

Quenched large deviations for Glauber evolution with Kac Interaction and Random Field. *

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Abstract

We study a spin-flip model with Kac type interaction, in the presence of a random field given by i.i.d. bounded random variables. The system, spatially inhomogeneous, evolves according to a non conservative (Glauber) dynamics. We show an almost sure (with respect to the random field) large deviations principle for the empirical magnetizations of this process. The rate functional depends on the statistical properties of the external random field, it is lower semicontinuous with compact level sets.

1 Introduction

We consider interacting spin-flip systems, in dimension d , with Kac type interaction in the presence of a random field given by i.i.d. bounded random variables. Kac potentials J_γ are two-body interactions with range γ^{-1} and strength γ^d , where γ is a dimensionless scaling parameter. When $\gamma \rightarrow 0$, i.e. very long range compared with the inter particle distance, the strength of the interaction becomes very weak, but in such a way that the total interaction between one particle and all the others is finite. Kac potentials were introduced in [KUH], and then generalized in [LP], to present a rigorous derivation of the van der Waals theory of a gas-liquid phase transition. There has been in the last decades an increasing interest in them. Indeed they induce the intermediate scale of interaction γ^{-1} (called mesoscopic) between the microscopic (lattice) one and a macroscopic one much bigger than the latter. They are suitable to interpolate not only between short and long range interactions, but, scaling space and time as functions of γ , one can hope to obtain more insights into the physics of the model. Recently they have been considered as models to describe social interactions and more general complex social systems, see for example [CDS] and references therein.

There has been several results on Kac Ising spin systems (without random field) in equilibrium and in non equilibrium Statistical Mechanics. We refer for a survey to the book [P]. Among the first papers dealing with dynamics issues were the papers [C], [CE]. They dealt with spin systems in a torus evolving according to a reversible and non conservative (Glauber) dynamics, with Kac interactions. In [C] the long time analysis

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of the spin system is studied, using large deviations techniques. In [CE] the main results are the infinite particle limits of the non-critical and critical fluctuation processes. In [DOPT] and [KS] (see also references therein), a complete description of the development and motion of interfaces (long time behaviour) has been derived: it is governed by the law of *motion by mean curvature*.

A natural extension of this analysis is its application to disordered systems. One of the simplest prototype models is obtained by adding random magnetic fields to an interacting spin system. Equilibrium statistical properties of these systems have been widely studied in the last decades, see [Bo] for a survey of results in this direction. The case of Kac type interaction has been investigated in $d = 1$ by [COP], [COPV] and [OP].

In this paper we study a reversible, nonconservative (Glauber) dynamics of ± 1 valued spins, interacting via a Kac potential and under the influence of an external random field. We assume the latter given by i.i.d. random variables taking values $a_i \in \mathbb{R}$ with probability p_i , for $i = 1, \dots, N$, with N a fixed integer. We do not require the Kac potential to be positive (that is we do not restrict the model to the ferromagnetic case).

Our main result is a quenched large deviations principle, almost sure with respect to the random field, for the empirical random magnetizations of this spin-flip process. The large deviations functional, which depends on the distribution of the random field, is lower semicontinuous, positive, with compact level sets. It is obtained via a contraction principle from the large deviations functional of the empirical *colored* magnetizations m_i (i.e. the magnetization over the sites where the random field takes value a_i), $i = 1, \dots, N$. As usual, the rate functional is determined by two distinct types of large deviations of the same order. The first one corresponds to large deviations from the initial state, the second one to the stochasticity of the evolution. Suppose $\mathcal{A} = \{\pi^\gamma(\cdot, dr) \simeq v(\cdot, r)dr, t \in [0, T]\}$ where $\pi^\gamma(\cdot, dr)$ is the local magnetization density, \simeq denotes closeness in some norm and v is a profile different from the solution of the nonlinear macroscopic equation giving the law of large numbers. We need to modify the measure of the process over the magnetization profiles so that event \mathcal{A} becomes typical. One possible choice is to drive the spin system by weak, slowly varying, space-time dependent external forces. This is the standard choice for spin systems evolving according to non conservative (Glauber) evolutions without randomness involved, or to conserved (Kawasaki) evolutions with gradient type interactions. For conserved non gradient systems, the force must be configuration dependent (see [Sp], p. 248), to take into account that for these systems the response in the current to an external force field is partially delayed. Namely, when an external random field is added to the Hamiltonian, in the conservative, non gradient case (see [MO]), one needs to take the external force weakly dependent on the field randomness. In the non conservative case, it turns out that the external force strongly depends on the field randomness. In other words, in dynamics with a conserved quantity, there is less freedom in choosing the class of perturbations than in non conservative dynamics.

