Università degli Studi Roma Tre - Corso di Laurea in Matematica

Tutorato di AM220

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Soluzioni del tutorato numero 8 (4 Maggio 2011)
Integrali, curve

I testi e le soluzioni dei tutorati sono disponibili al seguente indirizzo:
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1.
$$A=\left\{(x,y)\in\mathbb{R}^2:x^2+3y^2\le 1,0\le y\le x\right\}$$

Passando a coordinate ellittiche si ha $(x,y)=\Phi\left(\rho\cos\theta,\frac{\sqrt{3}}{3}\rho\sin\theta\right)$ e $\Phi^{-1}(A)=\left\{(\rho,\theta)\in[0,+\infty)\times[-\pi,\pi]:0\le\rho\le 1,0\le\theta\le\frac{\pi}{3}\right\}$, dunque essendo il determinante jacobiano pari a $\frac{\sqrt{3}}{3}\rho$, si ha

$$\int_A x^3dxdy=\int_0^{\frac{\pi}{3}}d\theta\int_0^1\frac{\rho^4\cos^3}{\sqrt{3}}\theta d\rho=\frac{\sqrt{3}}{3}\int_0^{\frac{\pi}{3}}\cos\theta\left(1-\sin^2\theta\right)d\theta\left[\frac{\rho^5}{5}\right]_0^1=$$

$$=\frac{\sqrt{3}}{15}\left[\sin\theta-\frac{\sin^3\theta}{3}\right]_0^{\frac{\pi}{3}}=\frac{3}{40}$$

2.
$$A=\left\{(x,y,z)\in\mathbb{R}^3:x^2+y^2\le\sin^2 z,z\in[0,\pi]\right\}$$

Passando a coordinate cilindriche si ha $(x,y,z)=\Phi(\rho\cos\theta,\rho\sin\theta,t)$ e $\Phi^{-1}(A)=\{(\rho,\theta,t)\in[0,+\infty)\times[-\pi,\pi]\times\mathbb{R}:\rho\le\sin t,t\in[0,\pi]\}$, dunque essendo il determinante jacobiano pari a ρ, si ha

$$Vol(A)=\int_{-\pi}^{\pi}d\theta\int_0^{\pi}dt\int_0^{\sin t}\rho d\rho=2\pi\int_0^{\pi}\frac{\sin^2 t}{2}=\pi\int_0^{\pi}\frac{1-\cos(2t)}{2}dt=\pi\left[\frac{t}{2}-\frac{\sin(2t)}{4}\right]_0^{\pi}=\frac{\pi^2}{2}$$

3.
$$A=\left\{(x,y)\in\mathbb{R}^2:x^3\le y\le 1-x^3,x\ge 0\right\}$$

Con il cambio di variabile $(x,y)=\Psi^{-1}(u,v)$, ove $\Psi(x,y)=\left(y-x^3,y+x^3\right)$, si ha $|\det J_{\Psi^{-1}}(u,v)|=\frac{1}{|\det J_\Phi(x,y)|}=\frac{1}{\begin{vmatrix}1&-3x^2\\1&3x^2\end{vmatrix}}=\frac{1}{6x^2}$; inoltre, $\Psi(A)=$ $=\left\{(u,v)\in\mathbb{R}^2:0\le u,v\le 1,u\le v\right\}$, e dunque

$$\int_A\left(x^2y^2-x^8\right)e^{x^6+2x^3y+y^2}dxdy=\frac{1}{6}\int_0^1dv\int_0^v uve^{v^2}du=\frac{1}{12}\int_0^1v^3e^{v^2}dv\overset{(t=v^2)}{=}\frac{1}{24}\int_0^1te^t=$$

$$\frac{1}{24}\left(\left[te^t\right]_0^1-\int_0^1e^tdt\right)=\frac{1}{24}\left(e-\left[e^t\right]_0^1\right)=\frac{1}{24}$$

