Università degli Studi Roma Tre - Corso di Laurea in Matematica

Tutorato di AM220

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Soluzioni del tutorato numero 4 (23 Marzo 2011)

Teorema della funzione implicita

I testi e le soluzioni dei tutorati sono disponibili al seguente indirizzo:
http://www.lifedreamers.it/liuck

1.
$$F(x,y) = \sin(xy) - \cos(x) + e^y$$

(a) F è di classe C^2 in un intorno dell'origine, inoltre $F(0,0) = 0$ e $\frac{\partial F}{\partial y}(0,0) = [x\cos(xy) + e^y]_{(x,y)=(0,0)} = 1 \neq 0$, dunque per il teorema della funzione implicita $\exists r, \rho > 0, g \in C^2(B_r(0), B_\rho(0))$ tali che $F(x, g(x)) \equiv 0\ \forall x \in B_r(0)$

(b) Supponendo $r, \rho \leq 1$, si ha
$$|F(x,0)| = |1 - \cos(x)| \leq \frac{x^2}{2} \leq \frac{r^2}{2} \leq \frac{r}{2}$$
dunque, posto $T = \frac{1}{\frac{\partial F}{\partial y}(0,0)} = 1$, per avere $\sup_{x \in B_r(0)} |F(x,0)| \leq \frac{r}{2} \leq \frac{\rho}{2\|T\|} = \frac{\rho}{2}$
è sufficiente porre $r = \rho$; inoltre,
$$\left|1 - T\frac{\partial F}{\partial y}(x,y)\right| = |1 - x\cos(xy) - e^y| \leq |1 - e^y| + |x\cos(xy)| \leq 3|y| + |x| \leq 3\rho + r \leq 4\rho$$
dunque per avere $\sup_{(x,y)\in B_r(0)\times B_\rho(0)} \left|1 - T\frac{\partial F}{\partial y}(x,y)\right| \leq 4\rho \leq \frac{1}{2}$ è sufficiente prendere $\rho = \frac{1}{8}$, e di conseguenza $r = \frac{1}{8}$.

(c) Essendo $\sin(xg(x)) - \cos(x) + e^{g(x)} \equiv 0\ \forall x \in B_r(0)$, allora
$$0 = \left[\frac{d}{dx}\left(\sin(xg(x)) - \cos(x) + e^{g(x)}\right)\right]_{x=0} = [(xg'(x) + g(x))\cos(xg(x)) + \sin(x) +$$
$$+ g'(x)e^{g(x)}\Big]_{x=0} = g(0) + g'(0)e^{g(0)} = g'(0) \Rightarrow g'(0) = 0$$
analogamente
$$0 = \left[\frac{d^2}{dx^2}\left(\sin(xg(x)) - \cos(x) + e^{g(x)}\right)\right]_{x=0} = [(xg''(x) + 2g'(x))\cos(xg(x)) -$$
$$-(xg'(x) + g(x))^2\sin(xg(x)) + \cos(x) + \left(g''(x) + g'(x)^2\right)e^{g(x)}\Big]_{x=0} = 2g'(0) + 1 +$$
$$+ \left(g''(0) + g'(0)^2\right)e^{g(0)} = 1 + g''(0) \Rightarrow g''(0) = -1$$
dunque lo sviluppo di Taylor al secondo ordine di g è
$$g(x) = g(0) + g'(0)x + \frac{g''(0)}{2}x^2 + o\left(x^2\right) = -\frac{x^2}{2} + o\left(x^2\right)$$

2.

$$F(x_1,x_2,y)=\arctan(x_1x_2y)+\log\left(\frac{\cos(x_1+x_2)}{y}\right)$$

(a) F è di classe C^2 in un intorno di $(0,0,1)$, inoltre $F(0,0,1)=0$ e $\dfrac{\partial F}{\partial y}(0,0,1)=\left[\dfrac{x_1x_2}{x_1^2x_2^2y^2+1}-\dfrac{1}{y}\right]_{(x_1,x_2,y)=(0,0,1)}=-1$, dunque per il teorema della funzione implicita $\exists r,\rho>0, g\in C^2(B_r((0,0)),B_\rho(1))$ tali che $F(x_1,x_2,g(x_1,x_2))\equiv 0\ \forall x\in B_r((0,0))$.