We distinguish between sites where the random field takes different values; on each of them we take a deterministic space-time dependent external force. This allows to write the large deviations functional in a closed form with respect to the local *colored* magnetization. We carried out explicitly the computations for $N = 2$; the general case follows. The simplest case to have in mind is $a_1 = 1$, $a_2 = -1$, $p_1 = p_2 = 1/2$ and $J \geq 0$; then, when β (which is proportional to the inverse temperature) is such that $\beta \geq \beta_c = (\int J(r) dr)^{-1}$, interesting phenomena appear when studying the long time behaviour of the spin system. This is related to the fact the underlying spins systems at equilibrium undergo to phase transition, even in one dimension in the limit $\gamma \downarrow 0$. In this paper we will study the dynamics of the spin system for finite time: in this regime,

the evolution does not depend crucially on the value of β . We will then set $\beta = 1$.

Technical difficulties arise from the lack of regularity of certain functionals (related to the definition of large deviations functional) at the boundary of the local magnetization: they become infinite for the colored particle system when $p_i \leq |m_i| \leq 1$, which indicates that these boundaries are not rare enough in the large deviations regime.

The random Curie-Weiss model (RCW), which describes a mean field interaction, has given rise to many results on short and long time dynamics. In [DD], short time dynamics has been studied. More precisely the large deviations for the empirical measures in the product space of magnetization trajectories and realizations of the random field are given. From this result one could derive annealed large deviations for the RCW but not quenched ones. In [MP] and [FMP], long time dynamics, convergence to equilibrium when the random field takes only the two values $\pm\epsilon$ are considered. In [BEGK], the RCW model is analyzed when the random field takes finitely many values, as an example of the use of the potential theoretical approach to metastability. Furthermore, in [BBI], the previous results are extended to continuous distributions of the field, and precise asymptotics of metastable characteristics are derived.

There are no available results for short and long time dynamics of the random field Kac model. We make here a first step in addressing this problem.

In Section 2 we present the model, main definitions and results. In Section 3 we define the large deviations functional and state its main properties. In this section we use some results already established by [C] in the model without randomness. In Section 4 we prove the hydrodynamic behavior for the colored particle process. The class of time dependent, random perturbations, needed to derive the lower bound of the large deviations principle, is introduced in Section 5, where the perturbed process is studied. In Section 6 we derive the upper bound and in Section 7 the lower bound of the large deviations principle. The lower bound is obtained first for smooth paths and then it is extended to a large class of trajectories by techniques introduced in [QRV]. For the upper bound we need, then, to show that the process concentrates on this class of trajectories. This is obtained by introducing an energy functional via an exponential martingale which excludes the trajectories that are not in the considered class. In the appendix (Section 8), we present technical results needed along the paper.

2 The model and the main results

The space of configurations: Let Λ be the d -dimensional torus of diameter 1. For $0 < \gamma < 1$ such that $\gamma^{-1} \in \mathbb{N}$, $\Lambda_\gamma = \mathbb{Z}^d / \gamma^{-1} \mathbb{Z}^d$ is the d -dimensional discrete torus of diameter γ^{-1} . We denote by $\mathcal{S}_\gamma \equiv \{-1, +1\}^{\Lambda_\gamma}$ the configuration space and by $\sigma = (\sigma(x))_{x \in \Lambda_\gamma}$ a spin configuration, where for each $x \in \Lambda_\gamma$, $\sigma(x) \in \{-1, 1\}$.

The disorder: It is described by a collection of i.i.d. random variables $\alpha = \{\alpha(x), x \in \mathbb{Z}^d\}$, taking two values, i.e. $\alpha(x) \in \{a_1, a_2\}$. The corresponding product measure on $\Omega = \{a_1, a_2\}^{\mathbb{Z}^d}$ is denoted by \mathbb{P} (and \mathbb{E} is the expectation with respect to \mathbb{P}),

$$\mathbb{P}\{\alpha(x) = a_i\} = p_i, \quad i = 1, 2. \quad (2.2)$$

For γ^{-1} an odd integer, α induces in a natural way a random field on Λ_γ , also denoted by α .

The Kac potential: We consider a pair interaction among particles given by a Kac potential of the form

$$J_\gamma(x, y) \equiv \gamma^d J(\gamma(x - y)), \quad (x, y) \in \Lambda_\gamma \times \Lambda_\gamma, \quad (2.3)$$

where $J : \Lambda \rightarrow \mathbb{R}$ is a symmetric function, that is $J(r) = J(-r)$, such that $\int J(r) dr = 1$ (normalization). The interaction J might have any sign. Denote by $\mathcal{C}(\Lambda)$ (resp. $\mathcal{C}^1(\Lambda)$, $\mathcal{C}^2(\Lambda)$) the space of continuous (resp. continuously differentiable, twice continuously differentiable) real functions on Λ . We assume $J \in \mathcal{C}^1(\Lambda)$.