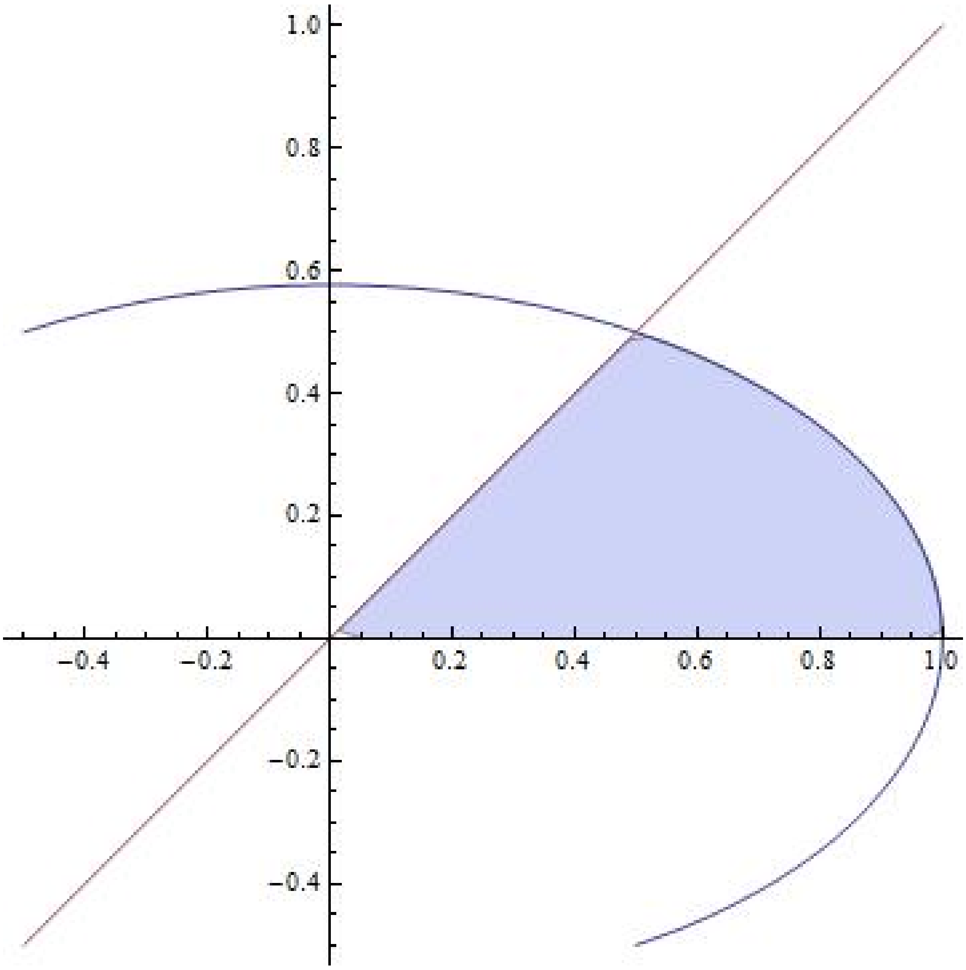

Figure 1: $A=\left\{(x,y)\in\mathbb{R}^2 : x^2+3y^2\leq 1, 0\leq y\leq x\right\}$

4.

$$A=\left\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2\leq z^2\right\}$$

Essendo l'integrale improprio, è sufficiente integrare su $A_{r,R}=\{(x,y,z)\in A : r^2\leq x^2+y^2+z^2\leq R^2\}$ e passare al $\lim\limits_{r\to 0, R\to+\infty}$: in coordinate sferiche, si ha $\Phi^{-1}(A_{r,R})=$

$=\left\{(\rho,\theta,\varphi)\in[0,+\infty)\times[-\pi,\pi]\times[0,\pi] : \rho\in[r,R], \varphi\in\left[0,\frac{\pi}{4}\right]\cup\left[\frac{3}{4}\pi,\pi\right]\right\}$,

dunque

$$\int_A \frac{dxdydz}{z^2\left(x^2+y^2+z^2+1\right)}=\lim_{r\to 0, R\to+\infty}\int_{-\pi}^{\pi}d\theta\int_{\left[0,\frac{\pi}{4}\right]\cup\left[\frac{3}{4}\pi,\pi\right]}d\varphi\int_r^R\frac{\rho^2\sin\varphi}{\rho^2\cos^2\varphi\left(\rho^2+1\right)}=$$

$$=\lim_{r\to 0, R\to+\infty}2\int_{-\pi}^{\pi}d\theta\int_0^{\frac{\pi}{4}}\frac{\sin\varphi}{\cos^2\varphi}d\varphi\int_r^R\frac{d\rho}{\rho^2+1}=\lim_{r\to 0, R\to+\infty}4\pi\left[\frac{1}{\cos\varphi}\right]_0^{\frac{\pi}{4}}\left[\arctan\rho\right]_r^R=$$

$$= \lim_{r\to 0, R\to+\infty} 4\pi\left(\sqrt{2}-1\right)(\arctan R - \arctan r) = 2\left(\sqrt{2}-1\right)\pi^2$$

5.
$$\gamma(t) = (t+\sin t, \cos t) \quad t\in[0,2\pi]$$

γ non è una curva regolare perché

$$|\dot\gamma(t)| = \sqrt{(1+\cos t)^2 + (-\sin t)^2} = \sqrt{2+2\cos t}$$

si annulla in $t=\pi\in(0,2\pi)$, comunque essendo regolare a tratti si può calcolare la sua lunghezza:

$$l(\gamma) = \int_0^{2\pi}\sqrt{2+2\cos t}dt = 2\int_0^{2\pi}\sqrt{\cos^2\left(\frac{t}{2}\right)}dt = 4\int_0^{\pi}\cos\left(\frac{t}{2}\right) = 8\left[\sin\left(\frac{t}{2}\right)\right]_0^{\pi} = 8$$

