Università degli Studi Roma Tre - Corso di Laurea in Matematica

Tutorato di AM220

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Soluzioni del tutorato numero 9 (11 Maggio 2011)

Integrali, curve

I testi e le soluzioni dei tutorati sono disponibili al seguente indirizzo:
http://www.lifedreamers.it/liuck

1. $\gamma(t) = \left(2t, 3t^2, 3t^3\right)$ per $t \in [0,1]$

   (a)

$$l(\gamma) = \int_0^1 |\dot{\gamma}(t)|dt = \int_0^1 \sqrt{2^2 + (6t)^2 + \left(9t^2\right)^2}dt = \int_0^1 \sqrt{4 + 36t^2 + 81t^4} = \int_0^1 9t^2+2dt =$$

$$= \left[3t^3 + 2t\right]_0^1 = 5$$

   (b)

$$\int_\gamma \frac{\sqrt[3]{z}\log\left(\frac{x}{2}+1\right)}{3y+2}d\ell = \int_0^1 \frac{\sqrt[3]{3t^3}\log(t+1)}{9t^2+2}9t^2+2dt = \sqrt[3]{3}\int_0^1 t\log(t+1)dt =$$

$$= \sqrt[3]{3}\left(\left[\frac{t^2}{2}\log(t+1)\right]_0^1 - \int_0^1 \frac{t^2}{2(t+1)}dt\right) = \sqrt[3]{3}\left(\frac{\log 2}{2} - \frac{1}{2}\int_0^1 t - 1 + \frac{1}{t+1}\right) =$$

$$= \sqrt[3]{3}\left(\frac{\log 2}{2} - \frac{1}{2}\left[\frac{t^2}{2} - t + \log(t+1)\right]_0^1\right) = \sqrt[3]{3}\left(\frac{\log 2}{2} - \frac{1}{2}\left(\log 2 - \frac{1}{2}\right)\right) = \frac{\sqrt[3]{3}}{4}$$

2. $\gamma(t) = \left(\frac{\sqrt{5}}{5}\cos t, \sin t, \frac{2\sqrt{5}}{5}\cos t\right)$ per $t \in [-\pi, \pi]$

   (a)

$$l(\gamma) = \int_{-\pi}^{\pi} \sqrt{\left(-\frac{\sqrt{5}}{5}\sin t\right)^2 + (\cos t)^2 + \left(-\frac{2\sqrt{5}}{5}\sin t\right)^2}dt = \int_{-\pi}^{\pi} \sqrt{\sin^2 t + \cos^2 t}dt = 2\pi$$

   (b)

$$\int_\gamma \frac{|y|}{y^2 + 5z^2}d\ell = \int_{-\pi}^{\pi} \frac{|\sin t|}{\sin^2 t + 4\cos^2 t}\sqrt{\cos^2 t + \sin^2 t}dt = 2\int_0^{\pi} \frac{\sin t}{3\cos^2 t + 1}dt \overset{(s=\cos t)}{=}$$

$$= 2\int_{-1}^{1} \frac{ds}{3s^2+1} = 2\left[\frac{\sqrt{3}}{3}\arctan\left(\sqrt{3}t\right)\right]_{-1}^{1} = \frac{4}{9}\sqrt{3}\pi$$

3. $\gamma(t) = \left(e^{-t}\cos t, e^{-t}\sin t\right)$ per $t \in [0, +\infty)$

(a)

$$l(\gamma)=\int_0^{+\infty}\sqrt{(-e^{-t}\cos t-e^{-t}\sin t)^2+(-e^{-t}\sin t+e^{-t}\cos t)^2}dt=$$

$$=\int_0^{+\infty}\sqrt{2e^{-2t}\left(\cos^2 t+\sin^2 t\right)}dt=\sqrt{2}\int_0^{+\infty}e^{-t}dt=\sqrt{2}\left[-e^{-t}\right]_0^{+\infty}=\sqrt{2}$$

(b)

$$\int_\gamma y^2 d\ell=\sqrt{2}\int_0^{+\infty}e^{-3t}\cos^2 tdt=\frac{\sqrt{2}}{2}\left(\int_0^{+\infty}e^{-3t}dt+\int_0^{+\infty}e^{-3t}\cos(2t)dt\right)=$$

$$=\frac{\sqrt{2}}{2}\left(\left[\frac{-e^{-3t}}{3}\right]_0^{+\infty}+\left[\frac{e^{-3t}\sin(2t)}{2}\right]_0^{+\infty}+\frac{3}{2}\int_0^{+\infty}e^{-3t}\sin(2t)\right)=$$

$$=\frac{\sqrt{2}}{6}+\frac{3}{4}\sqrt{2}\int_0^{+\infty}e^{-3t}\sin(2t)dt=\frac{\sqrt{2}}{6}+\frac{3}{4}\sqrt{2}\left(\left[-\frac{e^{-3t}\cos(2t)}{2}\right]_0^{+\infty}-\frac{3}{2}\int_0^{+\infty}e^{-3t}\cos(2t)dt\right)=$$

$$=\frac{13}{24}\sqrt{2}-\frac{9}{8}\sqrt{2}\int_0^{+\infty}e^{-2t}\cos(2t)dt\Rightarrow\int_0^{+\infty}e^{-3t}\cos(2t)dt=\frac{1}{3}-\frac{4}{13}\int_0^{+\infty}e^{-3t}dt\Rightarrow$$

$$\Rightarrow\int_\gamma y^2 d\ell=\frac{\sqrt{2}}{2}\left(\frac{9}{13}\int_0^{+\infty}e^{-3t}dt+\frac{1}{3}\right)=\frac{\sqrt{2}}{2}\left(\frac{3}{13}+\frac{1}{3}\right)=\frac{11}{39}\sqrt{2}$$