(b) Supponendo $r,\rho\le\dfrac{1}{2}$, si ha $-1\le x_1+x_2\le 1\Rightarrow \dfrac{1}{2}=\cos\left(\dfrac{\pi}{3}\right)\le\cos(1)\le\cos(x_1+x_2)\le 1$, quindi

$$|F(x_1,x_2,1)|=|\arctan(x_1x_2)+\log(\cos(x_1+x_2))|\le|\arctan(x_1x_2)|+|\log(\cos(x_1+x_2))|\le$$

$$\le|x_1x_2|+2|\cos(x_1+x_2)-1|\le\frac{x_1^2+x_2^2}{2}+(x_1+x_2)^2\le\frac{r^2}{2}+4r^2\le\frac{9}{2}r$$

pertanto, posto $T=\dfrac{1}{\frac{\partial F}{\partial y}(0,0,1)}=-1$, per avere $\sup\limits_{x\in B_r((0,0))}|F(x_1,x_2,0)|\le\dfrac{9}{2}r\le\dfrac{\rho}{2\|T\|}=\dfrac{\rho}{2}$ è sufficiente prendere $r=\dfrac{\rho}{9}$; inoltre,

$$\left|1-T\frac{\partial F}{\partial y}(x_1,x_2,y)\right|=\left|1+\frac{x_1x_2}{x_1^2x_2^2y^2+1}-\frac{1}{y}\right|\le\frac{|x_1x_2|}{x_1^2x_2^2y^2+1}+\left|1-\frac{1}{y}\right|\le|x_1x_2|+\left|\frac{y-1}{y}\right|\le$$

$$\le\frac{x_1^2+x_2^2}{2}+\frac{|y-1|}{\frac{1}{2}}\le\frac{r^2}{2}+2\rho\le\frac{r}{2}+2\rho\le\frac{13}{2}\rho$$

dunque per avere $\sup\limits_{(x,y)\in B_r(0,0)\times B_\rho(1)}\left|1-T\dfrac{\partial F}{\partial y}(x_1,x_2,y)\right|=\dfrac{13}{2}\rho\le\dfrac{1}{2}$ è sufficiente prendere $\rho=\dfrac{1}{13}$ e, di conseguenza, $r=\dfrac{1}{117}$.

(c) Essendo $\arctan(x_1x_2g(x_1,x_2))+\log\left(\dfrac{\cos(x_1+x_2)}{g_1(x_1,x_2)}\right)\equiv 0\ \forall x\in B_r((0,0))$, allora

$$0=\left[\frac{d}{dx_1}\left(\arctan(x_1x_2g(x_1,x_2))+\log\left(\frac{\cos(x_1+x_2)}{g_1(x_1,x_2)}\right)\right)\right]_{(x_1,x_2)=(0,0)}=$$

$$=\left[\frac{x_2g(x_1,x_2)+x_1x_2\frac{\partial g}{\partial x_1}(x_1,x_2)}{x_1^2x_2^2g(x_1,x_2)^2+1}-\frac{\sin(x_1+x_2)}{\cos(x_1+x_2)}-\frac{\frac{\partial g}{\partial x_1}(x_1,x_2)}{g(x_1,x_2)}\right]_{(x_1,x_2)=(0,0)}=$$

$$=-\frac{\partial g}{\partial x_1}(0,0)\Rightarrow\frac{\partial g}{\partial x_1}(0,0)=0$$

analogamente

$$0=\left[\frac{d}{dx_2}\left(\arctan(x_1x_2g(x_1,x_2))+\log\left(\frac{\cos(x_1+x_2)}{g_1(x_1,x_2)}\right)\right)\right]_{(x_1,x_2)=(0,0)}=$$