6.
$$\gamma(t) = \left(\cos^3 t, \sin^3 t\right) \quad t\in\left[0,\frac{\pi}{2}\right]$$

(a) γ è una curva regolare perché

$$|\gamma(t)| = \sqrt{\left(-3\sin t\cos^2 t\right)^2 + \left(3\cos t\sin^2 t\right)^2} = 3|\sin t||\cos t|\sqrt{\cos^2 t+\sin^2 t} =$$

$$= 3\sin t\cos t \neq 0 \ \forall t\in\left(0,\frac{\pi}{2}\right)$$

(b)
$$l(\gamma)\int_0^{\frac{\pi}{2}} 3\sin t\cos t = \left[\frac{3}{2}\sin^2 t\right]_0^{\frac{\pi}{2}} = \frac{3}{2}$$

(c)

$$\int_\gamma e^{y^{\frac{2}{3}}}d\ell = \int_0^{\frac{\pi}{2}} 3\sin t\cos t e^{\sin^2 t}dt = \left[\frac{3}{2}e^{\sin^2 t}\right]_0^{\frac{\pi}{2}} = \frac{3}{2}(e-1)$$

7. Sia $A = \left\{(x,y)\in\mathbb{R}^2 : 1\le x^2+y^2\le 4, y\ge\sqrt{3}|x|\right\}$. Passando in coordinate polari, si ha $\Phi^{-1}(A) = \left\{(\rho,\theta)\in[0,+\infty)\times[-\pi,\pi] : 1\le\rho\le 2, \frac{\pi}{3}\le\theta\le\frac{2}{3}\pi\right\}$

$$\int_A\frac{dxdy}{y} = \int_{\frac{\pi}{3}}^{\frac{2}{3}\pi}d\theta\int_0^1\frac{d\rho}{\sin\theta} = \int_{\frac{\pi}{3}}^{\frac{2}{3}\pi}\frac{\sin\theta}{1-\cos^2\theta}d\theta = \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{2}{3}\pi}\frac{\sin\theta}{1-\cos\theta}d\theta + \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{2}{3}\pi}\frac{\sin\theta}{1+\cos\theta}d\theta =$$

$$= \frac{1}{2}[\log|1-\cos\theta|]_{\frac{\pi}{3}}^{\frac{2}{3}\pi} + \frac{1}{2}[-\log|1+\cos\theta|]_{\frac{\pi}{3}}^{\frac{2}{3}\pi} = \frac{1}{2}\left(\log\frac{3}{2}-\log\frac{1}{2}\right) + \frac{1}{2}\left(\log\frac{3}{2}-\log\frac{1}{2}\right) = \log 3$$

Il bordo di A è dato dall'unione di quattro curve regolari che possono essere parametrizzate rispettivamente da

$$\gamma_1 = (\cos t, \sin t) \quad t\in\left[\frac{\pi}{3},\frac{2}{3}\pi\right]$$

$$\gamma_2 = \left(t, \sqrt{3}t\right) \quad t \in \left[\frac{1}{2}, 1\right]$$

$$\gamma_3 = (2\cos t, 2\sin t) \quad t \in \left[\frac{\pi}{3}, \frac{2}{3}\pi\right]$$

$$\gamma_4 = \left(-t, \sqrt{3}t\right) \quad t \in \left[\frac{1}{2}, 1\right]$$

Su ogni curva si ha

$$|\dot{\gamma_1}(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1 \Rightarrow \int_{\gamma_1} \frac{d\ell}{y} = \int_{\frac{\pi}{3}}^{\frac{2}{3}\pi} \frac{dt}{\sin t} = \log 3$$

$$|\dot{\gamma_2}(t)| = \sqrt{(1)^2 + \left(\sqrt{3}\right)^2} = 2 \Rightarrow \int_{\gamma_2} \frac{d\ell}{y} = \int_{\frac{1}{2}}^{1} \frac{2dt}{\sqrt{3}t} = \left[\frac{2\sqrt{3}}{3}\log|t|\right]_{\frac{1}{2}}^{1} = \frac{2\sqrt{3}}{3}\log 2$$

$$|\dot{\gamma_3}(t)| = \sqrt{(-2\sin t)^2 + (2\cos t)^2} = 2 \Rightarrow \int_{\gamma_3} \frac{d\ell}{y} = \int_{\frac{1}{2}}^{1} \frac{2dt}{\sqrt{3}t} = \left[\frac{2\sqrt{3}}{3}\log|t|\right]_{\frac{1}{2}}^{1} = \frac{2\sqrt{3}}{2}\log 2$$

