Dispense ultima lezione Teorie Logiche 1 (a.a. 2020/2021)

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Il testo che segue è largamente incompleto, molto schematico, scritto parzialmente in inglese e parzialmente in italiano. Si tratta di una traccia per fissare meglio il contenuto della lezione di lunedì 24 maggio 2021 ma sono solo appunti molto approssimativi. Vengono presentati tre risultati per il frammento moltiplicativo MLL della Logica Lineare:

1. l'interpretazione di una struttura di prova è invariante per eliminazione del taglio (Teorema 1, di cui si fornisce solo l'enunciato e lo schema della dimostrazione)

2. due esperienze di una struttura di prova che sia aciclica nei grafi di correttezza hanno risultati coerenti tra loro: ne consegue che l'interpretazione di una tale struttura di prova è una clique dello spazio coerente che interpreta il suo sequente conclusione (Teorema 2)

3. se l'interpretazione di una struttura di prova è una clique dello spazio coerente che interpreta il suo sequente conclusione, allora la struttura di prova è aciclica nei grafi di correttezza (Teorema 3)

Diciamo che una struttura di prova (PS in inglese) soddisfa AC oppure è AC-corretta quando ogni suo grafo di correttezza è aciclico.

L'intepretazione dei proof-net: le esperienze

Definizione 1 (experiment)**.** *The MLL case*
Let S be a PS. An experiment e of S (we sometimes write $e : S$) is an application which associates with every edge a of type A of S an element $e(a)$ (of some set), in such a way that the following conditions hold:

- *If $a = a_1$ is the conclusion of an axiom link with conclusions the edges a_1 and a_2, then $e(a_1) = e(a_2)$.*
- *If a is the premise of a cut link with premises a and b, then $e(a) = e(b)$.*
- *If a is the conclusion of a $\wp$ (resp. $\otimes$) link with left premise a_1 and right premise a_2, then $e(a) = (x_1, x_2)$, where $e(a_1) = x_1$ and $e(a_2) = x_2$.*

If $c_1, \dots, c_n$ are the edges conclusions of S, then the result of *e, denoted by $|e|$, is the sequence*[1] *$< e(c_1), \dots, e(c_n) >$.*

Osservazione 1. *Every labeling of the axiom links defines an experiment of a* cut-free *PS. In presence of cuts, this is not the case.*

Definizione 2. *Let R be a PS with $c_1, \dots, c_n$ as edges conclusion,*

$$[\![R]\!] := \{< e(c_1), \dots, e(c_n) > \ e \text{ is an experiment of } R\}.$$

$[\![R]\!]$ is said to be the interpretation or the semantics of R.

Osservazione 2. *The definition of interpretation does not make much sense until we don't specify the domain.*
Of course, the interpretation can be defined on PS which are not AC-correct.

Teorema 1. *If R and R' are two PS such that $R \to R'$, then $[\![R]\!] = [\![R']\!]$.*

Proof. We need to prove that for every cut $x = (ax), (\wp/\otimes)$ such that $R \xrightarrow{[x]} R'$ one has:

- for every experiment e of R there exists an experiment e' of R' such that $|e| = |e'|$
- for every experiment e' of R' there exists an experiment e of R such that $|e| = |e'|$.

□

Notazione: Date due esperienze e_1, e_2 di una SP R, e data a edge of R, we write $a :\frown$ (risp. $a :\smile, a :=$) meaning $e_1(a) \frown e_2(a)$ (risp. $e_1(a) \smile e_2(a)$, $e_1(a) = e_2(a)$).

