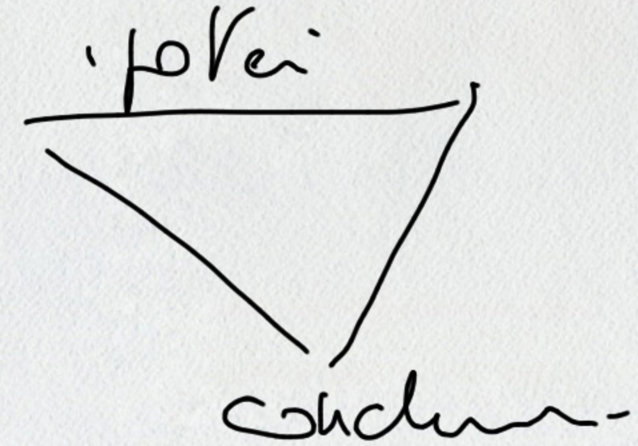
La logica classica non ha una "buona" semantica denotazionale

true false

invariante : una legge, una proprietà di una classe di oggetti che rimane stabile, inalterata, per trasformazioni
sotto

così una dimostrazione

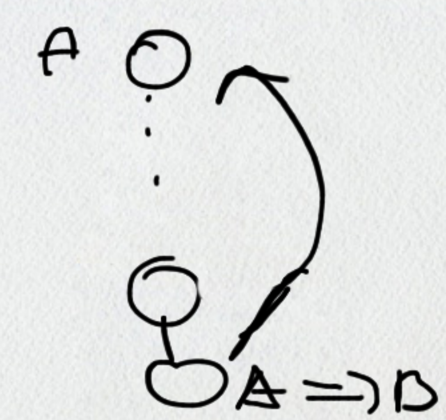


alberi?

$$\dfrac{\begin{array}{c}[A]\\ \vdots \\ B\end{array}}{A\Rightarrow B}$$

– deduzione naturale

$$\dfrac{\dfrac{A\vee B \quad \begin{array}{c}[A]\\ \vdots \\ C\end{array} \quad \begin{array}{c}[B]\\ \vdots \\ C\end{array}}{C} \qquad D}{X}\, r$$



$$\delta \qquad \dfrac{\dfrac{A \vee B \qquad \begin{matrix}[A]\\ \vdots\\ C\end{matrix} \qquad \begin{matrix}[B]\\ \vdots\\ C\end{matrix}}{C}\,E\vee \qquad \begin{matrix}\vdots\\ D\end{matrix}}{E}\,2 \qquad\qquad \delta \quad \begin{matrix}A \vee B\\ \vdots\\ E\end{matrix}$$

$\Downarrow$ conmutar

?

$$\delta' \qquad \dfrac{A \vee B \qquad \dfrac{\begin{matrix}[A]\\ \vdots\\ C\end{matrix} \quad \begin{matrix}\vdots\\ D\end{matrix}}{E} \qquad \dfrac{\begin{matrix}[B]\\ \vdots\\ C\end{matrix} \quad \begin{matrix}\vdots\\ D\end{matrix}}{E}}{E}\,E\vee \qquad\qquad \delta' \quad \begin{matrix}A \vee B\\ \vdots\\ E\end{matrix}$$

LK Calcolo di sequenti $\vdash$

eliminazione del taglio

$A\vdash A \cdots \quad B\vdash B$

$$\mathrm{RAA}\left\{\dfrac{\dfrac{\begin{array}{c}\vdots\ \pi_1\\ \Gamma,\neg A\vdash\end{array}}{\Gamma\vdash\neg\neg A}\qquad \neg\neg A\vdash A}{\Gamma\vdash A}\,\mathrm{cut}\right.$$

$\vdots$

$A\vdash A \cdots \quad B\vdash B \cdots$

$\Gamma\vdash A$

$\vdots$

$\Gamma,\neg A\vdash$

$$\dfrac{\dfrac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A}\,w \qquad \dfrac{\Gamma' \vdash \Delta'}{A, \Gamma' \vdash \Delta'}\,w}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\text{cut}$$

$$\dfrac{\Gamma \vdash \Delta}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,w \qquad\qquad \dfrac{\Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,w$$