The Energy: Given a realization α of the magnetic fields, define for all $\gamma, \theta > 0$, $\sigma \in \mathcal{S}_\gamma$, the Hamiltonian

$$H^{\gamma, \alpha}(\sigma) = -\frac{1}{2} \sum_{(x, y) \in \Lambda_\gamma \times \Lambda_\gamma} J_\gamma(x, y) \sigma(x) \sigma(y) - \theta \sum_{x \in \Lambda_\gamma} \alpha(x) \sigma(x), \quad (2.4)$$

and the Gibbs measure $\mu^{\gamma, \alpha, \beta}$ associated to $H^{\gamma, \alpha}$ at inverse temperature β , with normalization constant $Z^{\gamma, \alpha, \beta}$:

$$\mu^{\gamma, \alpha, \beta}(\sigma) = \frac{1}{Z^{\gamma, \alpha, \beta}} \exp[-\beta H^{\gamma, \alpha}(\sigma)].$$

The Glauber dynamics: Denote by σ^x the configuration obtained from σ by flipping the spin at site x :

$$\sigma^x(z) = \begin{cases} -\sigma(x) & \text{if } z = x \\ \sigma(z) & \text{otherwise,} \end{cases}$$

so that the energy difference resulting from a spin flip at x is

$$H^{\gamma, \alpha}(\sigma^x) - H^{\gamma, \alpha}(\sigma) = 2\sigma(x) [(J_\gamma \star \sigma)(x) + \theta\alpha(x)], \quad (2.5)$$

where without loss of generality we have assumed $J(0) = 0$, and we define the discrete convolution \star between function J_γ and a configuration σ by

$$(J_\gamma \star \sigma)(x) = \gamma^d \sum_{y \in \Lambda_\gamma} J(\gamma(x - y)) \sigma(y). \quad (2.6)$$

We consider a Markovian evolution on \mathcal{S}_γ , whose generator $\mathcal{L}^{\gamma, \alpha}$ acts on cylinder functions f as

$$\mathcal{L}^{\gamma, \alpha} f(\sigma) = \sum_{x \in \Lambda_\gamma} c_x^{\gamma, \alpha}(\sigma) [f(\sigma^x) - f(\sigma)], \quad (2.7)$$

where, for $x \in \Lambda_\gamma$,

$$c_x^{\gamma, \alpha}(\sigma) = \frac{\exp[-(\beta/2)(H^{\gamma, \alpha}(\sigma^x) - H^{\gamma, \alpha}(\sigma))]}{2 \cosh[(\beta/2)(H^{\gamma, \alpha}(\sigma^x) - H^{\gamma, \alpha}(\sigma))]} \quad (2.8)$$

Then $\mathcal{L}^{\gamma, \alpha}$ viewed as an operator on $L^2(\mu^{\gamma, \alpha, \beta})$ is self-adjoint. Since temperature is kept fixed in all the paper and does not play any role we set for simplicity $\beta = 1$. We fix a time $T > 0$, and we will study the process $(\sigma_t)_{t \in [0, T]}$ with infinitesimal generator given in (2.7).

The measures spaces: Let \mathcal{M}_1 be the set of signed Borel measures μ on the Borel σ -field of Λ with total variation norm bounded by 1. We equip \mathcal{M}_1 with the weak τ^* topology induced by $\mathcal{C}(\Lambda)$ via $\langle \mu, G \rangle = \int G d\mu$

(for $G \in \mathcal{C}(\Lambda)$). We denote by $\rho(\cdot, \cdot)$ the distance which makes (\mathcal{M}_1, τ^*) a metrizable compact space, see [Bill]: that is, given $(H_k)_{k \in \mathbb{N}}$ a dense subset in the unit ball of $\mathcal{C}(\Lambda)$ for $\mu_i \in \mathcal{M}_1$, $i = 1, 2$,

$$\rho(\mu_1, \mu_2) = \sum_{k \geq 0} 2^{-k} | \langle \mu_1 - \mu_2, H_k \rangle |. \quad (2.9)$$

Let $0 < q \leq 1$, and

$$\mathcal{M}_q^{ac} = \left\{ \mu \in \mathcal{M}_1 : \mu \ll \lambda \quad \text{and} \quad \left| \frac{d\mu}{d\lambda} \right| \leq q \quad \lambda - a.s. \right\}, \quad (2.10)$$

where λ is the Lebesgue measure on Λ . We identify $\mu \in \mathcal{M}_q^{ac}$ with its Radon-Nikodym derivative $\frac{d\mu}{d\lambda}$, and, by an abuse of notation, we write $\langle \mu, G \rangle = \langle \frac{d\mu}{d\lambda}, G \rangle$. Since \mathcal{M}_q^{ac} is a closed ball of \mathcal{M}_1 , it is τ^* compact.