$$|\dot{\gamma_4}(t)| = \sqrt{(-1)^2 + \left(\sqrt{3}\right)^2} = 2 \Rightarrow \int_{\gamma_4} \frac{d\ell}{y} = \int_{\frac{\pi}{3}}^{\frac{2}{3}\pi} \frac{2dt}{2\sin t} = \log 3$$

e dunque

$$\int_{\partial A} \frac{d\ell}{y} = \int_{\gamma_1} \frac{d\ell}{y} + \int_{\gamma_2} \frac{d\ell}{y} + \int_{\gamma_3} \frac{d\ell}{y} + \int_{\gamma_4} \frac{d\ell}{y} = \frac{4\sqrt{3}}{3}\log 2 + 2\log 3$$

8.

$$A_M = \{x \in \mathbb{R}^n : \langle Mx, x\rangle \le 1\}$$

Essendo M simmetrica e definita positiva, per il teorema spettrale $\exists N$ ortogonale tale che $D = N^{-1}MN$ è diagonale; posto $x = \Phi(y) = Ny$, essendo $N^T = N^{-1}$, si ha $\langle MNy, Ny\rangle = \left\langle N^{-1}MNy, y\right\rangle = \langle Dy, y\rangle = \lambda_1 y_1^2 + \cdots + \lambda_n y_n^2$, se $\lambda_1, \ldots, \lambda_n$ sono gli elementi sulla diagonale di D (cioè gli autovalori di M); quindi,

$$\Phi^{-1}(A_M) = \{y \in \mathbb{R}^n : \langle MNy, Ny\rangle \le 1\} = \left\{y \in \mathbb{R}^n : \lambda_1 y_1^2 + \ldots \lambda_n y_n^2 \le 1\right\}$$

Infine, con il cambio di variabile $y = \Psi(z) = \left(\frac{z_1}{\sqrt{\lambda_1}}, \ldots, \frac{z_n}{\sqrt{\lambda_n}}\right)$ si ottiene

$$\Psi^{-1}\left(\Phi^{-1}(A_M)\right) = \left\{z \in \mathbb{R}^n : z_1^2 + \ldots z_n^2 \le 1\right\}$$

e dunque, essendo $|\det J_\Phi| = |\det N| = 1$ e $|\det J_\Psi| = \frac{1}{\sqrt{\lambda_1 \ldots \lambda_n}} = \frac{1}{\sqrt{\det M}}$

$$|A_M| = \int_M dx = \int_{\Phi^{-1}(A_M)} dy = \int_{\Psi^{-1}(\Phi^{-1}(A_M))} \frac{dz}{\sqrt{\det M}} = \frac{B_n}{\sqrt{\det M}}$$