$f(a)$

c

d

qual'è l'oggetto canonico

$$\dfrac{\dfrac{\dfrac{\pi}{\vdash B}}{\vdash B, C}\,wR \qquad \dfrac{\dfrac{\pi'}{\vdash B}}{C \vdash B}\,wL}{\dfrac{\vdash B, B}{\vdash B}\,c}\,cut(C)$$

$\rightsquigarrow$

alg1 $\qquad$ alg2

$$\dfrac{\dfrac{\pi}{\vdash B}}{\cancel{\vdash B, B}}\,w \;\cancel{\;c\;}\; \cancel{\vdash B} \qquad \equiv \qquad \dfrac{\dfrac{\pi'}{\vdash B}}{\cancel{\vdash B, B}}\,w \;\cancel{\;c\;}\; \cancel{\vdash B}$$

$t \qquad\qquad t$

<u>at design time</u>

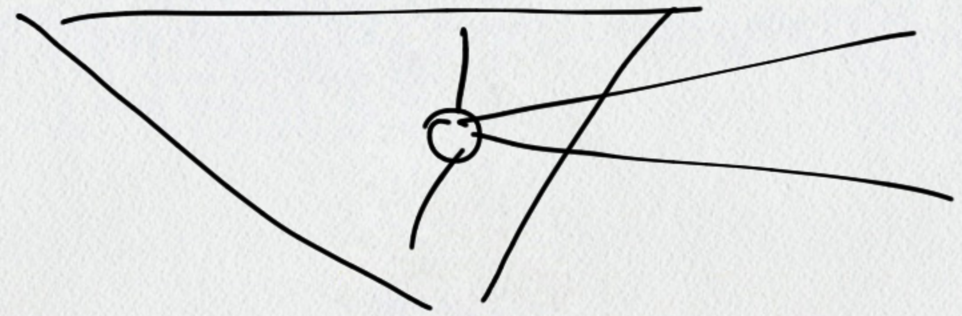
$$\dfrac{\vdash \Gamma}{\vdash \Gamma, \Gamma'} \qquad \dfrac{\vdash \Gamma'}{\vdash \Gamma', \Gamma}$$

$$\{a, a\} = \{a\}$$

Proprietà delle sottoformule

$$\begin{array}{c}\Pi \\ \vdots \\ \Gamma \vdash \Delta\end{array}$$

normale
cut-free



$\Gamma \vdash \Delta$

$\Gamma' \vdash \Delta'$

$\downarrow$

F

sono sottoformule di
qualche formula G
$G \in \Gamma$, oppure in Δ

$$\frac{}{\Gamma \vdash \Delta}$$? quali le altre regole?

Esempio

$$
\dfrac{\dfrac{\dfrac{A \vdash A}{\vdash \neg A, A}\,\neg R \qquad \dfrac{A \vdash A}{\vdash \neg A, A}\,\neg R}{\vdash \neg A \wedge \neg A,\ A}\,\wedge^a_R}{\vdash \neg A \wedge \neg A,\ A \vee B}\,\vee^1_a R \quad ?
$$

in LK

$$
\cancel{\dfrac{\dfrac{\dfrac{?}{\vdash \neg A, B} \qquad \dfrac{?}{\vdash \neg A, B}}{\vdash \neg A \wedge \neg A,\ B}\,\wedge}{\vdash \neg A \wedge \neg A,\ A \vee B}\,\vee^2_a R}
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{\vdash \neg A, A}\,\neg R \qquad \dfrac{A \vdash A}{\vdash \neg A, A}\,\neg R}{\vdash \neg A \wedge \neg A,\ A, A}\,\wedge_m R}{\vdash \neg A \wedge \neg A,\ A, A, B}\,WR}{\vdash \neg A \wedge \neg A,\ A, B}\,CR}{\vdash \neg A \wedge \neg A,\ A \vee B}\,\vee_m R
$$

