CRITICAL POINT METHODS FOR INDEFINITE NONLINEAR ELLIPTIC EQUATIONS AND HAMILTONIAN SYSTEMS

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Introduction

There is much current interest in nonlinear indefinite problems: by searching (possibly positive) solutions to equations of the kind $Lu = p(x, u)$ in the suitable Sobolev space, depending on the boundary conditions, one can treat several problems arising in differential geometry and mathematical biology. Also in the study of Hamiltonian Systems there is a rich literature dealing with indefinite problems.

We will deal at first with elliptic equations of the type

\[ (P) \begin{cases} -\Delta u(x) - \lambda u(x) = p(x, u) & \text{in } \Omega \subset \mathbb{R}^N, \\ u(x)|_{\partial \Omega} = 0 \end{cases} \]

and discuss some results obtained via variational methods, in particular minimax theorems, in case that $p$ is superlinear, up to the critical exponent, as well as asymptotically linear.

The variational approach consists of finding the solutions of $(P)$ by searching the critical points of a suitable associated functional. More precisely the natural energy functional for $(P)$ is given by

\[ I(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \frac{\lambda}{2} \int_{\Omega} u^2 - \int_{\Omega} P(x, u) \quad u \in H^1_0(\Omega) \quad (0.1) \]

where $P(x, u) = \int_0^u p(x, \xi)d\xi$. The Sobolev space where the functional lives is chosen according to the boundary conditions, in the present case the Dirichlet boundary condition suggests the space $H^1_0(\Omega)$. We are dealing with a functional which is neither bounded from below nor from above, so the main goal is to prove that $I$ satisfies a minimax theorem, i.e. it has a critical point of minimax kind (neither a minimum, nor a maximum). Usually one has to show that the functional has a certain geometric structure and that the set of critical points has a kind of compactness property,
which is often the main difficulty to overcome. This is summarized in the following Palais–Smale (PS) condition:

\[
\text{The functional } I \in E \text{ satisfies the Palais–Smale condition if any sequence } u_n \in E \text{ such that } I(u_n) < C \text{ and } \|I'(u_n)\| \to 0 \text{ admits a strongly converging subsequence.}
\]

The nonlinearities we will treat, neither homogeneous nor positive, hide a range of difficulties. First of all let us remark that, when dealing with homogeneous functions, even changing sign, it is very simple to relate the functional to its derivative, so the proof of the PS condition becomes easier. If \( p \) is not homogeneous and changes sign, it is necessary to decompose it as the sum of its positive and negative part, then treat them separately. To show that the functional has the right geometric structure requires some additional hypotheses, with respect to the case of homogeneous functions. Another difficulty arises if one considers an arbitrary choice of \( \lambda \). Indeed, if \( \lambda < \lambda_1 \), where \( \lambda_1 \) is the first eigenvalue of the Laplacian in \( H^1_0(\Omega) \), the quadratic part of the functional is coercive and it is easy to prove that the functional possesses a Mountain Pass structure (see Section 2.2), independently of the change of sign of the nonlinearity. Everything changes if \( \lambda \geq \lambda_1 \): as it will be shown with details in Section 2.3, if \( \lambda \in [\lambda_k, \lambda_{k+1}] \) with \( k > 1 \), one has to exploit the variational characterization of the eigenvalues of the Laplacian in order to show that the functional has a Linking structure. The case \( \lambda = \lambda_1 \) is very interesting because one can prove that \( I \) either possesses a Mountain Pass or a Linking structure, according to the hypotheses required on the function \( p \). The solution found by means of Mountain Pass theorem are positive, while those of Linking type may change sign.

A meaningful example of an application of indefinite problems to some problems arising in biology, in particular population dynamics, is the following (for further details see \([32, 29]\) and the references therein):
let us consider the parabolic model

\[
(PM) \quad \begin{cases} 
    u_t - \Delta u = \lambda u + a(x)|u|^p \quad (x, t) \in \Omega \times (0, \infty), \\
    u(\cdot, t)|_{\Gamma} = 0 \quad t > 0, \\
    u(\cdot, 0) = u_0 \geq 0 \quad \text{in} \quad \Omega
\end{cases}
\]

One is interested in the equilibrium states of \((PM)\), which are provided by the solution to the elliptic equation

\[
(EE) \quad -\Delta u = \lambda u + a(x)|u|^p \quad \text{with Dirichlet boundary conditions.}
\]

In mathematical biology the weight function \(a\) is usually taken to be positive, but it has a biological meaning even in the indefinite case. The density of a single species inhabiting \(\Omega\) at time \(t\) is given by the function \(u(\cdot, t)\), \(\lambda\) is the growth rate, \(u_0\) is the initial population density and \(a\) represents the effects of crowding in the species in \(\Omega_-\) and the symbiosis effects due to the intraspecific cooperation in \(\Omega_+\), where \(\Omega_{\pm} = \{x \in \Omega: a^\pm(x) > 0\}\). In other words the growth of a population has a positive aspect, which is symbiosis, and a negative one, that is an excess of crowding which could lead to a lack of the resources necessary to the population to survive. Observe that we are dealing with a single specie, so it could sound strange to talk about symbiosis, which occurs when there are two or more species in the same domain. In this case we can think of cooperation, i.e. symbiosis, for example between the old and the young individuals.

Results about existence, multiplicity and stability of positive solutions can be found in \([2, 3, 16, 17, 48, 49]\) and the references therein. From the mathematical results carried out in the above papers it can be shown that \((PM)\) possesses a stable positive steady state if and only if

\[
\int_\Omega ae_1^{p+2} < 0,
\]

where \(e_1\) is a positive eigenfunction related to \(\lambda_1\), and that \((EE)\) does not admit a stable positive solution if \(\lambda \leq \lambda_1\). From the biological point of view this means that, if \(\lambda \leq 0\) or \(\lambda > 0\) but the habitat is not sufficiently large so that \(\lambda > \lambda_1\), then the species
u does not have enough room to avoid extinction (as a consequence of its variational
characterization, see page 3, $\lambda_1$ is smaller as the domain $\Omega$ becomes bigger). In [32]
it was proved that the positive stable solution is unique for $\lambda \in (\lambda_1, \Lambda^*)$, and this
is surprising because for each $\lambda \in (\lambda_1, \Lambda^*)$ there are at least two positive solutions,
so only one is stable. The uniqueness of the stable solution is a universal property
independent of the shape of $\Omega$ and of the nodal behaviour of $a$. So, if $\lambda \leq \lambda_1$, the
unique non-negative stable solution will be the trivial one. Therefore the species will
go towards extinction, and this can be avoided if the initial population lies on the
stable manifold of some of these positive states.

On the other hand a suitable model for the case of two or more different species living in
the same domain could be a Hamiltonian system. The existence of a periodic solution
is meaningful in mathematical biology for it represents a kind of clock: like seasons,
many natural phenomena are periodic (see Chapter 4 for Hamiltonian Systems and
periodic solutions).

It is well known that, under a suitable assumption of superlinear subcritical growth at
infinity for $p$ in the $u$ variable, and in case that $P(x, \xi) = \int_0^\xi p(x, s)ds$ is positive, ($P$)
has positive solutions if $\lambda < \lambda_1$. These solutions are found through the Mountain-Pass
Theorem by Ambrosetti-Rabinowitz [8], while, for $\lambda \geq \lambda_1$, Rabinowitz proved (see [51])
the existence of Linking type solutions, which in general may change sign. Always in
the case that $P$ is positive, an extensive literature has been developed if the growth
$\beta$ of $P$ is critical, that is the exponent of its superquadratic behaviour at zero and at
infinity is given by the Sobolev critical exponent $\beta = 2^* = \frac{2N}{N-2}$ ($N \geq 3$).

The motivation which led to investigate elliptic problems involving the critical ex-
ponent comes from the fact that problems with lack of compactness can be easily
found in geometry and physics. The problem of describing the possible scalar curva-
tures associated with Riemannian metrics on a given manifold (in particular Yamabe’s
problem) is a very famous example:

let $(M, g)$ be a Riemannian manifold of dimension $\geq 3$ with scalar curvature $k$ and let
K be a given function on M. The problem of the scalar curvature consists in finding a new metric $g^*$ on M such that $K$ is the scalar curvature of $g^*$ and $g^*$ is pointwise conformal to $g$ (i.e. $g^* = u^{\frac{4}{N-2}}g$ for some $u > 0$ on $M$). The equation generated by this problem is

$$\frac{4(N-1)}{N-2} \Delta u - ku + K u^{\frac{N+2}{N-2}} = 0$$

(0.2)

In [48, 49] the author treats the more general equation $\Delta u + \lambda u + hw^p = 0$, either in case that $h \geq 0$ or it can change sign on $M$, and searches positive solutions. So after constructing the equation, which requires a deep knowledge of differential geometry, one can free the problem from geometry and treat it only from the analytical point of view. In other words one can consider the general nonlinear equation (P) on a Riemannian manifold $M$, with $p(x, u) = -hu^p$ and find positive solutions with Dirichlet or Neumann boundary conditions.

Other results concerning the analysis on manifolds and the problem of scalar curvature can be found in [12],[30],[41]. Moreover a very strong dependence on the shape of the domain, a kind of “sensibility” with respect to very small changes of the domain, is a peculiarity of the critical exponent, and some physical phenomena show this aspect too. Always in the setting of the critical exponent we recall the pioneering paper by Brezis and Nirenberg (see [21]) where it was proved that there exists $\lambda^* \geq 0$ such that, if $\lambda^* < \lambda < \lambda_1$, then there exists a positive solution to (P). Some existence results, for $\lambda > \lambda_1$, were established in [23],[24], where $P(x, \xi) = |\xi|^{2^*}$. When we start considering indefinite nonlinearities of the kind $P(x, \xi) = W(x)|\xi|^{2^*}$, with $W$ changing sign, the only results at our knowledge are due to Alama and Tarantello (see Theorem 4.1 of [2]).

As it will be shown in Sections 2.4, 2.5 we are able to treat the case $P(x, \xi) = W(x)F(\xi)$ with $F$ having critical growth both at 0 and at infinity.

On the other hand, in the framework of the subcritical growth (i.e. $\beta < 2^*$), some important results were obtained in [16],[17], in case that $P(x, \xi) = W(x)F(\xi)$, with $W$ changing sign, $F(\xi) = |\xi|^\beta$ as well as for more general elliptic operators including
the Laplacian, and for more general choices than the power function \( |\xi|^\beta \). For a general elliptic operator a problem like \((P)\) cannot be attacked with variational techniques, so in [17] the authors used degree theory.

In the same period Alama and Tarantello stated in [2] some more general results (always for the pure Laplacian case) in order to find positive solutions of \((P)\) in case that \( \lambda_1 < \lambda < \Lambda^* \), with \( \Lambda^* \) suitably near to \( \lambda_1 \), possibly changing sign solutions for any \( \lambda \neq \lambda_k \) and multiplicity results. In [16] the authors found a necessary and sufficient condition for the solvability of \((P)\) with \( F(\xi) = |\xi|^p \). This condition will appear when we will treat the main results of this thesis. In the following, many other interesting papers were devoted to the existence or nonexistence of (possibly infinitely many) solutions of problem \((P)\), either in the case that \( \lambda \in [\lambda_1, \Lambda^*] \), or also for every \( \lambda \), in case the nonlinear term satisfies some oddness assumption (see [3],[1], [14],[2],[13]).

A recent result concerning all the possible choices of \( \lambda \) different from any eigenvalue of the Laplacian, under some rather general assumptions, has been stated in [53]. The main idea is to consider a sequence of truncated problems and apply a local Linking Theorem. Following the line of [53], and assuming that \( W \) is a changing sign homogeneous polynomial function of degree two, a result of existence of a nonzero solution is given in [28] in case \( \lambda \neq \lambda_k, \forall k \). An existence result result concerning a more general equation where the linear term \( \lambda u \) is multiplied by a changing sign weight function can be found in

There is a rich literature dealing with asymptotically linear problems too, either with nonlinearities which are indefinite or definite in sign. We first recall the paper of Amann and Laetsch [4], in which the positivity is essential in order to apply the theory of ordered Banach spaces. If \( f \) is allowed to change sign we find in the papers of Hess [38], Ambrosetti-Hess [6] and Hess-Kato [39] some results of existence via topological degree arguments.

Firstly we point out that the condition

\[ 0 < \theta F(x, s) \leq f(x, s)s \quad \text{for } \theta > 2, \ |s| > M \ \text{sufficiently large}, \quad (0.3) \]
obviously does not hold when dealing with asymptotically linear nonlinearities, for it implies that $f$ is superlinear. On the other hand (0.3) is crucial, because it is known that it guarantees that every (PS) sequence is bounded. When dealing with asymptotically linear functions, the existence of unbounded Palais–Smale sequences cannot be excluded (for more details see section 3.4).

Existence results on bounded domains, of (not necessarily positive) solutions have been obtained via variational methods in [26]. Let us finally mention that problem ($P$) has been recently studied also on $\mathbb{R}^N$ in [62],[58],[40],[43], assuming in an essential way that $f$ has to be positive.

One of the physical motivations for considering asymptotically linear problems arises from the study of guided modes of an electromagnetic field in a nonlinear medium, satisfying some suitable constitutive assumptions (see, for example [56], [57]). For example, nonlinearities of the form

$$f(s) = \frac{|s|^2}{1 + \gamma |s|^2} s, \quad \gamma > 0$$

(0.4)

were found to describe the variation of the dielectric constant of gas vapors where a laser beam propagates, and those of the form

$$f(s) = \left(1 - \frac{1}{e^{\gamma |s|^2}}\right) s$$

(0.5)

were used in the context of laser beams in plasma (see [59] and the references therein). On the other hand, the change of sign of the nonlinearity is meaningful for example in selection–migration models in population genetics (see [31, 22, 32, 29]).

Variational methods are very diffused also when searching periodic solutions to Hamiltonian Systems and this kind of approach was introduced by Rabinowitz in a famous paper of 1978 (cfr. [50]). In case of second order Systems, that is equations of the kind $\ddot{x} + U'(x) = 0$, periodic solutions can be found by searching the critical points of the natural associated functional in the Sobolev space $H_T^1 = \{v \in H^1([0,T]; \mathbb{R}^N) : v(0) = v(T)\}$. We are interested in the system

$$\begin{align*}
(HS) \quad \ddot{x} + A(t)x + b(t)V'(x) &= 0,
\end{align*}$$
where $A(t)$ is a continuous $T$–periodic (for some fixed $T > 0$) matrix valued function, $b(t)$ is a continuous $T$–periodic real function and $V \in C^2(\mathbb{R}^N, \mathbb{R})$. Recently many authors have studied the problem of existence and multiplicity for periodic and subharmonic solutions of $(HS)$ either in case that the quadratic term is identically zero or in case that it is definite in sign. However there is no general result when $A(t)$ is indefinite in sign: as far as the author knows, the only existence results have been stated in [10], in case that the matrix $A$ satisfies the integral condition
\[
\int_0^T \langle A(t)\xi, \xi \rangle \, dt > 0 \quad \forall \xi \in \mathbb{R}^N, \ |\xi| = 1
\]
and $b(t)$ is such that $\int_0^T b(t) \, dt > 0$. The case $A(t) \equiv 0$ and $B$ changing sign presents already some difficulties, above all a lack of compactness. Only at the end of 80’s there were found some results assuring existence of $T$-periodic solutions under some suitable conditions on the negative part of $B$ (see [15, 33, 42, 50]). In some cases, as in [42], imposing some homogeneity and convexity assumptions on the potential $V$, the method used is the well–known duality principle by Clarke and Ekeland, introduced at the end of 70’s for first order Systems. Also homogeneity helps to simplify proofs: if $V$ is homogeneous, under the only hypotheses that $b$ has zero mean on $[0, T]$, and considering a superquadratic $U$, Palais–Smale condition becomes an almost trivial matter. Adding a positive quadratic term $A(t)x$ some existence results were obtained in [9, 27, 34]. In Chapter 4 we will present a result of existence of subharmonic solutions (i.e. $kT$–periodic with minimal period $kT$) under some evenness condition imposed on $V$.

In Chapter 2 we will show some existence results for problem $(P)$ in case that the nonlinearity is superlinear, including the critical exponent. In the spirit of [2], for $\lambda \leq \lambda_1$, we find a positive solution of Mountain Pass type without homogeneity assumption at infinity on $f$. We exploit a technique introduced by Girardi and Matzeu in [33] for Hamiltonian Systems to prove the Palais Smale condition. The major idea appearing in [33] was to weak the homogeneity assumption on $f$ by requiring that the potential related to $f$ should differ from a homogeneous function by a quadratic term which is in some sense controlled by the negative part of $W$. Furthermore, when
\( \lambda \geq \lambda_1 \), we extend the results by Rabinowitz for Linking type solutions of \((P)\) in case that \( P(x,\xi) = W(x)F(\xi) \) with \( W \) changing sign and \( F \) having superquadratic growth \( \beta < 2^* \). On the other hand we also consider, for the same choice of \( W \) changing sign, the case of critical growth. Suitably reinforcing the assumptions given for the subcritical growth, we obtain some existence results for any \( \lambda \geq \lambda_1 \). In this case we are able to prove, as well as in the subcritical case, the boundedness of the Palais–Smale sequences \( \{u_n\} \), but it is well known that the lack of compactness of the embedding of \( H_0^1 \) into \( L^{2^*} \) does not allow to prove the compactness of \( \{u_n\} \) in the critical case.

However, using some known techniques, it is possible to estimate the level \( c \) of the associated functional \( I \) where the Palais–Smale condition fails. Indeed we will prove that, under a suitable level, the PS condition is restored (see Section 2.4). At this stage, using the same Linking structure of the subcritical case, we are able to deduce the existence of at least one solution if we show that its energy level is less than some suitable constant \( c^* \). In order to prove this estimate we were inspired by the arguments carried out in [23]. We point out that the same geometrical conditions which provide a ”Linking structure” for the associated functional \( I \) play a crucial role to show that \( I \) is actually controlled from above by the level \( c^* \).

In Chapter 3 we will treat the asymptotically linear case, with \( \lambda = 0 \), and we extend a result obtained in [61]. Here we are able to treat more general nonlinearities, and our main improvement consists of allowing \( p \) to change sign. Moreover, if \( p(x,s) \geq 0 \) for \( s \geq 0 \) and \( p(x,s) \equiv 0 \) for \( s \leq 0 \), the result of [61] is included in our present result. While in [61] the results are derived using a particular version of the Mountain Pass Theorem, for our result we are able to prove, using the classical Theorem of Ambrosetti–Rabinowitz (see [8]), that problem \((P)\) has always a positive solution under the required assumptions. We used some fundamental properties of the principal eigenvalues of a linear operator with indefinite weight function due to Hess and Kato [39] in order to prove that, under some suitable assumptions, we are able to exclude the presence of unbounded Palais–Smale sequences, and thus to obtain an existence result for \((P)\). While in the superquadratic case (i.e. when (0.3) is satisfied)
all the Palais–Smale sequences are bounded, this fact is not anymore true when the nonlinearity is asymptotically linear. So we establish a relationship between the existence of unbounded Palais–Smale sequences and of positive solutions to an eigenvalue problem. Finally, in section 3.6, in order to illustrate the meaning of our assumptions, we give some examples of nonlinearities for which the existence result in section 3.5 applies.

Chapter 4 will be devoted to a result of existence of periodic solutions in case of a changing sign potential Hamiltonian System. The technique used is based on the consideration of the Nehari’s manifold $M$, suitably connected to the functional $I$ associated with problem (P) (see also [7], where the manifold $M$ was used for the first time in the variational approach to Hamiltonian systems). Precisely the symmetry assumptions on the potential allow to use some constrained minimum arguments and a result of the Ljusternik-Schrinelmann cathegory theory, in order to find periodic solutions, subharmonic and homoclinic solutions to (P), starting by the critical points of $I$ on $M$. In the particular case that $A(t)$ is negative definite, we also find the existence of subharmonic solutions, which have some symmetry properties, and of an homoclinic solution, obtained by a limit procedure starting from a suitable translation of these subharmonics. A future extension of this result will be the existence for periodic solutions for the system $\ddot{x}(t) + \lambda x + b(t)V'(x) = 0$, where $\lambda$ is a real parameter.
Chapter 1
Preliminaries

We briefly introduce the most significant tools that will be used.

**Theorem 0.1  (The Mountain Pass Theorem [8],[51])**

Let $E$ be a real Banach space and $J \in C^1(E, \mathbb{R})$. Suppose $J$ satisfies (PS) condition, $J(0) = 0$ and the following assumptions

(J1) there exist constants $\rho, \alpha > 0$ such that $J|_{\partial B_{\rho}} \geq \alpha$, and

(J2) there exists an $e \in E \setminus \partial B_{\rho}$ such that $J(e) \leq 0$.

Then $J$ possesses a critical value $c \geq \alpha$ which can be characterized as

$$c = \inf_{g \in \Gamma} \max_{u \in g([0,1])} J(u),$$

where

$$\Gamma = \{g \in C([0,1], E): g(0) = 0, g(1) = e\}.$$

Actually the above version, which is the standard one, of the Mountain Pass Theorem will be used in Chapter 3 in order to prove the existence result for equations with asymptotically linear nonlinearities, while, in Chapter 2, Section 2.2, we shall use a slightly different version of the Mountain Pass Theorem, which generalizes the standard one by weakening condition (J1). To be more precise, setting

$$X = \{D \subset E: D \text{ is open}, 0 \in D, e \not\in \overline{D}\}$$

if one requires :
(J1*) there exists \( B \in X \) such that \( J|_{\partial B} \geq 0 \), and

(J2*) there exists an \( e \in E \setminus \overline{B} \) such that \( J(e) \leq 0 \).