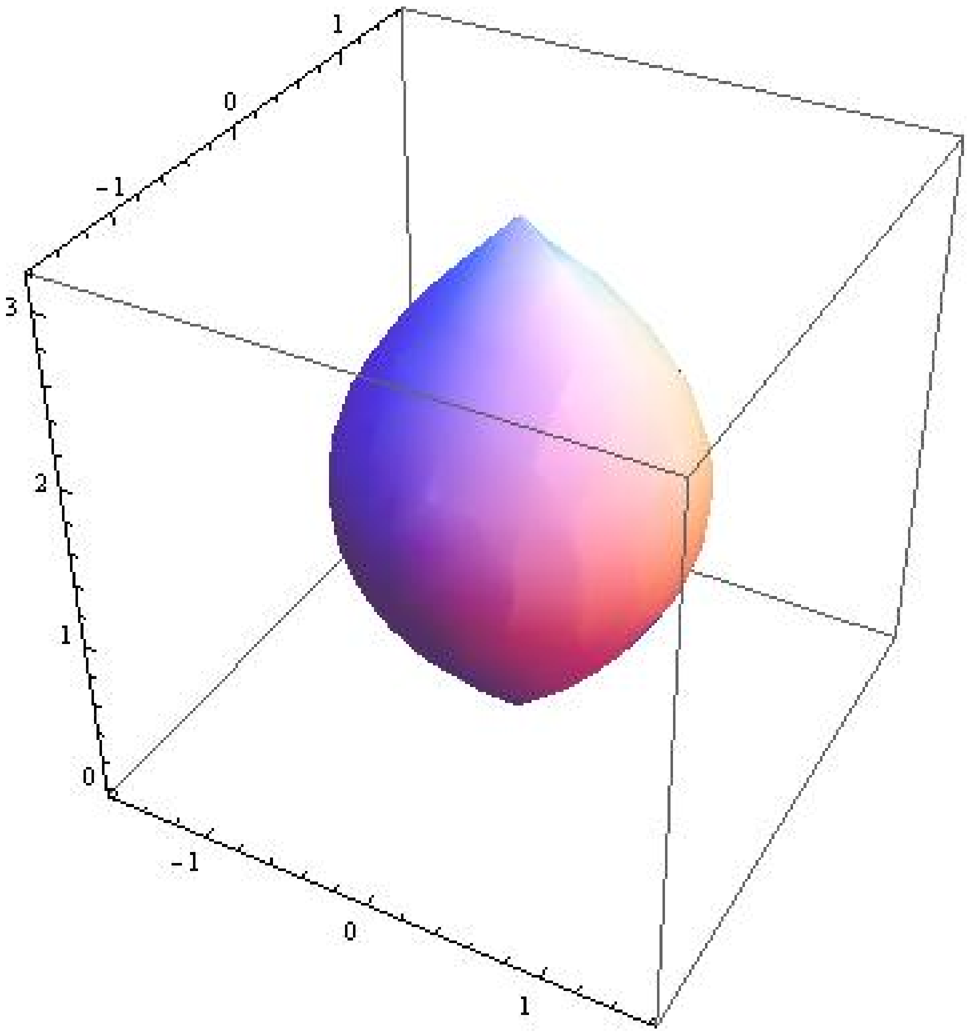

Figure 2: $A = \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq \sin^2 z, z \in [0, \pi]\right\}$