If $\sigma \in \mathcal{S}_\gamma$ we define the *empirical measure* $\pi^\gamma(\sigma) \in \mathcal{M}_1$ by

$$\pi^\gamma(\sigma)(dr) = \gamma^d \sum_{x \in \Lambda_\gamma} \sigma(x) \delta_{\gamma x}(dr), \quad (2.11)$$

where $\delta_{\gamma x}$ is the Dirac measure concentrated on point γx . Remark that if we denote by $\mu * G$ the convolution of a measure μ and a function G over Λ , namely $(\mu * G)(r') = \int_\Lambda G(r' - r) \mu(dr)$, then we can rewrite

$$(J_\gamma \star \sigma)(x) = (\pi^\gamma(\sigma) * J)(\gamma x). \quad (2.12)$$

We denote by $D([0, T], \mathcal{M}_1)$ the space of functions from $[0, T]$ to \mathcal{M}_1 that are right continuous with left limits, endowed with the Skorohod topology, see [Bill].

The initial condition: Let $(\sigma^\gamma)_\gamma$ be a sequence of configurations such that $\pi^\gamma(\sigma^\gamma)$ converges in the weak topology to the measure $m_0 \lambda$, for a continuous function $m_0 : \Lambda \rightarrow [-1, 1]$. This means that

$$\lim_{\gamma \rightarrow 0} \rho(\pi^\gamma(\sigma^\gamma), m_0 \lambda) = 0. \quad (2.13)$$

We denote by $P_{\sigma^\gamma}^{\gamma, \alpha}$ the law (and by $E_{\sigma^\gamma}^{\gamma, \alpha}$ the expectation) of the process $(\sigma_t)_{t \in [0, T]}$ on the Skorohod space $D([0, T], \mathcal{S}_\gamma)$ starting at time $t = 0$ from the deterministic initial configuration σ^γ , and by $Q_{\sigma^\gamma}^{\gamma, \alpha}$ the law on $D([0, T], \mathcal{M}_1)$ of the corresponding empirical measure process $(\pi_t^\gamma)_{t \in [0, T]}$, where π_t^γ stands for $\pi^\gamma(\sigma_t)$.

We first obtain the ‘‘law of large numbers’’.

Theorem 2.2 *Assume $(\sigma^\gamma)_\gamma, m_0$ satisfy (2.13). For all $t \geq 0$,*

$$\lim_{\gamma \rightarrow 0} \rho(\pi_t^\gamma, m(t, \cdot) \lambda) = 0, \quad \mathbb{P}\text{-a.s.}, \quad (2.14)$$

where $m(\cdot, \cdot)$ is the unique weak solution of

$$\begin{cases} \partial_t m(t, r) = -m(t, r) + \sum_{i=1,2} p_i \tanh [(J * m(t, \cdot))(r) + a_i \theta] \\ m(0, \cdot) = m_0(\cdot). \end{cases} \quad (2.15)$$

Furthermore, for all $G \in \mathcal{C}^{0,1}([0, T] \times \Lambda)$, $\delta > 0$,

$$\lim_{\gamma \rightarrow 0} P_{\sigma^\gamma}^{\gamma, \alpha} \left[\sup_{t \in [0, T]} | \langle \pi_t^\gamma, G(t, \cdot) \rangle - \langle m(t, \cdot), G(t, \cdot) \rangle | \geq \delta \right] = 0. \quad (2.16)$$

By an abuse of notation we write from now on $(J * m)(t, r)$ instead of $(J * m(t, \cdot))(r)$.

Remark 2.3 . The Cauchy problem (2.15) in this setup is well posed with a unique global solution, because the right hand side of (2.15) is uniformly Lipschitz, and because the set $\{m \in L^\infty(\Lambda) : \|m\|_\infty \leq 1\}$ is left invariant, since $|\tanh z| \leq 1$ for all z . Furthermore the solution is differentiable in time.

Next we state the *quenched* large deviations principle for $Q_{\sigma^\gamma}^{\gamma, \alpha}$. Different choices of initial conditions could be treated as well. The only difference would be an extra term to add to the large deviation functional $\tilde{I}_{m_0}(\cdot)$, taking into account the deviation from the initial profile at time $t = 0$. The functional $\tilde{I}_{m_0}(\cdot)$ depends on the distribution of the random field but not on its realization; it is obtained through a contraction principle, as explained in the introduction. Its explicit formulation, given in (2.29), relies on several intermediate steps. Let

$$\mathcal{D}(\tilde{I}_{m_0}) = \{\pi \in D([0, T], \mathcal{M}_1) : \tilde{I}_{m_0}(\pi) < \infty\}. \quad (2.17)$$

Theorem 2.4 Assume $(\sigma^\gamma)_\gamma, m_0$ satisfy (2.13). For all closed subsets $\mathcal{F} \subset D([0, T], \mathcal{M}_1)$ and open subsets $\mathcal{A} \subset D([0, T], \mathcal{M}_1)$, we have

$$\limsup_{\gamma \rightarrow 0} \gamma^d \log Q_{\sigma^\gamma}^{\gamma, \alpha}(\mathcal{F}) \leq - \inf_{\pi \in \mathcal{F}} \tilde{I}_{m_0}(\pi), \quad IP - a.s., \quad (2.18)$$

$$\liminf_{\gamma \rightarrow 0} \gamma^d \log Q_{\sigma^\gamma}^{\gamma, \alpha}(\mathcal{A}) \geq - \inf_{\pi \in \mathcal{A}} \tilde{I}_{m_0}(\pi), \quad IP - a.s. \quad (2.19)$$