Teorema 2. *If R is a proof-structure satisfying (AC) with conclusion Γ, then $[\![R]\!] \in \wp\Gamma$ (here $\wp\Gamma$ is the coherent space interpreting the formula $\wp\Gamma$). That is: given any two experiments $e_1, e_2 : R$ one has $|e_1| \overset{\frown}{\smile} |e_2|$ $(\wp\Gamma)$.*

Proof. We are actually going to prove a stronger result, namely that if $|e_1| \overset{\smile}{\frown} |e_2|$ $(\wp\Gamma)$ then $e_1 = e_2$ which immediately yields the result we look for, and Corollary 1.
Consider the (true: no pending edges) graph G_R obtained from R by adding a terminal node to every conclusion. As we added no edges, the two functions e_1 and e_2 are still defined on every edge of G_R. We define from G_R a directed graph R_0 as follows:

[1] One should say that a PS is given together with an order on its conclusions, so the sequence $< e(c_1), \dots, e(c_n) >$ is uniquely determined by e and π.

1. if $a :=$ erase the edge a, and erase also the links which have now become isolated
2. for the "switching links" (contributing to correctness graphs) $\wp$ with premises a_1, a_2, if there is a premise a_i of l such that $a_i : \smile$ and for the other premise we have $a_j : \frown$, then we disconnect a_i from the link so that a_i becomes a new conclusion called "switched conclusion"
3. we direct all the edges b following the coherence relation on b between e_1 and e_2: if $b : \smile$ then $\uparrow b$ and if $b : \frown$ then $\downarrow b$.

We call R_0 the graph thus obtained to which we add a terminal node for every switched conclusion. First notice that R_0 is a (true: no pending edges) directed acyclic graph (dag):

- R_0 is still a graph, since we have only erased edges. And we have erased nodes only when they had become isolated: it is still the case that every edge connects two nodes. R_0 is oriented since e_1 and e_2 are still defined on every edge of R_0 (we added no new edge).
- since R is AC-correct, the unique possibility to have a (directed) cycle in R_0 is to bounce in a switching link ($\wp$)[2]. But this is forbidden by items 2 and 3: by item 3, in order to bounce we need that the two premises have a different (and strict) "coherent label", which means that one is labelled $\smile$ and the other one is labelled $\frown$; in this case by item 2 the edge labelled $\smile$ is not premise of a switching link (in the directed graph R_0).

If R_0 is empty, then by item 1 for every edge a with depth 0 in R we have $e_1(a) = e_2(a)$, which implies the result. It is then enough to prove that R_0 is empty. Suppose by contradiction it isn't. We are going to prove that whatever link l of R_0 one chooses[3], l is not a sink: there is always a way out from l. Since R_0 is a dag, if it is not empty there is a sink, and if this sink is not a link of R it has to be a conclusion node. This node cannot be connected to a switching conclusion (those conclusions are all oriented "up'), and is thus connected to a conclusion of R, which must then be oriented down. But this means by item 3 that $c : \frown$, thus contradicting the hypothesis $|e_1| \stackrel{\smile}{\frown} |e_2|$. The only possibility is then that R_0 is empty and we are done.

To conclude, we show that every link l of R_0 is not a sink:

- if l is an axiom or a cut (and it has not been erased) it is obvious: the labels of the two conclusions/premises are the same and the types of the conclusions/premises are dual
- $l = \otimes$: if for the conclusion a we have $a : \frown$ we can exit from there. If $a :=$, then $a_i :=$ for both the premises and l is not a node of R_0 by item 1. Then $a : \smile$ and for one of the two premises a_i of l we have $a_i : \smile$ and we can exit from there
- $l = \wp$ i.e. a switching link: if for the conclusion a we have $a : \frown$ we can exit from there. Otherwise $a : \stackrel{\smile}{\frown}$ and then for every premise a_i of l we have $a_i : \stackrel{\smile}{\frown}$. If for every premise a_i of l we have $a_i :=$, then $a :=$ and l is not a node of R_0 by item 1. Otherwise there is a premise a_i of l in R such that $a_i : \smile$ and by item 1 this premise is still present in R_0 (remember that for every premise a_i of l we have $a_i : \stackrel{\smile}{\frown}$, so that no premise has been disconnected by item 1), and we can exit from there.

