**Università degli Studi Roma Tre - Corso di Laurea in Matematica**

# Tutorato di AM220

**A.A. 2010-2011 - Docente: Prof.ssa S. Mataloni**

**Tutore: Luca Battaglia**

Soluzioni del tutorato numero 10 (18 Maggio 2011)

Superfici

I testi e le soluzioni dei tutorati sono disponibili al seguente indirizzo:
*http://www.lifedreamers.it/liuck*

1.

$$\Phi(u,v)=(\cosh u\cos v,\cosh u\sin v,u)\quad (u,v)\in[0,\log 2]\times[-\pi,\pi]$$

$$\Phi_u\wedge\Phi_v=\begin{vmatrix}\overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k}\\ \sinh u\cos v & \sinh u\sin v & 1\\ -\cosh u\sin v & \cosh u\cos v & 0\end{vmatrix}=(-\cosh u\cos v,-\cosh u\sin v,\cosh u\sinh u)\Rightarrow$$

$$\Rightarrow\|\Phi_u\wedge\Phi_v\|=\cosh u^2\Rightarrow Area(\Sigma)=\int_0^{\log 2}\int_{-\pi}^{\pi}\cosh^2 u du dv=2\pi\int_0^{\log 2}\frac{e^{2u}+e^{-2u}+2}{4}du=$$

$$2\pi\left[\frac{e^{2u}-e^{-2u}}{8}+\frac{u}{2}\right]_0^{\log 2}=\pi\left(\frac{15}{16}+\log 2\right)$$

2.

$$\Phi(u,v)=(u,v,e^u)\quad (u,v)\in[0,1]\times[0,3]$$

$$\Phi_u\wedge\Phi_v=\begin{vmatrix}\overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k}\\ 1 & 0 & e^u\\ 0 & 1 & 0\end{vmatrix}=(-e^u,0,1)\Rightarrow\|\Phi_u\wedge\Phi_v\|=\sqrt{e^{2u}+1}\Rightarrow$$

$$\Rightarrow\int_\Sigma z^2 d\sigma=\int_0^1\int_0^3 e^{2u}\sqrt{e^{2u}+1}dudv\overset{(t=e^{2u})}{=}3\int_1^{e^2}\frac{\sqrt{t+1}}{2}=\left[(t+1)^{\frac{3}{2}}\right]_1^{e^2}=\left(e^2+1\right)^{\frac{3}{2}}-2\sqrt{2}$$

3.

$$\Phi(u,v)=(u\cos v,u\sin v,v)\quad (u,v)\in[0,\pi]\times[-\pi,\pi]$$

$$\Phi_u\wedge\Phi_v=\begin{vmatrix}\overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k}\\ \cos v & \sin v & 0\\ -u\sin v & -u\cos v & 1\end{vmatrix}=(\sin v,-\cos v,u)\Rightarrow\|\Phi_u\wedge\Phi_v\|=\sqrt{u^2+1}\Rightarrow$$

$$\Rightarrow\int_\Sigma\frac{yz}{\sqrt{x^2+y^2+1}}d\sigma=\int_0^\pi\int_{-\pi}^{\pi}uv\sin v dudv=\left[\frac{u^2}{2}\right]_0^\pi\left([-v\cos v]_{-\pi}^{\pi}+\int_{-\pi}^{\pi}\cos v\right)=$$

$$=\frac{\pi^2}{2}\left(2\pi+[\cos v]_{-\pi}^{\pi}\right)=\pi^3$$

4. $\Sigma$ è parametrizzata da

$$\Phi(u,v) = ((\cos u+2)\cos v, (\cos u+2)\sin v, \sin u) \quad (u,v) \in [-\pi,\pi]\times[-\pi,\pi]$$

$$\Phi_u \wedge \Phi_v = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ -\sin u \sin v & -\sin u \sin v & \cos u \\ -(\cos u+2)\sin v & (\cos u+2)\cos v & 0 \end{vmatrix} =$$

$$= (-\cos u(\cos u+2)\cos v, -\cos u(\cos u+2)\sin v, -\sin u(\cos u+2)) \Rightarrow \|\Phi_u\wedge\Phi_v\| = \cos u+2 \Rightarrow$$

$$\Rightarrow Area(\Sigma) = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \cos u + 2dudv = 2\pi[\sin u + 2]_{-\pi}^{\pi} = 8\pi^2$$

$$\int_\Sigma x^2+y^2+z^2 d\sigma = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}(4\cos u+5)(\cos u+2)dudv = 2\pi\int_{-\pi}^{\pi} 2\cos(2u)+13\cos u+12du =$$

$$= 2\pi[\sin(2u) + 13\sin u + 12u]_{-\pi}^{\pi} = 48\pi^2$$

5.

$$\Sigma = \left\{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + \frac{z^2}{4} = 1\right\}$$