$$=\left[\frac{x_1g(x_1,x_2)+x_1x_2\frac{\partial g}{\partial x_2}(x_1,x_2)}{x_1^2x_2^2g(x_1,x_2)^2+1}-\frac{\sin(x_1+x_2)}{\cos(x_1+x_2)}-\frac{\frac{\partial g}{\partial x_2}(x_1,x_2)}{g(x_1,x_2)}\right]_{(x_1,x_2)=(0,0)}=$$

$$= -\frac{\partial g}{\partial x_2}(0,0) \Rightarrow \frac{\partial g}{\partial x_2}(0,0) = 0$$

$$0 = \left[\frac{d^2}{dx_1^2}\left(\arctan(x_1x_2g(x_1,x_2)) + \log\left(\frac{\cos(x_1+x_2)}{g_1(x_1,x_2)}\right)\right)\right]_{(x_1,x_2)=(0,0)} =$$

$$= \left[\frac{2x_2\frac{\partial g}{\partial x_1}(x_1,x_2) + x_1x_2\frac{\partial^2 g}{\partial x_1^2}(x_1,x_2)}{x_1^2x_2^2g(x_1,x_2)^2+1} - \right.$$

$$-\frac{2x_1x_2g(x_1,x_2)\left(x_2g(x_1,x_2)+x_1x_2\frac{\partial g}{\partial x_1}(x_1,x_2)\right)^2}{\left(x_1^2x_2^2g(x_1,x_2)^2+1\right)^2} - \frac{1}{\cos^2(x_1+x_2)} -$$

$$\left.-\frac{\frac{\partial^2 g}{\partial x_1^2}(x_1,x_2)g(x_1,x_2) - \left(\frac{\partial g}{\partial x_1}(x_1,x_2)\right)^2}{g(x_1,x_2)^2}\right]_{(x_1,x_2)=(0,0)} = -1-\frac{\partial^2 g}{\partial x_1^2}(x_1,x_2) \Rightarrow \frac{\partial^2 g}{\partial x_1^2}(x_1,x_2) = -1$$

$$0 = \left[\frac{d^2}{dx_1dx_2}\left(\arctan(x_1x_2g(x_1,x_2)) + \log\left(\frac{\cos(x_1+x_2)}{g_1(x_1,x_2)}\right)\right)\right]_{(x_1,x_2)=(0,0)} =$$

$$= \left[\frac{g(x_1,x_2) + x_2\frac{\partial g}{\partial x_2}(x_1,x_2) + x_1\frac{\partial g}{\partial x_1}(x_1,x_2) + x_1x_2\frac{\partial^2 g}{\partial x_1\partial x_2}(x_1,x_2)}{x_1^2x_2^2g(x_1,x_2)^2+1} - \right.$$

$$-\frac{x_1x_2g(x_1,x_2)\left(x_2g(x_1,x_2)+x_1x_2\frac{\partial g}{\partial x_1}(x_1,x_2)\right)\left(x_1g(x_1,x_2)+x_1x_2\frac{\partial g}{\partial x_2}(x_1,x_2)\right)}{\left(x_1^2x_2^2g(x_1,x_2)^2+1\right)^2} -$$

$$\left.-\frac{1}{\cos^2(x_1+x_2)} - \frac{\frac{\partial^2 g}{\partial x_1x_2}(x_1,x_2)g(x_1,x_2) - \left(\frac{\partial g}{\partial x_1}(x_1,x_2)\right)\left(\frac{\partial g}{\partial x_2}(x_1,x_2)\right)}{g(x_1,x_2)^2}\right]_{(x_1,x_2)=(0,0)} =$$

$$= -\frac{\partial^2 g}{\partial x_1\partial x_2}(0,0) \Rightarrow \frac{\partial^2 g}{\partial x_1\partial x_2}(0,0) = 0$$