algoritmi
- eliminazione del taglio
- proof search

  prolog

⋮

$$
\dfrac{\dfrac{\vdash A, A, A}{\vdash A, A}}{\vdash A} \quad ?
$$

rappresentazione canonica delle invarianti di sintassi

Logica Lineare, Jean-Yves GIRARD, 1987 TCS 1987

1 — formule vero/falso → risorse

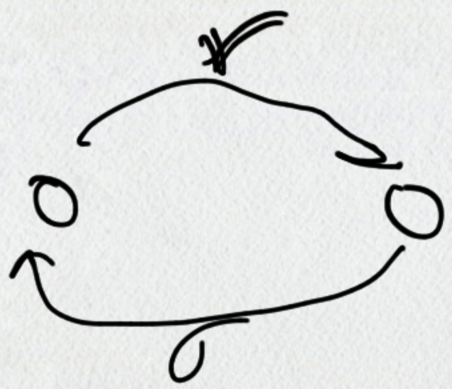
2 — regole additive / moltiplicative

3 — "controllo" delle regole strutturali

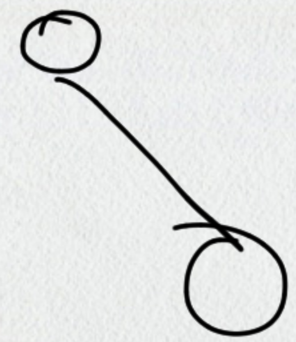
4 — dimostrazione graf Proof nets

$!A$ $?A$

modalità



graf



albero

$A \Rightarrow B \qquad A \multimap B = A^{\perp} \parr B$

par $\vee$ moltipl

$\wedge$ $\vee$

$\otimes$ $\&$ $\oplus$ $\parr$

Calcolo dei Sequenti di LL

$\Gamma \vdash \Delta$

$\swarrow$

$A_1, \ldots, A_n \vdash B_1, \ldots, B_m$

<u>Identità</u>

$a \vdash a$

$\alpha \vdash \alpha \qquad \alpha$: prop. atomica $\qquad A = \alpha \mid A \bullet B$

<u>cut</u>

$$\frac{\Gamma \vdash \Delta, A \qquad A, \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\text{cut}$$

$\perp$ = negazione lineare

$\neg A = A^{\perp\perp\perp}$

<u>regole della negazione lineare</u>:

$$\frac{\Gamma \vdash \Delta, A}{A^{\perp}, \Gamma \vdash \Delta}\,\perp_L \qquad \frac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, A^{\perp}}\,\perp_R$$

Regole logiche : additive / moltiplicative

$\Gamma \vdash \Delta$

$A_1 \ldots A_n \vdash B_1 \ldots B_m$

$A_1 \otimes \ldots \otimes A_n \vdash B_1 ⅋ \ldots ⅋ B_m$

<u>Moltiplicative</u> :

<u>congiunzione moltiplicativa</u> "$\otimes$" tensor (times) $\times$

prodotto

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \otimes B \vdash \Delta} \otimes_L \qquad \frac{\Gamma \vdash \Delta, A \qquad \Gamma' \vdash \Delta', B}{\Gamma, \Gamma' \vdash \Delta, \Delta', A \otimes B} \otimes_R$$

elemento neutro del $\otimes$ è "1" <u>"vero moltiplicativo"</u>

$$\frac{\Gamma \vdash \Delta}{\Gamma, 1 \vdash \Delta} 1_L \qquad \frac{}{\vdash 1} 1_R$$

- connettivi
- vero / falso
- proposizioni

in LK

<u>Disgiunzione ⅋ par</u>

$$\frac{\Gamma, A \vdash \Delta \qquad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A ⅋ B \vdash \Delta, \Delta'} ⅋_L \qquad \frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A ⅋ B} ⅋_R$$

elemento neutro $\perp$ bottom

$$\frac{}{\perp \vdash} \perp_L \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \perp} \perp_R$$

Regole logiche additive

congiunzione additiva $\&$ "with"

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, A \& B \vdash \Delta}\,\&_L^1 \qquad \frac{\Gamma, B \vdash \Delta}{\Gamma, A \& B \vdash \Delta}\,\&_L^2 \qquad \frac{\Gamma \vdash \Delta, A \quad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \& B}\,\&_R$$

elem. neutro della "$\&$" $\top$ (top) vero additivo $\quad \dfrac{}{\Gamma \vdash \top, \Delta}\,\top_R$