The functional $\tilde{I}_{m_0}(\cdot)$, defined in (2.29) below, is positive for $\pi \in D([0, T], \mathcal{M}_1)$, lower semicontinuous with compact level sets and, see Definition 3.1,

$$\mathcal{D}(\tilde{I}_{m_0}) \subset \{m \in \mathcal{C}([0, T], \mathcal{M}_1^{ac}) : m(t) \text{ absolutely continuous for } t \in [0, T]\}.$$

Subsection 2.1: The colored particle systems.

To derive the large deviations functional we introduce *random* empirical measures $\bar{\pi}^\gamma = (\pi_1^\gamma, \pi_2^\gamma)$. For $\alpha \in \Omega, x \in \Lambda_\gamma, i = 1, 2$, set

$$\alpha_i(x) = \mathbb{I}_{\{\alpha(x)=a_i\}}, \quad (2.20)$$

$$\pi_i^\gamma(\sigma)(dr) = \gamma^d \sum_{x \in \Lambda_\gamma} \alpha_i(x) \sigma(x) \delta_{\gamma x}(dr). \quad (2.21)$$

Though we do not write it explicitly, $\pi_i^\gamma(\sigma) \in \mathcal{M}_1$ depends on the randomness. Moreover the knowledge of $\pi_i^\gamma(\sigma)$ for $i = 1, 2$ determines $\pi^\gamma(\sigma) = \pi_1^\gamma(\sigma) + \pi_2^\gamma(\sigma)$. We denote by $\bar{Q}_{\sigma^\gamma}^{\gamma, \alpha}$ the law on $D([0, T], \mathcal{M}_1 \times \mathcal{M}_1)$ of the empirical measure process $(\bar{\pi}_t^\gamma)_{t \in [0, T]} = (\pi_{1,t}^\gamma, \pi_{2,t}^\gamma)_{t \in [0, T]}$ under $P_{\sigma^\gamma}^{\gamma, \alpha}$. We denote, for $\bar{G} = (G_1, G_2) \in (\mathcal{C}(\Lambda))^2$,

$$\langle \bar{\pi}_t^\gamma, \bar{G} \rangle = \sum_{i=1,2} \gamma^d \sum_{x \in \Lambda_\gamma} G_i(\gamma x) \alpha_i(x) \sigma_t(x) \quad (2.22)$$

and, for $\bar{m} = (m_1, m_2) \in (L^\infty(\Lambda))^2$, by an abuse of notation,

$$\langle \bar{m}, \bar{G} \rangle = \langle (m_1 \lambda, m_2 \lambda), \bar{G} \rangle = \sum_{i=1}^2 \int_{\Lambda} G_i(r) m_i(r) dr. \quad (2.23)$$

Theorem 2.5 *Assume $(\sigma^\gamma)_\gamma, m_0$ satisfy (2.13). For all $t \in [0, T]$, $\delta > 0$ and $\bar{G} \in (C^1(\Lambda))^2$,*

$$\lim_{\gamma \rightarrow 0} P_{\sigma^\gamma}^{\gamma, \alpha} [\langle \bar{\pi}_t^\gamma, \bar{G} \rangle - \langle \bar{m}(t, \cdot), \bar{G} \rangle \geq \delta] = 0 \quad \mathbb{P} - a.s.,$$

where $\bar{m} = (m_1, m_2)$ is the unique weak solution of

$$\begin{cases} \partial_t m_i(t, r) = -m_i(t, r) + p_i \tanh[\beta((J * m)(t, r) + a_i \theta)], \\ m = m_1 + m_2; \quad m_i(0, \cdot) = p_i m_0(\cdot), \quad i = 1, 2. \end{cases} \quad (2.24)$$

Remark 2.6 . Similarly to Remark 2.3, the Cauchy problem (2.24) in this setup is well posed with a unique global solution; here, the set $\{\bar{m} \in (L^\infty(\Lambda))^2 : \|m_i\|_\infty \leq p_i, i = 1, 2\}$ is left invariant. The solution is differentiable in time. The case $J \geq 0$, $a_1 = 1$, $a_2 = -1$, $p_1 = p_2 = 1/2$ is analyzed in [COP4].