$$0 = \left[\frac{d^2}{dx_2^2}\left(\arctan(x_1x_2g(x_1,x_2)) + \log\left(\frac{\cos(x_1+x_2)}{g_1(x_1,x_2)}\right)\right)\right]_{(x_1,x_2)=(0,0)} =$$

$$= \left[\frac{2x_1\frac{\partial g}{\partial x_2}(x_1,x_2) + x_1x_2\frac{\partial^2 g}{\partial x_2^2}(x_1,x_2)}{x_1^2x_2^2g(x_1,x_2)^2+1} - \right.$$

$$-\frac{2x_1x_2g(x_1,x_2)\left(x_1g(x_1,x_2)+x_1x_2\frac{\partial g}{\partial x_2}(x_1,x_2)\right)^2}{\left(x_1^2x_2^2g(x_1,x_2)^2+1\right)^2} - \frac{1}{\cos^2(x_1+x_2)} -$$

$$\left.-\frac{\frac{\partial^2 g}{\partial x_2^2}(x_1,x_2)g(x_1,x_2) - \left(\frac{\partial g}{\partial x_2}(x_1,x_2)\right)^2}{g(x_1,x_2)^2}\right]_{(x_1,x_2)=(0,0)} = -1-\frac{\partial^2 g}{\partial x_2^2}(x_1,x_2) \Rightarrow \frac{\partial^2 g}{\partial x_2^2}(x_1,x_2) = -1$$

pertanto

$$g(x_1,x_2) = g(0,0) + \left\langle\left(\frac{\partial g}{\partial x_1}(0,0), \frac{\partial g}{\partial x_1}(0,0)\right), (x_1,x_2)\right\rangle +$$

$$+\frac{1}{2}\left\langle (x_1,x_2), \begin{pmatrix} \frac{\partial^2 g}{\partial x_1^2}(0,0) & \frac{\partial^2 g}{\partial x_1 \partial x_2}(0,0) \\ \frac{\partial^2 g}{\partial x_1 \partial x_2}(0,0) & \frac{\partial^2 g}{\partial x_2^2}(0,0) \end{pmatrix} (x_1,x_2) \right\rangle + o\left(x_1^2+x_2^2\right) = -\frac{x_1^2+x_2^2}{2} + o\left(x_1^2+x_2^2\right)$$

3.

$$F(x,y_1,y_2) = \left(\sin(y_1) + xe^{y_1} - 1, y_1^2 + \sinh(xy_2) + \log x\right)$$

(a) F è di classe C^2 in un intorno di $(0,0,1)$, inoltre $F(0,0,1) = (0,0)$ e $\frac{\partial F}{\partial y}(0,0,0) = \left[\begin{pmatrix} \cos(y_1) + xe^{y_1} & 0 \\ 2y_1 & -x\cosh(xy_2) \end{pmatrix}\right]_{(x,y_1,y_2)=(1,0,0)} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ è invertibile (con $T = \left(\frac{\partial F}{\partial y}(0,0,0)\right)^{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}$), dunque per il teorema della funzione implicita $\exists r, \rho > 0, g \in C^2(B_r(0), B_\rho((0,0)))$ tali che $F(x, g_1(x), g_2(x)) \equiv 0\ \forall x \in B_r(0)$

(b) Supponendo $r \le \frac{1}{2}, \rho \le 1$, si ha

$$\|F(x,0,0)\| = \sqrt{(x-1)^2 + (\log x)^2} \le |x-1| + |\log x| \le r + 2|x| \le 3r$$

e dunque, per avere $\sup_{x \in B_r(1)} \|F(x,0,0)\| = 3r \le \frac{\rho}{4} = \frac{\rho}{4\|T\|_\infty} \le \frac{\rho}{2\|T\|}$,
è sufficiente prendere $r = \frac{\rho}{12}$; inoltre,

$$\mathbb{I}_2 - T\frac{\partial F}{\partial y}(x,y_1,y_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(y_1) + xe^{y_1} & 0 \\ 2y_1 & -x\cosh(xy_2) \end{pmatrix} =$$

