LK <u>logica classica</u>

$$
\dfrac{\dfrac{\begin{matrix}\Pi_1 \\ \vdots \\ \vdash B\end{matrix}}{\vdash B, C}\,w_R \qquad \dfrac{\begin{matrix}\Pi_2 \\ \vdots \\ \vdash B\end{matrix}}{C \vdash B}\,w_L}{\vdash B}\,\text{cut}
\qquad \longrightarrow \quad \begin{matrix}\Pi_1 \\ \text{oppure} \\ \Pi_2\end{matrix} \quad ? \quad \text{non det.}
$$

$$\Downarrow$$

<u>Logica Lineare</u>

- regole <u>moltiplicative</u> <u>additive</u>
- esponenziali $!$, $?$ op. unari $!A$ $?A$
- formule come risorse computazionali
- <u>proof nets</u> dimostrazioni come graf (<u>invariante</u>)

dualità

De Morgan

- $(A \otimes B)^\perp \dashv\vdash A^\perp ⅋ B^\perp$ $\qquad$ $(A ⅋ B)^\perp \dashv\vdash A^\perp \otimes B^\perp$
- $(A \oplus B)^\perp \dashv\vdash A^\perp \,\&\, B^\perp$ $\qquad$ $(A \,\&\, B)^\perp \dashv\vdash A^\perp \oplus B^\perp$
- $\underbrace{1^\perp \dashv\vdash \perp}$ $\qquad$ $\underbrace{\perp^\perp \dashv\vdash 1}$ $\qquad$ $(\ )^\perp$ duale, negazione lineare

$\perp$ perp bottom (falso moltip.)

False

$$\dfrac{}{0^\perp \vdash T} \qquad \dfrac{\dfrac{}{0, T \vdash}\,0_L}{T \vdash 0^\perp}\,\perp_R$$

$$\dfrac{\dfrac{}{\vdash 0, T}}{T^\perp \vdash 0}\,\perp_L \qquad \dfrac{}{0 \vdash T}\,0_L$$

$$0^\perp = T$$

$$T^\perp = 0$$

dualità esponenziali

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}}{\vdash A^{\perp}, A}\ \perp_R}{\vdash ?A^{\perp}, A}\ \text{derel?}}{\vdash ?(A^{\perp}), !A}\ \text{prom !}}{(!A)^{\perp} \vdash ?(A^{\perp})}\ \perp_L$$

$$(!A)^{\perp} \dashv\vdash ?(A^{\perp})$$

equiv.

$$\equiv$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}}{A, A^{\perp} \vdash}}{!A, A^{\perp} \vdash}\ !_L}{!A, ?A^{\perp} \vdash}\ \text{prom ? } (?_L)}{?A^{\perp} \vdash (!A)^{\perp}}\ \perp_R$$

eserc.

$$(?A)^{\perp} \dashv\vdash !(A^{\perp})$$

th negazione lineare è involutiva $\quad A \dashv\vdash (A^\perp)^\perp \qquad A \equiv A^{\perp\perp}$

dim per ind. sulla complessità logica di A
$\downarrow$
$\#$ di connettivi di A

passo induttivo
es.

$$\underbrace{A \otimes B}_{(n+1)} \dashv\vdash (A^\perp \parr B^\perp)^\perp = A^{\perp\perp} \otimes B^{\perp\perp} = A \otimes B$$

DeMorgan $\qquad$ ip. ind. su $\quad B^{\perp\perp} \equiv B \quad A^{\perp\perp} \equiv A$

base $\quad$ atomo $\quad a \equiv a^{\perp\perp}$ def.

$$\neg\neg A \equiv A$$

$$(A \otimes B)^\perp \overset{\dashv\vdash}{\equiv} A^\perp \parr B^\perp$$
DeMorg

$$((A \otimes B)^\perp)^\perp = (A^\perp \parr B^\perp)^\perp$$

$$\text{de Morgan} = \underbrace{A^{\perp\perp}}_{A} \otimes \underbrace{B^{\perp\perp}}_{B}$$

Elemento Neutro

1 elem. neutro del $\otimes$

$A \equiv A \otimes 1$

$$\dfrac{\dfrac{\overline{A \vdash A}}{A, 1 \vdash A}\,1_L}{A \otimes 1 \vdash A}\,\otimes_L \qquad\qquad \dfrac{\overline{A \vdash A} \quad \overline{\vdash 1}}{A \vdash A \otimes 1}\,\otimes_R$$

$\bot$ è elem. neutro del $⅋$

$$\dfrac{\dfrac{\overline{A \vdash A}}{A \vdash A, \bot}\,\bot_R}{A \vdash A ⅋ \bot}\,⅋_R \qquad\qquad \dfrac{\overline{A \vdash A} \quad \overline{\bot \vdash}\,\bot_L}{A ⅋ \bot \vdash A}\,⅋_L$$