$\top$ non ha regole di intro a sx

<u>disgiunzione</u> $\oplus$ plus

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \oplus B \vdash \Delta}\,\oplus_L \qquad \frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A \oplus B}\,\oplus_R^1 \qquad \frac{\Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \oplus B}\,\oplus_R^2$$

elem. neutro del plus $\oplus$, è 0 (zero) $\quad \dfrac{}{\Gamma, 0 \vdash \Delta}\,0_L$

0 non ha regole di intro a dx

Regole Exponenziali

modalità che ci consentono "controllo" delle regole strutturali Weakening / Contraction
indebolim. contrazione.

"!" bang $!A$

? why not $?A$

$!A \quad ?A$

$!\Gamma = !A_1 \dots !A_n$

$?\Delta = ?B_1 \dots ?B_m$

dereliction (abbandono)

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, !A \vdash \Delta}\ !_d$$

$$\frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, ?A}\ ?_d$$

$$\frac{\dfrac{\Gamma \vdash A}{\Gamma \vdash A, A} \quad \dfrac{A \vdash \Delta}{A, A \vdash \Delta}}{}\ cr$$

promotion (promozione)

$$\frac{!\Gamma \vdash ?\Delta, B}{!\Gamma \vdash ?\Delta, !B}\ !_R$$

$$\frac{!\Gamma, A \vdash ?\Delta}{!\Gamma, ?A \vdash ?\Delta}\ ?_L$$

Regole strutturali : W / C / EX

scambio:

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, B, A \vdash \Delta}\,ExL \qquad \frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, B, A}\,ExR$$

Weakening:

$$\frac{\Gamma \vdash \Delta}{\underbrace{\Gamma, !A}_{\uparrow} \vdash \Delta}\,WL \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, ?A}\,WR$$

$$\frac{?A, !C}{?A, !C \vdash ?\Delta}$$

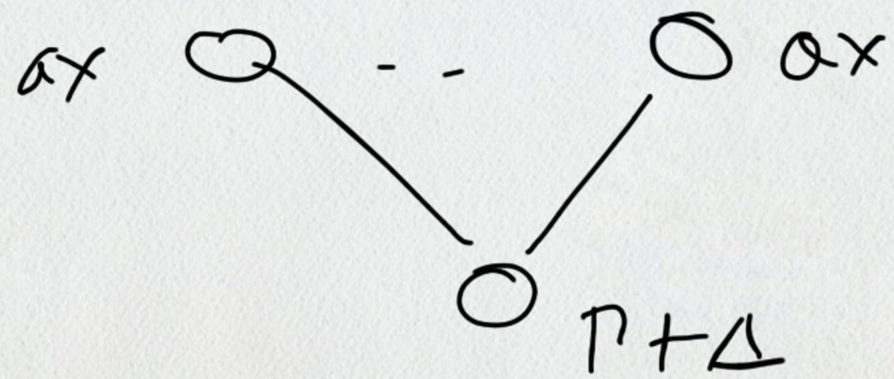
Contraction

$$\frac{\Gamma, !A, !A \vdash \Delta}{\Gamma, !A \vdash \Delta}\,CL \qquad \frac{\Gamma \vdash \Delta, ?A, ?A}{\Gamma \vdash \Delta, ?A}\,CR$$

Fine

dimostrazione di $\Gamma \vdash \Delta$ è un albero con axiomi nelle foglie

e $\Gamma \vdash \Delta$ ritratto alla radice



$\Gamma \vdash \Delta$ è dimostrabile in LL se $\exists$ un albero con foglie gli axiomi e

$\Gamma \vdash \Delta$ come conclusione.

$$\overline{A \vdash A} \qquad A \text{ qualsiasi}$$

$$\dfrac{\dfrac{\dfrac{\overline{A \vdash A}}{\vdash A, A^{\perp}}\perp_L \qquad \dfrac{\overline{B \vdash B}}{\vdash B, B^{\perp}}\perp_L}{\vdash A \otimes B, A^{\perp}, B^{\perp}}\otimes_R}{\vdash A \otimes B, A^{\perp} \parr B^{\perp}}\parr_R$$

ausiliarie

principale

$$\dfrac{\dfrac{\overline{A \vdash A} \qquad \overline{B \vdash B}}{A \parr B \vdash A, B}\parr_L}{A \parr B \vdash A \parr B}\parr_R$$