è parametrizzata da

$$\Phi(u,v) = (\cos u \sin v, \sin u \sin v, 2\cos v) \quad (u,v) \in [-\pi,\pi]\times[0,\pi]$$

$$\Phi_u\wedge\Phi_v = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ -\sin u \cos v & \cos u \sin v & 0 \\ \cos u \cos v & \sin u \cos v & -2\sin v \end{vmatrix} = \left(-2\cos u \sin^2 v, -2\sin u\sin^2 v, -\cos v \sin v\right) \Rightarrow$$

$$\Rightarrow \|\Phi_u\wedge\Phi_v\| = \sin v\sqrt{3\sin^2 v + 1} \Rightarrow Area(\Sigma) = \int_{-\pi}^{\pi}\int_0^{\pi} \sin v\sqrt{4-3\cos^2 v}dudv \overset{(t=\cos v)}{=}$$

$$= 2\pi\int_{-1}^{1}\sqrt{4-3t^2}dt \overset{\left(s=\arccos\left(\frac{\sqrt{3}}{2}t\right)\right)}{=} \frac{8}{3}\sqrt{3}\pi\int_{\frac{\pi}{6}}^{\frac{5}{6}\pi}\sin^2 sds = \frac{8}{3}\sqrt{3}\pi\int_{\frac{\pi}{6}}^{\frac{5}{6}\pi}\frac{1-\cos(2s)}{2} =$$

$$= \frac{8}{3}\sqrt{3}\pi\left[\frac{s}{2} - \frac{\sin(2s)}{4}\right]_{\frac{\pi}{6}}^{\frac{5}{6}\pi} = \pi\left(\frac{8}{9}\sqrt{3}\pi + 2\right)$$

$$\int_\Sigma \sqrt{16-3z^2}d\sigma = \int_{-\pi}^{\pi}\int_0^{\pi}\sin v\sqrt{16-12\cos^2 v}\sqrt{4-3\cos^2 v}dudv = 4\pi\int_0^{\pi}\sin v\left(4-3\cos^2 v\right)dv =$$

$$= 4\pi\left[-4\cos v + \cos^3 v\right]_0^{\pi} = 24\pi$$

6.

$$A = \left\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \le 1, x^2+y^2 \le 3z^2, z \ge 0\right\}$$

$$\partial A = \Sigma_1 \cup \Sigma_2$$

ove

$$\Sigma_1 = \left\{(x,y,z) \in \mathbb{R} : x^2+y^2+z^2 \le 1, x^2+y^2 = 3z^2, z \ge 0\right\}$$

è parametrizzata da

$$\Phi_1(u,v)=\left(u\cos v, u\sin v, \frac{\sqrt{3}}{3}u\right) \quad (u,v)\in\left[0,\frac{\sqrt{3}}{2}\right]\times[-\pi,\pi]$$

con

$$\Phi_{1,u}\wedge\Phi_{1,v}=\begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \cos v & \sin v & \frac{\sqrt{3}}{3} \\ -u\sin v & u\cos v & 0 \end{vmatrix}=\left(-\frac{\sqrt{3}}{3}u\cos v, -\frac{\sqrt{3}}{3}u\sin v, u\right)\Rightarrow \|\Phi_{1,u}\wedge\Phi_{1,v}\|=\frac{2\sqrt{3}}{3}u\Rightarrow$$

$$\Rightarrow Area(\Sigma_1)=\int_0^{\frac{\sqrt{3}}{2}}\int_{-\pi}^{\pi}\frac{2\sqrt{3}}{3}u\,du\,dv=2\pi\left[\frac{\sqrt{3}}{3}u^2\right]_0^{\frac{\sqrt{3}}{2}}=\frac{\sqrt{3}}{2}\pi$$

e

$$\Sigma_2=\left\{(x,y,z)\in\mathbb{R}: x^2+y^2+z^2=1, x^2+y^2\leq 3z^2, z\geq 0\right\}$$

è parametrizzata da

$$\Phi_2(u,v)=(\cos u\sin v, \sin u\sin v, \cos v) \quad (u,v)\in[-\pi,\pi]\times\left[0,\frac{\pi}{3}\right]$$

con

$$\Phi_{2,u}\wedge\Phi_{2,v}=\begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ -\sin u\sin v & \cos u\sin v & 0 \\ \cos u\cos v & \sin u\cos v & -\sin v \end{vmatrix}=\left(-\cos u\sin^2 v, -\sin u\sin^2 v, -\cos v\sin v\right)\Rightarrow$$

$$\Rightarrow \|\Phi_{2,u}\wedge\Phi_{2,v}\|=\sin v\Rightarrow Area(\Sigma_2)=\int_{-\pi}^{\pi}\int_0^{\frac{\pi}{3}}\sin v\,du\,dv=2\pi[-\cos v]_0^{\frac{\pi}{3}}=\pi$$

Dunque,

$$Area(\partial A)=Area(\Sigma_1)+Area(\Sigma_2)=\left(\frac{\sqrt{3}}{2}+1\right)\pi$$

7.

$$\Sigma=\left\{(x,y,z)\in\mathbb{R}^3: x^2+y^2\leq 1, 0\leq x, 0\leq y, z=x^2-y^2\right\}$$