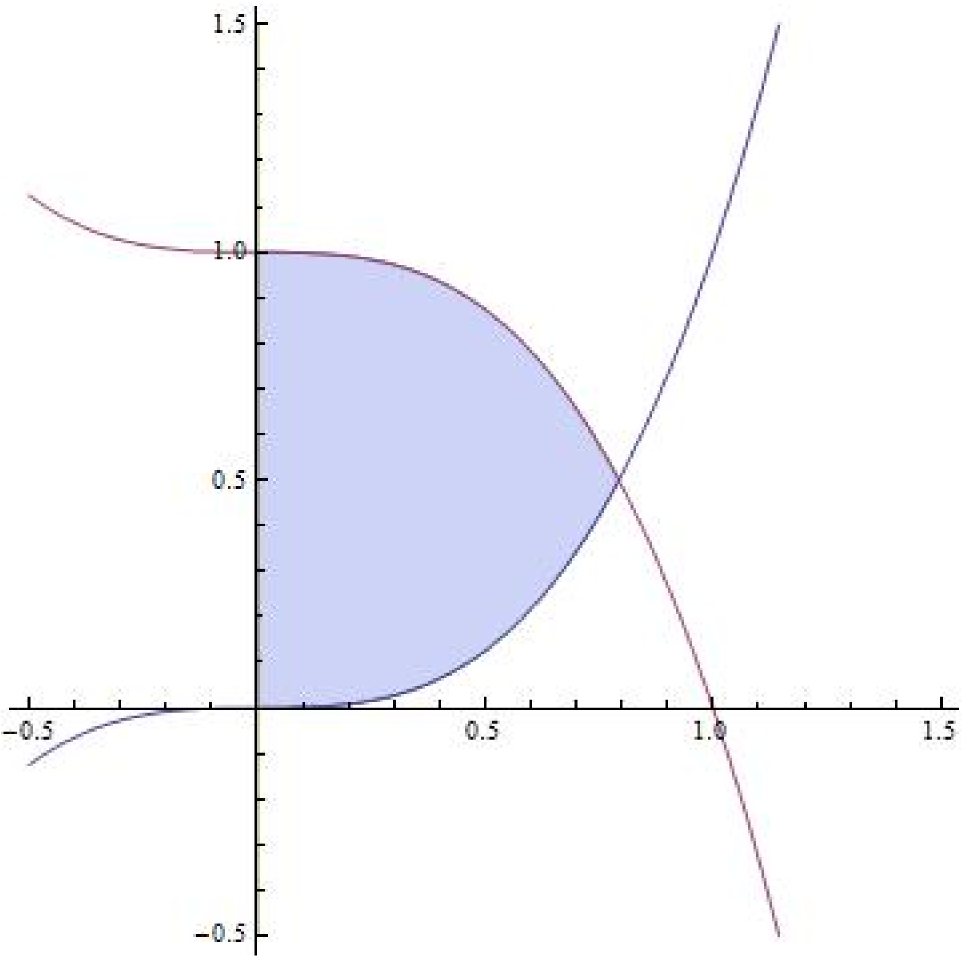

Figure 3: $A = \{(x,y) \in \mathbb{R}^2 : x^3 \leq y \leq 1 - x^3, x \geq 0\}$

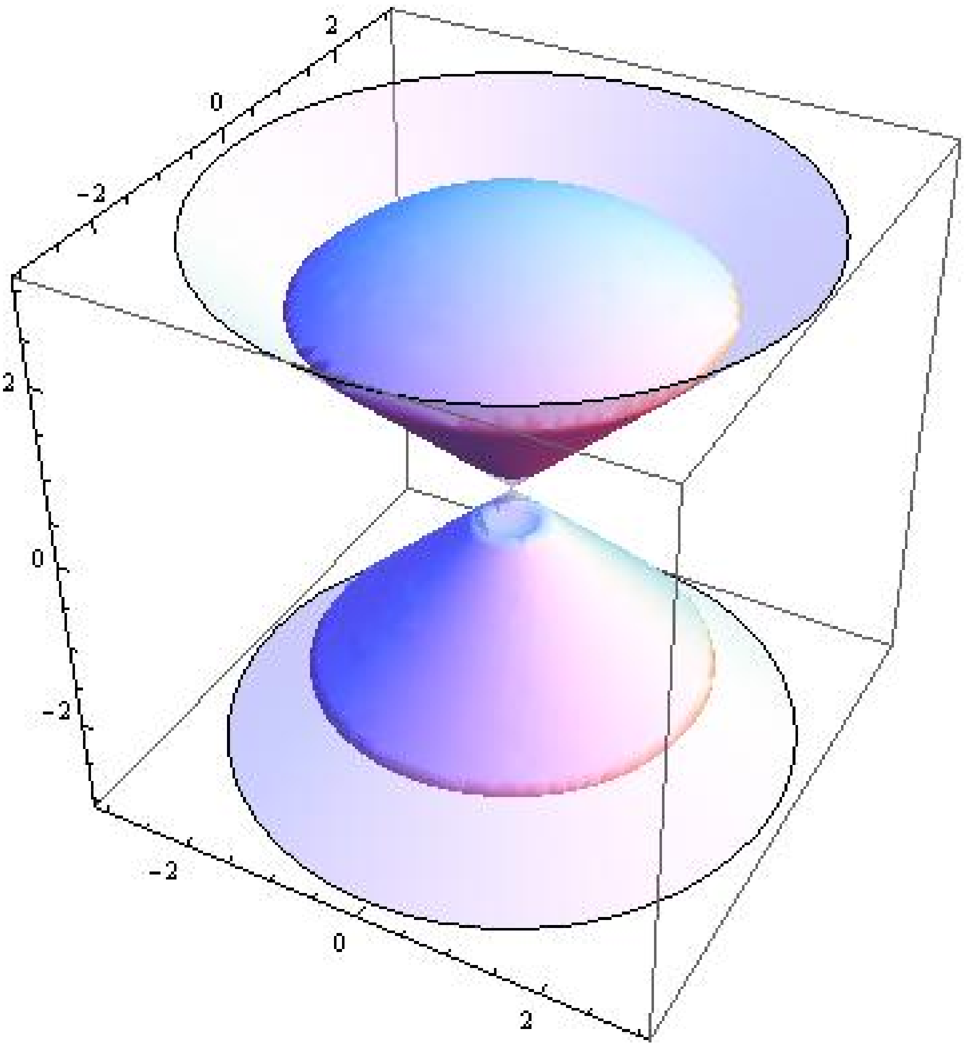

Figure 4: $A = \left\{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 \leq z^2\right\}$

