Università degli Studi Roma Tre - Corso di Laurea in Matematica

Tutorato di AM220

A.A. 2010-2011 - Docente: Prof.ssa S. Mataloni
Tutore: Luca Battaglia

Soluzioni del tutorato numero 12 (27 Maggio 2011)
Teoremi di Gauss-Green, divergenza, Stokes

I testi e le soluzioni dei tutorati sono disponibili al seguente indirizzo:
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1.

$$\omega=\left(x^2+y^2\right)dx+\left(x^2-y^2\right)dy \quad A=\left\{(x,y)\in\mathbb{R}^2: x^2+y^2\le 2, x\ge 0\right\}$$

In coordinate polari, si ha $\Phi^{-1}(A)=\left\{(\rho,\theta)\in[0,+\infty)\times[-\pi,\pi]:\rho\le\sqrt{2},\theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\right\}$, dunque

$$\int_A\frac{\partial}{\partial x}\left(x^2-y^2\right)-\frac{\partial}{\partial y}\left(x^2+y^2\right)dxdy=\int_A 2x-2ydxdy=2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\int_0^{\sqrt{2}}\rho^2(\cos\theta-\sin\theta)d\rho=$$

$$=2[\sin t+\cos t]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\frac{\rho^3}{3}\right]_0^{\sqrt{2}}=\frac{8}{3}\sqrt{2}$$

mentre $\partial^+A=\gamma_1\cup\gamma_2$ con

$$\gamma_1=\left(\sqrt{2}\cos t,\sqrt{2}\sin t\right)\quad t\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\qquad\gamma_2=\left(0,\sqrt{2}-t\right)\quad t\in\left[0,2\sqrt{2}\right]$$

pertanto

$$\int_{\partial^+A}\omega=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\langle\omega(\gamma_1(t)),\dot{\gamma_1}(t)\rangle dt+\int_0^{2\sqrt{2}}\langle\omega(\gamma_2(t)),\dot{\gamma_2}(t)\rangle dt=$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left\langle\left(1,2\cos^2 t-2\sin^2 t\right),\left(-\sqrt{2}\sin t,\sqrt{2}\cos t\right)\right\rangle dt+$$

$$+\int_0^{2\sqrt{2}}\left\langle\left(\left(\sqrt{2}-t\right)^2,-\left(\sqrt{2}-t\right)^2\right),(0,-1)\right\rangle dt=$$

$$=\sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos t\left(1-2\sin^2 t\right)-\sin tdt+\int_0^{2\sqrt{2}}\left(\sqrt{2}-t\right)^2dt=$$

$$=\sqrt{2}\left[2\sin t-\frac{4}{3}\sin^3 t+\cos t\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}+\left[-\frac{\left(\sqrt{2}-t\right)^3}{3}\right]_0^{2\sqrt{2}}=\frac{8}{3}\sqrt{2}$$

2.

$$\gamma(t)=\left(\cos^3 t,\sin t\right)\quad t\in[-\pi,\pi]$$

Se A è l'insieme racchiuso da γ, dal teorema di Gauss-Green si ha

$$Area(A)=\int_A\frac{\partial}{\partial x}x-\frac{\partial}{\partial y}0dxdy=\int_\gamma xdy=\int_{-\pi}^{\pi}\left\langle\left(0,\cos^3 t\right),\left(3\cos^2 t\sin t,\cos t\right)\right\rangle=$$

$$= \int_{-\pi}^{\pi} \cos^4 t dt = \int_{-\pi}^{\pi} \frac{1 + 2\cos(2t) + \cos^2(2t)}{4} dt = \int_{-\pi}^{\pi} \frac{3}{8} + \frac{\cos(2t)}{2} + \frac{\cos(4t)}{8} =$$

$$= \left[\frac{3}{8}t + \frac{\sin(2t)}{4} + \frac{\sin(4t)}{32}\right]_{-\pi}^{\pi} = \frac{3}{4}\pi$$

3.

