,v(x)) - p(x,v(x))| < \eta, \; \forall v \in B\).
This assumption holds, in particular, if

\[ p_n(x, \xi) \to p(x, \xi) \]

as \( n \to \infty \), a.e. in \( \Omega \) and uniformly on bounded intervals of \( \mathbb{R} \) (see [2], Remark 4.2).

**Remark 8.** Hypothesis \((H8)\) is not empty. Indeed, it is enough to take \( \tilde{v} \) in Remark 3 sufficiently regular. In order to have \((H5)\), we can assume that the sequence \((\mu_n)\) of Remark 3 is converging.

First of all, we will prove that the sequence \((u_n)\) is bounded from above and from below in \( H^1_0(\Omega) \). Then, denoting by \( u \) the weak limit of a subsequence of \((u_n)\) in \( H^1_0(\Omega) \), we will introduce the following auxiliary problem

\[
(A_n) \quad \begin{cases}
\pi_n \in H^1_0(\Omega), & \pi_n \leq \psi_n \text{ in } \Omega \\
\langle A_n \pi_n, v - \pi_n \rangle - \lambda \int_\Omega u(x)(v - \pi_n)(x)dx \\
& \geq \int_\Omega p(x, u(x))(v - \pi_n)(x)dx \\
\forall v \in H^1_0(\Omega), & v \leq \psi_n \text{ in } \Omega.
\end{cases}
\]

After proving that the sequence \((\pi_n)\) of solutions of problems \((A_n)\) weakly converges in \( H^1_0(\Omega) \) to a solution \( \bar{\pi} \) of the following problem

\[
(A) \quad \begin{cases}
\bar{\pi} \in H^1_0(\Omega), & \bar{\pi} \leq \psi \text{ in } \Omega \\
\langle A\bar{\pi}, v - \bar{\pi} \rangle - \lambda \int_\Omega u(x)(v - \bar{\pi})(x)dx \\
& \geq \int_\Omega p(x, u(x))(v - \bar{\pi})(x)dx \\
\forall v \in H^1_0(\Omega), & v \leq \psi \text{ in } \Omega,
\end{cases}
\]

we will conclude the proof of Theorem 2 by showing that \( u = \bar{\pi} \).

**Proof.** (of Theorem 2) Let us proceed by steps.

**Step 1.** There exists \( K_5 > 0 \) such that \( |u_n| \geq K_5 \), for any \( n \in \mathbb{N} \).

**Proof.** Let us fix \( n \in \mathbb{N} \). As \( u_n \) is a solution of \((P_n)\), taking \( v \equiv 0 \) as test function, one has

\[
\langle A_n u_n, u_n \rangle - \lambda \int_\Omega u_n^2(x)dx \leq \int_\Omega p_n(x, u_n(x))u_n(x)dx. \quad (24)
\]

By (24), (3), \((F2)\) and the compact embedding of \( H^1_0(\Omega) \) into \( L^{s+1}(\Omega) \) one gets

\[
\tilde{c}_1 |u_n|^2 \leq \int_\Omega p_n(x, u_n(x))u_n(x)dx \\
\leq a_1 |u_n|_{L^1(\Omega)} + a_2 |u_n|_{L^{s+1}(\Omega)}^{s+1} \leq \tilde{a}_1 |u_n| + \tilde{a}_2 |u_n|^{s+1},
\]
where \( \tilde{a}_1, \tilde{a}_2 \) are positive constants independent of \( n \).

Step 1 follows from the fact that \( s > 1 \).

**Step 2.** There exists \( K_6 > 0 \) such that \( \|u_n\| \leq K_6 \), for any \( n \in \mathbb{N} \).

**Proof.** Let us fix \( n \in \mathbb{N} \). By Step 10 of Theorem 1, we have that there exists \( (u_{n_k}^\epsilon) \) such that \( u_{n_k}^\epsilon \to u_n \) weakly in \( H_0^1(\Omega) \) as \( \epsilon_k \to 0 \). The weakly lower semicontinuity of \( \| \cdot \| \) and Step 8 of Theorem 1 yield

\[
K_3 \geq \liminf_{\epsilon_k \to 0} \| u_{n_k}^\epsilon \| \geq \| u_n \|,
\]

where \( K_3 \) is independent of \( n \).