0 è elem. neutro del $\oplus$

$$\dfrac{\overline{A \vdash A} \quad \overline{0 \vdash A}\,0_L}{A \oplus 0 \vdash A}\,\oplus_L \qquad\qquad \dfrac{\overline{A \vdash A}}{A \vdash A \oplus 0}\,\oplus_1$$

$\top$ è elem neutro del & "with"

$$\dfrac{\overline{A \vdash A}}{A \& \top \vdash A}\,\&^1_L \qquad\qquad \dfrac{\overline{A \vdash A} \quad \overline{A \vdash \top}\,\top_R}{A \vdash A \& \top}\,\&_R$$

idempotenza degli additivi

$$\frac{\overline{A \vdash A} \quad \overline{A \vdash A}}{A \oplus A \vdash A}\oplus_L \qquad \frac{\overline{A \vdash A}}{A \vdash A \oplus A}\oplus_R^{i} \qquad \frac{A \vdash A}{A \& A \vdash A}\&_L^{i} \qquad \frac{\overline{A \vdash A} \quad \overline{A \vdash A}}{A \vdash A \& A}\&_R$$

distributività dei moltiplicativi su gli additivi

$$\frac{\dfrac{\dfrac{\dfrac{\overline{A \vdash A} \quad \overline{B \vdash B}}{A, B \vdash A \otimes B}\otimes_R}{A, B \vdash (A \otimes B) \oplus (A \otimes C)}\oplus_R^{1} \qquad \dfrac{\dfrac{\overline{A \vdash A} \quad \overline{C \vdash C}}{A, C \vdash A \otimes C}\otimes_R}{A, C \vdash (A \otimes B) \oplus (A \otimes C)}\oplus_R^{2}}{A, \underline{B \oplus C} \vdash (A \otimes B) \oplus (A \otimes C)}\oplus_L}{A \otimes (B \oplus C) \vdash (A \otimes B) \oplus (A \otimes C)}\otimes_L$$

$$2 \cdot (3+4) = (2 \cdot 3) + (2 \cdot 4)$$

$$2 \cdot 7 = 6 + 8 = 14$$

Proprietà esponenziali

$$
\dfrac{
\dfrac{\dfrac{\dfrac{\overline{A \vdash A}}{A \& B \vdash A}\,\&^1_L}{!(A\&B) \vdash A}\,!_L}{!(A\&B) \vdash !A}\,p!
\qquad
\dfrac{\dfrac{\dfrac{\overline{B \vdash B}}{A \& B \vdash B}\,\&^2_L}{!(A\&B) \vdash B}\,!_L}{!(A\&B) \vdash !B}\,p!
}{!(A\&B) \vdash !A \otimes !B}\,\otimes_R
$$

$+$ esercizio

$$\dashv$$

$$\wedge_m \qquad \wedge_a \qquad \ldots$$

$$
\dfrac{\dfrac{
\dfrac{\dfrac{\overline{A \vdash A}}{A \vdash ?A}\,?_d}{A \vdash ?A, ?B}\,W_R
\qquad
\dfrac{\dfrac{\overline{B \vdash B}}{B \vdash ?B}\,?_d}{B \vdash ?A, ?B}\,W_R
}{A \oplus B \vdash ?A, ?B}\,\oplus_L}{\dfrac{?(A \oplus B) \vdash ?A, ?B}{?(A\oplus B) \vdash ?A \parr ?B}\,\parr_R}\,p?
$$

$+$ esercizio

$$\otimes \not\equiv \&$$

$$\parr \not\equiv \oplus \qquad \vee$$

$$A \parr B \not\equiv A \oplus B \quad \underline{\text{NO}}$$

One sided Sequent Calculus $\emptyset \vdash \Gamma, \Delta$ un calcolo più compatto

$\sim 1/2$ regole

theorem: $\Gamma \vdash \Delta$ è dim in LL $\overset{\Rightarrow}{\Longleftrightarrow}$ iff $\vdash \Gamma^\perp, \Delta$

notazione compatta

$$\Gamma = A_1, \dots A_n$$

$$\Gamma^\perp = A_1^\perp, \dots, A_n^\perp$$

$(\Longrightarrow)$

$$\dfrac{\dfrac{\underbrace{A_1 \dots A_n}\quad \vdots}{\Gamma \vdash \Delta}}{\dfrac{\vdots}{\vdash \Gamma^\perp, \Delta}}$$

neg. a destra via De Morgan

$\Longleftarrow$

$$\dfrac{\dfrac{\dfrac{\vdots}{\vdash \Gamma^\perp, \Delta}}{\Gamma^{\perp\perp} \vdash \Delta}\text{negazione a left De Morgan}}{\Gamma \vdash \Delta}\text{involuzione } ()^{\perp\perp}$$