è parametrizzata da

$$\Phi(u,v)=\left(u\cos v, u\sin v, u^2\left(\cos^2 v-\sin^2 v\right)\right) \quad (u,v)\in[0,1]\times\left[0,\frac{\pi}{2}\right]$$

$$\Phi_u\wedge\Phi_v=\begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \cos v & \sin v & 2u\left(\cos^2 v-\sin^2 v\right) \\ -u\sin v & u\cos v & -4u^2\sin v\cos v \end{vmatrix}=\left(-2u^2\cos v, -2u^2\sin v, u\right)\Rightarrow$$

$$\Rightarrow \|\Phi_u\wedge\Phi_v\|=u\sqrt{4u^2+1}\Rightarrow Area(\Sigma)=\int_0^1\int_0^{\frac{\pi}{2}}u\sqrt{4u^2+1}\,du\,dv \overset{(t=4u^2)}{=}$$

$$=\frac{\pi}{16}\int_0^4\sqrt{t+1}\,dt=\frac{\pi}{16}\left[\frac{2}{3}(t+1)^{\frac{3}{2}}\right]_0^4=\frac{5\sqrt{5}-1}{24}\pi$$

8.
$$A = \left\{(x,y,z) \in \mathbb{R}^3 : 2x^2 + z^2 = 1, |y| \le z\right\}$$

è parametrizzata da

$$\Phi(u,v) = \left(\frac{\sqrt{2}}{2}\cos u, v, \sin u\right) \qquad (u,v) \text{ tali che } -\sin u \le v \le \sin u, 0 \le u \le \pi$$

$$\Phi_u \wedge \Phi_v = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ -\frac{\sqrt{2}}{2}\sin u & 0 & \cos u \\ 0 & 1 & 0 \end{vmatrix} = \left(-\cos u, 0, -\frac{\sqrt{2}}{2}\sin u\right) \Rightarrow \|\Phi_u \wedge \Phi_v\| = \frac{\sqrt{2\left(\cos^2 + 1\right)}}{2} \Rightarrow$$

$$\Rightarrow \int_\Sigma |x| d\sigma = \int_0^\pi \int_{-\sin u}^{\sin u} \frac{|\cos u|\sqrt{(\cos^2+1)}}{2} dudv = \int_0^{\frac{\pi}{2}} du = 2\int_0^{\frac{\pi}{2}} \sin u \cos u \sqrt{\cos^2 u + 1} du \overset{(t=\cos^2 u)}{=}$$

$$= \int_0^1 \sqrt{t+1} dt = \left[\frac{2}{3}(t+1)^{\frac{3}{2}}\right]_0^1 = \frac{2}{3}\left(2\sqrt{2} - 1\right)$$