Then \( c \) defined in theorem (0.1) is a critical value of \( J \) with \( c \geq 0 \). If \( c = 0 \) there exists a critical value of \( J \) on \( \partial B \).

Condition (J1*) requires that 0 is a (not strict) local minimum for our functional, but not that the functional must be strictly positive on the boundary of a ball \( B \). It is actually a weaker condition with respect to (J1), indeed let us look at the following example:

Let \( v \in l^1 \), \( v = \{v_n\}_{n \in \mathbb{N}} \), and let \( J \) be the functional \( J(v) = \sum_{n=1}^{+\infty} \frac{|v_n|}{n} \). Then \( J \) is always positive, but \( \inf_{\|v\|=\varepsilon} J(v) = 0 \), for any \( \varepsilon > 0 \). Taking \( v = \varepsilon e_k \), one gets \( J(\varepsilon e_k) = \frac{\varepsilon}{k} \to 0 \) as \( k \to +\infty \). So we cannot say that \( J|_{\partial B_{\varepsilon}} > \alpha > 0 \) for no choice of a ball \( B_{\varepsilon} \) of radius \( \varepsilon > 0 \).

**Theorem 0.2 (The Linking Theorem [51])** Let \( E \) be a real Banach space with \( E = E_1 \oplus E_2 \), where \( E_2 \) is finite dimensional. Suppose \( J \in C^1(E; \mathbb{R}) \) satisfies the Palais Smale condition and the further assumptions

(J3) \( \exists \rho, \alpha > 0 \) such that \( J(v) \geq \alpha \forall v \in E_1: \|v\| = \rho \),

(J4) \( J(v) \leq 0 \ \forall v \in E_2 \),

(J5) \( \exists \tilde{v} \in E_1 \) and \( R > \rho \) such that \( J(v) \leq 0 \forall v \in E_2 \oplus \text{span}\{\tilde{v}\} \) with \( \|v\| \geq R \).

Then \( J \) possesses a critical point \( \overline{u} \neq 0 \) such that

\[
\overline{c} = J(\overline{u}) = \inf_{h \in \Gamma} \max_{v \in \overline{Q}} J(h(v)) \tag{0.1}
\]

where

\[
Q \equiv (B_R \cap E_2) \oplus \{r\tilde{v}: 0 < r < R\}
\]

and

\[
\Gamma = \{h \in C(\overline{Q}, E): h = \text{id on } \partial Q\}.
\]
Chapter 2

Superlinear nonlinearities

2.1 Introduction and main results

Let us consider a semilinear Dirichlet problem of the kind

\[
\begin{cases}
-\Delta u(x) - \lambda u(x) = W(x)F(u) & \text{in } \Omega \subset \mathbb{R}^N, \\
\quad u(x)|_{\partial \Omega} = 0
\end{cases}
\]

where \( \lambda \) is a real parameter, \( F \) is a sufficiently regular function on \( \overline{\Omega} \times \mathbb{R} \), \( W \in C(\overline{\Omega}) \) is a changing sign function and \( \Omega \) is a bounded domain of \( \mathbb{R}^N \), \( (N \geq 2) \) with a smooth boundary \( \partial \Omega \).

Let \( 0 < \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_k \leq \lambda_{k+1} \leq \ldots \) be the sequence of eigenvalues of the operator \( -\Delta \) with respect to the zero boundary conditions on \( \Omega \). Each eigenvalue \( \lambda_k \) has a finite multiplicity, which we choose to be coinciding with the number of its different indexes. So let us call \( X_k \) the \((k\text{-dimensional})\) subspace of the Sobolev space \( H^1_0(\Omega) \) spanned by the eigenfunctions related to \( \{\lambda_1, \ldots, \lambda_k\} \) with \( \lambda_k < \lambda_{k+1} \). As it is well-known, these eigenvalues have a variational characterization, that is:

\[
\lambda_{k+1} = \min_{v \in X_k^+ \setminus \{0\}} \frac{\int_{\Omega} |\nabla v|^2 dx}{\int_{\Omega} v^2 dx} \quad (1.1)
\]

Finally \( f \in C^0(\mathbb{R}) \) and put

\[
F(t) = \int_0^t f(\xi)d\xi \quad \forall t \in \mathbb{R}.
\]

and, for \( N \geq 3 \), \( 2^* = \frac{2N}{N-2} \). In Section 2.2 we will prove Theorem 1.1, which shows an existence result, contained in [47], of a positive solution in case that \( \lambda \leq \lambda_1 \);
Section 2.3 is devoted to an existence result of possibly changing sign solutions when $\lambda \in [\lambda_k \lambda_{k+1}]$, obtained in [37]. Finally Sections 2.4, 2.5 deal with the critical case; more precisely one must show that the Palais-Smale condition holds below a certain level and then estimate the Linking type critical level of the functional. The result obtained in theorem 1.7 (see [37]) concerns the existence of possibly changing sign solutions.

**Theorem 1.1** [47] Let $f$ be a continuous real valued function on $\mathbb{R}$ such that the following conditions hold

\[ f(t)t \geq \beta F(t) \quad \forall t \in \mathbb{R}. \]  
\[ |f(t)| \leq C |t|^\beta - 1 \quad \forall t \in \mathbb{R}, \text{ for some } \beta \in (2, 2^*). \]  
\[ \lim_{u \to 0} \frac{f(u)}{|u|^\beta - 2 \cdot u} = a > 0 \]  
\[ W^-(f(u)u - \beta F(u)) \leq \gamma |u|^2, \quad |u| \geq R, \]  
\[ \text{for some } \gamma \in \left(0, \left(\frac{\beta}{2} - 1\right) (\lambda_2 - \lambda_1)\right) \]  

where $\lambda_2$ is the second eigenvalue of $-\Delta$ in $H_0^1$, and $W^-(x) = -\min\{W(x), 0\}$, $W^- = \max_{x \in \Omega} W^-(x)$, where $W^- = \max\{W^-(x) : x \in \Omega\}$.

Furthermore let us assume

\[ \text{meas}\{x \in \Omega: W(x) = 0\} = 0, \]  
\[ W^+(x) \not\equiv 0, \]  
\[ \int_{\Omega} W(x)e_1^\beta < 0 \]
2.1. INTRODUCTION AND MAIN RESULTS

where $e_1$ is a positive eigenfunction related to $\lambda_1$. Then $(P_\lambda)$ has a positive solution $u_\lambda$ for any $\lambda \in (0, \lambda_1]$.

**Remark 1.2** The exponent $\beta$ appearing in (1.3), (1.4), (1.9) can be different from the one appearing in (1.2), (1.5), (1.6), but in order to simplify the reading of calculations we will assume all the exponents to be equal to $\beta$.

**Remark 1.3** A condition similar to (1.5) was introduced in [33] in the context of periodic solutions of Hamiltonian Systems and plays a crucial role in the proof of Palais Smale condition. Indeed in case that $W^-$ and $(F'(u)u - \beta F(u))$ are both strictly positive (otherwise (1.5) is trivially verified) the inequality gives a relation between the negative part of $W$ and the difference in homogeneity at infinity of the function $F$. For example one can choose $F(u) = a_1 |u|^\beta + a_2 |u|^\theta$, where here $a_1 > 0$ can be chosen as an arbitrary positive number, while $a_2 > 0$ still can be taken as an arbitrary positive number if $\theta < 2$ or as $\gamma$ in (1.5) in case $\theta = 2$.

**Remark 1.4** Note that (1.2) implies, for $\beta \in (2, 2^\ast)$

$$F(t) \geq C|t|^\beta \quad \text{for } |t| > R \text{ sufficiently large.} \quad (1.10)$$

**Theorem 1.5** [37] Let $f, W$ verify assumptions (1.2), (1.3), (1.7), (1.8) and (1.5) with

$$\gamma \in \left(0, \left(\frac{\beta}{2} - 1\right)(\lambda_{k+1} - \lambda)\right) \quad (1.11)$$

Furthermore let us assume that

$$\int_\Omega W(x)F(v(x))dx \geq 0 \quad \forall v \in X_k, \quad (1.12)$$

$$\exists \overline{v} \in X_k^\perp \setminus \{0\} : \int_\Omega W(x)F(v(x))dx \geq C_0 \int_\Omega |v(x)|^\beta dx, \forall v \in X_k \oplus \text{span}\{\overline{v}\}, \quad (1.13)$$

with $\|v\| \geq R$.

Then problem $(P_\lambda)$ admits a nontrivial solution $u$ for any $\lambda \geq \lambda_1$. 

Remark 1.6 It has been proved in [2] and in [16] that, in case \( f(u) = u|u|^{\beta-2} \), (1.9) is a necessary and sufficient condition for the existence of a positive solution to \((P_\lambda)\). So in our case we can state that the solution found in Theorem 1.5 is surely not always positive because (1.9) does not hold.

Theorem 1.7 [37] Let all the assumptions of Theorem 1.5 be satisfied with \( \beta = 2^* \), and let \( N \geq 5 \). Let \( F \) be convex and let us require that

\[
\lim_{\varepsilon \to 0} \varepsilon^{N+2} f\left(s\varepsilon^{\frac{2-N}{2}}\right) = |s|^{2^*-2}s, \quad \text{uniformly w.r. to } s \in \mathbb{R},
\]

(1.15)

\[
\lim_{\mu \to 0} \mu^{\frac{N-2}{4}} \int_{0}^{1} \frac{\rho^2 - 1}{(1 + \rho^2)^{\frac{N}{2}}} \left[ f\left(\frac{\mu^{\frac{2-N}{4}}}{(1 + \rho^2)^{\frac{N}{2}}}\right) - \frac{\mu^{-\frac{N+2}{4}}}{(1 + \rho^2)^{\frac{N}{2}}}\right] \rho^{N-1} d\rho = 0.
\]

(1.16)

Then problem \((P_\lambda)\) admits a nontrivial solution \( u \).

Remark 1.8 Let us point out that, in general, one cannot exclude that the solution \( u \) in Theorems 1.5, 1.7 can change its sign, differently from the case of Mountain Pass solutions.

Remark 1.9 If one chooses \( f(t) = t|t|^{2^*-2} \) then the "technical" conditions (1.15), (1.16) as well as, obviously, all the other conditions (1.2)-(1.5) (even in case that \( 2^* \) is replaced by \( \beta \in (2, 2^*) \)), are indeed satisfied.

Remark 1.10 A condition of the same kind as (1.16) appeared in [21], section 2 (2.44).

We are going to find the solution to problem \((P_\lambda)\) by looking for (nontrivial) critical points of the functional

\[
I(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \frac{\lambda}{2} \int_{\Omega} u^2 dx - \int_{\Omega} W(x)F(u)dx \quad u \in H^1_0(\Omega).
\]

(1.17)
2.2. The case $\lambda \leq \lambda_1$

We will prove Theorem 1.1 and the idea is to apply to the functional $I$ the Mountain Pass Theorem (cfr. Chapter "Preliminaries", and [8]). In the following proof $f(u)$ will be considered only for $u \geq 0$ and extended as an odd function for $u < 0$. It is not restrictive, since the solution found will be actually positive. A basic proposition is the following result, in which we consider the case $\lambda = \lambda_1$, because if $\lambda < \lambda_1$ the argument is the same.

**Proposition 2.1** Under assumptions (1.2), (1.3), (1.5), (1.7), the functional $I(u)$ satisfies the Palais Smale condition on $H^1_0$, that is

$$\text{any } \{u_n\} \subset H^1_0, \text{ such that } \{I(u_n)\} \text{ is bounded and } \{I'(u_n)\} \to 0, \quad (2.1)$$

possesses a converging subsequence.

**Remark 2.2** Observe that, for the proof of the Palais-Smale condition, assumption (1.9) is not necessary.

**Proof**

Let $\{u_n\} \subset H^1_0$ be a Palais Smale sequence, namely

$$\exists c_1, c_2: c_1 \leq \frac{1}{2} (\|\nabla u_n\|^2_2 - \lambda_1 \|u_n\|^2_2) - \int_\Omega W(x)F(u_n) \leq c_2 \forall n \in \mathbb{N} \quad (2.2)$$

$$\sup_{\varphi \in H^1_0, \|\varphi\|=1} \left\{ \int_\Omega \nabla u_n \nabla \varphi - \lambda_1 \int_\Omega u_n \varphi - \int_\Omega W(x)f(u_n)\varphi \right\} \to 0 \text{ as } n \to \infty \quad (2.3)$$

We are going to prove that $u_n$ is bounded in $H^1_0$. By assumption (1.2) from (2.2) and (2.3) we have, for some constant $c_R > 0$ depending on the number $R$ appearing in (1.5),

$$\int_\Omega |\nabla u_n|^2 = \lambda_1 \int_\Omega u_n^2 + \int_\Omega W(x)f(u_n)u_n - \varepsilon_n (\|\nabla u_n\|^2_2) \geq \lambda_1 \int_\Omega u_n^2 + \beta \int_\Omega W^+(x)F(u_n) - \gamma \int_{\Omega \cap \{|u| > R\}} u_n^2 +$$

$$\geq \lambda_1 \int_\Omega u_n^2 + \beta \int_\Omega W^+(x)F(u_n) - \gamma \int_{\Omega \cap \{|u| > R\}} u_n^2 +$$
\[
-\beta \int_{\Omega \cap \{|u| > R\}} W^-(x) F(u_n) + c_R - \varepsilon_n (\|\nabla u_n\|_2) \geq
\]
\[
\geq \lambda_1 \int_{\Omega} u_n^2 + \beta \int_{\Omega} W(x) F(u_n) - \gamma \int_{\Omega} u_n^2 + c_R - \varepsilon_n (\|\nabla u_n\|_2) \geq
\]
\[
\geq \lambda_1 \int_{\Omega} u_n^2 + \beta \left( \frac{1}{2} \int_{\Omega} |\nabla u_n|^2 - \frac{\lambda_1}{2} \int_{\Omega} u_n^2 - c_2 \right) - \gamma \int_{\Omega} u_n^2 + c_R - \varepsilon_n (\|\nabla u_n\|_2),
\]
where \(\varepsilon_n\) is an infinitesimal sequence of positive numbers. So we get

\[
\left( \frac{\beta}{2} - 1 \right) \left( \int_{\Omega} |\nabla u_n|^2 - \lambda_1 \int_{\Omega} u_n^2 \right) - \gamma \int_{\Omega} u_n^2 - \varepsilon_n (\|\nabla u_n\|_2) \leq \beta c_2 - c_R. \tag{2.5}
\]

Let us split now \(u_n\) into the sum \(u_n = k_n e_1 + \tilde{u}_n\), where \(\tilde{u}_n \in (X_1)^\perp\), and \(X_1 \equiv [e_1]\).

Observe that, by the definition of \(\lambda_1\) and by the variational characterization of \(\lambda_2\),

\[
\int_{\Omega} |\nabla u_n|^2 - \lambda_1 \int_{\Omega} u_n^2 = \int_{\Omega} |\nabla \tilde{u}_n|^2 - \lambda_1 \int_{\Omega} \tilde{u}_n^2 + k_n^2 \int_{\Omega} (\|\nabla e_1\|_2 - \lambda_1 e_1^2) \tag{2.6}
\]
\[
= \int_{\Omega} |\nabla \tilde{u}_n|^2 - \lambda_1 \int_{\Omega} \tilde{u}_n^2 \geq \left( 1 - \frac{\lambda_1}{\lambda_2} \right) \int_{\Omega} |\nabla \tilde{u}_n|^2.
\]

So (2.5), (2.6) yield

\[
\left( \frac{\beta}{2} - 1 \right) \left( 1 - \frac{\lambda_1}{\lambda_2} \right) \int_{\Omega} |\nabla \tilde{u}_n|^2 - \varepsilon_n (\|\nabla u_n\|_2) \leq \gamma \int_{\Omega} (k_n e_1 + \tilde{u}_n)^2 + C
\]

By the variational characterization of \(\lambda_1\), it follows

\[
\left[ \left( \frac{\beta}{2} - 1 \right) \left( 1 - \frac{\lambda_1}{\lambda_2} \right) - \frac{\gamma}{\lambda_2} \right] \int_{\Omega} |\nabla \tilde{u}_n|^2 - \varepsilon_n (\|\nabla u_n\|_2) \leq
\]
\[
\leq C + \gamma k_n^2 \int_{\Omega} e_1^2.
\]

By assumption (1.6) and taking into account that \(\varepsilon_n \to 0\), we can conclude that

\[
\int_{\Omega} |\nabla \tilde{u}_n|^2 \leq const(1 + k_n^2), \tag{2.7}
\]

One can prove that (a subsequence of) \(\{\tilde{u}_n\}_{[k_n]}\) weakly converges in \(H^1_0\) to some \(h\) and that

\[
h(x) + e_1(x) \quad \text{does not coincide with} \quad 0 \text{ a.e. in } \Omega. \tag{2.8}
\]
2.2. THE CASE $\lambda \leq \lambda_1$

Let us prove (2.8). The weak convergence of $\{\tilde{u}_n\}$ implies that $\{\tilde{u}_n\} \to h$ strongly in $L^p$, $p \in [2, 2^*)$. If, by contradiction, $h \equiv e_1$ a.e. in $\Omega$, then

$$\int_{\Omega} \tilde{u}_n e_1 = 0 \implies \int_{\Omega} h e_1 = 0$$

so taking $h = -e_1$ we get a contradiction. It remains to prove that $\{|k_n|\}$ is bounded.

For this purpose let us rewrite (2.3) in the form

$$-\Delta (k_n e_1 + \tilde{u}_n) - \lambda_1 (k_n e_1 + \tilde{u}_n) = W(x) f(k_n e_1 + \tilde{u}_n) + g_n, \quad (2.9)$$

with $g_n \to 0$ in $H^{-1}$ and, by contradiction, suppose that $|k_n| \to \infty$, so we can divide (2.9) by $|k_n|$, for $n$ large enough, and find that

$$-\frac{\Delta \tilde{u}_n}{|k_n|} - \frac{\lambda_1 \tilde{u}_n}{|k_n|} = W(x) \frac{f(k_n e_1 + \tilde{u}_n)}{|k_n|} + \frac{g_n}{|k_n|}, \quad \text{for } n \text{ large enough.} \quad (2.10)$$

Multiplying (2.10) by $\psi_n = (k_n e_1 + \tilde{u}_n)|k_n|^{-1} \psi$, where $\psi$ is a regular function with compact support in $\Omega$, and integrating by parts, we get

$$\frac{1}{|k_n|} \int_{\Omega} \nabla \tilde{u}_n \nabla \psi_n - \frac{1}{|k_n|} \int_{\Omega} \lambda_1 \tilde{u}_n \psi_n = \frac{1}{|k_n|} \int_{\Omega} W(x) f(k_n e_1 + \tilde{u}_n) \psi_n + o(1) \quad (2.11)$$

for $n$ large enough. At this point, the Holder inequality and (2.7) yield that

$$\frac{1}{|k_n|} \int_{\Omega} \nabla \tilde{u}_n \nabla \psi_n \leq \frac{1}{|k_n|} \|\nabla \tilde{u}_n\|_2 \|\nabla \psi_n\|_2 \leq \text{const} \frac{|k_n|}{|k_n|} \|\nabla \psi_n\|_2 \leq \text{const}$$

and

$$\frac{\lambda_1}{|k_n|} \int_{\Omega} \tilde{u}_n \psi_n \leq \frac{\lambda_1}{|k_n|} \|\tilde{u}_n\|_2 \|\psi_n\|_2 \leq \frac{\lambda_1}{\lambda_2 |k_n|} \|\nabla \tilde{u}_n\|_2 \|\psi_n\|_2 \leq \text{const} \frac{\lambda_1 |k_n|}{\lambda_2 |k_n|} \|\psi_n\|_2 \leq \text{const}.$$ 

On the other side, from (1.7) and (2.8) it follows that either

$$\int_{\text{supp } W^+} |h(x) + e_1(x)|^\beta > 0, \text{ or } \int_{\text{supp } W^-} |h(x) + e_1(x)|^\beta > 0.$$ 

In the first case one chooses $\psi$ as a regular nonnegative function with $\text{supp } \psi \subset \text{supp } W^+$ such that

$$\int_{\text{supp } W^+} W^+(x) \psi(x) |h(x) + e_1(x)|^\beta > 0, \quad (2.12)$$
CHAPTER 2. SUPERLINEAR NONLINEARITIES

Then, by (1.5), (2.12):

\[ \frac{1}{|k_n|} \int_{\Omega} W(x)f(k_ne_1 + \tilde{u}_n)\psi_n \geq \frac{1}{|k_n|} \int_{\text{supp}W^+} W^+(x)|k_ne_1 + \tilde{u}_n|^\beta \psi_n - C = \]

\[ = k_n^{\beta - 2} \int_{\text{supp}W^+} W^+(x)|e_1 + \frac{\tilde{u}_n}{k_n}|^\beta \psi - C \rightarrow +\infty \]

which yields a contradiction with the boundedness of \( \frac{1}{|k_n|} \int_{\Omega} \nabla \tilde{u}_n \nabla \psi_n \) and \( \frac{1}{|k_n|} \int_{\Omega} \lambda_1 \tilde{u}_n \psi_n \).