$A^{\perp\perp} \equiv A$

Calcolo dei Sequenti ad una sola parte (destra) — LL^1

Identità group

$$\frac{}{\vdash \alpha, \alpha^\perp}\,ax \quad \text{atomi}$$

$$\frac{\vdash \Gamma, A \qquad \vdash \Delta, A^\perp}{\vdash \Gamma, \Delta}\,cut$$

moltiplicative

$$\frac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}\,\otimes \qquad \frac{}{\vdash 1}\,(1) \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\,\parr \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \bot}\,(\bot)$$

additive

$$\frac{\vdash \Gamma, A_{i=1,2}}{\vdash \Gamma, A_1 \oplus A_2}\,\oplus_i \qquad \frac{\overbrace{\vdash \Gamma, A \qquad \vdash \Gamma, B}^{=}}{\vdash \Gamma, A \& B}\,\& \qquad \frac{}{\vdash \Gamma, \top}\,(\top)$$

esponenziali:

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\,?d \qquad \frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !\underline{A}}\,p!$$

strutturali

$$\frac{\vdash \Gamma, A, B, \Delta}{\vdash \Gamma, B, A, \Delta}\,ex \qquad \frac{\vdash \Gamma}{\vdash \Gamma, ?\underline{A}}\,w \;\text{weakening} \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?\underline{A}}\,c \;\text{contraction}$$

Teorema: $\Gamma \vdash \Delta$ è dimostrabile in LL^2 (a due parti) con le regole sinistra e destra $\overset{\text{iff}}{\Longleftrightarrow}$ $\vdash \Gamma^\perp, \Delta$ è dimostrabile in LL^1 ad una sola parte con le sole regole destra.

$LL^1 \subseteq LL^2$

Proof: $\Leftarrow$ part $\vdash \Gamma^\perp, \Delta$ in $LL^1 \Longrightarrow \Gamma \vdash \Delta$ in LL^2

$$\dfrac{\dfrac{\dfrac{\vdots}{\vdash \Gamma^\perp, \Delta \text{ in } LL^1 \subseteq LL^2}}{\Gamma^{\perp\perp} \vdash \Delta}\ \text{negazione a left, De Morgan}}{\Gamma \vdash \Delta \text{ in } LL^2}\ \text{involuz. } (\perp\perp)$$

$\perp L$

$\Rightarrow$ part (the only part) $\pi : \Gamma \vdash \Delta$ in LL_2 $\Longrightarrow \pi' : \vdash \Gamma^\perp, \Delta$ in LL^1

per induzione sull'altezza di π

base $\quad \overline{\alpha \vdash \alpha}$ in LL^2 $\Rightarrow$ $\overline{\vdash \alpha^\perp, \alpha}\, ax$ in LL^1 De Morgan

passo induttivo

per casi

$$\pi : \dfrac{\overset{\pi_1}{\Gamma \vdash \Delta, A, B}}{\Gamma \vdash \Delta, A \otimes B}\,\otimes \quad \text{in } LL^2 \qquad \text{ip ind su } \pi_1 \ \exists \pi_1' \qquad \dfrac{\vdash \Gamma^\perp, \Delta, A, B}{\vdash \Gamma^\perp, \Delta, A \otimes B}\,\otimes_{LL^1}$$

$$\pi : \dfrac{\overset{\pi_1}{\Gamma, A \vdash \Delta} \qquad \overset{\pi_2}{\Gamma, B \vdash \Delta}}{\Gamma, A \oplus B \vdash \Delta}\,\oplus_L \qquad \text{ip-ind su } \pi_i \ i=1,2 \qquad \dfrac{\dfrac{\pi_1'}{\vdash \Gamma^\perp, A^\perp, \Delta} \qquad \dfrac{\pi_2'}{\vdash \Gamma^\perp, B^\perp, \Delta}}{\vdash \Gamma^\perp, A^\perp \& B^\perp, \Delta}\,\&$$

per De Morgan $= (A \oplus B)^\perp = A^\perp \& B^\perp$

in LL^2

$$\pi \ \dfrac{\overset{\pi_1}{\Gamma, A \vdash \Delta}}{\Gamma, A \& B \vdash \Delta} \qquad \text{ip. ind} \qquad \dfrac{\overset{\pi_1'}{\vdash \Gamma^\perp, A^\perp, \Delta}}{\vdash \Gamma^\perp, A^\perp \oplus B^\perp, \Delta}\,\oplus^1 \quad \text{in } LL^1$$

via De Morgan $(A^\perp \oplus B^\perp) = (A \& B)^\perp$

caso promotion left

$$\pi \quad \dfrac{\overset{\pi_1 \quad \text{in } LL^2}{!\Gamma, A \vdash ?\Delta}}{!\Gamma, \underline{?A} \vdash ?\Delta}\, p? \longrightarrow \text{ ip ind su } \Pi_1 \qquad \exists\, \pi_1' \quad \dfrac{\vdash ?(\Gamma^\perp), A^\perp, ?\Delta}{\vdash ?(\Gamma^\perp), !(A^\perp), ?\Delta}\, p!$$