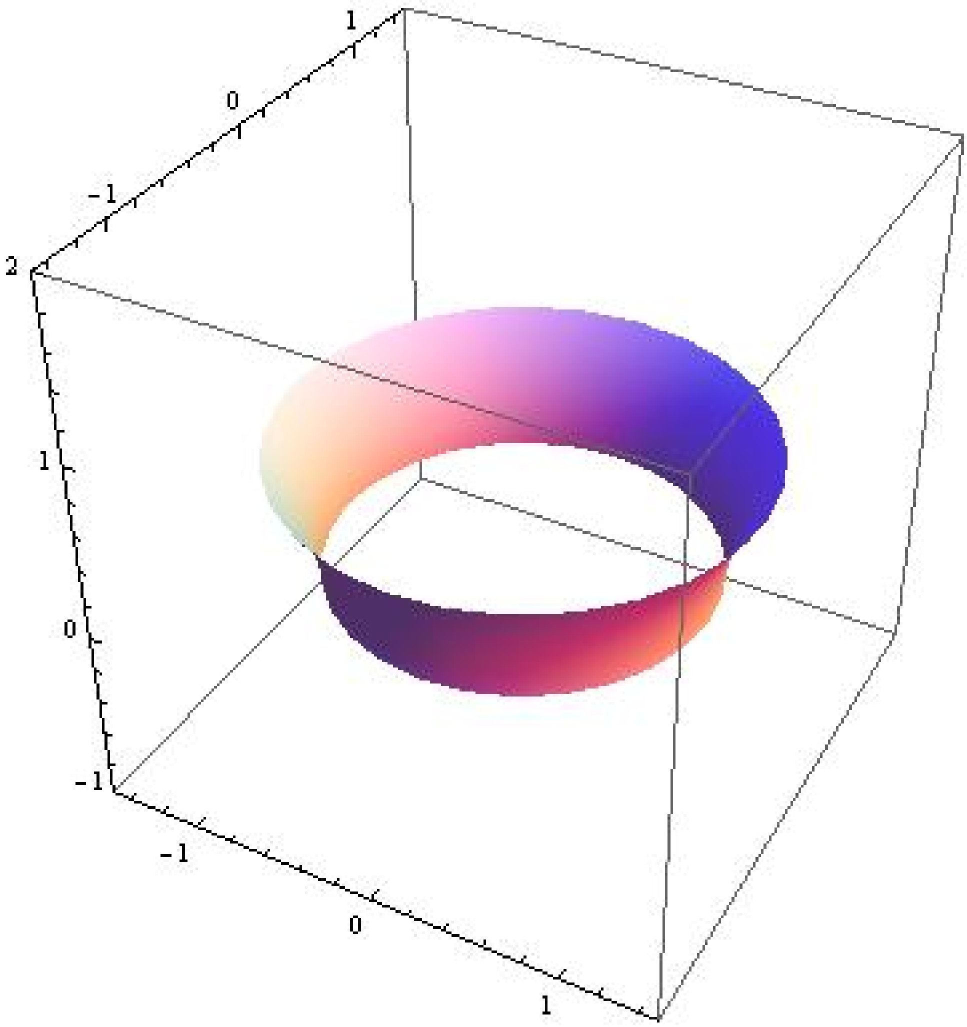


Figure 1: $\Sigma = Im(\Phi)$ con $\Phi(u,v) = (\cosh u \cos v, \cosh u \sin v, u)$