To derive Theorem 2.9 below, we need a stronger type of convergence:

Corollary 2.7 *For all $\bar{G} \in (C^{0,1}([0, T] \times \Lambda))^2$, $\delta > 0$,*

$$\lim_{\gamma \rightarrow 0} P_{\sigma^\gamma}^{\gamma, \alpha} \left[\sup_{t \in [0, T]} |\langle \bar{\pi}_t^\gamma, \bar{G}(t, \cdot) \rangle - \langle \bar{m}(t, \cdot), \bar{G}(t, \cdot) \rangle| \geq \delta \right] = 0. \quad (2.25)$$

Remark 2.8 . Theorem 2.5 and Corollary 2.7 imply Theorem 2.2 since if $\bar{G} = (G, G)$,

$$\langle \bar{\pi}_t^\gamma, \bar{G} \rangle = \langle \pi_{1,t}^\gamma, G \rangle + \langle \pi_{2,t}^\gamma, G \rangle = \langle \pi_t^\gamma, G \rangle.$$

Next theorem states the large deviation principle for the colored system. Theorem 2.4 is based on this important intermediate result, interesting for itself.

Theorem 2.9 *Assume $(\sigma^\gamma)_\gamma, m_0$ satisfy (2.13). We have, for all open subset $\bar{\mathcal{A}}$ and closed subset $\bar{\mathcal{F}}$ in $D([0, T], \mathcal{M}_1 \times \mathcal{M}_1)$,*

$$\liminf_{\gamma \rightarrow 0} \gamma^d \log \bar{Q}_{\sigma^\gamma}^{\gamma, \alpha}(\bar{\mathcal{A}}) \geq - \inf_{\bar{\pi} \in \bar{\mathcal{A}}} I_{m_0}(\bar{\pi}), \quad \mathbb{P} - a.s. \quad (2.26)$$

$$\limsup_{\gamma \rightarrow 0} \gamma^d \log \bar{Q}_{\sigma^\gamma}^{\gamma, \alpha}(\bar{\mathcal{F}}) \leq - \inf_{\bar{\pi} \in \bar{\mathcal{F}}} I_{m_0}(\bar{\pi}), \quad \mathbb{P} - a.s. \quad (2.27)$$

where

$$I_{m_0}(\bar{\pi}) = \begin{cases} I_0(\bar{\pi}) & \text{if } \pi_i(0, \cdot) = p_i m_0(\cdot) \lambda, \quad i = 1, 2, \\ +\infty & \text{otherwise,} \end{cases} \quad (2.28)$$

and $I_0(\cdot)$, defined in (3.6) below, is lower semicontinuous with compact level sets.

Define, for a path $\pi \in D([0, T], \mathcal{M}_1)$,

$$\tilde{I}_{m_0}(\pi) = \inf \left(I_{m_0}(\bar{\pi}), \bar{\pi} = (\pi_1, \pi_2), \pi_i \in D([0, T], \mathcal{M}_1), i = 1, 2, \pi_1 + \pi_2 = \pi \right). \quad (2.29)$$

Since the map $(\pi_1, \pi_2) \mapsto \pi_1 + \pi_2$ is continuous in $D([0, T], \mathcal{M}_1 \times \mathcal{M}_1)$, by the contraction principle, see [V], [DZ], Theorem 2.9 proves Theorem 2.4. Therefore in the following sections we will focus on the colored particle system.

3 Rate functional

In this section we define the rate functional $I_0(\cdot)$ of the colored particle systems and state its main properties. Proofs of the latter, quite technical, are carried out in Section 8. Heuristics to define $I_0(\cdot)$ consists in finding for any path $\bar{\phi}$ on $[0, T]$ smooth enough an exponential change of probability under which the process $(\sigma_t)_{t \in [0, T]}$ is uniformly close to $\bar{\phi}$ on $[0, T]$. When there exists some potential $\bar{V}(t, \cdot)$, $t \in [0, T]$ smooth enough for $\bar{\phi}$ to be the solution of a *perturbed equation* (obtained by the law of large numbers from the process $(\sigma_t)_{t \in [0, T]}$, see (5.5)), then $I_0(\cdot)$ is related to the Radon-Nikodym derivative of the distribution of $(\sigma_t)_{t \in [0, T]}$ with respect to the distribution of the original process, see Theorem 5.4. In the general case, i.e. when there is no such $\bar{V}(t, \cdot)$, we are still able to provide an explicit representation of $I_0(\cdot)$ (this is similar to the results of [C]). We will then show that this representation of $I_0(\cdot)$ is equivalent to the usual definition of the rate functional, given through the macroscopic functional associated to the Radon-Nikodym derivative, see (3.8). We start by specifying the functional spaces on which we will define $I_0(\cdot)$. For $(p_1, p_2) \in [0, 1]^2$, we identify the set

$$B_{p_1, p_2} = \{\bar{u} = (u_1, u_2) : u_i \in L^\infty(\Lambda), \|u_i\|_\infty \leq p_i, i = 1, 2\} \quad (3.1)$$

with $\mathcal{M}_{p_1}^{ac} \times \mathcal{M}_{p_2}^{ac}$, see (2.10), and extend the distance ρ (see (2.9)) to elements of $\mathcal{M}_{p_1}^{ac} \times \mathcal{M}_{p_2}^{ac}$ by