$$\omega = \frac{x}{(x^2+y^2)^2} dx + \frac{y}{(x^2+y^2)^2} dy \qquad \gamma_R = (R\cos t, R\sin t) \quad t \in [-\pi, \pi]$$

(a) ω è chiusa perché

$$\frac{\partial}{\partial x} \frac{y}{(x^2+y^2)^2} = -\frac{4xy}{(x^2+y^2)^3} = \frac{\partial}{\partial y} \frac{x}{(x^2+y^2)^2}$$

(b)

$$\int_{\gamma_R} \omega = \int_{-\pi}^{\pi} \langle \omega(\gamma), \dot{\gamma} \rangle dt = \int_{-\pi}^{\pi} \langle (R\cos t, R\sin t), (-R\sin t, R\cos t) \rangle dt = 0$$

(c) Per mostrare l'esattezza di ω, è sufficiente far vedere che il suo integrale lungo ogni curva chiusa è nullo e, poiché ogni curva chiusa si decompone in un numero finito di curve semplici, ciò equivale a var vedere che l'integrale si annulla lungo ogni curva semplice chiusa γ; inoltre, poiché cambiando il verso di percorrenza della curva l'integrale cambia segno, si può supporre che γ venga percorsa in senso antiorario; prendendo R sufficientemente largo affinché la circonferenza γ_R non intersechi γ, è possibile applicare il teorema di Gauss-Green all'insieme A racchiuso tra le due curve:

$$0 = \int_A \frac{\partial}{\partial y} \frac{x}{(x^2+y^2)^2} - \frac{\partial}{\partial x} \frac{y}{(x^2+y^2)^2} dxdy = \int_{\partial^+ A} \omega = \int_{\gamma_R} \omega - \int_{\gamma} \omega = -\int_{\gamma} \omega \Rightarrow \int_{\gamma} \omega = 0$$

4.

$$\omega = -\frac{y^3}{4x^2+y^6} dx + \frac{3xy^2}{4x^2+y^6} dy \qquad \gamma_1(t) = \left(\frac{\cos t}{2}, \sqrt[3]{\sin t}\right) \quad t \in [-\pi, \pi]$$

$$\gamma_2(t) = (2\cos t, 2\sin t) \quad t \in [-\pi, \pi]$$

(a) ω è una chiusa perché

$$\frac{\partial}{\partial y} - \frac{y^3}{4x^2+y^6} = \frac{3y^2\left(y^6 - 4x^2\right)}{(4x^2+y^6)^2} = \frac{\partial}{\partial x} \frac{3xy^2}{4x^2+y^6}$$

(b)

$$\int_{\gamma_1} \omega = \int_{-\pi}^{\pi} \langle \omega(\gamma(t)), \dot{\gamma}(t) \rangle dt = \int_{-\pi}^{\pi} \left\langle \left(-\sin t, 3\cos t \sin^{\frac{2}{3}} t\right), \left(-\frac{\sin t}{2}, \frac{\cos t}{3\sin^{\frac{2}{3}} t}\right) \right\rangle dt =$$

$$= \int_{-\pi}^{\pi} \sin^2 t + \cos^2 t dt = 2\pi$$

(c) Poiché $||\gamma(t)|| \leq \sqrt{\frac{1}{4}+1} < 2$, è possibile applicare il teorema di Gauss-Green sull'insieme A racchiuso tra γ_1 e γ_2:

$$0 = \int_A \frac{\partial}{\partial x}\frac{3xy^2}{4x^2+y^6} - \frac{\partial}{\partial y} - \frac{y^3}{4x^2+y^6}dxdy = \int_{\partial^+ A}\omega = \int_{\gamma_2}\omega - \int_{\gamma_1}\omega = \int_{\gamma_2}\omega - 2\pi \Rightarrow \int_{\gamma_2}\omega = 2\pi$$

5.