$\square$

[2] This can be checked carefully case by case: if an oriented cycle does not bounce in a switching link l and passes through the 2 premisses of l, then there exists a (smaller) cycle which doesn't pass through both the premisses of l (one of the two is useless).

[3] We mean here link of LL, coming from R.

Corollario 1. *If R is typed and (AC)-correct and $e_1, e_2 : R$ are two experiments of R such that $|e_1| = |e_2|$, then $e_1 = e_2$.*

Proof. Since $|e_1| = |e_2|$ implies $|e_1| \stackrel{\smile}{\frown} |e_2|$, it is an immediate consequence of the proof of Theorem 2. $\square$

Path: non si può "tornare indietro": è proibito $\downarrow d \uparrow d$ così come $\uparrow d \downarrow d$.

Teorema 3. *Let R be a typed MLL cut-free PS and $\mathcal{X}$ a coherent space such that $x, y, z \in |\mathcal{X}|$ with $x \frown y$ and $x \smile z$. If for the interpretation of MLL obtained by associating with every atomic formula the coherent space $\mathcal{X}$ we have that $[\![R]\!]$ is a clique, then R is AC-correct.*

The proof of Theorem 3 follows from the following lemmas, where we call "switching path" a path of a correctness graph.

Lemma 1. *Let R be a MLL cut-free PS satisfying AC, and let c, c' be two different conclusions of R and Φ be a switching path of R with starting edge $\uparrow c$ and terminal edge $\downarrow c'$. Then there exists a switching path Φ' of R with starting edge $\uparrow c$ and terminal edge $\downarrow c'$ such that for every $d \geqslant c$*[4] *edge of R, one has $\downarrow d \notin \Phi$.*

Proof. Per andare da c a c' ad un certo punto bisognerà per forza (essendo $c \neq c'$) che Φ "abbandoni" l'insieme degli archi d tale che $d \geqslant c$, e bisognerà che lo faccia "definitivamente", cioè esiste $\uparrow d$, con $d \geqslant c$ tale che in Φ da $\uparrow d$ in poi passo solo da archi b che non sono sopra c. Prendo allora come Φ' la composizione dell'unico cammino switching Ψ che parte da $\uparrow c$ e termina in $\uparrow d$ e del sottocammino switching di Φ che va da $\uparrow d$ fino a $\downarrow c'$. Tale cammino è switching e gli archi che percorre sopra c li percorre tutti salendo, e dunque soddisfa la conclusione del lemma. $\square$

Lemma 2. *Let $\mathcal{X}$ a coherent space such that $x, y, z \in |\mathcal{X}|$ with $x \frown y$ and $x \smile z$, and consider the interpretation of MLL obtained by associating with every atomic formula the coherent space $\mathcal{X}$. Let R be a MLL cut-free PS satisfying AC, let c, c' be two conclusions of R, and let Φ be a directed switching path of R with starting edge $\uparrow c$ and terminal edge $\downarrow c'$ such that, for every $d \geqslant c$ edge of R, one has $\downarrow d \notin \Phi$.*
There exists two experiments e_1 and e_2 of R s.t. $c : \frown$, $c' : \smile$ and for every $d \neq c, c'$ we have $d : \stackrel{\smile}{\frown}$.

Proof. Since we are cut-free and in MLL, an experiment is entirely determined by its values on the axiom edges. Notice that whatever formula A of MLL we consider, there exists $\alpha, \beta, \gamma \in |\mathcal{A}|$ such that $\alpha \frown \beta$ and $\alpha \smile \gamma$[5]. Now we define e_1 and e_2 by declaring their values on the conclusions of the generic axiom link whose conclusions are labelled by $A, A^\perp$: $e_1(a) = \alpha$ (we associate the element of $|\mathcal{A}|$ inheriting the property we assume is satisfied by the space $\mathcal{X}$ interpreting every propositional variable). The values of e_2 depend on the way Φ crosses the axiom edges:

- if $\uparrow a \in \Phi$, then we set $e_2(a) = \beta$ (thus $a : \frown$)
- if $\downarrow a \in \Phi$, then we set $e_2(a) = \gamma$ (thus $a : \smile$)
- if $\uparrow a \notin \Phi$ and $\downarrow a \notin \Phi$, then we set $e_1(a) = e_2(a) = \alpha$.