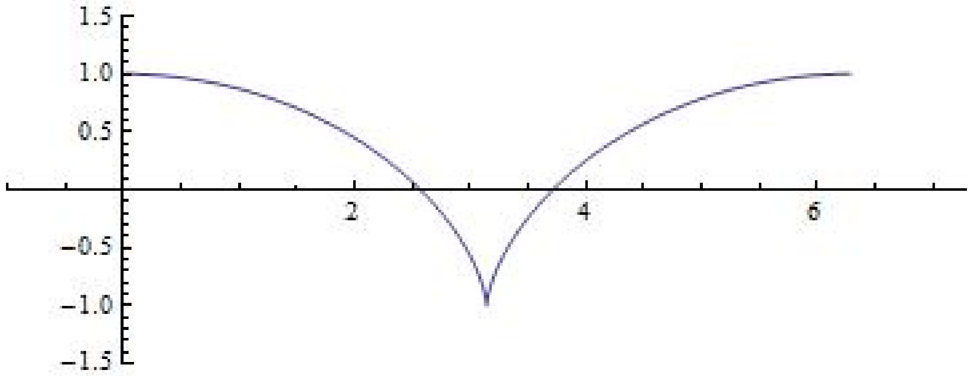

Figure 5: $\gamma(t) = (t + \sin t, \cos t)$

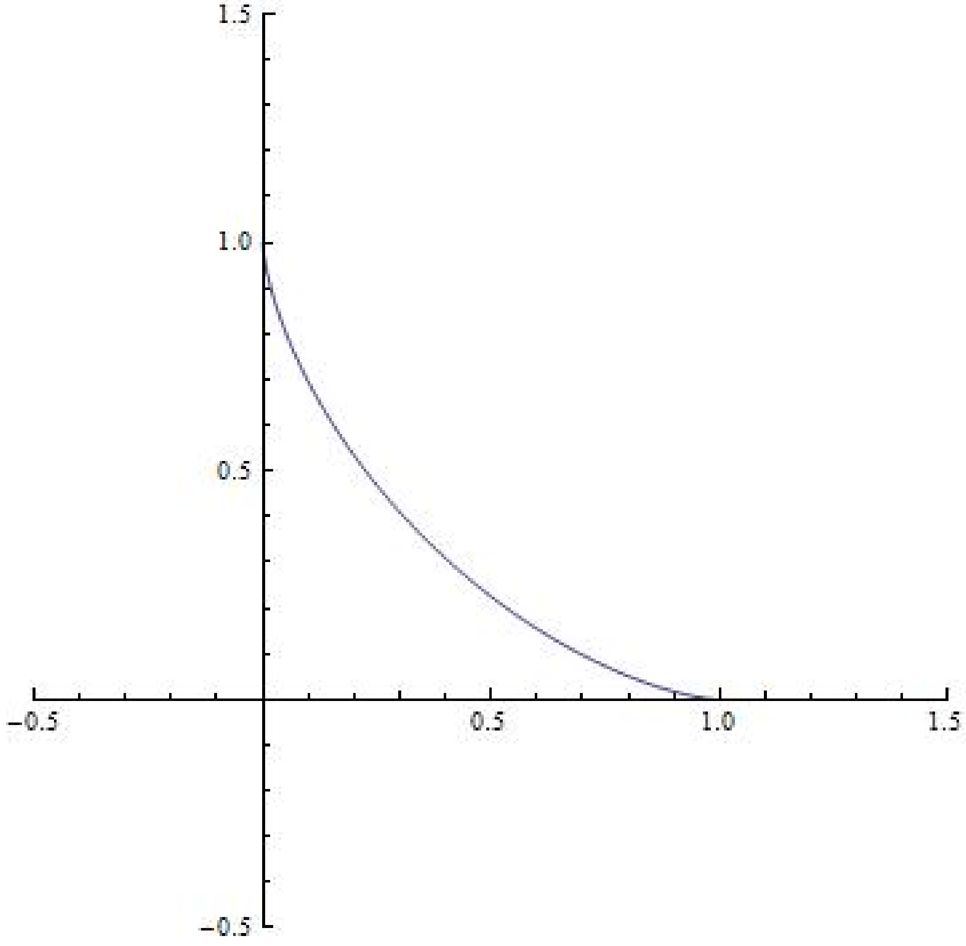

Figure 6: $\gamma(t) = \left(\cos^3 t, \sin^3 t\right)$

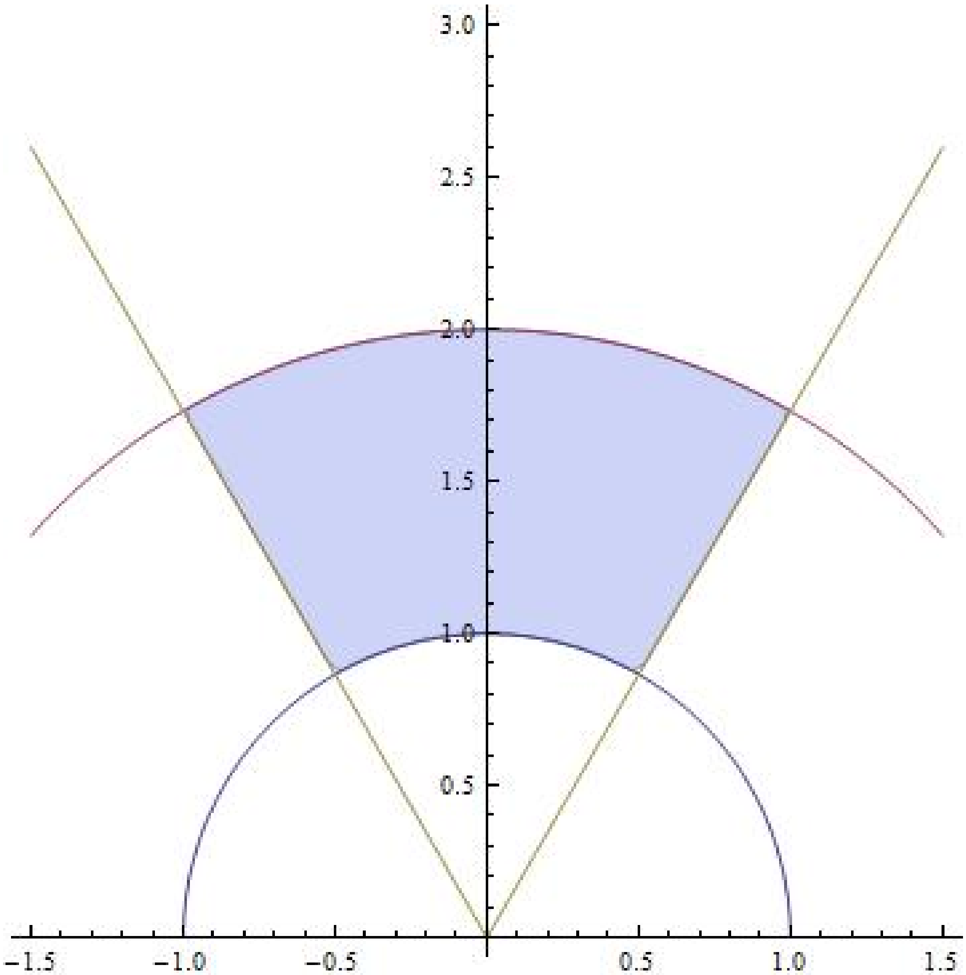