$$= \begin{pmatrix} 1 - \frac{\cos(y_1)+xe^{y_1}}{2} & 0 \\ 2y_1 & 1 - x\cosh(xy_2) \end{pmatrix}$$

e dunque, essendo

$$\left|1 - \frac{\cos(y_1)+xe^{y_1}}{2}\right| \le \frac{|1-\cos(y_1)|}{2} + \frac{|1-xe^{y}|}{2} \le \frac{y_1^2}{2} + \frac{|1-x| + |x||1-e^y|}{2} \le \frac{\rho^2}{2} + \frac{r + \frac{3}{2}3|y|}{2} \le$$

$$\le \rho + \frac{r}{2} + \frac{9}{4}\rho \le \frac{79}{24}\rho$$

$$|2y_1| \le 2\rho$$

$$|1 - x\cosh(xy_2)| \le |1-x| + |x||1-\cosh(xy_2)| \le r + \frac{3}{2}3|xy_2| \le r + \frac{9}{4}\left(x^2 + y_2^2\right) \le \frac{13}{4}r + \frac{9}{4}\rho \le \frac{121}{48}\rho$$

per avere $\sup_{(x,y_1,y_2) \in B_r(1) \times B_\rho((0,0))} \left\|\mathbb{I}_2 - T\frac{\partial F}{\partial y}(x,y_1,y_2)\right\| \le$
$\le 2 \sup_{(x,y_1,y_2) \in B_r(1) \times B_\rho((0,0))} \left\|\mathbb{I}_2 - T\frac{\partial F}{\partial y}(x,y_1,y_2)\right\| = \frac{79}{12}\rho \le \frac{1}{2}$ è sufficiente prendere $\rho = \frac{6}{79}$ e, di conseguenza, $r = \frac{1}{158}$.

(c) Essendo $\sin(g_1(x)) + xe^{g_1(x)} - 1 \equiv 0\ \forall x \in B_r(1)$, allora

$$0 = \left[\frac{d}{dx}\left(\sin(g_1(x)) + xe^{g_1(x)} - 1\right)\right]_{x=1} = \left[g_1'(x)\cos(g_1(x)) + (1 + xg_1'(x))e^{g_1(x)}\right]_{x=1} =$$

$$2g_1'(0) + 1 \Rightarrow g_1'(0) = -\frac{1}{2}$$

analogamente

$$0 = \left[\frac{d}{dx}\left(g_1(x)^2 + \sinh(xg_2(x)) + \log x\right)\right]_{x=1} = [2g_1'(x)g_1(x) + (g_2(x) + xg_2'(x))\cosh(xg_2(x)) +$$

$$+\frac{1}{x}\Big]_{x=1} = g_2'(0) + 1 \Rightarrow g_2'(0) = -1$$

$$0 = \left[\frac{d^2}{dx^2}\left(\sin(g_1(x)) + xe^{g_1(x)} - 1\right)\right]_{x=1} = \big[g_1''(x)\cos(g_1(x)) - g_1'(x)^2\sin(g_1(x)) +$$

$$+(xg_1''(x) + 2g_1'(x) + xg_1'(x)^2)e^{g_1(x)}\Big]_{x=1} = -\frac{3}{4} + 2g_1''(0) \Rightarrow g_1''(0) = \frac{3}{8}$$

$$0 = \left[\frac{d^2}{dx^2}\left(g_1(x)^2 + \sinh(xg_2(x)) + \log x\right)\right]_{x=1} = \big[2\left(g_1'(x)^2 + g_1(x)g_1''(x)\right) +$$

$$+(g_2(x) + xg_2'(x))^2\sinh(xg_2(x)) + (2g_2'(x) + xg_2''(x))\cosh(xg_2(x)) - \frac{1}{x^2}\Big]_{x=0} =$$

$$= g_2''(0) - \frac{5}{2} \Rightarrow g_2''(0) = \frac{5}{2}$$

dunque lo sviluppo di Taylor al secondo ordine di g è

$$g_1(x) = -\frac{x-1}{2} + \frac{3}{16}(x-1)^2 + o\left((x-1)^2\right)$$

$$g_2(x) = -(x-1) + \frac{5}{4}(x-1)^2 + o\left((x-1)^2\right)$$

4.