Now, for every edge d of R, we prove that:

1. if for some $d' \geqslant d$ we have $\uparrow d' \in \Phi$ or $\downarrow d' \in \Phi$, then $d :\neq$

[4]For a, b edges of R one can define the partial order $a \leqslant b$ iff b is "above" a: antisimmetry is a consequence of the absence of vicious cycles.

[5]Proof: exercise by induction on A. If $A = A_1 \otimes A_2$ or $A = A_1 \wp A_2$, then by IH $\alpha_i \frown \beta_i$ and $\alpha_i \smile \gamma_i$, thus $(\alpha_1, \alpha_2) \frown (\beta_1, \beta_2)$ and $(\alpha_1, \alpha_2) \smile (\gamma_1, \gamma_2)$.

2. if $\uparrow d \notin \Phi$, then $d : \overset{\smile}{\frown}$

3. if for every $d' \geqslant d$ we have $\downarrow d' \notin \Phi$, then $d : \overset{\frown}{\smile}$

Property 1 is a consequence of the fact that whenever $\uparrow a \in \Phi$ or $\downarrow a \in \Phi$, there exists an axiom edge $b \geqslant a$ such that $\uparrow b \in \Phi$ or $\downarrow b \in \Phi$ (in which case we have by definition $b :\neq$ and thus $a :\neq$): if $\uparrow a \in \Phi$ I need to go up until I can and if $\downarrow a \in \Phi$ then either a is conclusion of an axiom and we are done or it is the conclusion of a $\otimes$ or $\wp$ link such that one of its premises a_1 satisfies $\downarrow a_1 \in \Phi$ (intuitively $\downarrow a$ comes from an axiom conclusion $\downarrow b$).
Let's now prove properties 2 and 3 by induction on the number of nodes above d:

- if d is the conclusion of an axiom link such that $\downarrow d \in \Phi$ or $\uparrow d \in \Phi$, then if we call a and b the conclusions of the axiom we necessarily have $\downarrow a, \uparrow b \in \Phi$ or $\downarrow b, \uparrow a \in \Phi$: suppose we have for example $\downarrow a, \uparrow b \in \Phi$, then by definition of e_1, e_2 we have $a : \smile$ and $b : \frown$ and all the conditions are satisfied

- if d is the conclusion of a $\otimes$ link with left (resp. right) premise a (resp. b):

property 2: suppose $\uparrow d \notin \Phi$. If $\uparrow a \in \Phi$ (resp. $\uparrow b \in \Phi$), then since $\uparrow d \notin \Phi$ we necessarily have $\downarrow b \in \Phi$ (resp. $\downarrow a \in \Phi$): then $\uparrow b \notin \Phi$ and by induction hypothesis $b : \overset{\smile}{\frown}$ (resp. $a : \overset{\smile}{\frown}$) which by property 1 implies $b : \smile$ (resp. $a : \smile$) and thus $d : \smile$. If on the contrary $\uparrow a, \uparrow b \notin \Phi$, then by induction hypothesis $a : \overset{\smile}{\frown}$ and $b : \overset{\smile}{\frown}$, which implies $d : \overset{\smile}{\frown}$.

property 3: suppose for every $d' \geqslant d$ we have $\downarrow d' \notin \Phi$. Then obviously for every $d' \geqslant a$ (resp. for every $d' \geqslant b$) we have $\downarrow d' \notin \Phi$, and thus by IH $a : \overset{\frown}{\smile}$ (resp. $b : \overset{\frown}{\smile}$), which implies $d : \overset{\frown}{\smile}$.