In the second case, \( \int_{\text{supp}W^-} |h(x) + e_1(x)|^\beta > 0 \), the argument is quite similar: one has to replace \( W^+ \) by \( W^- \) and the previous \( \psi \) by a nonnegative function \( \tilde{\psi} \) such that \( \text{supp} \tilde{\psi} \subset \text{supp}W^- \) and such that \( \int_{\text{supp}W^-} W^-(x)\tilde{\psi}(x)|h(x) + e_1(x)|^\beta > 0 \). Thus we have proved that \( \int_{\Omega} |\nabla u_n|^2 \leq C \), so there exists a weakly converging subsequence, i.e. \( u_n \rightharpoonup w \) in \( H^1_0(\Omega) \). The weak convergence in \( H^1_0 \) yields strong convergence in \( L^p \ \forall p \in [2, 2^*) \), so by (2.3) and by (1.3) we have

\[ \int_{\Omega} |\nabla u_n|^2 \rightarrow \lambda_1 \int_{\Omega} w^2 + \int_{\Omega} W(x)f(w)w, \]  \hspace{1cm} (2.13)

while the definition of weak convergence yields

\[ \int_{\Omega} \nabla u_n \nabla w \rightarrow \int_{\Omega} |\nabla w|^2. \]  \hspace{1cm} (2.14)

Furthermore

\[ \int_{\Omega} \nabla u_n \nabla w \rightarrow \lambda_1 \int_{\Omega} w^2 + \int_{\Omega} W(x)f(w)w, \]  \hspace{1cm} (2.15)

by uniqueness property of the limit we conclude that

\[ \int_{\Omega} |\nabla u_n|^2 \rightarrow \int_{\Omega} |\nabla w|^2 \]

so \( u_n \) converges strongly to \( w \) in \( H^1_0(\Omega) \).

\[ \square \]

We are going to prove that 0 is a strict local minimum for the above functional (see Preliminaries, page 2). Observe that, if \( \lambda < \lambda_1 \), it can be easily proved that the functional has a Mountain Pass structure, so in the following we will consider only the case \( \lambda = \lambda_1 \). Actually the argument is the same one used in [2], but we prefer to show
all the details for reader’s convenience. Observe that each $u \in H^1_0(\Omega)$ can be written as $u = te_1 + v$, where $t \in \mathbb{R}$, $e_1$ is a positive eigenfunction related to the first eigenvalue of $-\Delta$ and $v$ is such that $\int_\Omega ve_1 = 0$. Directly from this decomposition we have

$$
\int_\Omega |\nabla u|^2 = t^2 \lambda_1 \int_\Omega e_1^2 + \int_\Omega v^2.
$$

Choosing $e_1$ such that $\int_\Omega e_1^2 = \frac{1}{\lambda_1}$ one gets, for all $u$ such that $\|\nabla u\|_2 < \frac{1}{2\|e_1\|_\infty}$,

$$
t^2 < \int_\Omega |\nabla u|^2 < \frac{1}{4\|e_1\|_\infty^2}. \tag{2.16}
$$

Then, by the variational characterization of the eigenvalues of the Laplacian one has, for a suitable function $R(t, v)$

$$
I(u) \geq \frac{1}{2} \left( 1 - \frac{\lambda_1}{\lambda_2} \right) \|\nabla v\|_2^2 - \int_\Omega W(x)F(te_1 + v) \geq \frac{1}{2} \left( 1 - \frac{\lambda_1}{\lambda_2} \right) \|\nabla v\|_2^2 - |t|^\beta \int_\Omega W(x)e_1^\beta + R(t, v) \tag{2.17}
$$

where, by (1.4),

$$
R(t, v) = \int_\Omega W(x) \left[ |te_1|^\beta - F(te_1) \right] + \int_\Omega W(x) \left[ F(te_1) - F(te_1 + v) \right] = \int_\Omega W(x) \left[ F(te_1) - F(te_1 + v) \right] + o(|t|^\beta). \tag{2.18}
$$

Let us find an estimate for $R(t, v)$.

Let $\theta = \theta(v, t, x)$ be a number such that

$$
|F(te_1) - F(te_1 + v)| = |f(te_1 + \theta v(x))v(x)|,
$$

In case that $|te_1 + \theta v(x)| \geq 1$, then

$$
|\theta v(x)| \geq 2|t|\|e_1\|_\infty - |t|\|e_1\|_\infty \geq |t|\|e_1\|_\infty,
$$

so, by (1.3),

$$
|f(te_1 + \theta v(x))| \leq C|te_1 + \theta v(x)|^{\beta - 1}|v(x)| \leq 2^{\beta - 2}C|\theta v(x)|^{\beta - 1}|v(x)| \leq 2^{\beta - 1}C|v(x)|^{\beta}, \tag{2.19}
$$

while, if $|te_1 + \theta v(x)| \leq 1$, using again (1.3), one gets
\[ |W(x)||f(te_1 + \theta v(x))v(x)| \leq C|te_1 + \theta v(x)|^{\beta-1}|v(x)| \leq \]
\[ \leq C \left[ |te_1|^{\beta-1} + |v(x)|^\beta \right] \leq \varepsilon |te_1|^\beta + C_\varepsilon |v(x)|^\beta, \]
where \( \varepsilon, C_\varepsilon \) are positive suitable constant numbers. Set \( A = -\int_\Omega W(x)e_1^\beta > 0 \). Putting together (2.19) and (2.20), and using (1.9), (2.17) becomes:

\[ I(u) \geq \frac{1}{2} \left( 1 - \frac{\lambda_1}{\lambda_2} \right) \| \nabla v \|^2 - t^\beta \int_\Omega W(x)e_1^\beta - |R(t,v)| \geq \]
\[ \geq \frac{1}{2} \left( 1 - \frac{\lambda_1}{\lambda_2} \right) \| \nabla v \|^2 + t^\beta A - 2C \int_{\Omega \cap \{|u|>1\}} |W(x)||v(x)|^\beta + \]
\[ - \int_{\Omega \cap \{|u|\leq1\}} [\varepsilon|te_1|^\beta + C_\varepsilon |v(x)|^\beta] + o(|t|^\beta) \geq \]
\[ \geq \frac{1}{2} \left( 1 - \frac{\lambda_1}{\lambda_2} \right) \| \nabla v \|^2 + t^\beta (A - C\varepsilon) - \text{const} \|v(x)\|^\beta + o(|t|^\beta). \]

Then, by the Sobolev embedding theorem, we deduce, for \( \varepsilon < \frac{A}{C} \),

\[ I(u) \geq \frac{1}{2} \left( 1 - \frac{\lambda_1}{\lambda_2} \right) \| \nabla v \|^2 + \text{const} \cdot |t|^\beta - \text{const} \cdot \|\nabla v\|^2 + o(|t|^\beta). \]

For \( \beta > 2 \) the last expression is strictly positive when \( \|\nabla v\|^2 \) is close to 0. Now let us verify, by a standard argument, that there exists \( \overline{\pi} \in H^1_0(\Omega) \), with \( \|\overline{\pi}\| > \rho, \rho \) large enough, such that \( I(\overline{\pi}) < 0 \). Let \( \overline{\pi} = t\phi, t \in \mathbb{R}, \phi \in C_0^\infty(\text{supp}W^+(x)) \), where \( W^+(x) = \max\{W(x),0\} \) (note that \( \phi \) is well defined thanks to (1.8)).

Using (1.4) one gets

\[ I(t\phi) = \frac{t^2}{2} \|\nabla \phi \|^2 - \lambda_1 \|\phi\|^2 - \int_\Omega W(x)F(t\phi) \leq \]
\[ \leq \frac{t^2}{2} \|\nabla \phi \|^2 - \int_{\text{supp}W^+} W^+ F(t\phi) \leq \]
\[ \leq \frac{t^2}{2} \|\nabla \phi \|^2 - C t^\beta \int_{\text{supp}W^+} W^+ |\phi|^\beta. \]

For \( \beta > 2 \), the last expression goes to \(-\infty\) as \( t \to +\infty \).

Finally one has to show that the solution of mountain Pass type is positive (the argument is the same of [2], theorem 1.6). The critical value

\[ c = \inf_{g \in \Gamma} \max_{t \in [0,1]} I(g(t)), \]
2.3. THE CASE \( \lambda > \lambda_1 \)

where
\[
\Gamma = \{ g \in C([0, 1], H^1_0(\Omega)) : g(0) = 0, \ I(g(1)) < 0 \}
\]

admits at least one corresponding positive critical point. Indeed if \( g \in \Gamma \), then \( |g| \in \Gamma \) and \( I(g) = I(|g|) \), because \( F(u) = F(|u|) \). So, for any \( n \in \mathbb{N} \) there exists \( g_n \in \Gamma \), \( g_n(t) \geq 0 \) (a. e. in \( \Omega \)) \( \forall t \in \Omega \), such that (by definition of infimum)
\[
c \leq \max_{t \in [0,1]} I(g_n(t)) < c + \frac{1}{n}.
\]

Now, by Ekeland’s variational principle (see [55]), it follows that there exists \( g_n^* \in \Gamma \) such that:
\[
\begin{align*}
&\bullet \ c \leq \max_{t \in [0,1]} I(g_n^*(t)) \leq \max_{t \in [0,1]} I(g_n^*(t)) < c + \frac{1}{n}, \\
&\bullet \ \max_{t \in [0,1]} \| g_n(t) - g_n^*(t) \| < \frac{1}{\sqrt{n}}, \\
&\bullet \ \exists t_n \in [0,1] : v_n = g_n^*(t_n) \text{ satisfies:} \\
&\bullet \ I(v_n) = \max_{t \in [0,1]} I(g_n^*(t)) \text{ and } \| I'(v_n) \| \leq \frac{1}{\sqrt{n}}.
\end{align*}
\]

Furthermore, up to subsequences, \( v_n \to v \) in \( H^1_0(\Omega) \) and \( g_n(t) \to v \) in \( H^1_0(\Omega) \). Since \( g_n(t) \geq 0 \) (a. e. in \( \Omega \)) we conclude that \( v \geq 0 \) (a. e. in \( \Omega \)). By the regularity of \( F \), knowing that \( F(0) = 0 \) and applying Lagrange’s Theorem one gets
\[
-\Delta u = \lambda_1 u + W(x)F(u) = \lambda_1 u + W(x)F'(\xi)u = \lambda_1 u + \overline{c}(x)u = c(x)u.
\]

So \( u \geq 0 \) verifies a linear equation of the kind \( -\Delta u + c(x)u = 0 \) where \( c(x) \in L^\infty \), by the strong maximum principle (see [36]) \( u > 0 \) or \( u \equiv 0 \), but the Mountain Pass Theorem assures that \( u \) is not the trivial solution.

\( \square \)

2.3 The case \( \lambda > \lambda_1 \)

In order to prove Theorem 1.5, we will first state that \( I \) satisfies the Palais–Smale condition. This proof follows the same arguments showed when proving PS condition.
for theorem 1.1, but in this case we need some additional hypotheses and calculations are more technical, so we will show the details of this proof too. On the other hand, in the case $\beta = 2^*$, it can be shown that there exist some levels of $I$ where the Palais–Smale condition does not hold. Nevertheless, for all $\beta \in [2, 2^*]$ the boundedness of Palais–Smale sequences is guaranteed. Now one can state the following

**Proposition 3.1** Under assumptions (1.2),(1.5),(1.7) and (1.11), any Palais–Smale sequence for the functional $I$ is bounded.

**Proof**

Arguing as in the proof of Theorem 1.1, one has that

\[
\int_\Omega |\nabla u_n|^2 dx = \lambda \int_\Omega u_n^2 dx + \int_\Omega W(x)f(u_n)u_n dx - \varepsilon_n (\|\nabla u_n\|_2) \geq \lambda \int_\Omega u_n^2 dx + \beta (\frac{1}{2} \int_\Omega |\nabla u_n|^2 dx - \frac{\lambda}{2} \int_\Omega u_n^2 dx - C_2) - \gamma \int_\Omega u_n^2 dx + CR - \varepsilon_n (\|\nabla u_n\|_2).
\]

Let us split now $u_n$ into the sum

\[u_n = v_n + w_n \quad \text{with} \quad v_n \in X_k, \ w_n \in X_k^\perp.\]

Thus, if $\{e_1, \ldots, e_k\}$ is an (orthogonal) basis for $X_k$, one has

\[v_n = \sum_{i=1}^k t_n^i e_i \quad \text{for some} \quad t_n^i \in \mathbb{R}, \ i = 1, \ldots, k,\]

so, by the variational characterization of $\lambda_{k+1}$, (see 1.1),

\[
\int_\Omega |\nabla u_n|^2 dx - \lambda \int_\Omega u_n^2 \geq \sum_{i=1}^k (\lambda - \lambda_i) (t_n^i)^2 \int_\Omega e_i^2 dx + \left(1 - \frac{\lambda}{\lambda_{k+1}}\right) \int_\Omega |\nabla w_n|^2. \quad (3.4)
\]

Then (3.1)–(3.4) yield,

\[
\left[\varepsilon_n + \left(\frac{\beta}{2} - 1\right) \left(1 - \frac{\lambda}{\lambda_{k+1}}\right)\right] \int_\Omega |\nabla w_n|^2 dx \leq \beta - \frac{1}{2} \sum_{i=1}^k \frac{(\lambda - \lambda_i)^2}{\lambda_{k+1}} \int_\Omega e_i^2 dx + \gamma \int_\Omega u_n^2 dx + C. \quad (3.5)
\]
2.3. THE CASE $\lambda > \lambda_1$

Observe that

$$\gamma \int_{\Omega} u_n^2 \, dx = \gamma \int_{\Omega} \sum_{i=1}^{k} (t_n^i e_i)^2 \, dx + \gamma \int_{\Omega} w_n^2 \, dx \leq (3.6)$$

$$\leq \gamma \int_{\Omega} \sum_{i=1}^{k} (t_n^i e_i)^2 \, dx + \gamma \int_{\Omega} |\nabla w_n|^2 \, dx.$$  

At this point,

$$\left[ \varepsilon_n + (\beta/2 - 1) \left( 1 - \frac{\lambda}{\lambda_{k+1}} \right) - \frac{\gamma}{\lambda_{k+1}} \right] \int_{\Omega} |\nabla w_n|^2 \, dx \leq (3.7)$$

$$\leq (\beta/2 - 1) \sum_{i=1}^{k} (\lambda - \lambda_i) (t_n^i)^2 \int_{\Omega} e_i^2 \, dx + \gamma \sum_{i=1}^{k} (t_n^i)^2 \int_{\Omega} (e_i)^2 \, dx + C.$$  

Taking into account the choice of $\gamma$ in (1.11), one easily deduces from (3.7) the relation

$$\int_{\Omega} |\nabla w_n|^2 \, dx \leq K_1 \sum_{i=1}^{k} (\lambda - \lambda_i) (t_n^i)^2 + K_2 \sum_{i=1}^{k} (t_n^i)^2 + K_3, (3.8)$$

for some positive constant numbers $K_1, K_2, K_3$.

Let us prove now that $\{t_n^i\}_n$ is a bounded sequence for $i = 1, \ldots, k$. By contradiction, let us suppose that, putting

$$T_n = \max_{\{i=1, \ldots, k\}} |t_n^i| \quad \forall n \in \mathbb{N},$$

the sequence $T_n$ is unbounded, so, at least for a subsequence, $\{T_n\} \to +\infty$ as $n \to \infty$. Therefore the sequence $\{w_n/T_n\}$ is bounded in $H^1_0(\Omega)$, so a subsequence, also named $\{w_n/T_n\}$, weakly converges in $H^1_0(\Omega)$. Let us put

$$h(x) = \lim_{n \to \infty} (T_n)^{-1} w_n \quad \text{weakly in } H^1_0(\Omega). \quad (3.9)$$

On the other side there exists an index $\mathcal{I} \in \{1, \ldots, k\}$ such that $|t_n^\mathcal{I}| = T_n$ for infinite indexes $n \in \mathbb{N}$. So, up to a subsequence,

$$\sum_{i=1}^{k} (t_n^i)(T_n)^{-1} e_i \to \sum_{i=1}^{k} \tau_i e_i \quad \text{in } H^1_0(\Omega)$$
with $|\tau_l| = 1$. We claim that
\[
\left| h(x) + \sum_{i=1}^{k} \tau_i e_i(x) \right|^{\beta} \neq 0 \text{ on a subset of } \Omega \text{ with positive measure.} \tag{3.10}
\]
Indeed, if (3.10) did not hold, taking into account that
\[
\int_{\Omega} \nabla w_n \nabla e_i dx = 0 \quad \forall n \in \mathbb{N}, \quad \forall i = 1, \ldots, k \tag{3.11}
\]
one would get from (3.9),(3.11), the relation
\[
\sum_{i=1}^{k} (\tau_i)^2 \int_{\Omega} |\nabla e_i|^2 dx = - \int_{\Omega} |\nabla h|^2 dx = \int_{\Omega} \lim_{n \to \infty} \frac{\nabla w_n}{T_n} \sum_{i=1}^{k} \tau_i \nabla e_i = 0,
\]
which would imply $\tau_i = 0$ for $i = 1, \ldots, k$, which is a contradiction with the fact that $|\tau_l| = 1$. Thus (3.10) is proved.

Let us choose now a sequence $\{\psi_n\}$ in $H_0^1(\Omega)$ defined as
\[
\psi_n = (T_n)^{-1} \sum_{i=1}^{k} (t_i^n e_i + w_n) \psi \tag{3.12}
\]
where $\psi$ is a suitable non-zero regular function with a compact support in $\Omega$ which will be better specified in the following. Since $\{\psi_n\}$ is bounded in $H_0^1(\Omega)$, from $\{I'(u_n)\} \to 0$ and (3.11), one gets in $H^{-1}(\Omega)$,
\[
(T_n)^{-1} \int_{\Omega} \nabla w_n \nabla \psi_n dx = \lambda(T_n)^{-1} \int_{\Omega} w_n \psi_n dx = \frac{1}{T_n} \int_{\Omega} W(x) f \left( \sum_{i=1}^{k} t_i^n e_i + w_n \right) \psi_n dx + \eta_n \quad \text{with } \eta_n \to 0 \text{ in } \mathbb{R}. \tag{3.13}
\]
Let us note that (1.7) implies that at least one of the relations
\[
\left| h(x) + \sum_{i=1}^{k} \tau_i e_i(x) \right|^{\beta} \big| W^+(x) > 0 \text{ on a subset of supp } W^+ \text{ with positive measure} \tag{3.14}
\]
\[
\left| h(x) + \sum_{i=1}^{k} \tau_i e_i(x) \right|^{\beta} \big| W^-(x) > 0 \text{ on a subset of supp } W^- \text{ with positive measure} \tag{3.15}
\]
must hold.
If, for example, (3.14) holds, one chooses $\psi \not\equiv 0$ in (3.12) as a regular non negative function, with $\text{supp } \psi \subset \text{supp } W^+$ such that one has

$$\int_{\text{supp } W^+} W^+(x)\psi(x) \left| h(x) + \sum_{i=1}^k \tau_i e_i(x) \right|^\beta \, dx > 0. \quad (3.16)$$

Then (1.10) and the very definition of $\psi_n$ yield, for some positive constant numbers $K_3, K_4$,

$$\left( T_n \right)^{-1} \int_\Omega W(x)f \left( \sum_{i=1}^k t^n_i e_i + w_n \right) \psi_n \, dx = (3.17)$$

$$= (T_n)^{-2} \int_{\text{supp } W^+} W^+(x)f \left( \sum_{i=1}^k t^n_i e_i + w_n \right) \psi(x) \, dx \geq K_3(T_n)^{-2} \int_{\text{supp } W^+} W^+(x)\psi(x) \left| \sum_{i=1}^k t^n_i e_i + w_n \right|^\beta \, dx - K_4 = K_3(T_n)^{\beta-2} \int_{\text{supp } W^+} W^+(x)\psi(x) \left| \sum_{i=1}^k t^n_i e_i + w_n \right|^\beta \, dx - K_4,$$

where, by (3.16)

$$b_n = \int_{\text{supp } W^+} W^+(x)\psi(x) \left| \sum_{i=1}^k \frac{t^n_i}{T_n} e_i + \frac{w_n}{T_n} \right|^\beta \, dx \to K_5 > 0 \text{ as } n \to +\infty \quad (3.18)$$

On the other hand, all the terms of the first member of (3.13) are bounded as $n \to \infty$, which is a contradiction with (3.18) since $\beta > 2$ in (3.17) and $(T_n)^{\beta-2} \to \infty$ as $n \to \infty$. In case that (3.15) holds, the argument is quite similar: only one has to replace $W^+$ with $W^-$ and $\psi$ as in (3.16) with a non–zero regular non positive function $\bar{\psi}$ with $\text{supp } \bar{\psi} \subset \text{supp } W^-$ and such that

$$\int_{\text{supp } W^-} W^-(x)\bar{\psi}(x) \left| h(x) + \sum_{i=1}^k \tau_i e_i(x) \right|^\beta \, dx < 0.$$ 

Therefore $\{t^n_i\}_n$ is a bounded sequence for $i = 1, \ldots, k$, hence $\{w_n\}$ is bounded in $H^1_0(\Omega)$ by (3.8), then $\{u_n\}$ given by (3.2),(3.3) is a bounded sequence in $H^1_0(\Omega)$. □

The proof of Theorem 1.5 relies on the Linking Theorem by Rabinowitz (see Preliminaries or [51]).
Remark 3.2 Actually the precise characterization of $c$ given by
\[
\bar{c} = J(\overline{u}) = \inf_{h \in \Gamma} \max_{v \in Q} J(h(v)) \tag{3.19}
\]
where
\[
Q \equiv (B_{\overline{R}} \cap E_2) \oplus \{r \tilde{v}: 0 < r < \overline{R}\}
\]
and
\[
\Gamma = \{ h \in C(\overline{Q}, E): h = \text{id} \text{ on } \partial Q \}.
\]
is not relevant when dealing with the subcritical case (except for the fact that it guarantees the nontriviality of the solution).