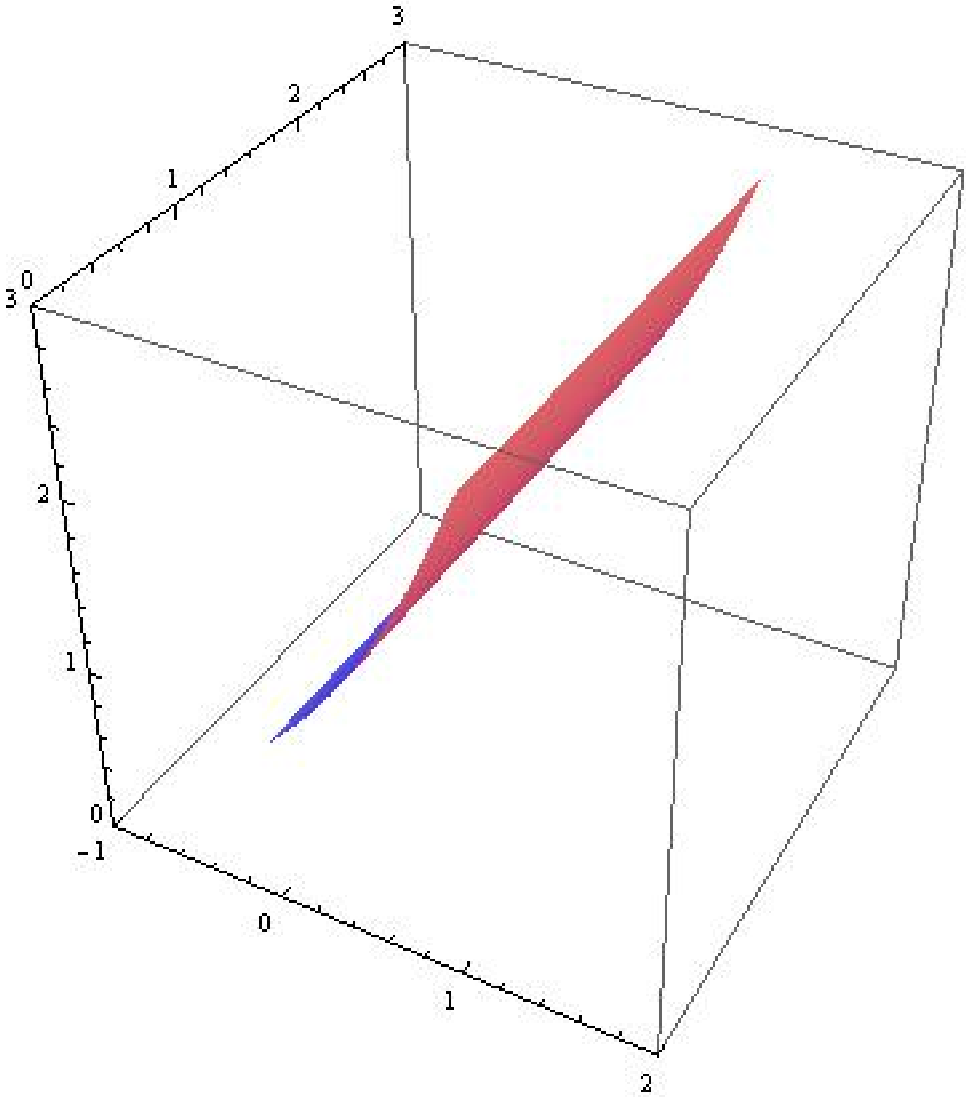


Figure 2: $\Sigma = Im(\Phi)$ con $\Phi(u, v) = (u, v, e^u)$

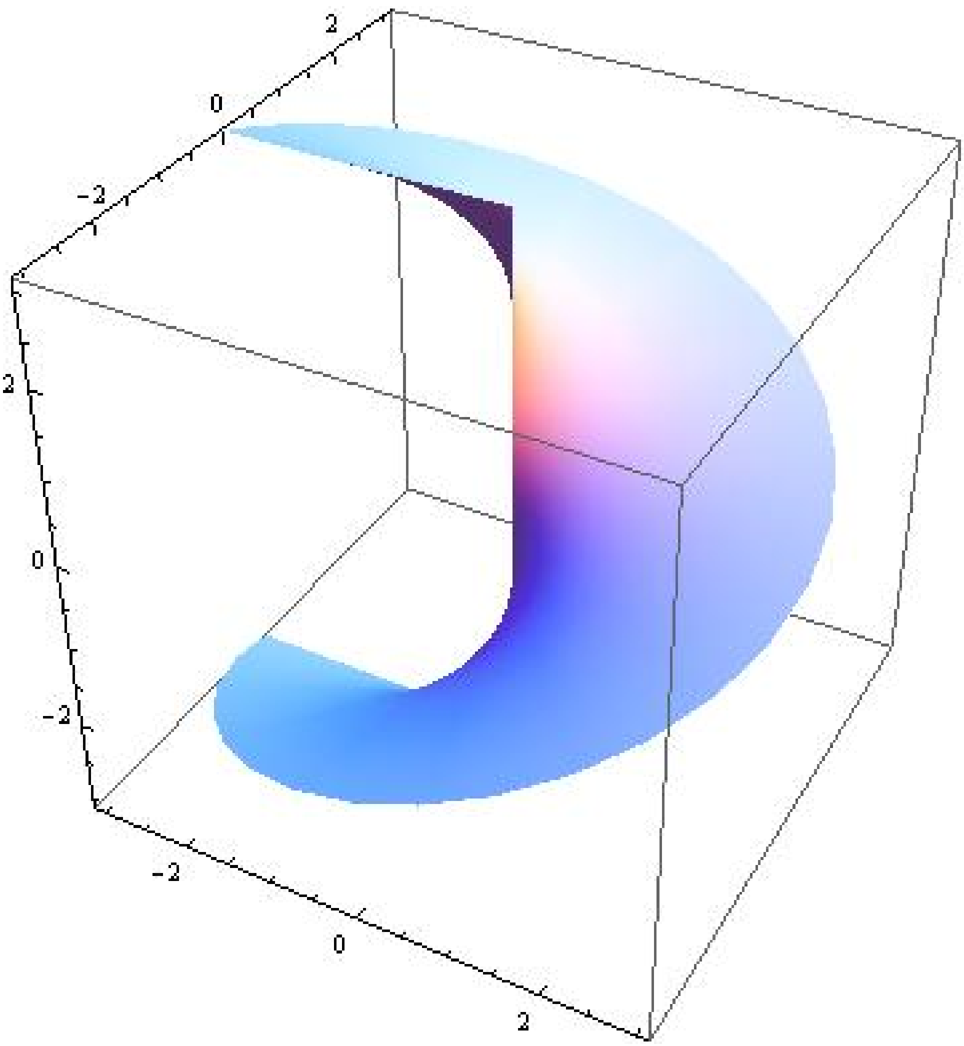


Figure 3: $\Sigma = Im(\Phi)$ con $\Phi(u,v) = (u\cos v, u\sin v, v)$

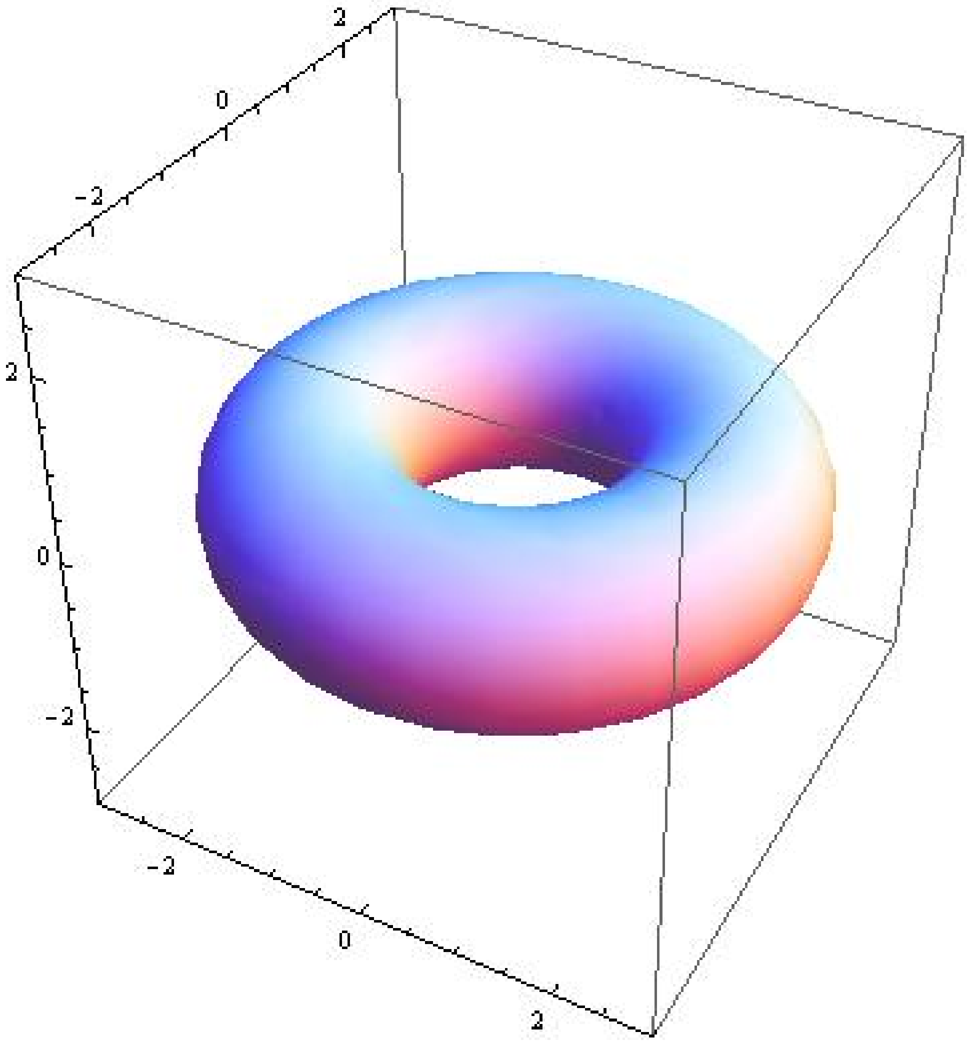