- if d is the conclusion of a $\wp$ link with left (resp. right) premise a (resp. b):

property 2: suppose $\uparrow d \notin \Phi$. Since Φ is a switching path, necessarily $\uparrow a \notin \Phi$ and $\uparrow b \notin \Phi$ (like in the $\otimes$ case, if $\uparrow a \in \Phi$ and $\uparrow d \notin \Phi$ we should have $\downarrow b \in \Phi$, but in this case Φ would not be switching). Then by IH $a : \overset{\smile}{\frown}$ and $b : \overset{\smile}{\frown}$ which implies $d : \overset{\smile}{\frown}$

property 3: suppose for every $d' \geqslant d$ we have $\downarrow d' \notin \Phi$. Then obviously for every $d' \geqslant a$ (resp. for every $d' \geqslant b$) we have $\downarrow d' \notin \Phi$, and thus by IH $a : \overset{\frown}{\smile}$ (resp. $b : \overset{\frown}{\smile}$), which implies $d : \overset{\frown}{\smile}$.

From the properties 1, 2 and 3 we can conclude that $c : \frown$: indeed, since $\uparrow c \in \Phi$ and (by the hypothesis of the lemma) for every $d \geqslant c$ we have $\downarrow d \notin \Phi$, by property 3 this yields $c : \overset{\frown}{\smile}$, and by property 1 this means that $c : \frown$. On the other hand, since $\downarrow c' \in \Phi$ we have $\uparrow c' \notin \Phi$ and hence by property 2 $c' : \overset{\smile}{\frown}$, which implies by property 1 that $c' : \smile$. □

Let's now prove Theorem 3:

Proof. We prove, by induction on the number of links of R, that if R does not satisfy AC, then $[\![R]\!]$ is not a clique. If R does not satisfy AC, then it necessarily has a terminal $\otimes$-link or a terminal $\wp$-link:

- if R has a terminal $\wp$-link, then we can remove it thus obtaining R' which is still a PS which does not satisfy AC. Then by IH $[\![R']\!]$ is not a clique, and $[\![R]\!]$ neither (from two experiments e'_1 and e'_2 of R' such that $|e'_1| \smile |e'_2|$ it is easy to build two experiments e_1 and e_2 of R such that $|e_1| \smile |e_2|$[6])

[6]Do it! Distinguish the case $e'_1(a) \smile e'_2(a)$ above the $\wp$ and the other case.

- if R has a terminal $\otimes$-link l, then we can remove l thus obtaining R', for which there are two possibilities:
  - R' is still a PS which does not satisfy AC, in which case by IH $[\![R']\!]$ is not a clique and $[\![R]\!]$ neither (proceeding like in the $\wp$ case)
  - R' has become a PS satisfying AC: then there exists a switching path Φ of R' with starting edge a premise $\uparrow a$ of l and terminal edge the other premise $\downarrow b$ of l (since R has a switching cycle and R' hasn't, every switching cycle of R crosses l: we just select one of them). By Lemma 1, we can suppose that, for every $g \geqslant a$ edge of R', one has $\downarrow g \notin \Phi$. We can now apply Lemma 2 and find two experiments e'_1, e'_2 of R' such that for every conclusion $c \neq a, b$ of R' we have $c : \stackrel{\smile}{\frown}$ while $a : \frown$ and $b : \smile$: e'_1 and e'_2 immediately yield two experiments e_1, e_2 of R such that $e_1(d) \smile e_2(d)$ (where d is the conclusion of l), and for every conclusion $c \neq d$ we have $e_1(c) \stackrel{\smile}{\frown} e_2(c)$: thus $|e_1| \smile |e_2|$ and $[\![R]\!]$ is not a clique.

$\square$

Osservazione 3. *The absence of cuts is not really a limit (substitute with $\otimes$).*