$$F(x,y) = (\sin(\pi x), e^y) \qquad A = \left\{(x,y) \in \mathbb{R}^2 : x+y \leq 1, x \geq 0, y \geq 0\right\}$$

$$\int_A divF = \int_A \pi\cos(\pi x) + e^y dxdy = \int_0^1 dx \int_0^{1-x} \pi\cos(\pi x)dy + \int_0^1 dy \int_0^{1-y} e^y dx =$$

$$= \int_0^1 \pi(1-x)\cos(\pi x)dx + \int_0^1 (1-y)e^y dy = [(1-x)\sin(\pi x)]_0^1 + \int_0^1 \sin(\pi x)dx + [(1-y)e^y]_0^1 +$$

$$+ \int_0^1 e^y dy = \left[-\frac{\cos(\pi x)}{\pi}\right]_0^1 - 1 + [e^y]_0^1 = \frac{2}{\pi} + e - 2$$

Essendo $\partial^+ A = \gamma_1 \cup \gamma_2 \cup \gamma_2$ con

$$\gamma_1 = (t,0) \quad t \in [0,1] \qquad \gamma_2 = (1-t,t) \quad t \in [0,1] \qquad \gamma_3 = (0,1-t) \quad t \in [0,1]$$

la normale esterna è rispettivamente

$$\nu_1(t) = (0,-1) \qquad \nu_2(t) = (1,1) \qquad \nu_3(t) = (-1,0)$$

e dunque

$$\int_{\partial^+ A}\left\langle F, \frac{\nu}{\|\nu\|}\right\rangle d\ell = \int_0^1 \left\langle (\sin(\pi t), 1), \frac{(0,-1)}{\|\dot{\gamma_1}(t)\|}\right\rangle \|\dot{\gamma_1}(t)\|dt +$$

$$+ \int_0^1 \left\langle \left(\sin(\pi(1-t)), e^t\right), \frac{(1,1)}{\|\dot{\gamma_2}(t)\|}\right\rangle \|\dot{\gamma_2}(t)\|dt + \int_0^1 \left\langle \left(0, e^{1-t}\right), \frac{(-1,0)}{\|\dot{\gamma_3}(t)\|}\right\rangle \|\dot{\gamma_3}(t)\|dt =$$

$$= \int_0^1 -1 + e^t + \sin(\pi(1-t))dt = \left[-t + e^t - \frac{\cos(\pi(1-t))}{\pi}\right]_0^1 = \frac{2}{\pi} + e - 2$$

6.

$$F(x,y,z) = \left(x^2, y^2, z^2\right) \qquad A = \left\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 \leq z^2, 0 \leq z \leq 1\right\}$$

In coordinate cilindriche, si ha $\Phi^{-1}(A) = \{(\rho,\theta,t) \in [0,+\infty) \times [-\pi,\pi] \times \mathbb{R} : \rho \leq z, 0 \leq z \leq 1\}$, dunque

$$\int_A divF = \int_A 2x + 2y + 2zdxdydz = 2\int_{-\pi}^{\pi} \cos\theta + \sin\theta d\theta \int_0^1 dz \int_0^z \rho d\rho +$$

$$+2\int_{-\pi}^{\pi} d\theta \int_0^1 zdz \int_0^z \rho d\rho = 2[\sin\theta - \cos\theta]_{-\pi}^{\pi} \int_0^1 \frac{z^2}{2}dz + 4\pi\int_0^1 \frac{z^3}{2} = 4\pi\left[\frac{z^4}{8}\right]_0^1 = \frac{\pi}{2}$$