η-expansion

η-expansion of $A \otimes B$

$$\dfrac{\dfrac{A \vdash A \qquad B \vdash B}{A, B \vdash A \otimes B}\ \otimes_R}{A \otimes B \vdash A \otimes B}\ \otimes_L$$

A, B atomic

η-expansion of $A \& B$

$$\dfrac{\dfrac{A \vdash A}{A \& B \vdash A}\ \&_L^1 \qquad \dfrac{B \vdash B}{A \& B \vdash B}\ \&_L^2}{A \& B \vdash A \& B}\ \&_R$$

$$\dfrac{\dfrac{\overline{A \vdash A}}{A \vdash A \oplus B}\ \oplus_1^R \qquad \dfrac{\overline{B \vdash B}}{B \vdash A \oplus B}\ \oplus_2^R}{A \oplus B \vdash A \oplus B}\ \oplus_L$$

η-expansion of $A \oplus B$

$$\dfrac{\dfrac{\overline{A \vdash A}}{A, A^\perp \vdash}\ \perp_L}{A^\perp \vdash A^\perp}\ \perp_R \qquad \Bigg| \qquad \dfrac{\dfrac{A \vdash A}{\vdash A^\perp, A}\ \perp_R}{A^\perp \vdash A^\perp}\ \perp_L$$

$\eta(!A)$:

$$\dfrac{\dfrac{A \vdash A}{!A \vdash A}\, d!}{!A \vdash !A}\, !R$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A} \quad \dfrac{}{B \vdash B}}{A, B \vdash A \otimes B}}{A \otimes B \vdash A \otimes B}\, dR}{A \otimes B \vdash ?(A \otimes B)}\, ?L}{?(A \otimes B) \vdash ?(A \otimes B)}$$

$A \otimes B$

$\eta(?A)$

eta-expansion di A qualsiasi formula di LL

$$\eta(A) : \quad \dfrac{}{A \vdash A} \longrightarrow \begin{cases} \alpha \vdash \alpha & \text{axiom atomic} \\ \vdots & \\ A \vdash A & \end{cases}$$

$$\dfrac{}{A \vdash A}$$

A

$F \vdash F$

$$\delta \quad \dfrac{}{A \otimes B \vdash A \otimes B} \iff \dfrac{\dfrac{\dfrac{}{A \vdash A} \quad \dfrac{}{B \vdash B}}{A, B \vdash A \otimes B}}{A \otimes B \vdash A \otimes B}$$

$\Gamma \vdash \Delta$ è dim. da axiom qualsiasi

allora resta dimostrabile anche da axiom atomici

via η-expansion.

Dualità $\quad F \longmapsto$ duale $F^\perp \qquad\qquad F \dashv\vdash G$

due formule $F \equiv G$ sse $\exists$ una dim. di $F \vdash G \wedge G \vdash F$

Leggi De Morgan sul duale delle formule

$$\boxed{(A \otimes B)^\perp \overset{\equiv}{\dashv\vdash} A^\perp \parr B^\perp}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{\vdash A^\perp, A}\perp R \qquad \dfrac{B \vdash B}{\vdash B, B^\perp}\perp R}{\vdash A^\perp, B^\perp, A \otimes B}\otimes R}{\vdash A^\perp \parr B^\perp, A \otimes B}\parr R}{(A \otimes B)^\perp \vdash A^\perp \parr B^\perp}\perp L$$

$$\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{\vdash A, A^\perp}\perp R \qquad \dfrac{B \vdash B}{\vdash B, B^\perp}\perp R}{A, B, A^\perp \parr B^\perp \vdash}\parr L}{A \otimes B, A^\perp \parr B^\perp \vdash}\otimes L}{A^\perp \parr B^\perp \vdash (A \otimes B)^\perp}\perp R$$