On the contrary, in the critical case (see Proposition 5.6), this characterization plays a crucial role. Indeed one can prove that Theorem 0.2 still holds if the PS condition is verified only in a neighborhood of the critical level $c$. This fact will allow us to apply Theorem 0.2 after estimating level $c$.

At this point one can state the

Proof of Theorem 1.5

First of all one has to prove that the functional $I$ verifies the Palais-Smale condition on $H_0^1(\Omega)$. As $\beta < 2^*$, this is a consequence of Proposition 3.1 and the strong convergence of a subsequence of a Palais-Smale sequence. At this point one applies Theorem 0.2. Precisely let us take $J = I$, $E = H_0^1(\Omega)$, $E_1 = (X_k)^\perp$, $E_2 = X_k$. As for $(J3)$, it is easy to conclude that $(1.2),(1.3),(1.1)$ imply
\[
I(v) \geq \frac{1}{2} \left(1 - \frac{\lambda}{\lambda_{k+1}}\right) \rho^2 - C\rho^\beta \quad \forall \ v \in E_1, \|v\| = \rho,
\]
then, as $\beta > 2$ one gets
\[
I(v) \geq \alpha > 0 \quad \forall \ v \in E_1, \|v\| = \rho > 0 \text{ sufficiently small}.
\]
As for the proof of $(J4)$, it is an obvious consequence of $(1.9)$. Finally, let us verify $(J5)$. Let us choose $\tilde{v}$ as the function $\overline{v}$ as in $(1.13)$. From the equivalence of all the
2.4. PS SEQUENCES IN THE CRITICAL CASE

norms in the finite-dimensional space $E_2 \oplus \{\tilde{v}\}$, one obtains, for some $C > 0$,

$$I(v) \leq C (\|v\|^2 - \|v\|^\beta) \quad \forall v \in E_2 \oplus \{\tilde{v}\}, \|v\| \geq R$$

which yields $(J3)$ for $R > R$ and $\beta > 2$.

2.4 The behaviour of Palais–Smale sequences in the critical case.

In this section we consider the case $\beta = 2^* = \frac{2N}{N-2}$ and $N \geq 5$.

It is well known that, since the embedding of $H^1_0(\Omega)$ in $L^{2^*}(\Omega)$ is not compact, the Palais–Smale condition for the functional $I$ does not hold in general. In the next remark we state this more precisely.

Remark 4.1 Let us consider the sequence

$$u_n(x) = \phi(x) \frac{[N(N-2)]^{\frac{N-2}{4}}}{(\frac{1}{n^2} + |x-x_0|^2)^{\frac{N-2}{2}}} \cdot \frac{1}{n^{\frac{N-2}{2}}} \cdot \frac{1}{W(x_0)^{\frac{N-2}{4}}}$$

with $x_0 \in supp(W^+), \phi \in C_0^\infty(\Omega)$ and $\phi \equiv 1$ in $B(x_0, r)$ where $0 < r < \text{dist} (x_0, \partial \Omega)$.

It is easy to verify that $I(u_n) \to \frac{1}{N} S_{W(x_0)^{\frac{N-2}{4}}}^N \frac{1}{W(x_0)^{\frac{N-2}{4}}}$ (here $S$ denotes the best Sobolev constant), $I'(u_n) \to 0$ in $H^{-1}$ and $u_n$ does not admit any strongly converging subsequence in $H^1_0(\Omega)$, for it converges to $0$ almost everywhere and weakly to a constant. This shows that the Palais–Smale condition does not hold for the functional $I$ at the level $c = \frac{1}{N} S_{W(x_0)^{\frac{N-2}{4}}}^N \frac{1}{W(x_0)^{\frac{N-2}{4}}}$ for any $x_0 \in supp(W^+)$. 

In the next proposition, which is an easy generalization of some known results in literature (see e.g., [18], [44],[45],[54]), we prove that below the levels $c$ considered in Remark 3.2, the Palais–Smale condition holds for $I$. 
Proposition 4.2 Let us assume the same assumptions of Proposition 3.1. Suppose that $F$ is convex and (1.15) holds. If

$$c < \frac{1}{N} S^N \frac{1}{\|W^+\|_{\infty}^{\frac{N}{2}}} + I(u) = c^* + I(u),$$

(4.2)

for any solution $u$ of $(P_{\lambda})$, then the Palais–Smale condition holds at level $c$, i.e. any PS sequence $\{u_n\}$ s.t. $I(u_n) \to c$ admits a strongly converging subsequence in $H^1_0(\Omega)$.

Proof

We will prove that, if $\{u_n\}$ is a Palais–Smale sequence with $I(u_n) \to c$, which does not admit any subsequence strongly converging in $H^1_0(\Omega)$, then

$$c \geq I(u_0) + \frac{1}{N} S^N \frac{1}{\|W^+\|_{\infty}^{\frac{N}{2}}},$$

(4.3)

for some $u_0$ solution of $(P_{\lambda})$. We follow the same argument of [18], section 1. By Proposition 3.1 we have that

$$\|u_n\|_{H^1_0(\Omega)} \leq C \quad \text{and} \quad \left| \int_{\Omega} W(x)F(u_n)dx \right| \leq C, \quad \forall n \in \mathbb{N}. \quad (4.4)$$

So there exists $u_0 \in H^1_0(\Omega)$ such that, up to subsequences,

$$\begin{align*}
& u_n \rightharpoonup u_0 \quad \text{weakly in } H^1_0(\Omega) \\
& u_n \to u_0 \quad \text{strongly in } L^q(\Omega), \quad \text{for any } q \in [2, 2^*] \\
& u_n \to u_0 \quad \text{almost everywhere in } \Omega.
\end{align*} \quad (4.5)$$

Therefore $u_0$ is a solution of $(P_{\lambda})$. Let $v_n = u_n - u_0$. Since, by contradiction, $\{u_n\}$ does not admit any subsequence which converges strongly to $u_0$, we have that $\int_{\Omega} |\nabla v_n|^2dx \geq \alpha > 0$. At this point, proceeding as in [18], we can find two sequences $a_n \in \Omega$ and $\varepsilon_n \searrow 0$ such that

$$\lim_{n \to \infty} \frac{\text{dist}(a_n, \partial \Omega)}{\varepsilon_n} = +\infty \quad (4.6)$$

and

$$\int_{a_n + \varepsilon_n \Omega} |\nabla v_n|^2dx \geq \alpha > 0 \quad (4.7)$$
Set $\tilde{v}_n(x) = \varepsilon_n^{N-2} v_n(a_n + \varepsilon_n x)$ and $\Omega_n = \frac{\Omega - a_n}{\varepsilon_n}$. From (4.6) we obtain that, for any compact set $K \subset \mathbb{R}^N$, there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$ we have $K \subset \frac{\Omega - a_n}{\varepsilon_n}$.

By standard computations we get, for any compact $K \subset \mathbb{R}^N$,

$$-\Delta \tilde{v}_n = W(\varepsilon_n x + a_n)\varepsilon_n^{N-2} f\left(\frac{\tilde{v}_n}{\varepsilon_n}\right) + g_n, \quad \text{with } g_n \to 0 \text{ in } H^{-1}(K) \quad (4.8)$$

Since

$$\int_{\Omega_n} |\nabla \tilde{v}_n|^2 dx = \varepsilon_n^N \int_{\Omega} |\nabla \tilde{v}_n\left(\frac{z - a_n}{\varepsilon_n}\right)|^2 \frac{1}{\varepsilon_n^N} dz = \int_{\Omega} |\nabla v_n\left(\varepsilon_n\left(\frac{z - a_n}{\varepsilon_n}\right) + a_n\right)|^2 dz = \int_{\Omega} |\nabla v_n|^2 dz$$

one has

$$\int_{\Omega_n} |\nabla \tilde{v}_n|^2 dx \leq \int_{\Omega} |\nabla u_n|^2 dx + \int_{\Omega} |\nabla u_0|^2 dx \leq C,$$

so we deduce that

$$\tilde{v}_n \rightharpoonup U \text{ weakly in } D^{1,2}(\mathbb{R}^N) = \left\{ u \in L^2^*(\mathbb{R}^N) : \int_{\mathbb{R}^N} |\nabla u|^2 dx < +\infty \right\}. \quad (4.9)$$

Finally, from (1.15) we can pass to the limit in (4.8) and so $U$ solves the problem

$$-\Delta U = W(a_0)|U|^{2^*-2}U \quad \text{in } \mathbb{R}^N, \quad U \in D^{1,2}(\mathbb{R}^N) \quad (4.10)$$

with $a_0 = \lim_{n \to \infty} a_n, \quad a_0 \in \overline{\Omega}$.

Arguing as in [18] it is possible to deduce that $\tilde{v}_n \to U$ strongly in $H^1_{loc}(\mathbb{R}^N)$. From this and (4.7) we get that $U \not\equiv 0$.

Now we claim that $W(a_0) > 0$. By contradiction, if $W(a_0) \leq 0$, let $a_0$ be a maximum point for $U$, so $-\Delta U(a_0) \geq 0$. From (4.10) we deduce $U(a_0) \leq 0$. On the other hand

$$U(x) = -\frac{1}{(1 + |x|^2)^{\frac{N-2}{2}}}$$

and integration by parts yields

$$-\int_{\mathbb{R}^N} \Delta U U = W(a_0)^{N(N-2)} - \int_{\mathbb{R}^N} |u|^{2^*}$$

which leads to a contradiction. If $W(a_0) = 0$, $U$ is a harmonic function in $\mathbb{R}^N$ thus it does not admit any maximum or minimum in $\mathbb{R}^N$. But this yields a contradiction.
with \( U \in D^{1,2}(\mathbb{R}^N) \), for on every compact set of \( \mathbb{R}^N \) \( U \) admits maxima and minima \((U \) is continuous by [19]). Let us set

\[
\bar{U} = W(a_0)^{\frac{N-2}{4}} U
\]

which verifies

\[
\begin{cases}
-\Delta u = |u|^{\frac{4}{N-2}} u \text{ in } \mathbb{R}^N \\
u \in D^{1,2}(\mathbb{R}^N)
\end{cases}
\]

Since \( \tilde{v}_n \to \frac{\bar{U}}{W(a_0)^{\frac{N-2}{4}}} \) weakly in \( D^{1,2}(\mathbb{R}^N) \), we have

\[
\liminf_{n \to \infty} \int_{\Omega_n} |\nabla \tilde{v}_n|^2 dx \geq \frac{1}{W(a_0)^{\frac{N-2}{4}}} \int_{\mathbb{R}^N} |\nabla U|^2 dx,
\]

then

\[
\liminf_{n \to \infty} \left[ \int (|\nabla u_n|^2 - \lambda u_n^2) dx \right] \geq \liminf_{n \to \infty} \int |\nabla v_n|^2 dx + \int |\nabla u_0|^2 dx - \lambda \int u_0^2 dx \geq \int |\nabla u_0|^2 dx - \lambda \int u_0^2 dx + \frac{1}{W(a_0)^{\frac{N-2}{4}}} \int |\nabla U|^2 dx.
\]

Moreover, by Brezis-Lieb Lemma (see [20]), by the convexity of \( F \), one has

\[
\int_{\Omega} W(x)F(u_n)dx = \int_{\Omega} W(x)F(v_n)dx + \int_{\Omega} W(x)F(u_0)dx + o(1).
\]

Finally, using again (1.5) and by (2.3)

\[
\int_{\Omega} W(x)F(v_n)dx = \frac{N-2}{2N} \int_{\Omega} W(x)f(v_n)v_n dx + o(1) = \frac{N-2}{2N} \int_{\Omega} |\nabla v_n|^2 dx + o(1).
\]

Hence, by (4.13)-(4.16) we get

\[
c = \frac{1}{2} \int_{\Omega} |\nabla u_n|^2 dx - \frac{\lambda}{2} \int_{\Omega} u_n^2 dx - \int_{\Omega} W(x)F(u_n)dx = \frac{1}{2} \int_{\Omega} |\nabla v_n|^2 dx + \frac{1}{2} \int_{\Omega} |\nabla u_0|^2 dx + \frac{1}{2} \int_{\Omega} \nabla v_n \nabla u_0 dx - \frac{\lambda}{2} \int_{\Omega} v_n^2 dx - \frac{\lambda}{2} \int_{\Omega} v_n u_0 dx - \int_{\Omega} W(x)F(v_n)dx + \int_{\Omega} W(x)F(u_0)dx + o(1) \geq I(u_0) + \frac{1}{2} \int_{\Omega} |\nabla v_n|^2 dx - \frac{N-2}{2N} \int_{\Omega} |\nabla v_n|^2 dx + o(1) \geq
\]
2.5. ESTIMATE OF THE CRITICAL LEVEL

\[ I(u_0) + \frac{1}{\|W^+\|^2} \frac{1}{N} \int_{\mathbb{R}^N} |\nabla U|^2 dx \geq I(u_0) + \frac{1}{\|W^+\|^2} \frac{1}{N} \int_{\mathbb{R}^N} |\nabla U_0|^2 dx \]

where \(U_0\) is the unique positive solution of (4.12). The last inequality follows by the known fact that

\[ \inf_{u \in \mathcal{C}} \left( \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{1}{2^*} \int_{\mathbb{R}^N} |u|^{2^*} dx \right) \]

where \(\mathcal{C} = \{ u \text{ is a solution of (4.12)} \}\) is achieved at \(u = U_0\).

Since \(\int_{\mathbb{R}^N} |\nabla U_0|^2 dx = S^N_{\pi}\), then (4.3) follows. \(\square\)

2.5 Estimate of the critical level and proof of Theorem 1.7

In this section we will prove Theorem 1.7, starting by estimating the critical level of the functional (1.17), page 6, obtained via the Linking theorem.

Let us assume that \(\|W\|_\infty = W(0)\) and \(B(0,2) \subset supp \ W^+ \cap \Omega\). Let us denote by

\[ U_\mu(x) = \frac{|N(N-2)\mu|^\frac{N-2}{4}}{W(0)^\frac{2}{4-N}} |\mu + |x|^2|^\frac{N-2}{2}, \quad \mu > 0, \]

and \(\tilde{\Psi}_\mu(x) = \phi(x)U_\mu(x)\) where \(\phi \in C^\infty_0(B(0,2)), \phi \equiv 1\) in \(B(0,1)\).

In this section we assume \(N \geq 5\). Let \(X_k\) be the \(k\)-dimensional vector space considered in Theorem 1.5 and \(P_k\) the projector on \(X_k^\perp\) and set

\[ \Psi_\mu = P_k\tilde{\Psi}_\mu. \] (5.1)

One can get the following

**Lemma 5.1** The following estimates hold

\[ \int_\Omega |\nabla \tilde{\Psi}_\mu|^2 dx = \frac{S^N_{\pi}}{W(0)^\frac{N-2}{2}} + O(\mu^\frac{N-2}{2}). \] (5.2)
\[ \int_{\Omega} W(x) \Psi^{2*}_{\mu} dx = \int_{\Omega} W(x) \bar{\Psi}^{2*}_{\mu} dx + O(\mu^{\frac{N-2}{2}}), \quad (5.3) \]

\[ \int_{\Omega} |\bar{\Psi}_{\mu}|^{2*} - 1 dx = O(\mu^{\frac{N-2}{2}}), \quad (5.4) \]

\[ \int_{\Omega} \bar{\Psi}_{\mu} dx = O(\mu^{\frac{N-2}{4}}), \quad (5.5) \]

\[ \int_{\Omega} \bar{\Psi}_{\mu}^2 dx = k_1 \mu + O(\mu^{\frac{N-2}{4}}). \quad (5.6) \]

**Proof**

See [23], Remark 2.4.

**Lemma 5.2** Let us assume (1.4), (1.16) and let \( W \in C^3(\overline{\Omega}) \). Then the following relation holds

\[ \int_{\Omega} W(x) F(t \bar{\Psi}_{\mu}) dx = \frac{t^{2^*}}{2^* W(0)^{\frac{N-2}{2}}} S_N^N + c(N) \Delta W(0) \mu + o(\mu). \quad (5.7) \]

**Proof**

Let us look at the identity

\[ \int_{\Omega} W(x) F(t \bar{\Psi}_{\mu}) dx = \frac{t^{2^*}}{2^*} \int_{\Omega} W(x) \bar{\Psi}^{2*}_{\mu} dx + \int_{\Omega} W(x) \left( F(t \bar{\Psi}_{\mu}) - \frac{t^{2^*}}{2^*} \bar{\Psi}^{2*}_{\mu} \right) dx. \quad (5.8) \]

Set \( C_N = (N(N-2))^N \). Then, by Taylor’s formula (recalling that 0 is the maximum point for \( W \)), one has

\[ W(0)^{\frac{N}{2}} \int_{\Omega} W(x) \bar{\Psi}^{2*}_{\mu} dx = \mu^{\frac{N}{2}} C_N \int_{B(0,1)} \frac{W(x)}{(\mu + |x|^2)^N} dx = \quad (5.9) \]

\[ W(0)^{\frac{N}{2}} C_N \int_{B(0,1)} \frac{1}{(\mu + |x|^2)^N} dx + \mu^{\frac{N}{2}} C_N \sum_{i,j=1}^{N} \frac{\partial^2 W(0)}{\partial x_i \partial x_j} \int_{B(0,1)} \frac{x_i x_j}{(\mu + |x|^2)^N} dx + \]

\[ + \mu^{\frac{N}{2}} \frac{3!}{3} \sum_{i,j,k=1}^{N} \int_{B(0,1)} \frac{\partial^3 W(\xi)}{\partial x_i \partial x_j \partial x_k} \frac{x_i x_j x_k}{(\mu + |x|^2)^N} dx \]

for a suitable \( \xi \) belonging to the segment joining 0 and \( x \). We have the following asymptotic estimates

\[ \mu^{\frac{N}{2}} \int_{B(0,1)} \frac{1}{(\mu + |x|^2)^N} dx = C_N \int_{\mathbb{R}^N} \frac{1}{(\mu + |x|^2)^N} dx + O(\mu^{\frac{N}{2}}), \quad (5.10) \]
2.5. ESTIMATE OF THE CRITICAL LEVEL

\[
\frac{\mu^N}{2} \sum_{i,j=1}^{N} \frac{\partial^2 W(0)}{dx_i dx_j} \int_{B(0,1)} \frac{x_i x_j}{(\mu + |x|^2)^N} dx = (5.11)
\]

\[
= \frac{N}{2} \Delta W(0) \omega_N \int_{\mathbb{R}^N} \frac{x_1}{(\mu + |x|^2)^N} \mu dx + o(\mu),
\]

where \(\omega_N\) is the \((N - 1)\)-dimensional measure of \(S^{N-1}\), and

\[
\frac{\mu^N}{3!} \sum_{i,j,k=1}^{N} \int_{B(0,1)} \frac{\partial^3 W(\xi)}{dx_i dx_j dx_k} \frac{x_i x_j x_k}{(\mu + |x|^2)^N} dx = O(\mu^2). \quad (5.12)
\]

So (5.9) becomes

\[
\int_{\Omega} W(x) \tilde{\Psi}_\mu^{2^*} dx = \int_{\mathbb{R}^N} \frac{C_N}{(1 + |x|^2)^N} \frac{1}{W(0)} \frac{\Delta W(0)}{W(0)} \mu + o(\mu) = (5.13)
\]

\[
= \frac{1}{W(0)} \frac{\Delta W(0)}{W(0)} S^{\frac{N-1}{2}} + C(N) \frac{\Delta W(0)}{W(0)} \mu + o(\mu).
\]

Now, by (1.3)

\[
\int_{\Omega} W(x) \left( F(\tilde{\Psi}_\mu) - \frac{t^{2^*}}{2^*} \tilde{\Psi}_\mu^{2^*} \right) dx = \int_{B(0,1)} W(x) \left( F(\tilde{\Psi}_\mu) - \frac{t^{2^*}}{2^*} \tilde{\Psi}_\mu^{2^*} \right) dx. \quad (5.14)
\]

Let us prove now that

\[
\lim_{\mu \to 0} \frac{\int_{B(0,1)} W(x) \left( F(\tilde{\Psi}_\mu) - \frac{t^{2^*}}{2^*} \tilde{\Psi}_\mu^{2^*} \right) dx}{\mu} = 0. \quad (5.15)
\]

By (1.15) we have that \(\int_{B(0,1)} W(x) \left( F(\tilde{\Psi}_\mu) - \frac{t^{2^*}}{2^*} \tilde{\Psi}_\mu^{2^*} \right) dx \to 0\) as \(\mu \to 0\). Hence, by De l’Hopital rule

\[
\lim_{\mu \to 0} \frac{\int_{B(0,1)} W(x) \left( F(\tilde{\Psi}_\mu) - \frac{t^{2^*}}{2^*} \tilde{\Psi}_\mu^{2^*} \right) dx}{\mu} = \lim_{\mu \to 0} \int_{B(0,1)} W(x) \left[ f \left( t \left( \frac{C_N}{W(0)} \right)^{\frac{N-2}{4}} \frac{\mu^{\frac{N-2}{4}}}{(\mu + |x|^2)^{\frac{N-2}{2}}} \right) t \left( \frac{C_N}{W(0)} \right)^{\frac{N-2}{4}} \right] dx =
\]

\[
= \lim_{\mu \to 0} \int_{B(0,1)} W(x) \left[ f \left( t \left( \frac{C_N}{W(0)} \right)^{\frac{N-2}{4}} \frac{\mu^{\frac{N-2}{4}}}{(\mu + |x|^2)^{\frac{N-2}{2}}} \right) \right] \left. \frac{\rho^2 - 1}{(1 + \rho^2)^{\frac{N-1}{2}}} \right|_{\rho = \sqrt{\mu \rho}, \sqrt{\mu \theta}} \right. \left. \int_0^\infty W(\sqrt{\mu \rho}, \sqrt{\mu \theta}) \frac{\rho^2 - 1}{\left(1 + \rho^2\right)^{\frac{N-1}{2}}} \right. \left. \rho^{N-1} \right. dx =
\]

\[
\left[ f \left( t \left( \frac{C_N}{W(0)} \right)^{\frac{N-2}{4}} \frac{\mu^{\frac{2-N}{4}}}{(1 + \rho^2)^{\frac{N-2}{2}}} \right) \right] - \left( t^{\frac{N+2}{2}} \left( \frac{C_N}{W(0)} \right)^{\frac{N+2}{4}} \right) \frac{\mu^{\frac{N+2}{4}}}{(1 + \rho^2)^{\frac{N+2}{2}}} \rho^{N-1} dx.
\]
Setting $b_0 = t \left( \frac{C_N}{W(0)} \right)^{\frac{N-2}{2}}$, by (1.16) we obtain (5.15).