Poi, si ha $\partial A = \Sigma_1 \cup \Sigma_2$, con Σ_i parametrizzate da Φ_i

$$\Phi_1(u,v) = (v\cos u, v\sin u, v) \quad (u,v) \in [-\pi,\pi] \times [0,1]$$

$$\Phi_2(u,v) = (u\cos v, u\sin v, 1) \quad (u,v) \in [0,1]\times[-\pi,\pi]$$

dunque, poiché

$$\Phi_{1,u} \wedge \Phi_{1,v} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ -v\sin u & v\cos u & 0 \\ \cos u & \sin u & 1 \end{vmatrix} = (v\cos u, v\sin u, -v)$$

$$\Phi_{2,u} \wedge \Phi_{2,v} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 0 \end{vmatrix} = (0,0,u)$$

allora si ha

$$\int_{\partial A} \left\langle F, \frac{\nu}{\|\nu\|} \right\rangle d\sigma = \int_{-\pi}^{\pi} du \int_0^1 \left\langle \left(v^2\cos^2 u, v^2\sin^2 u, v^2\right), \frac{(v\cos u, v\sin u, -v)}{\|\Phi_{1,u} \wedge \Phi_{1,v}\|} \right\rangle \|\Phi_{1,u} \wedge \Phi_{1,v}\| dv +$$

$$+ \int_0^1 du \int_{-\pi}^{\pi} \left\langle (u\cos v, u\sin v, 1), \frac{(0,0,u)}{\|\Phi_{2,u} \wedge \Phi_{2,v}\|} \right\rangle \|\Phi_{2,u} \wedge \Phi_{2,v}\| dv =$$

$$= \int_{-\pi}^{\pi} \cos^3 v + \sin^3 v - 1 dv \int_0^1 u^3 du + 2\pi \int_0^1 u du =$$

$$= \frac{1}{4} \int_{-\pi}^{\pi} \cos u \left(1 - \sin^2 v\right) + \sin v \left(1 - \cos^2 v\right) - 1 dv + \pi =$$

$$= \frac{1}{4} \left[\sin v - \frac{\sin^3 v}{3} - \cos v + \frac{\cos^3 v}{3} - v\right]_{-\pi}^{\pi} + \pi = \frac{\pi}{2}$$

7.

$$\omega = \frac{z}{x+y+z+1} dx + \frac{x}{x+y+z+1} dy + \frac{y}{x+y+z+1} dz$$

$$\Sigma = \{(x,y,z) \in \mathbb{R}^3 : x+y+z = 1, x \geq 0, y \geq 0, z \geq 0\}$$

Σ è parametrizzata da

$$\Phi(u,v) = (u, v, 1-u-v) \quad (u,v) \text{ tali che } 0 \leq v \leq 1-u, 0 \leq u \leq 1$$

con

$$\Phi_u \wedge \Phi_v = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = (1,1,1)$$

mentre

$$rot\omega = \left(\frac{\partial}{\partial y} \frac{y}{x+y+z+1} - \frac{\partial}{\partial z} \frac{x}{x+y+z+1}, \frac{\partial}{\partial z} \frac{z}{x+y+z+1} - \frac{\partial}{\partial x} \frac{y}{x+y+z+1}, \right.$$

$$\left. \frac{\partial}{\partial x} \frac{x}{x+y+z+1} - \frac{\partial}{\partial y} \frac{z}{x+y+z+1} \right) = \left(\frac{2x+z+1}{(x+y+z+1)^2}, \frac{x+2y+1}{(x+y+z+1)^2}, \frac{y+2z+1}{(x+y+z+1)^2} \right)$$

quindi

$$\int_\Sigma \left\langle rot\omega, \frac{\nu}{\|\nu\|} \right\rangle d\sigma =$$

$$= \int_0^1 du \int_0^{1-u} \left\langle \left(\frac{u-v+2}{4}, \frac{u+2v+1}{4}, \frac{-2u-v+3}{4} \right), \frac{(1,1,1)}{\|\Phi_u \wedge \Phi_v\|} \right\rangle \|\Phi_u \wedge \Phi_v\| dv =$$

$$= \frac{3}{2} \int_0^1 du \int_0^{1-u} dv = \frac{3}{2} \int_0^1 1 - u du = \frac{3}{2} \left[-\frac{(1-u)}{2} \right]_0^1 = \frac{3}{4}$$