$!A^\perp = (?A)^\perp$ per ~~[illegible]~~

$$(!\Gamma)^\perp = ?\Gamma^\perp$$

$$(?A)^\perp = !(A^\perp)$$

$$(A \otimes B)^\perp = A^\perp \parr B^\perp$$

Cut-eliminazione   procedura effettiva   algoritmo

ragionamento per induzione

$\Pi : \vdash \Gamma$ ass $\neg \exists$ una $\Pi' \vdash \Gamma$ cut free

$$\frac{\dfrac{\neg\exists}{\vdots}}{\dfrac{\bot}{}}$$

Non è costruttivo!

$\neg\neg\exists \Rightarrow \exists$ una $\Pi' \vdash \Gamma$ cut free.

Eliminazione dei tagli per LL^{2}

in LK^{2}

$$\dfrac{\dfrac{\begin{array}{c}\Pi_1\\ \vdash \Gamma\end{array}}{\vdash \Gamma, A}\,w \qquad \dfrac{\begin{array}{c}\Pi_2\\ \vdash \Gamma'\end{array}}{\vdash \Gamma', \neg A}\,w}{\vdash \Gamma, \Gamma'}\,cut$$

Π_1 ? Π_2

fare una scelta!

in LL^{2}

$$\dfrac{\begin{array}{c}\Pi_1\\ \vdash \Gamma\end{array}}{\vdash \Gamma, ?A}\,w \qquad \dfrac{\begin{array}{c}\Pi_2\\ \vdash \Gamma'\end{array}}{\vdash \Gamma', ?A^{\perp}}\,w$$

$$(?A)^{\perp} = !(A^{\perp})$$

$\vdash \Gamma, ?A \qquad \vdash \Gamma', \underline{\underline{!A^{\perp}}}$

$?A$

<u>Definizioni</u> : cut rank (rango del Taglio)

~~A~~ è la formula del taglio rk(A) è la complessità logica della formula del taglio

$$\Pi: \frac{\vdash \Gamma, A \quad \vdash \Gamma', A^{\perp}}{\vdash \Gamma, \Gamma'} r$$

<u>rk(r)</u>

r regola del taglio

rk(r) è la complessità logica della formula tagliata.

dei connettivi eccetto ¬

cut r : (<u>rk(r)</u>, <u>lv(r)</u>)

<u>livello del taglio r</u>

è la <u>Taglia</u> (size) della dimostrazione Π

la dim è un albero quindi la size di Π è il numero di nodi dell'albero di Π multicut   cut-strutturale

$$\frac{\vdash \Gamma, C^{k} \qquad \vdash \Delta, (C^{\perp})^{l}}{\vdash \Gamma, \Delta}\ \text{scut}$$

$k, l \geq 0$

$$\frac{\vdash \Gamma, \underline{C_1 \dots C_k} \qquad \vdash \Delta, \underline{C_1^{\perp} \dots C_l^{\perp}}}{\vdash \Gamma, \Delta}\ \text{scut}$$

un indice tra k ed l differisce da 1 solo se le formule con questo indice diverso da 1 è una ?-formula.

$k > 1$

$$\frac{\vdash \Gamma, \underline{?C_1 \dots ?C_k} \qquad \vdash \Delta, \underline{!C^{\perp}}}{\vdash \Gamma, \Delta}$$

$$\frac{\vdash \Gamma, C^{1} \qquad \vdash \Delta, C^{\perp 1}}{\vdash \Gamma, \Delta}$$

Vari casi di riduzione

- riduzione con axiom $\overline{\vdash C, C^{\perp}}$
- riduzione logica quando $C, C^{\perp}$ sono conclusioni di regole logiche
- riduzione espressa C o $C^{\perp}$ è concl. di una reg. exp (p.d)
- riduzione strutturale/commutativa altri casi.

- quando voliamo un cut passo di riduzione

lo voliamo in uno o più cut di complessità inferiore

oppure lo "spostiamo" ad un livello inferiore

$$\text{cut } v < rk(z), lv(z) >$$

$\uparrow$ $\uparrow$