Figure 4: $\Sigma$ ottenuta ruotando $\gamma(t) = (\cos t + 2, \sin t)$

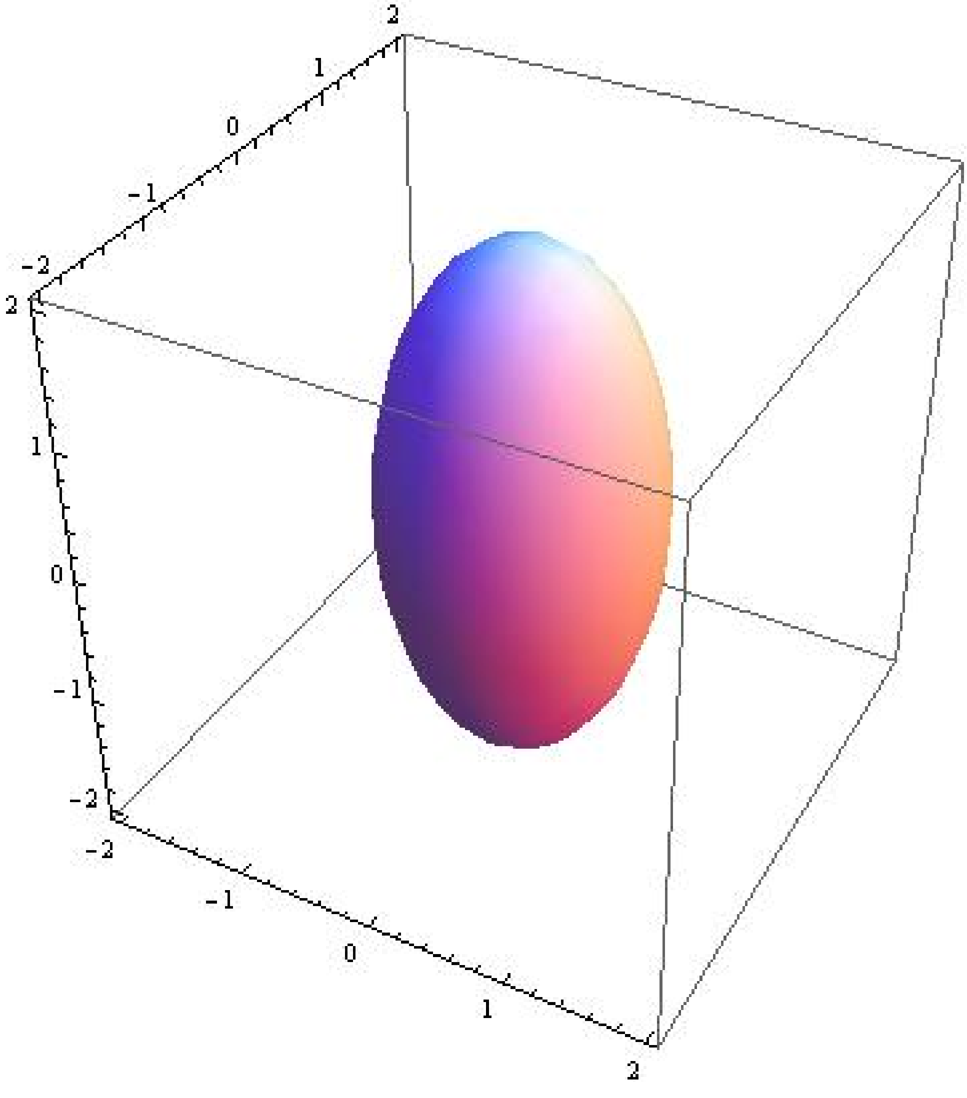


Figure 5: $\Sigma = \left\{ (x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + \frac{z^2}{4} = 1 \right\}$