$$\rho(\bar{\mu}, \bar{\nu}) = \sum_{i=1,2} \rho(\mu_i, \nu_i). \quad (3.2)$$

Definition 3.1 Let $\mathcal{AC}([0, T], B_{1,1}) \subset \mathcal{C}([0, T], B_{1,1})$ be the subset of absolutely continuous functions $\bar{\phi} = (\phi_1, \phi_2)$, that is, for $j = 1, 2$: for all $t' \in [0, T]$, $t \in [t', T]$, there exists $\dot{\phi}_j \in L^1([0, T] \times \Lambda)$ such that

$$\phi_j(t)(r) - \phi_j(t')(r) = \int_{t'}^t \dot{\phi}_j(s, r) ds, \quad \lambda - a.s.$$

By an abuse of notation, from now on we write $\phi_j(t, r)$ instead of $\phi_j(t)(r)$.

We will use the following characterization of absolutely continuous functions, see [DS].

Proposition 3.2 A function $\bar{\phi}$ belongs to $\mathcal{AC}([0, T], B_{1,1})$ if and only if: for all $\epsilon > 0$, there exists $\Delta > 0$ such that for all integer $k > 0$, rectangles A_1, \dots, A_k of Λ and $\{(s_i, t_i), 1 \leq i \leq k\}$ nonempty disjoint intervals of $[0, T]$,

$$\sum_{i=1}^k |t_i - s_i| \lambda(A_i) < \Delta \Rightarrow \sum_{i=1}^k \left| \int_{A_i} (\phi_j(t_i, r) - \phi_j(s_i, r)) dr \right| < \epsilon, \quad j = 1, 2.$$

To write $I_0(\cdot)$, we start by defining, for each $t \in [0, T]$, the following functionals, in which time is kept fixed, therefore we omit to write it. For $\bar{\pi} = (\pi_1, \pi_2) \in \mathcal{M}_1 \times \mathcal{M}_1$ (we write $\pi = \pi_1 + \pi_2$), $\bar{\mu} = (\mu_1, \mu_2) \in \mathcal{M}_1 \times \mathcal{M}_1$ and $\bar{V} = (V_1, V_2) \in (L^\infty(\Lambda))^2$ denote

$$\begin{aligned} F_{\bar{V}}(\bar{\mu}, \bar{\pi}) &= \sum_{i=1,2} \langle \mu_i, \tanh(\pi * J + a_i \theta) \sinh(2V_i) + \cosh(2V_i) - 1 \rangle \\ &\quad - \sum_{i=1,2} \langle \pi_i, \tanh(\pi * J + a_i \theta) [\cosh(2V_i) - 1] + \sinh(2V_i) \rangle. \end{aligned} \quad (3.3)$$

For $\bar{u} = (u_1, u_2) \in B_{1,1}$, $\bar{g} \in (L^1(\Lambda))^2$ denote

$$\mathcal{H}^*(\bar{u}, \bar{g}) = \sup_{\bar{V} \in (L^\infty(\Lambda))^2} [\langle \bar{V}, \bar{g} \rangle - \frac{1}{2} \Gamma_{\bar{V}}(\bar{u})], \quad (3.4)$$

where

$$\Gamma_{\bar{V}}(\bar{u}) = F_{\bar{V}}((p_1 \lambda, p_2 \lambda), (u_1 \lambda, u_2 \lambda)). \quad (3.5)$$

The function $\bar{g} \rightarrow \mathcal{H}^*(\bar{u}, \bar{g})$ is convex. Next lemma ensures that $\mathcal{H}^*(\bar{u}, \cdot)$ is the Fenchel-Legendre transform of $\Gamma_{(\cdot)}(\bar{u})$ when $\bar{u} \in B_{p_1, p_2}$, and we will derive in that case an explicit formula for $\mathcal{H}^*(\bar{u}, \bar{g})$.

Lemma 3.3 *As a function of $\bar{V} \in (L^\infty(\Lambda))^2$, $\Gamma_{\bar{V}}(\bar{u})$ is convex differentiable for $\bar{u} \in B_{p_1, p_2}$.*

Definition 3.4 *The dynamical rate functional $I_0 : D([0, T], \mathcal{M}_1 \times \mathcal{M}_1) \rightarrow \mathbb{R} \cup \{\infty\}$ is given by*

$$I_0(\bar{\pi}) = \begin{cases} I_0(\bar{\phi}) = \int_0^T \mathcal{H}^*(\bar{\phi}(s, \cdot), \dot{\bar{\phi}}(s, \cdot)) ds, & \text{for } \bar{\pi} = (\phi_1 \lambda, \phi_2 \lambda), \bar{\phi} = (\phi_1, \phi_2) \in \mathcal{AC}([0, T], B_{1,1}), \\ \infty & \text{otherwise.} \end{cases} \quad (3.6)$$