Then, by (5.8)–(5.13) and (5.15), we deduce

$$
\int_{\Omega} W(x) F(\bar{\Psi}) dx = \frac{t^{2^*}}{2^* W(0)^{\frac{N-2}{2}}} S_{\frac{N}{2}}^N + C(N) \Delta W(0) \mu + o(\mu).
$$

(5.16)

In order to get (5.7) we have to evaluate

$$
\left| \int_{\Omega} (F(\Psi) - F(\bar{\Psi}) \right| dx \leq \int_{\Omega} \left( \int_0^1 f(\Psi - \tau P_k \Psi) P_k \Psi d\tau \right) dx \leq c \int_0^1 d\tau \int_{\Omega} (|\Psi|^{2^* - 1} + \tau^{2^* - 1} |P_k \Psi|^{2^* - 1}) |P_k \Psi| dx
$$

$$
\leq c \int_{\Omega} |\Psi|^{2^* - 1} |P_k \Psi|_{\infty} dx + \int_{\Omega} |P_k \Psi|^{2^*} dx.
$$

Recalling that

$$
||P_k \Psi||_{\infty} \leq C \mu^{\frac{N-2}{4}}
$$

(see [23] formulae (2.14), and (5.4) of the present paper) one gets

$$
\left| \int_{\Omega} F(\Psi) - F(\bar{\Psi}) \right| dx \leq c \mu^{\frac{N-2}{2}}.
$$

(5.17)

From (5.16) and (5.17) we deduce (5.7).

**Lemma 5.3** If $\mu$ is sufficiently small, then, for $t \leq \text{const}$ and $N \geq 5$, one has

$$
I(t\Psi) = \frac{1}{2} \int_{\Omega} (|\nabla(t\Psi)|^2 - \lambda |t\Psi|^2) dx - \int_{\Omega} W(x) F(\Psi) dx =
$$

$$
\frac{1}{\|W^+\|_{\infty}^{\frac{N-2}{2}}} \frac{1}{N} S_{\frac{N}{2}}^N + (C(N) \Delta W(0) - \lambda k_1) \mu + o(\mu).
$$

(5.18)

**Proof**

We recall that $W(0) = \|W^+\|_{\infty}$. From (5.2),(5.6) and (5.7) we get for $N \geq 4$

$$
I(t\Psi) = \frac{t^2}{2} \frac{S_{\frac{N}{2}}^N}{W(0)^{\frac{N-2}{2}}} - \frac{t^{2^*}}{2^*} \frac{S_{\frac{N}{2}}^N}{W(0)^{\frac{N-2}{2}}} + (C(N) \Delta W(0) - \lambda k_1) \mu + o(\mu).
$$

Since $\frac{t^2}{2} - \frac{t^{2^*}}{2^*} \leq \frac{1}{N}$ and $\Delta W(0) \leq 0$, we get (5.18).
2.5. ESTIMATE OF THE CRITICAL LEVEL

Lemma 5.4 Let \( u \in X_k \oplus t\Psi_\mu \) and \( \mu \leq \text{const.} \). If \( u = u_k + t\Psi_\mu, \) \( u_k \in X_k, \) then

\[
\lim_{t \to \infty} I(u_k + t\Psi_\mu) = -\infty \text{ uniformly w.r. to } \mu.
\] (5.19)

Proof

Let \( u = \sum_{i=1}^{k} a_i e_i + t\Psi_\mu \) with \( a_i \in \mathbb{R}. \) Arguing as in the proof of Theorem 1.5, by (1.13) we get

\[
I(u) = \frac{1}{2} \int_{\Omega} \left( |\nabla (\sum a_i e_i + t\Psi_\mu)|^2 - \lambda |\sum a_i e_i + t\Psi_\mu|^2 \right) dx - \int_{\Omega} W(x)F(\sum a_i e_i + t\Psi_\mu)dx \leq \]

\[
\leq c_\lambda \left( \sum a_i^2 + t^2 \right) - c \int_{\Omega} |\sum a_i e_i + t\Psi_\mu|^2 dx \leq \]

\[
\leq c_\lambda \left( \sum a_i^2 + t^2 \right) - c \left( \sum a_i^2 + t^2 \right)^{2^*} \rightarrow -\infty \text{ as } t \rightarrow \infty.
\]

Lemma 5.5 Let \( u = u_k + t\Psi_\mu \in X_k \oplus t\Psi_\mu. \) Then, for \( \mu \) small and for any \( t \in \mathbb{R}\)

\[
\int_{\Omega} W(x)F(u)dx \geq \int_{\Omega} W(x)F(t\Psi_\mu)dx + \frac{C_0}{2} \int_{\Omega} W(x)F(u_k)dx - c_1 t^{2^*} \mu^{\frac{N(N-2)}{2(N+4)}}. \] (5.21)

Proof

We closely follow the argument of the proof of Lemma 2.2 in [23].

\[
\left| \int_{\Omega} W(x)F(u)dx - \int_{\Omega} W(x)F(t\Psi_\mu)dx - \int_{\Omega} W(x)F(u_k)dx \right| = \\
\left| \int_{\Omega} W(x)dx \int_{0}^{u_k + t\Psi_\mu} f(s)ds - \int_{\Omega} W(x)dx \int_{0}^{t\Psi_\mu} f(s)ds - \int_{\Omega} W(x)dx \int_{0}^{u_k} f(s)ds \right| = \\
\left| \int_{\Omega} W(x)dx \int_{t\Psi_\mu}^{u_k + t\Psi_\mu} f(s)ds - \int_{\Omega} W(x)dx \int_{0}^{u_k} f(s)ds \right|.
\]

Changing the variables as \( s = \tau' + t\Psi_\mu, \) and for some \( \theta \in [0, 1], \)

\[
\left| \int_{\Omega} W(x)F(u)dx - \int_{\Omega} W(x)F(t\Psi_\mu)dx - \int_{\Omega} W(x)F(u_k)dx \right| = \] (5.22)\[ \\
= \left| \int_{\Omega} W(x)dx \int_{0}^{u_k} f(\tau' + t\Psi_\mu) - f(\tau')d\tau' \right| = \\
\]

For Proposition 5.6

As in Lemma 2.5 of [23] let us first consider the case

and (5.21) follows by (1.9).

Here \( C_0 \) is the same constant appearing in (1.13). Then we have

\[
\int_{\Omega} W(x) F(u) dx \geq \int_{\Omega} W(x) F(t \Psi_\mu) dx + \int_{\Omega} W(x) F(u_k) dx - \frac{C_0}{2} \int_{\Omega} u_k^2 dx - c_1 t^2 \mu \frac{N(N-2)}{2N+4} \tag{5.23}
\]

and (5.21) follows by (1.9).

Now we can prove the above mentioned estimate for \( I \)

**Proposition 5.6** For \( \mu \) small enough we have, for \( N \geq 4 \)

\[
\sup_{v \in X_k \oplus [\Psi_\mu]} I(v) < \frac{1}{\|W^+\|_{\infty}^2} \frac{1}{N} S_N^N \tag{5.24}
\]

**Proof**

As in Lemma 2.5 of [23] let us first consider the case \( \lambda \neq \lambda_k \). Let us split \( u = u_k + t \Psi_\mu \) with \( u_k \in X_k \) and let \( \overline{\lambda} = \max\{\lambda_k \text{ such that } \lambda_k < \lambda\} \). By Lemma 5.4 we can suppose that \( t \) is bounded. Using (2.21) of [23] we have

\[
I(u) \leq \frac{\overline{\lambda} - \lambda}{2} \int_{\Omega} u_k^2 dx + \frac{1}{2} t^2 \int_{\Omega} (|\nabla \Psi_\mu|^2 - \lambda \Psi_\mu^2) dx - \int_{\Omega} W(x) F(u_k + t \Psi_\mu) dx + c_2 \left( \int_{\Omega} u_k^2 dx \right)^{\frac{1}{2}} \mu \frac{N-2}{4} \tag{5.25}
\]
Let us set \( A(u_k, \mu, c_2) = \frac{\lambda - \lambda}{2} \int_{\Omega} u_k^2 dx + c_2 (\int_{\Omega} u_k^2 dx)^{\frac{1}{2}} \mu^{\frac{N-2}{4}} \) and point out that
\[
A(u_k, \mu, c_2) \leq \frac{c_2^2}{2(\lambda - \lambda)} \mu^{\frac{N-2}{2}}.
\]

If \( \frac{c_0}{2} \int_{\Omega} W(x)F(u_k)dx > c_1 t^2 \mu^{\frac{N(N-2)}{2N+4}} \), by Lemma 5.5 we get
\[
\int_{\Omega} W(x)F(u_k + t\psi_\mu)dx > \int_{\Omega} W(x)F(t\psi_\mu)dx
\]
and then, by (5.18), (5.25) becomes
\[
I(u) \leq \frac{1}{2} t^2 \int_{\Omega} (|\nabla \psi_\mu|^2 - \lambda \psi_\mu^2) dx - \int_{\Omega} W(x)F(t\psi_\mu)dx + \frac{c_2^2}{2(\lambda - \lambda)} \mu^{\frac{N-2}{2}} = (5.26)
\]
\[
= \frac{1}{\|W+\|_{\infty}^{\frac{N-2}{2}}} \frac{1}{N} S_N^\mu + (C(N)\Delta W(0) - \lambda)\mu + o(\mu) < \frac{1}{\|W+\|_{\infty}^{\frac{N-2}{2}}} \frac{1}{N} S_N^\mu.
\]

On the other hand if \( \frac{c_0}{2} \int_{\Omega} W(x)F(u_k)dx \leq c_1 t^2 \mu^{\frac{N(N-2)}{2N+4}} \) we have, by (*) of (5.22)
\[
\int_{\Omega} W(x)F(t\psi_\mu)dx \leq \int_{\Omega} W(x)F(u)dx - \int_{\Omega} W(x)F(u_k)dx + C_0 \left( \int_{\Omega} u_k^2 dx \right)^{\frac{1}{2}} t^{2^* - 1} \mu^{\frac{N-2}{4}} + \frac{C_0}{4} \int_{\Omega} u_k^{2^*} dx + \tilde{C}_0 t^{2^*} \mu^{\frac{N}{2}} \leq (5.27)
\]
\[
\leq \int_{\Omega} W(x)F(u)dx - \frac{3}{4} \int_{\Omega} W(x)F(u_k)dx + C_0 \left( \int_{\Omega} u_k^2 dx \right)^{\frac{1}{2}} t^{2^* - 1} \mu^{\frac{N-2}{4}} + \tilde{C}_0 t^{2^*} \mu^{\frac{N}{2}} \leq \int_{\Omega} W(x)F(u)dx + C_0 \left( \int_{\Omega} u_k^2 dx \right)^{\frac{1}{2}} \mu^{\frac{N-2}{4}} + c_\mu \mu^{\frac{N}{2}} \leq \int_{\Omega} W(x)F(u)dx + c_\mu \frac{N^2 - 2N}{2N+4} + \mu^{\frac{N}{2}} = \int_{\Omega} W(x)F(u)dx + o(\mu).
\]

Hence (5.25) becomes
\[
I(u) \leq \frac{1}{2} t^2 \int_{\Omega} (|\nabla \psi_\mu|^2 - \lambda \psi_\mu^2) dx - \int_{\Omega} W(x)F(u_k + t\psi_\mu)dx + A(u_k, \mu, c_2) \leq (5.28)
\]
\[
I(t\Psi_\mu) + o(\mu) < \frac{1}{\|W^+\|_\infty} \frac{1}{N} S^N_2.
\]

If \( \lambda = \lambda_k \) the claim follows in an analogous way (see [23]). 

Now we are in the position to give the

**Proof of Theorem 1.7** We will apply Proposition 0.2 with the choices \( E_1 = X_k \oplus [\Psi_\mu] \) and \( E_2 = E_1^\perp \). Actually one has two possibilities.

The first one is that \( \overline{c} \) given by (3.19) is a critical value of the functional \( I \), i.e. there exists \( \overline{u} \) such that \( I(\overline{u}) = \overline{c} \) and \( \overline{u} \) is a nontrivial solution of \( (P_\lambda) \), thus Theorem 1.7 is proved.

Otherwise, if \( \overline{c} \) is not a critical value one would have the following statement

\[ \exists u_0 \neq 0, \text{ with } I(u_0) \neq \overline{c} \text{ s.t. } u_0 \text{ solves } (P_\lambda), \quad (5.29) \]

thus still Theorem 1.7 would be proved.

Actually, if (5.29) was false, then by Proposition 4.2 (where \( u \) in (4.2) would be given by the only trivial solution \( u = 0 \), so \( I(u) = 0 \)) one would get \( \overline{c} \) given by (3.19) as a level where any PS sequence admits strongly converging subsequences, since, by definition,

\[ \overline{c} \leq \sup\{I(v) : v \in X_k \oplus [\Psi_\mu]\} \]

and (5.24) holds. Therefore, as a consequence of Proposition 4.2 (see Remark 3.2, page 18), \( \overline{u} \) would be a solution of \( (P_\lambda) \) which contradicts the hypothesis we started by.

\[ \square \]
Chapter 3
Asymptotically linear nonlinearities

3.1 Introduction and main results

We discuss the existence of positive solutions to the problem

\[ \begin{cases} -\Delta u = f(x, u) & u \in H_0^1(\Omega) \\ u|_{\partial \Omega} = 0 \end{cases} \]

where \( \Omega \) is a bounded domain of \( \mathbb{R}^N \) with smooth boundary \( \partial \Omega \), \( f(x, s) \) is allowed to change sign and has an asymptotically linear behaviour w.r. to \( s \) at infinity. More precisely, setting the usual notation \( F(s) = \int_0^s f(x, \xi) d\xi \), we assume that \( f \) satisfies the following hypotheses:

1. \( f: \Omega \times \mathbb{R} \to \mathbb{R} \) is a Caratheodory function, \( f(x, 0) = 0 \) a.e. \( x \in \Omega \). (1.1)

2. There exists a constant \( C \in \mathbb{R} \) such that

\[ \left| \frac{f(x, s)}{s} \right| \leq C \text{ a.e. } x \in \Omega, \forall s \in (0, +\infty). \] (1.2)

Setting \( F^+ = \max\{F, 0\} \), there exists \( \alpha \in L^\infty(\Omega) \) such that

3. Setting \( F^+ = \max\{F, 0\} \), there exists \( \alpha \in L^\infty(\Omega) \) such that

\[ \begin{align*} 
(i) & \quad \frac{\alpha(x)}{2} = \limsup_{s \to 0^+} \left\{ \frac{F^+(x, s)}{s^2} \right\} \text{ uniformly a.e. } x \in \Omega, \\
(ii) & \quad \Lambda_\alpha := \inf_{u \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla u|^2 \, dx}{\int_\Omega \alpha u^2 \, dx} > 1. 
\end{align*} \]
The function $f$ is “asymptotically linear” in $s$ at infinity in the sense (1.4)

(i) there exists $\beta \in C(\overline{\Omega})$ such that $\beta(x) = \lim_{s \to \infty} \frac{f(x, s)}{s}$

uniformly a.e. $x \in \Omega$,

(ii) $\exists x_0 \in \Omega$ such that $\beta(x_0) > 0$,

(iii) setting $\Omega_\beta := \{x \in \Omega : \beta(x) > 0\}$, we assume

$$\Lambda_\beta := \inf_{u \in H_0^1(\Omega_\beta)} \frac{\int_{\Omega_\beta} |\nabla u|^2 dx}{\int_{\Omega_\beta} \beta u^2 dx} < 1.$$ 

Assumption (1.1) implies that $u \equiv 0$ is a solution of problem $(P_0)$. We are interested in the existence of nontrivial solutions.

**Remark 1.1** We point out that $\Lambda_\alpha = +\infty$ is allowed and this fact is equivalent to say that $\alpha(x) = 0$ a.e. $x \in \Omega$. So when $F^+$ has a “superquadratic” behaviour at the origin, assumption (1.3) is trivially satisfied.

**Remark 1.2** Let us note that $\Lambda_\alpha \geq \frac{\lambda_1}{\|\alpha\|_\infty}$. In particular, if

$$\lambda_1 > \|\alpha\|_\infty,$$  

we always have $\Lambda_\alpha > 1$. Condition (1.5) was assumed in [61] for positive nonlinearities.

If $\beta(x) > 0$ a.e. in $\Omega$, then by [61] $\Lambda_\beta$ is achieved in $H_0^1(\Omega)$ and the assumption $\Lambda_\beta < 1$ is also used in [61]. In this chapter we show a result contained in [46], obtained by
extending the case treated by Zhou in [61]. Here we are able to treat more general nonlinearities, and our main improvement consists of allowing $f$ to change sign. Moreover if $f(x,s) \geq 0$ for $s \geq 0$ and $f(x,s) \equiv 0$ for $s \leq 0$, the result of [61] is included by our present result. We first remark that, without loss of generality, we can assume that $f(x,s) = 0$ for all $s \leq 0$. Indeed, by setting

$$g(x,s) = \begin{cases} f(x,s) & s > 0 \\ 0 & s \leq 0 \end{cases}$$

we will easily verify in section 3.5 that every solution of

$$\begin{cases} -\Delta u = g(x,u) \\ u \in H^1_0(\Omega) \end{cases}$$

is in fact positive and thus a solution of ($P_0$). So in the sequel we restrict our attention to problem (1.7) with $g$ defined by (1.6) and $f$ satisfying (1.1–1.4).

Nontrivial solutions of ($P_0$) are obtained via variational methods. In other words we look for nontrivial critical points of the associated functional

$$I(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \int_{\Omega} G(x,u) dx \quad u \in H^1_0(\Omega),$$

where $G(x,s) = \int_0^s g(x,\xi) d\xi$. By the growth assumptions on $f$, and the fact that $s \mapsto g(x,s)$ is continuous, the functional $I$ is well defined, of class $C^1$, and one can easily verify that its critical points are solutions of problem (1.7). While in [61] the results are derived using a particular version of the Mountain Pass Theorem, in the present paper we are able to prove, using the classical Mountain Pass Theorem of Ambrosetti–Rabinowitz (see [8]), that problem ($P_0$) has always a positive solution under assumptions (1.1)–(1.4).

This chapter is organized as follows.