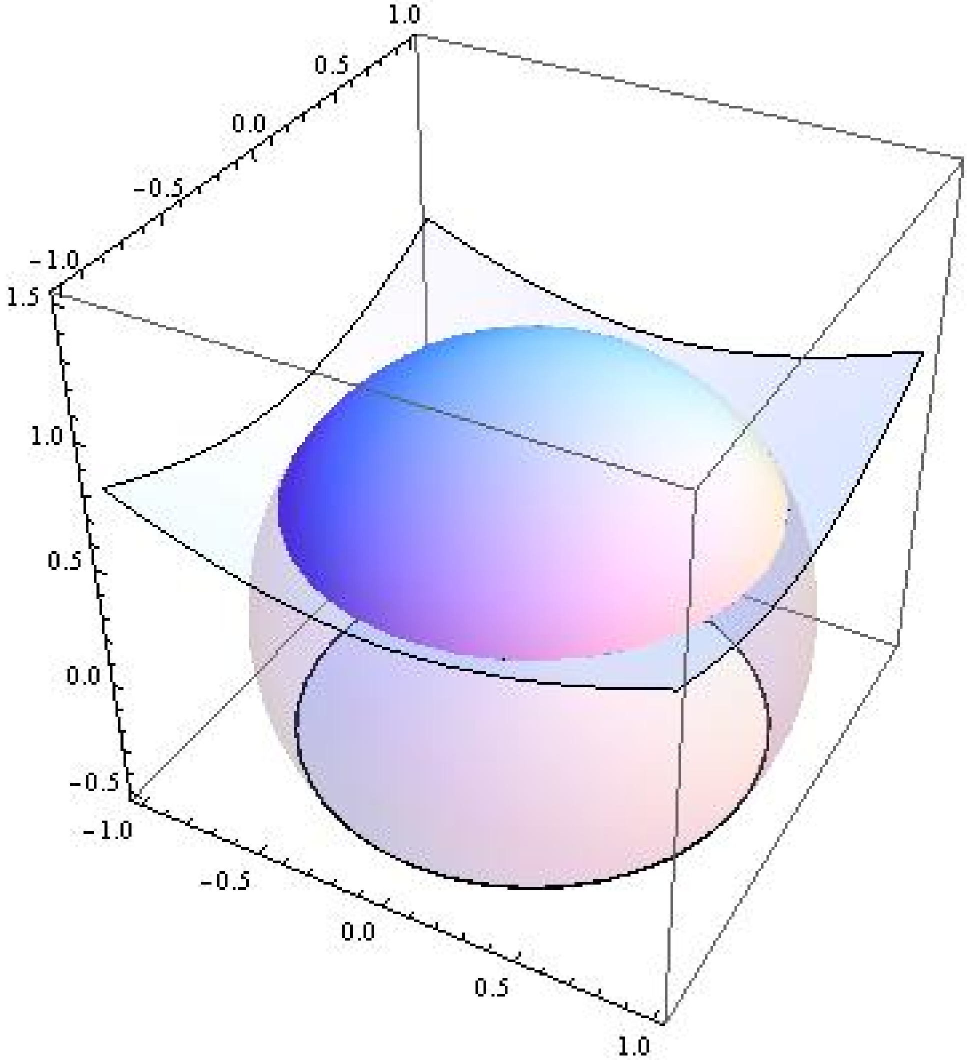


Figure 6: $\Sigma = \partial A$ con $A = \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1, x^2 + y^2 \leq 3z^2, z \geq 0\right\}$