$$F(x_1, x_2, y_1, y_2) = \left(\sqrt{y_1 + 1} - e^{\sin(x_1+x_2)}, \frac{y_2}{x_1^2 + 1} - \sin(x_2)\cos(y_2)\right)$$

(a) F è di classe C^1 in un intorno dell'origine, inoltre $F(0,0,0,0) = (0,0)$ e $\frac{\partial F}{\partial y}(0,0,0,0) = \left[\begin{pmatrix} \frac{1}{2\sqrt{y_1+1}} & 0 \\ 0 & \frac{1}{x_1^2+1} + \sin(x_2)\cos(y_2) \end{pmatrix}\right]_{(x_1,x_2,y_1,y_2)=(0,0,0,0)} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}$ è invertibile (con $T = \left(\frac{\partial F}{\partial y}(0,0,0,0)\right)^{-1} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$), dunque per il teorema della funzione implicita $\exists r, \rho > 0, g \in C^2(B_r((0,0)), B_\rho((0,0)))$ tali che $F(x_1, x_2, g_1(x_1,x_2), g_2(x_1,x_2)) \equiv 0 \ \forall x \in B_r((0,0))$

(b) Supponendo $r \leq 1, \rho \leq \frac{1}{2}$, si ha

$$\|F(x_1, x_2, 0, 0)\| = \sqrt{(1 - e^{\sin(x_1+x_2)})^2 + \sin(x_2)^2} \leq \left|e^{\sin(x_1+x_2)} - 1\right| + |\sin x_2| \leq 3|\sin(x_1+x_2)| + |x$$

e dunque, per avere

$$\sup_{x \in B_r((0,0))} \|F(x_1, x_2, 0, 0)\| \leq 7r \leq \frac{\rho}{4} = \frac{\rho}{4\|T\|_\infty} \leq \frac{\rho}{2\|T\|}$$

è sufficiente prendere $r = \dfrac{\rho}{28}$; inoltre,

$$\mathbb{I}_2 - T\frac{\partial F}{\partial y}(x_1,x_2,y_1,y_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2\sqrt{y_1+1}} & 0 \\ 0 & \frac{1}{x_1^2+1} + \sin(x_2)\cos(y_2) \end{pmatrix} =$$

$$= \begin{pmatrix} 1 - \frac{1}{\sqrt{y_1+1}} & 0 \\ 0 & 1 - \frac{1}{x_1^2+1} - \sin(x_2)\cos(y_2) \end{pmatrix}$$

e dunque, essendo

$$\left|1 - \frac{1}{\sqrt{y_1+1}}\right| = \frac{|y_1|}{\sqrt{y_1+1}} \le \frac{|y_1|}{\sqrt{1-\frac{1}{2}}} \le \sqrt{2}|y_1| \le \frac{3}{2}\rho$$

$$\left|1 - \frac{1}{x_1^2+1} - \sin(x_2)\cos(y_2)\right| \le \frac{x_1^2}{x_1^2+1} + |\sin(x_2)\cos(y_2)| \le x_1^2 + |\sin(x_2)| \le r^2 + |x_2| \le$$

$$\le r + r \le \frac{\rho}{12}$$

per avere $\sup_{(x_1,x_2,y_1,y_2)\in B_r((0,0))\times B_\rho((0,0))} \left\| \mathbb{I}_2 - T\frac{\partial F}{\partial y}(x_1,x_2,y_1,y_2)\right\| \le$