Section 3.2 provides some basic properties on the principal eigenvalues of a linear operator with indefinite weight function due to Hess and Kato [39]. In Section 3.3 we show that, under conditions (1.3)-(1.4), the functional $I$ exhibits a Mountain Pass structure. Section 3.4 is devoted to the study of unbounded Palais–Smale sequences. While in
the superquadratic case (i.e. when (0.3) is satisfied) all the Palais–Smale sequences are bounded, this fact is not anymore true when the nonlinearity is asymptotically linear. So we establish a relationship between the existence of unbounded Palais–Smale sequences and of positive solutions to the following eigenvalue problem

\[
\begin{aligned}
-\Delta u &= \beta(x)u \\
&\quad u > 0
\end{aligned}
\]

(1.9)

In Section 3.5, by showing that (1.4) implies the nonexistence of solutions to (1.9), we are able to exclude the presence of unbounded Palais–Smale sequences, and thus to obtain an existence result for (1.7), and so for \((P_0)\). Finally, in Section 3.6, in order to illustrate the meaning of our assumptions, we give some examples of nonlinearities for which the existence result in section 3.5 applies.

Notations: \(u^+ = \max\{u, 0\}\), \(u^- = \max\{-u, 0\}\), \(u = u^+ - u^-\),

\[\|u\| = \left(\int_{\Omega} |\nabla u|^2 dx\right)^{\frac{1}{2}}.\]

### 3.2 The Principal Eigenvalue

In this section we show some properties of principal eigenvalues and we recall a well-known result by Hess and Kato [39].

**Definition 2.1** Let \(D\) be a bounded smooth domain of \(\mathbb{R}^N\), \(h \in L^\infty(D)\). We say that \(\Lambda \in \mathbb{R}\) is a principal eigenvalue for the problem

\[
\begin{aligned}
-\Delta u &= \Lambda h(x)u \\
u &\in H_0^1(D)
\end{aligned}
\]

(2.1)

if there exists \(u \in H_0^1(D), u > 0\), solving (2.1).

**Proposition 2.2** Let \(D\) be a bounded smooth domain of \(\mathbb{R}^N\) such that

\(h \in L^\infty(D), D^+ := \{x \in D; h(x) > 0\}\) is an open set,
and let
\[ \Lambda := \inf_{u \in H^1_0(D)} \left\{ \int_D |\nabla u|^2 \, dx : \int_D h(x)u^2 \, dx = 1 \right\}. \]

Then one has
\begin{enumerate}
  \item \( S := \left\{ u \in H^1_0(D) : \int_D h(x)u^2 \, dx = 1 \right\} \neq \emptyset, \)
  \item \( \exists \hat{u} \in S \text{ such that } \int_D |\nabla \hat{u}|^2 \, dx = \Lambda, \)
  \item \( |\hat{u}| > 0. \)
\end{enumerate}

So \( \Lambda \) is a principal eigenvalue of (2.1).

**Proof**

Let us divide the proof in steps.

**Step 1**

Let \( u \in H^1_0(D^+) \) (well defined since \( D^+ \) is open), \( u \neq 0 \), then
\[ \int_{D^+} h(x)u^2 \, dx > 0 \quad (\text{since } \text{meas}(D^+) > 0). \] (2.2)

Choose \( \bar{u} \in H^1_0(D) \) as
\[ \bar{u}(x) = \begin{cases} \frac{u(x)}{\sqrt{\int_{D^+} h(x)u^2 \, dx}} & x \in D^+ \\ 0 & x \in D \setminus D^+ \end{cases} \] (2.3)

Then
\[ \int_{D^+} h(x)\bar{u}^2 \, dx = \int_{D^+} h(x)\frac{u^2}{\int_{D^+} h(x)u^2 \, dx} \, dx = 1. \] (2.4)

**Step 2**

Let \( \{u_n\} \in H^1_0(D) \) be a minimizing sequence, i.e.
\[ \int_D |\nabla u_n|^2 \, dx \to \Lambda \quad \text{and} \quad \int_D h(x)u_n^2 \, dx = 1, \] (2.5)

so \( \{u_n\} \) is bounded in \( H^1_0(D) \). Then
\[ u_n \rightharpoonup \hat{u} \text{ in } H^1_0(D) \quad \text{and} \quad u_n \to \hat{u} \text{ in } L^2(D). \] (2.6)
We have \( \hat{u} \in S \), indeed
\[
1 = \lim_{n \to \infty} \int_D h(x) u_n^2 dx = \int_D h(x) \hat{u}^2 dx
\]  
(2.7)
(from \( h \in L^{\infty} \) we deduce \( \int_D h(x) [u_n^2 - \hat{u}^2] dx \to 0 \)). Then, by the minimizing properties of \( \{ u_n \} \) and by the lower semicontinuity of the functional \( \int_D |\nabla u|^2 dx \), one has
\[
\Lambda = \lim \inf_{n \to \infty} \int_D |\nabla u_n|^2 dx \geq \int_D |\nabla \hat{u}|^2 dx \geq \Lambda.
\]  
(2.8)
So one gets \( \Lambda = \int_D |\nabla \hat{u}|^2 dx \).

**Step 3**

Let \( \hat{u} \) be a minimizer of
\[
\left\{ \int_D |\nabla u|^2 dx : \int_D h(x) u^2 dx = 1 \right\}.
\]

We show that \( |\hat{u}| \) is a minimizer too. If \( \hat{u} \in H_0^1(D) \), then \( |\hat{u}| \in H_0^1(D) \) (see Lemma 7.6, p 145, of [36]). Moreover
\[
\int_D h(x) \hat{u}^2 dx = \int_D h(x) |\hat{u}|^2 dx,
\]
so \( |\hat{u}| \in S \). Finally
\[
\Lambda = \int_D |\nabla \hat{u}|^2 dx = \int_D |\nabla |\hat{u}||^2 dx,
\]
so \( |\hat{u}| \) is a minimizer. By the strong maximum principle (see [36] Thm. 8.19, p. 198), \( \hat{u} > 0 \). \( \square \)

Finally let us recall the following result by Hess and Kato [39]:

**Theorem 2.3** Let \( h \in C(\bar{\Omega}) \) be such that \( h(x_0) > 0 \) for some \( x_0 \in \Omega \). Then the problem
\[
\left\{ \begin{array}{ll}
-\Delta u = \lambda h(x) u & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{array} \right.
\]  
(2.9)
admits a principal eigenvalue \( \lambda_1(h) > 0 \). Moreover, assume (2.9) holds with \( \lambda \geq \lambda_1(h) \). Then \( u > 0 \) only if \( \lambda = \lambda_1(h) \).

From this result we derive the following corollary
Corollary 2.4 Let $h \in C(\overline{D})$, and $\Lambda$ be defined as in Proposition 2.2. Let $(\lambda, u) \in (0, +\infty) \times H^1_0(D)$ be such that
\[
\begin{cases}
-\Delta u = \lambda h(x)u \\
u > 0
\end{cases}
\tag{2.10}
\]
Then $\lambda = \Lambda$.

### 3.3 Mountain Pass Structure

In this section we will prove that the functional $I$ has a Mountain Pass structure.

**Lemma 3.1** Let $\delta > 0$. For each $u \in H^1_0(\Omega)$ let us define
\[\Omega_\delta = \{ x \in \Omega : 0 < u(x) < \delta \}.\]
Then, assuming (1.1), (1.2), there exists a constant $C = C(\delta)$ such that
\[
\frac{\int_{\Omega \setminus \Omega_\delta} G(x, u)dx}{\int_{\Omega} |\nabla u|^2dx} \leq C\|u\|^{p-2}. \tag{3.1}
\]

**Proof**

Let us choose $p \in (2, 2^*)$. From (1.2) we deduce the existence of a constant $C' = C(\delta)$ such that
\[|G(x, s)| \leq C's^p \quad \forall \; |s| > \delta. \tag{3.2}\]
From this inequality and the Sobolev embedding $H^1_0(\Omega) \hookrightarrow L^p(\Omega)$, one gets
\[
\frac{\int_{\Omega \setminus \Omega_\delta} G(x, u)dx}{\int_{\Omega} |\nabla u|^2dx} \leq C\frac{\int_{\Omega \setminus \Omega_\delta} |u|^pdx}{\int_{\Omega} |\nabla u|^2dx} \leq C\left(\frac{\int_{\Omega} |\nabla u|^2dx}{\int_{\Omega} |\nabla u|^2dx}\right)^{\frac{p-2}{2}} = C\|u\|^{p-2}. \tag{3.3}
\]

**Remark 3.2** If $G$ is superquadratic near $s = 0$, then the conclusion of Lemma 3.1 holds also with $\delta = 0$. On the contrary, when $G$ has quadratic growth near the origin,
the above Lemma is no longer true with \( \delta = 0 \), as the following example shows: choose \( G(x, s) = s^2 \), let \( \psi \) be an eigenfunction related to the first eigenvalue of \(-\Delta\) in \( H^1_0(\Omega)\), then
\[
\frac{\int_{\Omega} |\varepsilon \psi|^2 \, dx}{\int_{\Omega} |\nabla (\varepsilon \psi)|^2 \, dx} = \frac{1}{\lambda_1} > 0.
\]

**Proposition 3.3** Under assumption (1.3) \( \exists \rho, M > 0 \) such that \( I|_{\partial B_\rho} > 0 \).

**Proof**

Let \( \varepsilon > 0 \). By (1.3), there exists \( \delta > 0 \) such that
\[
0 \leq \sup_{0 < s < \delta} \left\{ \frac{G^+(x, s)}{s^2} \right\} - \frac{\alpha(x)}{2} < \varepsilon \quad \text{a.e. } x \in \Omega.
\]

For each \( u \in H^1_0(\Omega) \), let \( \Omega_\delta \) be defined as in Lemma 3.1. Then one gets
\[
I(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx - \int_{\Omega_\delta} G(x, u) \, dx - \int_{\Omega \setminus \Omega_\delta} G(x, u) \, dx \geq \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx - \frac{1}{2} \int_{\Omega_\delta} G^+(x, u) \, dx - \int_{\Omega \setminus \Omega_\delta} G(x, u) \, dx \geq \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx - \int_{\Omega_\delta} \alpha u^2 \, dx \geq \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx - \int_{\Omega_\delta} \sup_{0 < s < \delta} \left\{ \frac{G^+(x, s)}{s^2} \right\} \, dx - \int_{\Omega \setminus \Omega_\delta} G(x, u) \, dx.
\]

From the following inequalities,
\[
\int_{\Omega_\delta} \frac{\alpha}{2} u^2 \, dx \leq \int_{\Omega} \frac{\alpha}{2} u^2 \, dx, \quad (observe that, by definition, \( \alpha(x) \geq 0 \)),
\]

\[
\int_{\Omega_\delta} \left[ \sup_{0 < s < \delta} \left\{ \frac{G^+(x, s)}{s^2} \right\} - \frac{\alpha}{2} \right] u^2 \, dx \leq \varepsilon \int_{\Omega_\delta} u^2 \, dx \leq \int_{\Omega} u^2 \, dx,
\]

we deduce
\[
I(u) \geq \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx - \frac{1}{2} \int_{\Omega} \alpha u^2 \, dx - \varepsilon \int_{\Omega_\delta} u^2 \, dx - \int_{\Omega \setminus \Omega_\delta} G(x, u) \, dx.
\]

So, from the definition of \( \Lambda_\alpha \), the characterization of \( \lambda_1 \) and Lemma 3.1, we get
\[
\frac{I(u)}{\int_{\Omega} |\nabla u|^2 \, dx} \geq \frac{1}{2} \left( 1 - \frac{1}{\lambda_1} \right) \frac{\int_{\Omega} \alpha u^2 \, dx}{\int_{\Omega} |\nabla u|^2 \, dx} - \varepsilon \frac{\int_{\Omega_\delta} u^2 \, dx}{\int_{\Omega} |\nabla u|^2 \, dx} - \frac{\int_{\Omega \setminus \Omega_\delta} G(x, u) \, dx}{\int_{\Omega} |\nabla u|^2 \, dx} \geq \frac{1}{2} - \frac{1}{2 \lambda_1} - \varepsilon \frac{1}{\lambda_1} - C\|u\|^{p-2}.
\]
By (1.3), since $\varepsilon$ is arbitrary, we deduce the existence of $\rho, M > 0$ such that
\[
I(u) \geq M \quad \forall \|u\| = \rho.
\]

\[\text{(3.10)}\]

Proposition 3.4 Assuming (1.1), (1.2), (1.4) one gets

\[
\exists u_0 \in H_0^1(\Omega) \text{ such that } I(u_0) < 0.
\]

(3.11)

Proof

Let $\Omega_\beta$ be defined as in (1.4). From Prop. 2.2, there exists $\phi \in H_0^1(\Omega_\beta)$ such that
\[
\int_{\Omega_\beta} |\nabla \phi|^2 dx = \Lambda, \quad \int_{\Omega_\beta} \beta(x)\phi^2 dx = 1, \text{ and } \phi > 0.
\]

(3.12)

Let $\phi$ be extended by 0 in $\Omega \setminus \Omega_\beta$. We have
\[
I\left(\frac{t\phi}{t^2}\right) = \frac{1}{2} \int_{\Omega_\beta} |\nabla \phi|^2 dx - \int_{\Omega_\beta} \frac{G(x, t\phi)}{t^2\phi^2} \phi^2 dx.
\]

(3.13)

Then by Fatou’s Lemma
\[
\liminf_{t \to \infty} I\left(\frac{t\phi}{t^2}\right) \leq \frac{1}{2} \int_{\Omega_\beta} |\nabla \phi|^2 dx - \int_{\Omega_\beta} \liminf_{n \to \infty} \frac{G(x, t\phi)}{t^2\phi^2} \phi^2 dx =
\]

\[
= \frac{1}{2} \int_{\Omega_\beta} |\nabla \phi|^2 dx - \int_{\Omega_\beta} \liminf_{t \to \infty} \left[ \frac{G(x, t\phi)}{t^2\phi^2} - \frac{\beta(x)}{2} \right] u^2 - \frac{\beta(x)}{2} \phi^2 dx =
\]

\[
= \frac{1}{2} \int_{\Omega_\beta} |\nabla \phi|^2 dx - \int_{\Omega_\beta} \liminf_{s \to \infty} \left[ \frac{G(x, s)}{s^2} - \frac{\beta(x)}{2} \right] \phi^2 dx - \int_{\Omega_\beta} \frac{\beta(x)}{2} \phi^2 dx =
\]

\[
= \frac{1}{2} \int_{\Omega_\beta} |\nabla \phi|^2 dx - \frac{1}{2} \int_{\Omega_\beta} \beta(x)\phi^2 = \frac{1}{2}(\Lambda - 1).
\]

By (1.4) $\Lambda < 1$, so $\liminf_{t \to \infty} I\left(\frac{t\phi}{t^2}\right) < 0$ and $I(t\phi) \to -\infty$ as $t \to \infty$. \hfill \Box

3.4 Palais–Smale Sequences

Proposition 4.1 Assume (1.1) and (1.2). Let $\{u_n\}$ be a Palais–Smale (PS) sequence for the functional (1.8), i.e.
\[
I(u_n) \to c, \quad \|I'(u_n)\| \to 0, \text{ as } n \to \infty.
\]

(4.1)

If $\{u_n\}$ is unbounded ($\|u_n\| \to \infty$), then, putting $w_n := \frac{u_n}{\|u_n\|}$, one has that $w_n \to w$ in $H_0^1(\Omega)$, with $w$ satisfying
(1) $w \not\equiv 0$

(2) $w > 0$

(3) $-\Delta w = \beta(x)w$ in $\Omega$.

**Proof**

We have $\|w_n\| = 1$, so the sequence $\{w_n\}$ is bounded in $H^1_0(\Omega)$, then

$$w_n \rightharpoonup w \text{ in } H^1_0(\Omega),$$

$$w_n \rightarrow w \text{ in } L^2(\Omega),$$

$$w_n \rightarrow w \text{ a.e. in } \Omega.$$ 

Let us divide the proof in steps.

*Step 1: $w \not\equiv 0$*

Arguing by contradiction, if $w \equiv 0$, then $w_n \rightarrow 0$ in $L^2$, and

$$\frac{DI_{(u_n)}(u_n)}{\|u_n\|^2} \rightarrow 0 \text{ (from definition of (PS) sequence)}. \quad (4.2)$$

So, a fortiori,

$$\frac{DI_{(u_n)}(u_n)}{\|u_n\|^2} \rightarrow 0. \quad (4.3)$$

This yields

$$\int_\Omega |\nabla w_n|^2 dx - \int_\Omega \frac{g(x, u_n)}{u_n} w_n^2 dx \rightarrow 0, \quad (4.4)$$

$$1 - \int_\Omega \frac{g(x, u_n)}{u_n} w_n^2 dx \rightarrow 0.$$ 

Since $\frac{g(x, u_n)}{u_n}$ is bounded and $w_n \rightarrow 0$ in $L^2$, we get the relation

$$1 = 0,$$

hence we must have $w \not\equiv 0$. 
Step 2: \( w > 0 \).

Knowing that
\[
\frac{DI_{u_n}(u_n)}{\| \phi \| \| u_n \|} \to 0, \quad \forall \phi \in H_0^1(\Omega)
\]
we deduce
\[
\frac{\int_\Omega \langle \nabla u_n, \nabla \phi \rangle \ dx - \int_\Omega g(x, u_n) \phi \ dx}{\| u_n \|} \to 0.
\]
Since \( g(x, s) = 0 \) for \( s \leq 0 \),
\[
\frac{\int_\Omega \langle \nabla u_n, \nabla \phi \rangle \ dx - \int_\Omega g(x, u_n^+) \phi \ dx}{\| u_n \|} \to 0,
\]
\[
\int_\Omega \langle \nabla w_n, \nabla \phi \rangle \ dx - \int_\Omega \frac{g(x, u_n^+)}{(u_n^+) \| u_n^+ \|} \phi \ dx \to 0.
\]
Since \( \frac{g(x, u_n^+)}{(u_n^+) \| u_n^+ \|} \) is bounded, it converges weakly in \( L^2 \) to some function \( \gamma \in L^\infty \). Then,
\[
\int_\Omega \langle \nabla w_n, \nabla \phi \rangle \ dx - \int_\Omega \frac{g(x, u_n^+)}{(u_n^+) \| u_n^+ \|} w_n^+ \phi \ dx \to 0,
\]
\[
\int_\Omega \langle \nabla w, \nabla \phi \rangle \ dx - \int_\Omega \gamma(x) w^+ \phi \ dx = 0 \quad \forall \phi \in H_0^1(\Omega).
\]
Choosing \( \phi = w^- \), one gets
\[
\int_\Omega \langle \nabla w, \nabla w \rangle \ dx - \int_\Omega \gamma(x) w^+ w^- \ dx = 0,
\]
\[
\int_\Omega |\nabla w^-|^2 \ dx = 0,
\]
which implies
\[
w^- = 0, \text{ then } w \geq 0.
\]
So \( w \geq 0 \) satisfies the equation
\[
-\Delta w = \gamma(x) w \quad \text{in } \Omega. \tag{4.5}
\]
Splitting \( \gamma(x) \) yields
\[
-\Delta w + \gamma^+(x) w = \gamma^-(x) w. \tag{4.6}
\]
By the strong maximum principle one has that either \( w > 0 \) or \( w \equiv 0 \). But, by Step 1, \( w \not\equiv 0 \). Then we can conclude that \( w > 0 \).
Step 3: \( w \) satisfies the equation \(-\Delta w = \beta(x)w\).

Since \( w > 0 \), \( u_n \to +\infty \) a.e. So

\[
\frac{g(x, u_n)}{u_n} \to \beta(x) \text{ a.e.,} \tag{4.7}
\]

\[
\frac{g(x, u_n)}{u_n} \to \gamma(x) \text{ in } L^2. \tag{4.8}
\]

this yields \( \beta(x) = \gamma(x) \), then \( w \) verifies the equation

\[
\int_{\Omega} \nabla w \nabla \phi dx = \int_{\Omega} \beta(x) w \phi dx,
\]

so

\[-\Delta w = \beta(x)w.\]

\[\blacksquare\]

**Proposition 4.2** Assume (1.1) and (1.2) and let \( \{u_n\} \) be an unbounded (PS) sequence. Then,

\[
\Lambda(\Omega) = \inf_{u \in H_0^1(\Omega)} \left\{ \int_{\Omega} |\nabla u|^2 dx: \int_{\Omega} \beta(x) u^2 dx = 1 \right\} = 1. \tag{4.9}
\]

**Proof**

By Proposition 4.1, \( w_n := \frac{u_n}{\|u_n\|} \to w \) with

\[
\begin{cases}
-\Delta w = \beta(x)w \\
w \in H_0^1(\Omega) \\
w > 0
\end{cases}
\]

(4.10)

So \( \lambda = 1 \) is a positive principal eigenvalue. By Proposition 2.2, \( \Lambda(\Omega) \) is also a positive principal eigenvalue. By Theorem 2.4, there exits a unique positive principal eigenvalue. Thus

\[
\Lambda(\Omega) = 1.
\]

\[\blacksquare\]

Proposition 4.2 implies that, if \( \Lambda(\Omega) \neq 1 \), then every (PS) sequence is bounded. In particular one has the following
Corollary 4.3 Assume (1.1) and (1.2). If $\Lambda_\beta < 1$, then every (PS) sequence is bounded.