Si ha poi $\partial^+ A = \gamma_1 \cup \gamma_2 \cup \gamma_3$ con

$$\gamma_1(t) = (1-t, t, 0) \quad t \in [0,1]$$

$$\gamma_2(t) = (t, 0, 1-t) \quad t \in [0,1]$$

$$\gamma_3(t) = (0, 1-t, t) \quad t \in [0,1]$$

pertanto

$$\int_{\partial^+ A} \omega = \int_0^1 \langle \omega(\gamma_1(t)), \dot{\gamma_1}(t) \rangle dt + \int_0^1 \langle \omega(\gamma_2(t)), \dot{\gamma_2}(t) \rangle dt + \int_0^1 \langle \omega(\gamma_3(t)), \dot{\gamma_3}(t) \rangle dt =$$

$$= \int_0^1 \left\langle \left(0, \frac{1-t}{2}, \frac{t}{2}\right), (-1,1,0) \right\rangle dt + \int_0^1 \left\langle \left(\frac{1-t}{2}, \frac{t}{2}, 0\right), (1,0,-1) \right\rangle dt +$$

$$+ \int_0^1 \left\langle \left(\frac{t}{2}, 0, \frac{1-t}{2}\right), (0,-1,1) \right\rangle dt = 3 \int_0^1 \frac{1-t}{2} dt = 3 \left[-\frac{(1-t)^2}{4} \right]_0^1 = \frac{3}{4}$$

8.

$$\omega = xy^2 dy + xz^2 dz \qquad \gamma(t) = (\cos t, \sin t, 0) \quad t \in [-\pi, \pi]$$

Una superficie chiusa Σ avente per bordo γ è il disco unitario contenuto nel piano xy, parametrizzato da

$$\Phi(u,v) = (u \cos v, u \sin v, 0) \quad (u,v) \in [0,1] \times [-\pi, \pi]$$

Poiché si ha

$$rot\omega = \left(\frac{\partial}{\partial y} xz^2 - \frac{\partial}{\partial z} xy^2, \frac{\partial}{\partial z} 0 - \frac{\partial}{\partial x} xz^2, \frac{\partial}{\partial x} xy^2 - \frac{\partial}{\partial y} 0 \right) = \left(0, -z^2, y^2\right)$$

e

$$\Phi_u \wedge \Phi_v = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \cos v & \sin v & 0 \\ -u \sin v & u \cos v & 0 \end{vmatrix} = (0,0,u)$$

allora

$$\int_A \left\langle rot\omega, \frac{\nu}{\|\nu\|} \right\rangle d\sigma = \int_0^1 du \int_{-\pi}^{\pi} \left\langle \left(0,0,u^2 \sin^2 v\right), \frac{(0,0,u)}{\|\Phi_u \wedge \Phi_v\|} \right\rangle \|\Phi_u \wedge \Phi_v\| dv =$$

$$= \int_0^1 u^3 du \int_{-\pi}^{\pi} \sin^2 v dv = \frac{1}{4} \int_{-\pi}^{\pi} \frac{1-\cos(2v)}{2} dv = \frac{1}{4} \left[\frac{v}{2} - \frac{\sin(2v)}{4} \right]_{-\pi}^{\pi} = \frac{\pi}{4}$$

mentre

$$\int_{\partial^+ A} \omega = \int_{-\pi}^{\pi} \langle \omega(\gamma(t)), \dot{\gamma}(t) \rangle dt = \int_{-\pi}^{\pi} \left\langle \left(0, \cos t \sin^2 t, 0\right), (-\sin t, \cos t, 0) \right\rangle dt =$$

$$= \int_{-\pi}^{\pi} \cos^2 t \sin^2 t dt = \int_{-\pi}^{\pi} \frac{\sin^2(2t)}{4} dt = \int_{-\pi}^{\pi} \frac{1-\cos(4t)}{8} dt = \left[\frac{t}{8} - \frac{\sin(2t)}{32} \right]_{-\pi}^{\pi} = \frac{\pi}{4}$$