$$\boxed{(A \parr B)^\perp \dashv\vdash A^\perp \otimes B^\perp}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{\vdash A^\perp, A}\,\perp_R \qquad \dfrac{B \vdash B}{\vdash B^\perp, B}\,\perp_R}{\vdash A^\perp \otimes B^\perp, A, B}\,\otimes_R}{\vdash A^\perp \otimes B^\perp, A \parr B}\,\parr_R}{(A \parr B)^\perp \vdash A^\perp \otimes B^\perp}\,\perp_L$$

$$\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{\vdash A, A^\perp}\,\perp_R \qquad \dfrac{B \vdash B}{\vdash B, B^\perp}\,\perp_R}{A \parr B, A^\perp, B^\perp \vdash}\,\parr_L}{A \parr B, A^\perp \otimes B^\perp \vdash}\,\otimes_L}{A^\perp \otimes B^\perp \vdash (A \parr B)^\perp}\,\perp_R$$

A, B atomici

$1^\perp \dashv\vdash \perp$

$\perp^\perp \dashv\vdash 1$

$$\dfrac{\dfrac{\dfrac{}{\vdash 1}\,1_R}{\vdash 1, \perp}\,\perp_R}{\perp^\perp \vdash 1}\,\perp_L \qquad \dfrac{\dfrac{\dfrac{}{\perp \vdash}\,\perp_L}{\perp, 1 \vdash}\,1_L}{1 \vdash \perp^\perp}\,\perp_R$$

$$\dfrac{\dfrac{\dfrac{}{\vdash 1}\,1_R}{\vdash \perp, 1}\,\perp_R}{1^\perp \vdash \perp}\,\perp_L \qquad \dfrac{\dfrac{\dfrac{}{\perp \vdash}\,\perp_L}{1, \perp \vdash}\,1_L}{\perp \vdash 1^\perp}\,\perp_R$$

De Morgan Leggi nelle <u>duali</u>

$$(A \oplus B)^\perp \dashv\vdash A^\perp \& B^\perp$$

$$\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{\vdash A^\perp, A}\,\perp R}{\vdash A^\perp, A \oplus B}\,\oplus R^1}{(A \oplus B)^\perp \vdash A^\perp}\,\perp L \qquad \dfrac{\dfrac{\dfrac{B \vdash B}{\vdash B^\perp, B}\,\perp L}{\vdash B^\perp, A \oplus B}\,\oplus R^2}{(A \oplus B)^\perp \vdash B^\perp}\,\perp L}{(A \oplus B)^\perp \vdash A^\perp \& B^\perp}\,\& R$$

$$\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{A, A^\perp \vdash}\,\perp L}{A, A^\perp \& B^\perp \vdash}\,\& L^1 \qquad \dfrac{\dfrac{B \vdash B}{B, B^\perp \vdash}\,\perp L}{B, A^\perp \& B^\perp \vdash}\,\& L^2}{A \oplus B, A^\perp \& B^\perp \vdash}\,\oplus L}{A^\perp \& B^\perp \vdash (A \oplus B)^\perp}\,\perp R$$

$$(A \& B)^{\perp} \dashv\vdash A^{\perp} \oplus B^{\perp}$$

$$\dfrac{\dfrac{\vdash A^{\perp}, A}{\vdash A^{\perp} \oplus B^{\perp}, A}\,\oplus^{1}_{R} \qquad \dfrac{\vdash B^{\perp}, B}{\vdash A^{\perp} \oplus B^{\perp}, B}\,\oplus^{2}_{R}}{\dfrac{\vdash A^{\perp} \oplus B^{\perp}, A \& B}{(A \& B)^{\perp} \vdash A^{\perp} \oplus B^{\perp}}\,\perp_{L}}\,\&_{R}$$

$$\dfrac{\dfrac{\dfrac{A \vdash A}{A, A^{\perp} \vdash}}{A \& B, A^{\perp} \vdash}\,\&^{1}_{L} \qquad \dfrac{\dfrac{B \vdash B}{B, B^{\perp} \vdash}}{A \& B, B^{\perp} \vdash}\,\&^{2}_{R}}{\dfrac{A \& B, A^{\perp} \oplus B^{\perp} \vdash}{A^{\perp} \oplus B^{\perp} \vdash (A \& B)^{\perp}}\,\perp_{R}}\,\oplus_{L}$$