Proof

$$\Lambda(\Omega) \leq \Lambda_\beta < 1.$$  

Remark 4.4 For nonlinearities satisfying condition (0.3), it is known that every (PS) sequence is bounded. Under assumptions (1.1), (1.2), the existence of unbounded (PS) sequences cannot be excluded. This can be shown by considering

$$G(s) = \begin{cases} 
\lambda_1 \frac{s^2}{2} & s \geq 0 \\
0 & s < 0 
\end{cases} \quad (4.11)$$

Taking $\psi_1$ a positive eigenfunction of $-\Delta$ related to $\lambda_1$, we see that the sequence $u_n = n\psi_1$ is an unbounded (PS) sequence for the functional

$$I(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 dx - \int_\Omega G(u) dx.$$  

Remark 4.5 When $f(x,s) \neq 0$ for $s \leq 0$, the analysis of unbounded (PS) sequence is more complicated. In particular the assumption that $\Lambda(\Omega) \neq 1$ in Proposition 4.2 is not sufficient to ensure the (PS) condition. To see this, one can consider the functional

$$I(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 dx - \lambda_k \int_\Omega u^2 dx,$$

where $\lambda_k$ is any eigenvalue of $(-\Delta, H^1_0(\Omega))$.

Example

For a nonlinearity of the kind

$$f(x,s) = \begin{cases} 
\lambda s + r(s) & s > 0 \\
0 & s \leq 0 
\end{cases}$$

with $r \in L^\infty(\Omega)$, we have $\Lambda_\beta = \frac{\lambda}{\lambda_1}$. In this case Corollary 4.3 asserts that for any $\lambda < \lambda_1$ every (PS) sequence of the functional (1.8) is bounded.
3.5 Existence Theorem

Our existence result will be based on the classical Mountain Pass Theorem (see [8, 51] and Preliminaries).

**Theorem 5.1** [46] Under assumptions (1.1)–(1.4), problem \((P_0)\) admits a positive solution.

**Proof**

By Propositions 3.3, 3.4, and Corollary 4.3, the functional

\[
I(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \int_{\Omega} G(x, u) dx
\]

satisfies the assumptions of the Mountain Pass Theorem. So there exists \(u \in H^1_0(\Omega), u \not\equiv 0\), such that

\[
\int_{\Omega} \nabla u \nabla \phi dx = \int_{\Omega} g(x, u) \phi dx \quad \forall \phi \in H^1_0(\Omega).
\]

Taking \(\phi = u^-,\) and since \(g(x, u) = 0\) for \(s \leq 0,\) we get

\[
\int_{\Omega} |\nabla u^-|^2 dx = 0.
\]

So \(u \geq 0,\) and, in particular, \(u\) is a solution of problem \((P_0).\) By strong maximum principle we have \(u > 0.\)

\[\square\]

**Remark 5.2** When \(\Lambda \beta \geq 1,\) without other restrictions on the nonlinearity \(f,\) we cannot ensure the existence of solutions to problem \((P_0).\) Actually, consider a nonlinearity satisfying (1.1)-(1.3) and

\[
0 \leq f(x, s) \leq \beta(x).
\]

Then, any solution of problem \((P_0)\) satisfies

\[
\int_{\Omega} |\nabla u|^2 dx = \int_{\Omega} f(x, u) u dx \leq \int_{\Omega} \beta(x) u^2 dx.
\]

So, \(\Lambda \beta \leq 1.\)
3.6 Some Examples

This section provides some examples for which Theorem 5.1 applies.

Example 1. Let us consider a nonlinearity $f$ satisfying (1.1),(1.2) and is such that the functions $\alpha$ and $\beta$ are independent of $x \in \Omega$, i.e.:

$$\limsup_{s \to 0^+} \frac{F^+ (x, s)}{s^2} = \alpha \quad \text{and} \quad \lim_{s \to \infty} \frac{f(x, s)}{s} = \beta,$$

with $\beta > \alpha \geq 0$. In this case, we have:

$$\Lambda_\alpha = \begin{cases} +\infty & \text{if } \alpha = 0 \\ \frac{\lambda_1}{\alpha} & \text{if } \alpha > 0 \end{cases}, \quad \Lambda_\beta = \frac{\lambda_1}{\beta}$$

and Theorem 5.1 gives: problem $(P_0)$ has a positive solution if

$$\lambda_1 \in (\alpha, \beta). \quad (6.1)$$

As a particular case, let us consider the nonlinearities (0.4) and (0.5) given in the introduction. Those nonlinearities clearly satisfy (1.1) and (1.2). For

$$f(s) = \frac{|s|^2}{1 + \gamma |s|^2} s, \quad \gamma > 0,$$

we find that

$$\alpha = 0 \quad \text{and} \quad \beta = \frac{1}{\gamma},$$

and condition (6.1) in this case is equivalent to

$$\lambda_1 \in \left(0, \frac{1}{\gamma}\right). \quad (6.2)$$

For the nonlinearity

$$f(s) = \left(1 - \frac{1}{e^{\gamma |s|^2}}\right) s,$$

one has

$$\alpha = 0 \quad \text{and} \quad \beta = 1,$$

and so condition (6.1) is equivalent to

$$\lambda_1 \in (0, 1). \quad (6.3)$$
Let us note that conditions (6.2) and (6.3) are always satisfied if the domain $\Omega$ is sufficiently large.

**Example 2.** Let us now consider a kind of nonlinearity $f$ which has been considered by Hess [38] and Ambrosetti-Hess [6]:

$$f(x,s) = \lambda (m_\infty s + r(s))$$

satisfying:

1) $m_\infty, \lambda > 0$;

2) $r : \mathbb{R} \rightarrow \mathbb{R}$ is continuous, $r(0) = 0$;

3) $\left| \frac{r(s)}{s} \right| \leq C \forall s \geq 0$;

4) $r'_+(0) = \lim_{s \rightarrow 0^+} \frac{r(s)}{s}$ exists and $r'_+(0) < 0$.

In this case,

$$\alpha = \begin{cases} 
\lambda (m_\infty + r'_+(0)) & \text{if } m_\infty + r'_+(0) > 0 \\
0 & \text{otherwise}
\end{cases} \quad \text{and} \quad \beta = \lambda m_\infty.$$

From condition (6.1) found in example 1, we have that problem $(P_0)$ has a positive solution if

$$\lambda_1 \in (\lambda [m_\infty + r'_+(0)], \lambda m_\infty).$$

In other words, setting

$$\lambda_\infty = \begin{cases} 
\frac{\lambda_1}{m_\infty + r'_+(0)} & \text{if } m_\infty + r'_+(0) > 0 \\
\infty & \text{otherwise}
\end{cases}$$

we see that problem $(P_0)$ has a positive solution if $\lambda \in (\frac{\lambda_1}{m_\infty}, \lambda_\infty)$.

In [38], using the Leray-Schauder degree, Hess already established this result assuming $r$ bounded. Actually, the existence result still holds if we just have that $\frac{r(s)}{s}$ is bounded (for $s \geq 0$). In particular, nonlinearities of the kind:

$$f(s) = \lambda (m_\infty s - \ln(1 + s))$$
are allowed.

**Example 3.** Consider a nonlinearity of the kind:

\[
f(x, s) = \lambda s + r(x, s)
\]

where \( r \) satisfies:

1) \( r : \Omega \times \mathbb{R} \to \mathbb{R} \) is Caratheodory (for example continuous) and \( r(x, 0) = 0 \) a.e. \( x \in \Omega \);

2) \( \left| \frac{r(x, s)}{s} \right| \leq C \forall s \geq 0; \)

3) The following limits exist:

\[
r_0 := \lim_{s \to 0^+} \frac{r(x, s)}{s}, \quad r_{\infty} := \lim_{s \to \infty} \frac{r(x, s)}{s},
\]

and \( r_0 < r_{\infty} \).

In this case, \( \alpha = \begin{cases} 
\lambda + r_0 & \text{if } \lambda + r_0 > 0 \\
0 & \text{otherwise}
\end{cases} \) and \( \beta = \lambda + r_{\infty} \).

So condition (6.1) becomes \( \lambda_1 \in (\lambda + r_0, \lambda + r_{\infty}) \). Thus, for this example, problem \((P_0)\) has a positive solution if \( \lambda \in (\lambda_1 - r_{\infty}, \lambda_1 - r_0) \).

We end this section by giving an explicit example where the function \( \beta \) changes sign.

**Example 4.** Let us consider the nonlinearity:

\[
f : (-\pi, \pi) \times \mathbb{R} \to \mathbb{R}, \quad (x, s) \mapsto \lambda \sin x \ r(s)
\]

where \( \lambda > 0 \) and \( r : \mathbb{R} \to \mathbb{R} \) is any continuous function satisfying:

\[
\lim_{s \to 0^+} \frac{r(s)}{s} = 0, \quad \lim_{s \to \infty} \frac{r(s)}{s} = 1.
\]

Then, we see easily that

\[
\Lambda_\alpha = \infty, \quad \beta(x) = \lambda \sin x, \quad \Omega_\beta = (0, \pi)
\]
and

\[ \Lambda_\beta = \frac{1}{\lambda} \inf_{u \in H^1_0(0,\pi)} \left\{ \frac{\int_0^\pi |u'|^2 \, dx}{\int_0^\pi \sin x \ u^2 \, dx} \right\}. \]

For \( \lambda \) big enough, this quantity is strictly less than 1. By taking the test function \( u(x) = \sin x \), we see that \( \Lambda_\beta \leq \frac{3\pi}{8} \frac{1}{\lambda} \). Thus, for this example of nonlinearity, problem \((P_0)\) has certainly a positive solution if \( \lambda > \frac{3\pi}{8} \).
Chapter 4

Hamiltonian Systems

4.1 Introduction and main results

Let us consider the second order Hamiltonian system

\[ \ddot{x} + A(t)x + b(t)V'(x) = 0 \]  \hspace{1cm} (1.1)

where \( A(t) \) is a continuous \( T \)-periodic (for some fixed \( T > 0 \)) matrix valued function, \( b(t) \) is a continuous \( T \)-periodic real function and \( V \in C^2(\mathbb{R}^N, \mathbb{R}) \). We deal with a potential, indefinite in sign, that satisfies some suitable evenness conditions and we prove a multiplicity result (see [11]) for \( T \)-periodic solutions of (1.1). If \( A(t) \) is negative definite, we also find the existence of subharmonic solutions, which have some symmetry properties, and of a homoclinic solution, obtained by a limit procedure starting from a suitable translation of these subharmonics. The techniques of the proofs are based on the consideration of the Nehari’s manifold \( M \), suitably connected to the functional \( I \) associated with problem (1.1) and the symmetry of the potential allows to use some constrained minimum arguments and a result of the Ljusternik-Schrînelmann category theory. Let us assume the following conditions:

\( A(t) \) is a symmetric continuous \( T \)-periodic \( N \times N \) matrix valued function,

\( b(t) \) is a continuous \( T \)-periodic real function such that :

\[ \exists t_0 \in [0,T] \text{ such that } b(t_0) > 0, \]  \hspace{1cm} (1.2)

\( V \in C^2(\mathbb{R}^N, \mathbb{R}) \) and there exists a constant \( \beta > 2 \) for which :

\[ \exists a_1 > 0 : \quad V(x) \geq a_1 |x|^{\beta} \quad \forall x \in \mathbb{R}^N \]  \hspace{1cm} (1.3)
\[ \exists a_2 > 0 : |V'(x)| \leq a_2 |x|^{\beta-1} \quad \forall x \in \mathbb{R}^N \quad (1.4) \]

\[ V'(x) \cdot x \geq \beta V(x) \quad \forall x \in \mathbb{R}^N \quad (1.5) \]

\[ V''(x) \cdot x \geq (\beta - 1) V'(x) \cdot x \quad \forall x \in \mathbb{R}^N \quad (1.6) \]

\[ V(x) = V(-x) \quad \forall x \in \mathbb{R}^N. \quad (1.7) \]

**Remark 1.1** It is well known that any given square symmetric matrix \( A(t) \) can be written as the sum of two matrices, i.e.

\[ A = A^+ + A^- \quad (1.8) \]

where \( A^+ \) and \( A^- \) are positive semidefinite and negative semidefinite respectively, more precisely the positive (negative) eigenvalues of \( A \) coincide with the eigenvalues of \( A^+ \) (\( A^- \)) different from zero.

For any fixed \( t \in \mathbb{R} \) let us put

\[ \Lambda^+ = \max_{t \in [0,T]} \left( \max_i \lambda_i^+(t) \right) \quad \left( \Lambda^- = \max_{t \in [0,T]} \left( \max_i \lambda_i^-(t) \right) \right) \quad (1.9) \]

where \( \lambda_i^+(t) \) (\( \lambda_i^-(t) \)) are the eigenvalues of the matrix \( A^+(t) \) (resp. \( A^-(t) \)) \((i = 1, \ldots, N\)) for \( T \in [0,T] \) (observe that, in general, we can’t assure that \( \Lambda^+ \) and \( \Lambda^- \) are different from zero). We will prove the following theorem:

**Theorem 1.2** [11] Let \( b(t) \in C^0([0,T], \mathbb{R}) \) be a \( T \)-periodic function satisfying (1.2), \( A(t) = [a_{ij}(t)] \) be a \( N \times N \) symmetric matrix, where \( a_{ij} \) is a \( T \)-periodic continuous function, for \( i, j = 1, \ldots, n \), and such that:

\[ \Lambda^+ < \frac{4}{T^2} \quad (1.10) \]

where \( \Lambda^+ \) is given by (1.8) and \( V \in C^2(\mathbb{R}^N, \mathbb{R}) \) be satisfying (1.3)–(1.7).

If we assume that \( b(t), A(t) \) and \( V(x) \) verify the following further conditions:

\[ b(t) = b(T-t) \quad \forall t \in \left[ 0, \frac{T}{2} \right], \quad (1.11) \]
4.1. INTRODUCTION AND MAIN RESULTS

\[ A(t) = A(T - t) \quad \forall t \in \left[0, \frac{T}{2}\right] , \]  
(1.12)

\[ B^- [V'(x) \cdot x - \beta V(x)] \leq c|x|^2, \quad \forall x \in \mathbb{R}^N \]  
(1.13)

and

\[ B^- [V''(x) x \cdot x - (\beta - 1)V'(x) \cdot x] \leq d|x|^2 \quad \forall x \in \mathbb{R}^N \]  
(1.14)

where the positive constants \( c \) and \( d \) are such that

\[ \max\{2c, d\} < (\beta - 2)(1 - \frac{T^2}{4}\Lambda^+)\frac{4\pi T^2}{T^2} , \]  
(1.15)

and \( B^- = \max_{t \in [0, T]} b^-(t) \), with \( b^-(t) = -\min\{0, b(t)\} \).

Then there exist infinitely many pairs of distinct \( T \)-periodic solutions \( x(t) \) of (1.1), which verify

\[ x \left( t + \frac{T}{2} \right) = -x \left( \frac{T}{2} - t \right) \quad \text{for any } t \in \left[0, \frac{T}{2}\right] . \]  
(1.16)

On the other hand, replacing (1.11), (1.12) and (1.15) respectively with

\[ b(t) = b(t + \frac{T}{2}) \quad \forall t \in \left[0, \frac{T}{2}\right] , \]  
(1.17)

\[ A(t) = A(t + \frac{T}{2}) \quad \forall t \in \left[0, \frac{T}{2}\right] , \]  
(1.18)

\[ \max\{2c, d\} < (\beta - 2)(1 - \frac{T^2}{4}\Lambda^+)\frac{4\pi}{T^2} , \]  
(1.19)

the solutions \( x(t) \) verify another kind of property, i.e.

\[ x(t) = -x(t + \frac{T}{2}) \quad \forall t \in \left[0, \frac{T}{2}\right] . \]  
(1.20)

Remark 1.3 Let us observe that condition (1.15) implies (1.19), but they must be considered separately because they come from different symmetry conditions.

Remark 1.4 If \( A(t) \equiv 0 \), first part of Theorem 1.2 yields Theorem 3 of \([33]\) as a particular case.
CHAPTER 4. HAMILTONIAN SYSTEMS

Starting from the critical points of the functional \( I: H^1_0 \left[ 0, \frac{T}{2} \right] \rightarrow \mathbb{R} \) defined by
\[
I(x) = \frac{1}{2} \int_0^{\frac{T}{2}} \dot{x}^2 - \frac{1}{2} \int_0^{\frac{T}{2}} \langle A(t)x, x \rangle - \int_0^{\frac{T}{2}} b(t)V(x),
\]
we get the solutions of the problem
\[
\begin{align*}
\ddot{x} + A(t)x + b(t)V'(x) &= 0 \\
x(0) &= x\left(\frac{T}{2}\right) = 0 \quad (1.21)
\end{align*}
\]
Precisely, putting
\[
\tilde{x}(t) = \begin{cases} 
x(t) & t \in \left[0, \frac{T}{2}\right] \\
-x(T-t) & t \in \left[\frac{T}{2}, T\right]
\end{cases}
\]
one can easily check that if \( x \) solves (1.21), \( \tilde{x} \) is a \( T \)-periodic solution of (1.1). Observe that the expression
\[
\|x\|_A = \left[ \int_0^{\frac{T}{2}} \dot{x}^2 - \int_0^{\frac{T}{2}} \langle A^-(t)x, x \rangle \right]^{\frac{1}{2}} \quad \forall x \in H^1_0 = H^1_0 \left( \left[0, \frac{T}{2}\right] ; \mathbb{R}^N \right)
\]
defines a norm in \( H^1_0 \) which is equivalent to the usual norm
\[
\|x\|_{H^1_0} = \left( \int_0^{\frac{T}{2}} |\dot{x}|^2 \right)^{\frac{1}{2}}.
\]
Thus the functional \( I \) can be written in the form:
\[
I(x) = \frac{1}{2} \|x\|^2_A - \frac{1}{2} \int_0^{\frac{T}{2}} \langle A^+(t)x, x \rangle - \int_0^{\frac{T}{2}} b(t)V(x).
\]
On the other hand, if we consider the space \( H^1_{0,k} = H^1_0([0, kT]; \mathbb{R}^N) \) endowed with the norm of \( H^1_k = \{ x \in H^1(0, kT; \mathbb{R}^N) : x(0) = x(kT) \} \), provided \( \Lambda^- \) defined in (1.9) is strictly negative, we obtain that
\[
\|x\|_{L^2_k} \leq \frac{1}{\lambda} \|x\|_A, \quad \text{with} \quad \lambda = \min\{1, -\Lambda^-\}, \quad (1.22)
\]
where \( \| \cdot \|_{L^2_k} \) is the usual \( L^2([0, kT]) \) norm. Taking these notations one can state the following existence result for subharmonic solutions:

**Theorem 1.5** [11] Let \( b(t) \in C^0([0, T], \mathbb{R}) \) be a \( T \)-periodic real function verifying (1.2) and (1.11) and \( A(t) \equiv A^-(t) \), with \( A^-(t) \) negative definite, satisfying (1.12).
Moreover let \( V \in C^2(\mathbb{R}^N, \mathbb{R}) \) verify (1.3)–(1.14) where
\[
\max\{2c, d\} < (\beta - 2)\lambda,
\]
4.2. PROOF OF THE THEOREMS

with \( \lambda \) given by (1.22). Then for all \( k \in \mathbb{N} \) there exists a \( kT \)-periodic solution \( x_k \) of (1.1), having minimal period \( kT \) and verifying

\[
x_k \left( t + \frac{kT}{2} \right) = -x_k \left( \frac{kT}{2} - t \right) \quad \text{for any} \ t \in \left[ 0, \frac{kT}{2} \right].
\]

In particular, for each couple of numbers \( k, h \), with \( k \neq h \), the solutions \( x_h \) and \( x_k \) are geometrically distinct.

Starting from Theorem 1.5, the following result holds:

**Theorem 1.6** [11] Under the same hypothesis of Theorem 1.5 there exists at least one homoclinic solution of (1.1), i.e. a nontrivial solution \( x \in C^2(\mathbb{R}, \mathbb{R}^N) \) of (1.1) which satisfies

\[
\lim_{|t| \to +\infty} x(t) = 0 \quad \text{and} \quad \lim_{|t| \to +\infty} \dot{x}(t) = 0.
\]

**4.2 Proof of the Theorems**

Theorem 1.2 is obtained as application of the following proposition, based on the theory of Ljusternik- Schr"{a}nemann category:

**Proposition 2.1** [5] Let \( X \) be a closed \( C^1 \) manifold of a Hilbert space \( E \), symmetric with respect to the origin \( 0 \) of \( E \) and such that \( 0 \notin X \). Let there exist a closed infinite dimensional subspace \( \tilde{E} \) of \( E \) such that the manifold \( \tilde{X} = X \cap \tilde{E} \) is homeomorphic to the unit sphere \( S_\infty \) of \( \tilde{E} \), through an even homeomorphism. Suppose \( j \in C^1(X, \mathbb{R}) \) satisfies the Palais-Smale condition and

\[
J \text{ is bounded from below on } X \quad (2.1)
\]

\[
J \text{ is even.} \quad (2.2)
\]

Then \( J \) has infinitely many pairs of critical points.