**Step 3.** The sequence \( (\pi_n)_n \) of solutions of the auxiliary problems \( (A_n) \) weakly converges in \( H_0^1(\Omega) \) to a solution \( \pi \) of problem \( (A) \).

**Proof.** By Lions-Stampacchia Theorem (see [5]), for any fixed \( n \in \mathbb{N} \) there exists a unique solution \( \pi_n \) of problem \( (A_n) \). Arguing as in Theorem 1 of [3] with \( g_n(\cdot) = \lambda u(\cdot) + p(\cdot, u(\cdot)) \), we obtain that

\[
\pi_n \to \pi \text{ weakly in } H_0^1(\Omega),
\]

as \( n \) goes to \( \infty \), where \( \pi \) is a solution of problem \( (A) \).

**Remark 9.** Let us observe that Boccardo and Capuzzo Dolcetta in [3] require that \( g_n \in L^2(\Omega) \) and \( \|A_n \psi_n\|_{L^2(\Omega)} \) is bounded for any \( n \in \mathbb{N} \), while we assume the same kind of hypotheses in \( L^{2^*}(\Omega) \). We remark that one can use the same arguments of [3] in case that \( L^2(\Omega) \) is replaced by \( L^p(\Omega) \), where \( p \in [1, 2^*) \).

**Step 4.** There exists a subsequence of \( (u_n)_n \) weakly converging in \( H_0^1(\Omega) \) to a function \( u \) which is a nontrivial non-negative solution of problem \( (P) \).

**Proof.** First of all, let us prove that

\[
p_n(\cdot, u_n(\cdot)) \to p(\cdot, u(\cdot)) \text{ in } L^{2^*/(2^* - 1)}(\Omega). \tag{25}
\]

By Step 2, up to subsequences, \( u_n \to u \) in \( L^{2^*/(2^* - 1)}(\Omega) \), \( u_n \to u \) a.e. in \( \Omega \) as \( n \) goes to \( \infty \) and there exists \( h \in L^{2^*/(2^* - 1)}(\Omega) \) such that

\[
|u_n(x)| \leq h(x) \text{ a.e. } x \in \Omega, \text{ for any } n \in \mathbb{N}. \tag{26}
\]

Therefore, by \( (F1) \) and \( (H7) \), one has

\[
p_n(x, u_n(x)) \to p(x, u(x)) \tag{27}
\]

a.e. in \( \Omega \), as \( n \) goes to \( \infty \). Hence (25) follows by (27), (26) and \( (F2) \).

Now let us prove that

\[
\|u_n - \pi_n\| \to 0 \text{ as } n \text{ goes to } \infty. \tag{28}
\]

By \( (B3) \) and using the fact that \( u_n \) is solution of \( (P_n) \) with \( v = \pi_n \) and \( \pi_n \) is solution of \( (A_n) \) with \( v = u_n \), we obtain:
\[ \|u_n - \bar{u}_n\|^2 \leq \frac{1}{c_1} (A_n (u_n - \bar{u}_n), u_n - \bar{u}_n) \]
\[ \leq \frac{\lambda}{c_1} \int_{\Omega} (u_n - u)(u_n - \bar{u}_n)dx + \frac{1}{c_1} \int_{\Omega} \left[ p_n(x, u_n(x)) - p(x, u(x)) \right] (u_n - \bar{u}_n)dx. \] 

(29)

By (25), Steps 2, 3 and the continuous embedding of \( H_0^1(\Omega) \) into \( L^{2^*}(\Omega) \), taking the limit in (29) as \( n \) goes to \( \infty \), one gets (28).

On the other hand (28) and Step 3 yield
\[ u_n \rightharpoonup \bar{u} \quad \text{weakly in} \quad H_0^1(\Omega). \]

Then \( u \equiv \bar{u} \), so \( u \) is a solution of problem \((P)\).

Finally, arguing as in Step 10 of Theorem 1, one can prove that \( u \geq 0 \) and \( u \not\equiv 0 \).

\( \square \)

References


A stability result for Mountain Pass type