$\le 2 \sup_{(x_1,x_2,y_1,y_2)\in B_r((0,0))\times B_\rho((0,0))} \left\| \mathbb{I}_2 - T\frac{\partial F}{\partial y}(x_1,x_2,y_1,y_2)\right\| = 3\rho \le \dfrac{1}{2}$

è sufficiente prendere $\rho = \dfrac{1}{6}$ e, di conseguenza, $r = \dfrac{1}{168}$

(c) Essendo $\sqrt{g_1(x_1,x_2)+1} - e^{\sin(x_1+x_2)} \equiv 0 \ \forall x \in B_r((0,0))$, allora

$$0 = \left[\frac{d}{dx_1}\left(\sqrt{g_1(x_1,x_2)+1} - e^{\sin(x_1+x_2)}\right)\right]_{(x_1,x_2)=(0,0)} = \left[\frac{\frac{\partial g_1}{\partial x_1}(x_1,x_2)}{2\sqrt{g_1(x_1,x_2)+1}} - \right.$$

$$\left. - \cos(x_1+x_2)e^{\sin(x_1+x_2)}\right]_{(x_1,x_2)=(0,0)} = \frac{1}{2}\frac{\partial g_1}{\partial x_1}(0,0) - 1 \Rightarrow \frac{\partial g_1}{\partial x_1}(0,0) = 2$$

analogamente

$$0 = \left[\frac{d}{dx_2}\left(\sqrt{g_1(x_1,x_2)+1} - e^{\sin(x_1+x_2)}\right)\right]_{(x_1,x_2)=(0,0)} = \left[\frac{\frac{\partial g_1}{\partial x_2}(x_1,x_2)}{2\sqrt{g_1(x_1,x_2)+1}} - \right.$$

$$\left. - \cos(x_1+x_2)e^{\sin(x_1+x_2)}\right]_{(x_1,x_2)=(0,0)} = \frac{1}{2}\frac{\partial g_1}{\partial x_2}(0,0) - 1 \Rightarrow \frac{\partial g_1}{\partial x_2}(0,0) = 2$$

$$0 = \left[\frac{d}{dx_1}\left(\frac{g_2(x_1,x_2)}{x_1^2+1} - \sin(x_2)\cos(g_2(x_1,x_2))\right)\right]_{(x_1,x_2)=(0,0)} =$$

$$= \left[\frac{\left(x_1^2+1\right)\frac{\partial g_2}{\partial x_1}(x_1,x_2) - 2x_1 g_2(x_1,x_2)}{\left(x_1^2+1\right)^2} + \sin(x_2)\frac{\partial g_2}{\partial x_1}(x_1,x_2)\sin(g_2(x_1,x_2))\right]_{(x_1,x_2)=(0,0)=}$$

$$= \frac{\partial g_2}{\partial x_1}(0,0) \Rightarrow \frac{\partial g_2}{\partial x_1}(0,0) = 0$$

$$0 = \left[\frac{d}{dx_2}\left(\frac{g_2(x_1,x_2)}{x_1^2+1} - \sin(x_2)\cos(g_2(x_1,x_2))\right)\right]_{(x_1,x_2)=(0,0)} =$$

$$\left[\frac{\frac{\partial g_2}{\partial x_2}(x_1,x_2)}{x_1^2+1} - \cos(x_2)\cos(g_2(x_1,x_2)) + \sin(x_2)\frac{\partial g_2}{\partial x_1}(x_1,x_2)\sin(g_2(x_1,x_2))\right]_{(x_1,x_2)=(0,0)} =$$

$$= \frac{\partial g_2}{\partial x_2}(0,0) - 1 \Rightarrow \frac{\partial g_2}{\partial x_2}(0,0) = 1$$

dunque lo sviluppo di Taylor al primo ordine di g è

$$g_1(x_1,x_2) = 2x_1 + 2x_2 + o\left(\sqrt{x_1^2+x_2^2}\right)$$

$$g_2(x_1,x_2) = x_2 + o\left(\sqrt{x_1^2+x_2^2}\right)$$