**Proof of Theorem 1.2**
Let us introduce the set
\[
M = \left\{ x \in H_0^1 \setminus \{0\} : \|x\|^2_A = \int_0^T \langle A^+(t)x, x \rangle + \int_0^T b(t)V'(x) \cdot x \right\}.
\] (2.3)

Taking
\[
h(x) = \|x\|^2_A - \int_0^T \langle A^+(t)x, x \rangle - \int_0^T b(t)V'(x) \cdot x,
\]
we can write
\[
M = \left\{ x \in H_0^1 \setminus \{0\} : h(x) = 0 \right\}.
\]

Let us prove that \( M \) has the following properties:

**\( \text{(m.1)} \) \( M \neq \emptyset \)**

**\( \text{(m.2)} \) \( M \) is a closed \( C^1 \)-manifold.**

Indeed if one considers an element \( \overline{x} \in H_0^1 \setminus \{0\} \), with \( \|\overline{x}\|_A = 1 \) such that \( \text{supp} (\overline{x}) \subset \text{supp} (b^+) \), (observe that (1.2) implies \( \text{supp} (b^+) \neq \emptyset \)), then from (1.4) it follows that for any constant \( r > 0 \)
\[
h(r\overline{x}) \geq r^2\|\overline{x}\|^2_A - \Lambda^+r^2\|\overline{x}\|^2_{L^2} - a_2r^\beta \int_0^T b^+(t)|\overline{x}|^\beta.
\]

Putting \( B^+ = \max_{t \in [0,T]} b^+(t) \), by the continuous embedding of \( H_0^1 \) in \( L^\beta \) we obtain
\[
h(r\overline{x}) \geq r^2 \left( 1 - \Lambda^+\frac{T^2}{4} \right) - a_2B^+r^\beta \cdot \text{const}
\] (2.4)

As (1.10) holds and \( \beta > 2 \), \( h(r\overline{x}) \geq 0 \) for \( r \) sufficiently small.

On the other hand, by (1.3), (1.5) and the positivity property of \( A^+ \), we have:
\[
h(r\overline{x}) \leq r^2 - \beta \int_0^T b^+(t)V(r\overline{x}) \leq r^2 - a_1\beta r^\beta \int_0^T b^+(t)|\overline{x}|^\beta := \alpha.
\]

As \( \beta > 2 \), for \( r \) sufficiently large, \( \alpha \) is negative, so \( h(r\overline{x}) \leq 0 \), and, together with (2.4), this proves (m.1). Let us show now (m.2). By the continuous embedding of \( H_0^1 \) in \( L^\beta \) and (1.4), for any \( x \) in \( M \) we have:
\[
0 = \|x\|^2_A - \int_0^T \langle A^+(t)x, x \rangle - \int_0^T b(t)V'(x) \cdot x \geq
\]
4.2. PROOF OF THE THEOREMS

\[
\geq \|x\|_A^2 \left( 1 - \Lambda^+ \frac{T^2}{4} \right) - \text{const} \cdot B^+ a_2 \|x\|_A^\beta
\]

So

\[
\|x\|_A^{\beta-2} \geq \left( 1 - \Lambda^+ \frac{T^2}{4} \right) \cdot \text{const} \cdot \frac{1}{a_2 B^+} := \gamma, \tag{2.5}
\]

where \(\gamma\) is positive as (1.10) holds.

Moreover (1.6) and (1.14) imply that, for any \(x \in M\):

\[
\langle h'(x), x \rangle = \|x\|_A^2 - \int_0^{\frac{T}{2}} \langle A^+(t)x, x \rangle - \int_0^{\frac{T}{2}} b(t)V''(x)x \cdot x \leq
\]

\[
\leq \|x\|_A^2 - \int_0^{\frac{T}{2}} \langle A^+(t)x, x \rangle - (\beta - 1) \int_0^{\frac{T}{2}} b^+(t)V'(x) \cdot x + \frac{T}{4} b^-(t)V''(x)x \cdot x \leq \left[ 2 - \beta + (\beta - 2)\Lambda^+ \frac{T^2}{4} + \frac{T^2}{4} \right] \|x\|_A^2. \tag{2.6}
\]

Therefore, by (1.15), one has \(\langle h'(x), x \rangle \leq c_1 \|x\|_A^2\), with

\[
c_1 = \left[ 2 - \beta + (\beta - 2)\Lambda^+ \frac{T^2}{4} + \frac{T^2}{4} \right] < 0,
\]

so (2.5) and (2.6) imply the \(C^1\)-regularity of \(M\). Finally, in order to prove that \(M\) is closed, let us take a sequence \(x_n \subset M\) strongly converging to some \(x\), knowing that the convergence in \(H^1\) implies uniform convergence, one has

\[
h(x_n) = \int_0^{\frac{T}{2}} |\dot{x}_n|^2 - \int_0^{\frac{T}{2}} \langle A^-(t)x_n, x_n \rangle - \int_0^{\frac{T}{2}} \langle A^+(t)x_n, x_n \rangle - \int_0^{\frac{T}{2}} b(t)V'(x_n)x_n \rightarrow
\]

\[
\rightarrow \int_0^{\frac{T}{2}} |\dot{x}|^2 - \int_0^{\frac{T}{2}} \langle A^-(t)x, x \rangle - \int_0^{\frac{T}{2}} \langle A^+(t)x, x \rangle - \int_0^{\frac{T}{2}} b(t)V'(x)x = h(x)
\]

so \(x \in M\).

At this point, taking \(X = M\) and \(\tilde{E} = \{u \in E : \text{supp} (u) \subset \text{supp} (b^+)\}\), we can apply Proposition 2.1 to the functional \(I\) after verifying that it is bounded from below.
and verifies Palais-Smale condition on $M$. If we denote with $I_M$ the restriction of $I$ to $M$, (1.5) and (1.13) imply that, for any $x \in M$,

$$ I_M(x) = \frac{1}{2} \|x\|_A^2 - \frac{1}{2} \int_0^T \langle A^+(t)x, x \rangle - \int_0^T b(t)V(x) \geq $$

$$ \geq \frac{1}{2} \|x\|_A^2 - \frac{1}{2} \int_0^T \langle A^+(t)x, x \rangle - \frac{1}{\beta} \int_0^T b(t)V'(x) \cdot x - \frac{1}{\beta} \int_0^T B^{-1}[V'(x) \cdot x - \beta V(x)] \geq $$

$$ \geq \frac{\beta - 2}{2\beta} \|x\|_A^2 - \frac{\beta - 2}{2\beta} \int_0^T \langle A^+(t)x, x \rangle - \frac{c T^2}{\beta} \|x\|_A^2 \geq $$

$$ \geq \left[ \frac{\beta - 2}{2\beta} \left( 1 - \Lambda^+ \frac{T^2}{4} \right) - \frac{c T^2}{\beta} \right] \|x\|_A^2, \quad (2.7) $$

where, by hypothesis (1.15),

$$ c < (\beta - 2) \left( 1 - \Lambda^+ \frac{T^2}{4} \right) \frac{2}{T^2}. $$

Therefore

$$ I_M(x) \geq \text{const} \|x\|_A^2 > 0, \quad (2.8) $$

i.e $I_M$ is coercive; in particular it is bounded from below.

The Palais–Smale condition for the functional $I_M$ is given by the following lemma:

**Lemma 2.2** The functional

$$ I_M(x) = \frac{1}{2} \|x\|_A^2 - \frac{1}{2} \int_0^T \langle A^+(t)x, x \rangle - \int_0^T b(t)V(x) $$

verifies Palais–Smale condition.

**Proof**

Let $\{x_n\} \subset M$ be a PS sequence, i.e. $\{I_M(x_n)\} < K$ and $\{I'_M(x_n)\} \to 0$, in other words, there exists $a_n \in \mathbb{R}$ such that

$$ I'(x_n) + a_n h'(x_n) = w_n \to 0 \text{ in } E'. $$

The boundedness of $I_M(x_n)$ and (2.8) imply the boundedness of $x_n$, so

$$ \langle I'(x_n), x_n \rangle + a_n \langle h'(x_n), x_n \rangle \to 0. \quad (2.9) $$
For \( x_n \in M \), it follows that \( \langle I'(x_n), x_n \rangle = 0 \ \forall n \in \mathbb{N} \), so \( a_n \to 0 \). By (2.9) and the boundedness of \( h'(x) \) we conclude that

\[
I'(x_n) \to 0.
\] (2.10)

Now let us prove that \( x_n \) admits a strongly converging subsequence. Let \( x_{nk} \) (for the sake of simplicity \( x_{nk} = x_n \)) be a subsequence of \( x_n \) which weakly converges to some \( \tilde{x} \in E \). By (2.10) and the boundedness of \( x_n \) we get

\[
\|x_n\|^2_A + \int_0^T \langle A^+(t)x_n, x_n \rangle - \int_0^T b(t)V'(x_n)x_n \to 0
\] (2.11)

which yields

\[
\|x_n\|^2_A \to -\int_0^T \langle A^+(t)\tilde{x}, \tilde{x} \rangle + \int_0^T b(t)V'(\tilde{x})\tilde{x}
\] (2.12)

On the other hand

\[
\langle x_n, \tilde{x} \rangle \to -\int_0^T \langle A^+(t)\tilde{x}, \tilde{x} \rangle + \int_0^T b(t)V'(\tilde{x})\tilde{x}
\] (2.13)

By (2.12), (2.13) and the definition of weak convergence we can conclude that \( x_n \) converges strongly to \( \tilde{x} \).

\[\square\]

For the second statement of Theorem 1.2 we will search the solutions of (1.1) in the space

\[
H^1_{odd} = \left\{ v \in H^1(0, T; \mathbb{R}^N) : v = \sum_{k=2h-1} v_k^{(1)} \cos \left( \frac{2k\pi t}{T} \right) + v_k^{(2)} \sin \left( \frac{2k\pi t}{T} \right), \right. 
\]

\[
v_k^{(1)}, v_k^{(2)} \in \mathbb{R}^N, h \in \mathbb{Z} \right\}
\]

In fact the functions belonging to \( H^1_{odd} \) are \( T \)-periodic, \( \frac{T}{2} \) antiperiodic (i.e. satisfy a property of type (1.20)), with zero mean and verify the Wirtinger inequality. Moreover they don’t necessarily satisfy (1.16), but in some particular cases, if \( b, A \) verify a suitable condition of symmetry, the solutions found in \( H^1_{odd} \) could coincide with those found in \( H^1_0 \).
Arguing as in the first part of this proof, we can apply Proposition 2.1 and find out that there exist infinitely many pairs of distinct $T$–periodic solutions in $H^1_{\text{odd}}$. In order to end the proof we need to show that the critical points of the functional $I$ on $H^1_{\text{odd}}$ are critical on $H^1_{T}$ too. Indeed, let us consider the decomposition of $H^1_{T}$ given by

$$H^1_{T} = H^1_{\text{odd}} \oplus H^1_{\text{even}},$$

where $H^1_{\text{even}}$ is the subspace of $H^1_{T}$ of functions which have only even terms in their Fourier expansion. Since (1.7) implies that $V'(x) = -V'(-x)$ it is easy to check that, if $x \in H^1_{\text{odd}},$

$$\langle f'(x), y \rangle = 0 \quad \forall y \in H^1_{\text{even}}.$$

\[\square\]

In order to prove Theorem 1.5 we will use a standard minimizing argument, i.e. the following

**Proposition 2.2** Let $X$ be a reflexive space and $K \subset X$ a closed subset. Let $G$ be a continuous functional on $X$, bounded from below on $K$ and satisfying the Palais-Smale condition on $K$. Then $F$ admits a minimum value on $K$.

**Proof of Theorem 1.5**

Putting

$$M_k = \left\{ x \in H^1_0 \left( 0, \frac{kT}{2}; \mathbb{R}^N \right) \setminus \{0\} : \|x\|_A^2 - \int_0^{\frac{kT}{2}} \langle A^+(t)x, x \rangle + + \int_0^{\frac{kT}{2}} b(t)V'(x) \cdot x = 0 \right\}$$

we can verify that $G = I_{M_k}$ is bounded from below and satisfies the Palais-Smale condition on $M_k$. Moreover, since the $H^1$–convergence implies the uniform convergence we can take the limit under the integral sign and obtain that $I_{M_k}$ is continuous. So we can apply Proposition 2.2 and find out that $I_{M_k}$ has a minimum $x_k$. 
Now one has to show that the solution corresponding to \( x_k \) has minimal period \( kT \).

Suppose, by contradiction, that for some \( h \in \mathbb{N} \), \( h \geq 2 \), \( \frac{kT}{h} \) is the minimal period of \( x_k \).

Then there would exists \( t_0 \in [0, \frac{kT}{h}] \) such that \( x_k(t_0) = 0 \), indeed if \( x_k \) had constant sign on the interval \([0, \frac{kT}{h}]\) it would have the opposite sign in \([\frac{kT}{h}, \frac{2kT}{h}]\), so \( x_k \) could not be \( \frac{kT}{h} \)-periodic. Then we consider the function

\[
\tilde{\varphi}(t) = \begin{cases} 
\eta \sin \left[ \frac{2\pi}{t_2-t_1}(t-t_1) \right] & \text{if } t \in [t_1, t_2] \\
0 & \text{if } t \in [0, T] \setminus [t_1, t_2]
\end{cases}
\]

where \( \eta \in \mathbb{R}^N \), and \( |\eta| = 1 \).

By construction \( \tilde{\varphi} \) belongs to \( M_k \), because it verifies the relation \( \langle I'(x_k), x_k \rangle = 0 \) in the interval \([0, t_0]\), and trivially in the interval \([t_0, \frac{kT}{2}]\). So one has

\[
I_{M_k}(x_k) = I_{M_k}(\tilde{\varphi}) + I_{M_k(t>t_0)}(x_k) > I_{M_k}(\tilde{\varphi}),
\]

which contradicts the minimality property of \( x_k \).

\[\square\]

**Proof of Theorem 1.6**

First of all observe that condition (1.2) implies that there exists an interval \([t_1, t_2] \subset [0, \frac{T}{2}]\) such that \( b(t) > 0 \), for any \( t \in [t_1, t_2] \). Let us consider the function

\[
\tilde{\varphi}(t) = \begin{cases} 
\eta \sin \left[ \frac{2\pi}{t_2-t_1}(t-t_1) \right] & \text{if } t \in [t_1, t_2] \\
0 & \text{if } t \in [0, T] \setminus [t_1, t_2]
\end{cases}
\]

where \( \eta \in \mathbb{R}^N \), and \( |\eta| = 1 \).

By construction \( \tilde{\varphi} \) belongs to \( H^1_0(0, \frac{T}{2}; \mathbb{R}^N) \). Arguing as in the proof of the nonemptyness of \( M \), we claim that there exists a constant \( r > 0 \) such that \( \phi = r \frac{\tilde{\varphi}}{||\tilde{\varphi}||} \) belongs to \( M \).

If we consider the element \( \psi \in H^1_0(0, kT; \mathbb{R}^N) \) given by:

\[
\psi(t) = \begin{cases} 
\phi(t) & \text{if } t \in [0, T] \\
0 & \text{if } t \in [T, kT]
\end{cases}
\]

then, by Theorem 1.5 and (2.7), we can construct a sequence \( \{x_k\} \) of subharmonics of (1.1) (corresponding to the minima of \( I_M \)) such that

\[
0 < \alpha \|x_k\|_A^2 \leq f(x_k) \leq \frac{1}{2} \|\psi\|_A^2 - \int_{t_1}^{t_2} b(t) V(\psi) = L,
\]

(2.14)
where \( L \) is a constant independent of \( k \).

Now let us prove that the sequence of subharmonics converge to a homoclinic solution of (1.1), i.e. a nontrivial solution \( x \) of (1.1) which satisfies

\[
x(t) \to 0 \text{ and } \dot{x}(t) \to 0 \text{ as } |t| \to +\infty.
\]

Let us consider the sequence of subharmonics we just constructed. By (2.14) and considering that \( \langle I'(x_k), x_k \rangle = 0 \), putting \( \| \cdot \|_k = \| \cdot \|_{H^1_{kT}} \), one has

\[
L \geq I(x_k) = I(x_k) + \frac{1}{\beta} \langle I'(x_k), x_k \rangle \geq \frac{\beta - 2}{2\beta} \| x_k \|^2_k - \frac{1}{\beta} \int_0^{kT} B^- [V'(x_k)x_k +

- \beta V(x_k)] \geq \frac{\beta - 2}{2\beta} \| x_k \|^2_k - \frac{1}{\beta} \int_0^{kT} \beta V(x_k)\| x_k \|^2_k = \left[ \frac{\beta - 2}{2\beta} - \frac{\text{const}}{\beta} \right] \| x_k \|^2_k
\]

that is

\[
\| x_k \|_k \leq \left( \frac{2\beta L}{\beta - 2 - \text{const}} \right)^{\frac{1}{2}} \equiv \tilde{L}
\]

with \( \tilde{L} \) independent of \( k \). With the following lemmas we prove that the sequence \( \{ x_k \} \) is bounded from above and from below, and this fact will assure that the limit procedure necessary to find the homoclinic solution, will lead neither to the trivial solution nor to an unbounded solution.

**Lemma 2.2** There exists a constant \( \eta > 0 \), independent of \( k \) such that

\[
\| x_k \|_{L^\infty} \geq \eta_1 \quad \forall k \in N.
\]

**Proof**

As \( x_k \) is \( kT \)-periodic solution of (1.1), thanks to (1.4) and to the estimate

\[
\| u \|_{L^p_k} \leq C_p \| u \|_k = C_p \| u \|_{H^1_{kT}}
\]

(for the proof see [27]) one has, if \( \| x_k \|_{L^\infty} \) is small enough,

\[
\| x_k \|^2_k \leq \int_0^{kT} \| x_k \|^2 \leq \int_0^{kT} |x_k|^2 \leq \int_0^{kT} \| x_k \|^{\beta - 2} |x_k|^2 \leq \int_0^{kT} \| x_k \|^{\beta - 2} \| x_k \|^2_k
\]

\[
\leq B^+_k a_2 C^2 \| x_k \|_{L^\infty} \| x_k \|^2_k
\]
where $B_k^+ = \max_{t \in [0, kT]} b(t) > 0$.

This implies
\[
\|x_k\|_{L^\infty} \geq \left( \frac{1}{a_2 B_k^+ C_2^2} \right)^{\frac{1}{\beta - 2}} \equiv \eta_1.
\]

\[\square\]

**Lemma 2.3** There exists $\eta_2 > 0$, independent of $k$, such that
\[
\|x_k\|_{C^2[0, kT]} \leq \eta_2 \quad \forall k \in \mathbb{N}.
\] (2.18)

**Proof**

It follows by (2.15) and by the fact that $x_k$ solves (1.1).

\[\square\]

By the properties of $x_k$, as $k$ varies in $\mathbb{N}$, one can construct a sequence $\{\tilde{x}_k\} \subset H^1_k$ which verifies $\tilde{x}_k(t) = x_k(t + r_k T)$, where the sequence $r_k \subset \mathbb{N}$ is such that
\[
-\infty < \max_{t \in [0, T]} |x_k(t + r_k T)| = \max_{t \in \mathbb{R}} |x_k(t)| < +\infty.
\]

By construction the functions $\tilde{x}_k$ satisfy the same estimates of $x_k$, so, applying Ascoli’s Theorem we obtain that, up to subsequences, $\tilde{x}_k$ converges to a solution $x$ of (1.1) in $C^2_{loc}$.

One has to verify that $x$ is a homoclinic solution, which means that $x$ is a nontrivial solution of (1.1) such that $x \to 0$ and $\dot{x} \to 0$ as $|t| \to +\infty$.

First of all observe that $x$ is not identically zero as it is the limit of functions whose maxima are greater than some positive number. Furthermore $x \in W^{1,2}$: indeed for any $D > 0$, as $x_k$ is bounded in $H^1_{kT}$,
\[
\int_{-D}^{D} [|\dot{x}|^2 + \langle A(t)x, x \rangle] = \lim_{k \to +\infty} \int_{-D}^{D} [|\dot{x}_k|^2 + \langle A(t)x_k, x_k \rangle] \leq \limsup_{k \to +\infty} \|x_k\|_k^2 \leq \tilde{L}^2,
\]
so
\[
\int_{\mathbb{R}} [|\dot{x}|^2 + \langle A(t)x, x \rangle] < +\infty.
\] (2.19)

This implies that
\[
\int_{|t| \geq m} (|\dot{x}|^2 + |x|^2) \to 0 \text{ as } m \to +\infty.
\] (2.20)
On the other side, in [52] it was proved that, for any \( v \in W^{1,2} \),

\[
|v(t)| \leq 2 \left[ \int_{t-\frac{1}{2}}^{t+\frac{1}{2}} (|\dot{v}|^2 + |v|^2) \right]^{\frac{1}{2}}.
\]  

(2.21)

Therefore

\[
x(t) \to 0 \text{ as } |t| \to +\infty.
\]

(2.22)

By (1.4) one gets \( V'(0) = 0 \), which, together with (2.22) yields

\[
\int_{m}^{m+1} |b(t)V'(u)| \to 0 \text{ as } m \to +\infty.
\]

Consequently, as \( x \) is a solution of (1.1) and satisfies (2.20), one has

\[
\int_{m}^{m+1} |\ddot{x}| \to 0 \text{ as } m \to +\infty.
\]

Therefore, replacing \( x \) by \( \dot{x} \) in (2.21) one obtains

\[
\dot{x}(t) \to 0 \text{ as } |t| \to +\infty.
\]
Bibliography


[27] Y.H. Ding, M. Girardi, Periodic and homoclinic solutions to a class of Hamiltonian systems with the potential changing sign, Dynamical Systems and Appl. 2 (1993), 131–145.