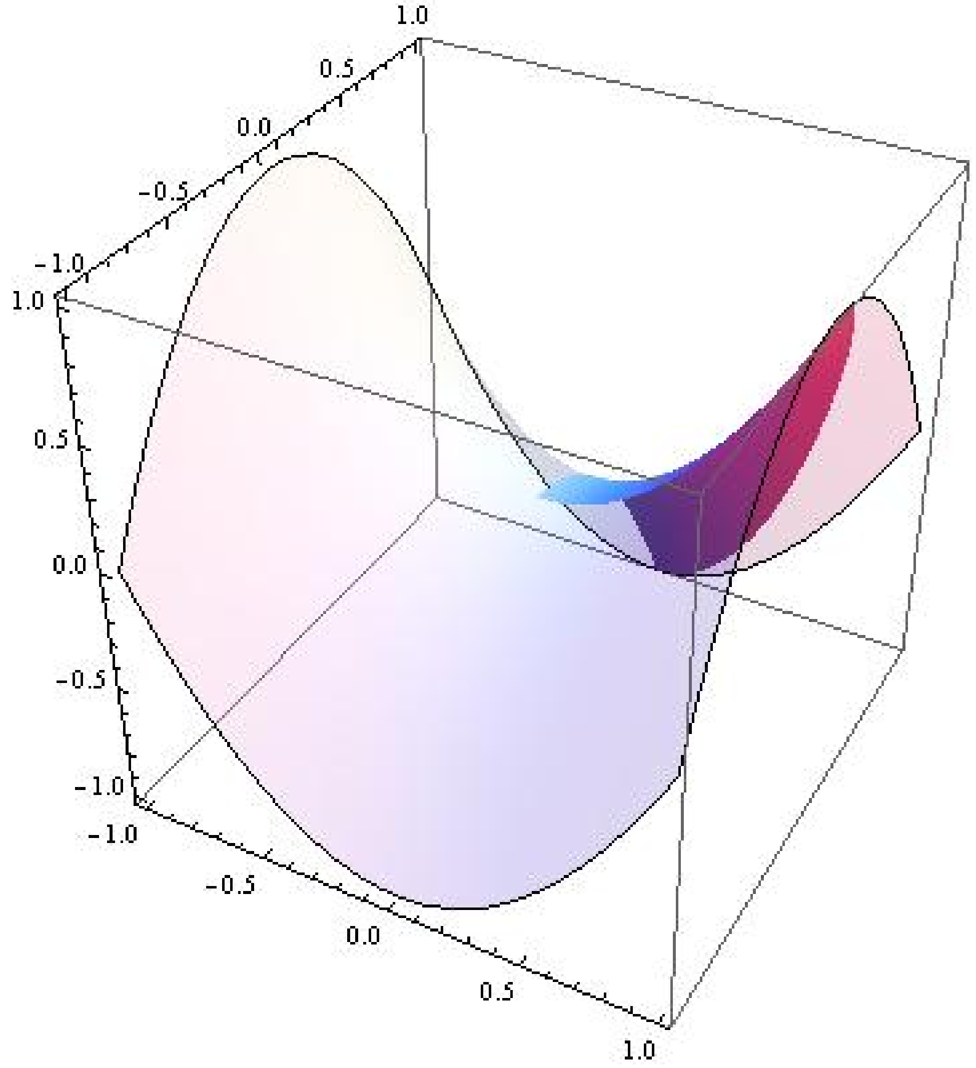


Figure 7: $\Sigma = \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq 1, 0 \leq x, 0 \leq y, z = x^2 - y^2\right\}$

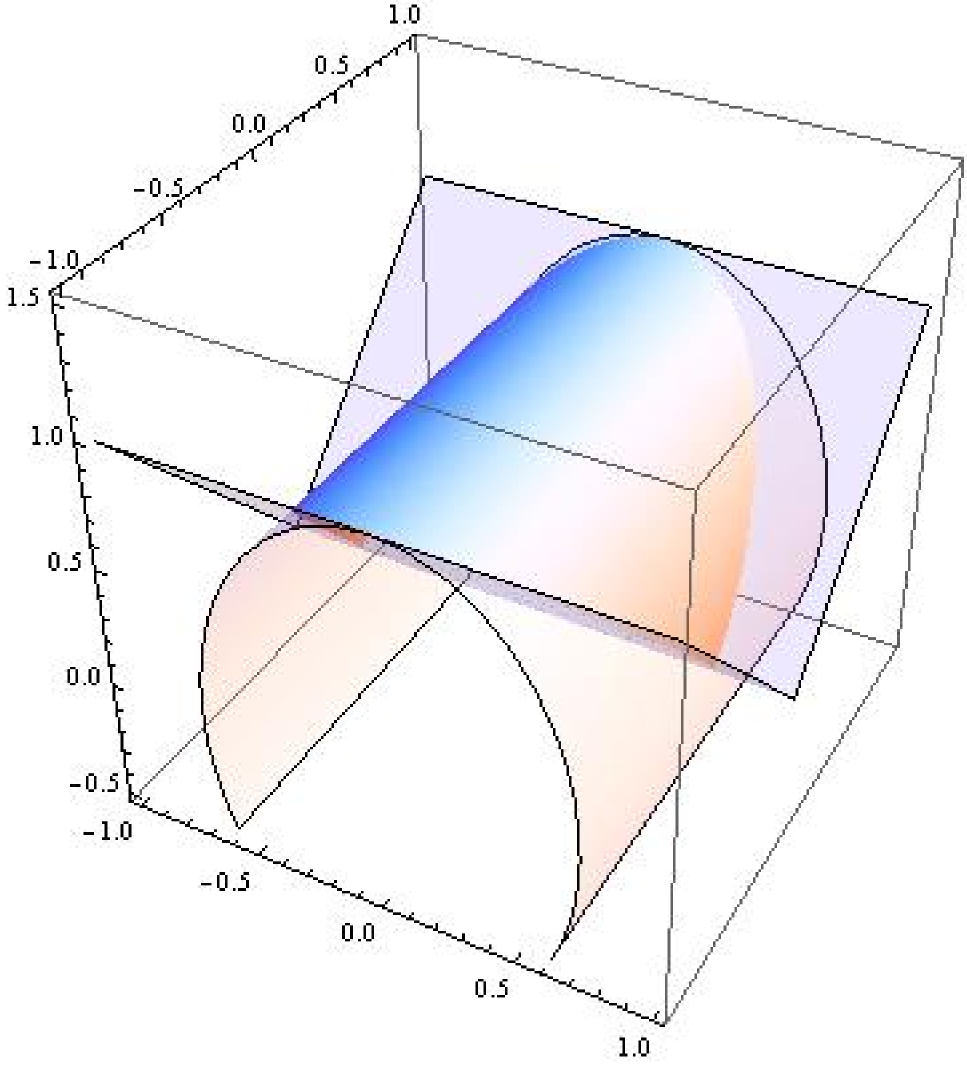


Figure 8: $\Sigma = \{(x, y, z) \in \mathbb{R}^3 : 2x^2 + z^2 = 1, |y| \leq z\}$