4. Sia $\gamma(t)=(\cos t,3\sin t)$ per $t\in[-\pi,\pi]$

(a)

$$\int_\gamma\sqrt{8x^2+1}d\ell=\int_{-\pi}^{\pi}\sqrt{8\cos^2 t+1}\sqrt{(-\sin t)^2+(3\cos t)^2}dt=\int_{-\pi}^{\pi}8\cos^2 t{+}1dt=$$

$$=\int_{-\pi}^{\pi}4\cos(2t)+5dt=\left[2\sin(2t)+5t\right]_{-\pi}^{\pi}=10\pi$$

(b) $\left(\frac{\sqrt{3}}{2},\frac{3}{2}\right)=\gamma\left(\frac{\pi}{6}\right)$, dunque essendo $\dot{\gamma}\left(\frac{\pi}{6}\right)=\left(-\frac{1}{2},\frac{3\sqrt{3}}{2}\right)$, la retta tangente alla curva in questo punto ha equazioni parametriche $\begin{cases}x=\frac{\sqrt{3}}{2}-\frac{t}{2}\\ y=\frac{3\sqrt{3}}{2}t+\frac{3}{2}\end{cases}$ e cartesiane $y=6-3\sqrt{3}x$

5. $\gamma=\gamma_1\cup\gamma_2\cup\gamma_3$, dove

$$\gamma_1(t)=(t,t)\quad t\in[0,1]$$

$$\gamma_2(t)=(t,2-t)\quad t\in[1,2]$$

$$\gamma_3(t)=(t,0)\quad t\in[0,2]$$

dunque

$$\int_\gamma xe^y d\ell=\int_{\gamma_1}xe^y d\ell+\int_{\gamma_2}xe^y d\ell+\int_{\gamma_3}xe^y d\ell=\int_0^1 te^t\sqrt{1^2+1^2}dt+\int_1^2 te^{2-t}\sqrt{1^2+(-1)^2}dt+$$

$$+\int_0^2 t\sqrt{1^2}dt = \sqrt{2}\int_0^1 te^t dt + \sqrt{2}e^2\int_1^2 te^{-t}dt + \int_0^2 tdt = \sqrt{2}\left(\left[te^t\right]_0^1 - \int_0^1 e^t dt\right) +$$

$$= \sqrt{2}e^2\left(\left[-te^{-t}\right]_1^2 + \int_1^2 e^{-t}dt\right) + \left[\frac{t^2}{2}\right]_0^2 = \sqrt{2}\left(e - \left[e^t\right]_0^1\right) + \sqrt{2}e^2\left(\frac{1}{e} - \frac{2}{e^2} + \left[-e^{-t}\right]_1^2\right) + 2 =$$

$$= 2\sqrt{2}e - 2\sqrt{2} + 2$$

6. $A = \left\{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 \leq z^2, 1 \leq z \leq 2\right\}$.
   Innanzi tutto, l'insieme è simmetrico rispetto alla variabile y, dunque le funzioni y e y^3, essendo dispari, avranno integrale nullo su A; in coordinate cilindriche, $\Phi^{-1} = \{(\rho,\theta,t) \in [0,+\infty) \times [-\pi,\pi] \times \mathbb{R} : \rho \leq t, 1 \leq t \leq 2\}$, dunque

$$\int_A y\left(y^2 + y + 1\right)dxdydz = \int_A y^2 dxdydz = \int_{-\pi}^{\pi} d\theta \int_1^2 dt \int_0^t \rho^3 \sin^2\theta d\rho =$$

$$= \int_{-\pi}^{\pi} \frac{1-\cos(2\theta)}{2}d\theta \int_1^2 dt \left[\frac{\rho^4}{4}\right]_0^t = \frac{1}{4}\left[\frac{\theta}{2} - \frac{\sin(2\theta)}{4}\right]_{-\pi}^{\pi} \int_1^2 t^4 dt = \frac{\pi}{4}\left[\frac{t^5}{5}\right]_1^2 = \frac{31}{20}\pi$$

7. $A = \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \geq 1, x + y \leq 2, x \geq 0, y \geq 0\right\}$.
   In coordinate polari, $\Phi^{-1}(A) = \left\{(\rho,\theta) \in [0,+\infty) \times [-\pi,\pi] : 1 \leq \rho \leq \frac{2}{\cos\theta + \sin\theta}, 0 \leq \theta \leq \frac{\pi}{2}\right\}$, dunque

$$\int_A \frac{x}{x^2+y^2}dxdy = \int_0^{\frac{\pi}{2}} d\theta \int_1^{\frac{2}{\cos\theta+\sin\theta}} \cos\theta d\rho = \int_0^{\frac{\pi}{2}} \frac{2\cos\theta}{\cos\theta+\sin\theta} - \cos\theta d\theta =$$

$$= 2\int_0^{\frac{\pi}{2}} \frac{d\theta}{1+\tan\theta} - \int_0^{\frac{\pi}{2}} \cos\theta d\theta \overset{s=\arctan s}{=} 2\int_0^{+\infty} \frac{ds}{(1+s)(1+s^2)} - [\sin t]_0^{\frac{\pi}{2}} =$$

$$= \int_0^{+\infty} \frac{1}{1+s} - \frac{t}{1+s^2}ds + \int_0^{+\infty} \frac{ds}{1+s^2} - 1 = \left[\log\frac{1+s}{\sqrt{1+s^2}}\right]_0^{+\infty} + [\arctan s]_0^{+\infty} - 1 = \frac{\pi}{2} - 1$$