Figure 7: $A = \left\{(x, y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2 \leq 4, y \geq \sqrt{3}|x|\right\}$

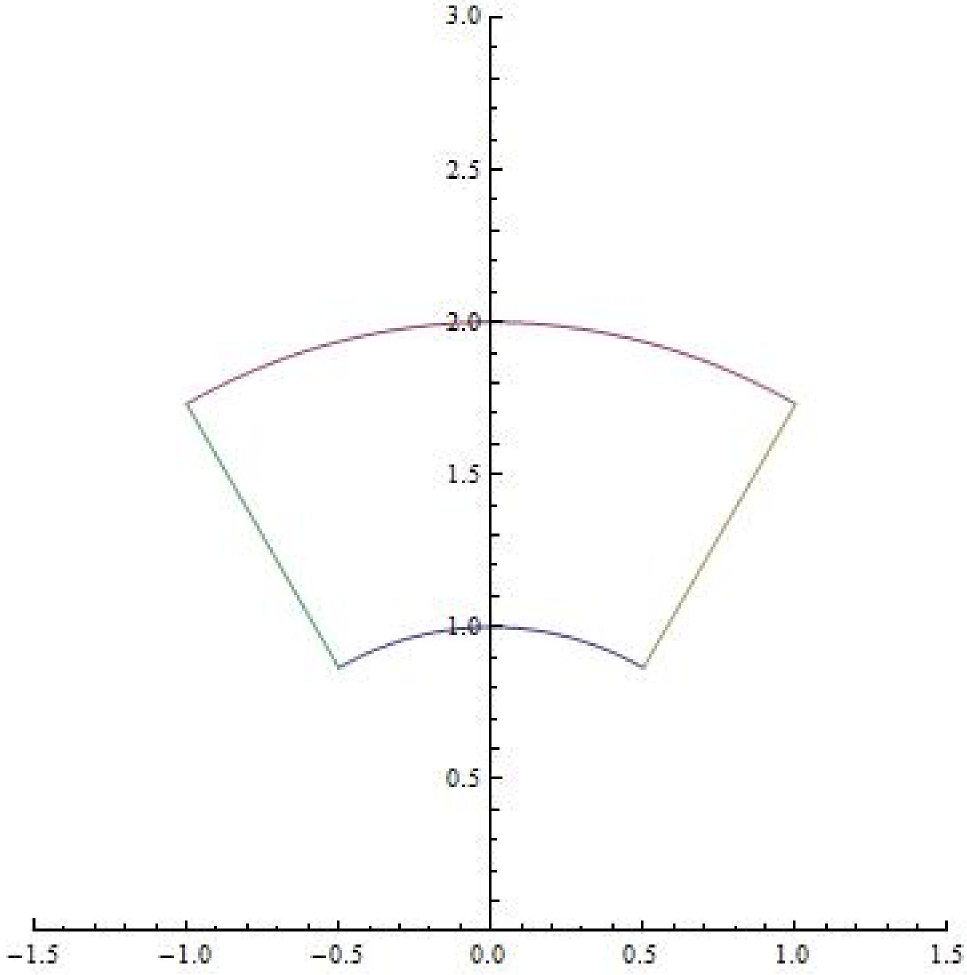

Figure 8: $\partial A = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$