To derive properties of the large deviations functional it is convenient to have different representations of I_0 . To this aim let $\bar{V} = (V_1, V_2) \in (L^\infty([0, T] \times \Lambda))^2$. We define, for $\bar{\pi} \in D([0, T], \mathcal{M}_1 \times \mathcal{M}_1)$ (cf. (3.5)),

$$K_{\bar{V}}(\bar{\pi}) = \begin{cases} K_{\bar{V}}(\bar{\phi}), & \text{for } \bar{\pi} = (\phi_1 \lambda, \phi_2 \lambda), \bar{\phi} = (\phi_1, \phi_2) \in \mathcal{AC}([0, T], B_{1,1}), \\ \infty & \text{otherwise,} \end{cases} \quad (3.7)$$

where

$$\begin{aligned} K_{\bar{V}}(\bar{\phi}) &= \int_0^T \langle \bar{V}(s, \cdot), \dot{\bar{\phi}}(s, \cdot) \rangle ds - \frac{1}{2} \int_0^T \Gamma_{\bar{V}(s, \cdot)}(\bar{\phi}(s, \cdot)) ds, \\ J_0(\bar{\pi}) &= \sup_{\bar{V} \in (L^\infty([0, T] \times \Lambda))^2} K_{\bar{V}}(\bar{\pi}). \end{aligned} \quad (3.8)$$

$$J_1(\bar{\pi}) = \begin{cases} J_1(\bar{\phi}) = \int_0^T \int_{\Lambda} \mathcal{H}(\bar{\phi}(t, r), \dot{\bar{\phi}}(t, r)) dr dt, & \text{for } \bar{\pi} = (\phi_1 \lambda, \phi_2 \lambda), \bar{\phi} = (\phi_1, \phi_2) \in \mathcal{AC}([0, T], B_{1,1}), \\ \infty & \text{otherwise,} \end{cases} \quad (3.9)$$

where for $\bar{u} = (u_1, u_2) \in B_{1,1}$, $\bar{g} = (g_1, g_2) \in (L^1([0, T] \times \Lambda))^2$, $(t, r) \in [0, T] \times \Lambda$,

$$\mathcal{H}(\bar{u}, \bar{g})(t, r) = \mathcal{H}(\bar{u}(t, r), \bar{g}(t, r)) = \sum_{i=1}^2 H_i(\bar{u}, g_i)(t, r), \quad (3.10)$$

$$H_i(\bar{u}, g_i)(t, r) = \sup_{v_i \in \mathbb{R}} \left\{ g_i(t, r) v_i - \frac{1}{2} B_i(\bar{u}(t, r), v_i) \right\}, \quad i = 1, 2, \quad (3.11)$$

$$\begin{aligned} B_i(\bar{u}(t, r), v_i) = & (p_i - u_i(t, r)) \frac{e^{(J * u)(t, r) + a_i \theta}}{2 \cosh[(J * u)(t, r) + a_i \theta]} [e^{2v_i} - 1] \\ & + (p_i + u_i(t, r)) \frac{e^{-[(J * u)(t, r) + a_i \theta]}}{2 \cosh[(J * u)(t, r) + a_i \theta]} [e^{-2v_i} - 1]. \end{aligned} \quad (3.12)$$

When $\bar{u}(t, r) \in [-p_1, p_1] \times [-p_2, p_2]$, $\sum_{i=1}^2 B_i(\bar{u}(t, r), \cdot)$ is convex so that $\mathcal{H}(\bar{u}(t, r), \cdot)$ is its Fenchel-Legendre transform. We now give an explicit representation of $\mathcal{H}(\cdot, \cdot)$. To simplify notations denote

$$\begin{aligned} A_i &= A_i(u, \theta)(t, r) = (J * u)(t, r) + a_i \theta, \\ D_i &= D_i(\bar{u}, g_i, \theta)(t, r) = g_i(t, r) \cosh[A_i(u, \theta)(t, r)] + R_i(\bar{u}, g_i, \theta)(t, r), \\ R_i &= R_i(\bar{u}, g_i, \theta)(t, r) = \sqrt{(g_i(t, r) \cosh[A_i(u, \theta)(t, r)])^2 + p_i^2 - u_i^2(t, r)}. \end{aligned} \quad (3.13)$$

Note that $D_i(\bar{u}, g_i, \theta)(t, r) \geq 0$ regardless of the sign of $g_i(t, r)$. When (t, r) is kept fixed we omit to write it.

Proposition 3.5

(a) If $|u_1| > p_1$ or $|u_2| > p_2$, then $\mathcal{H}(\bar{u}, \bar{g}) = +\infty$.

(b) When $|u_i| < p_i$, then

$$H_i(\bar{u}, g_i) = \frac{g_i}{2} \left(\log \frac{D_i}{p_i - u_i} - A_i \right) + \frac{p_i}{2} - \frac{u_i}{2} \tanh A_i - \frac{R_i}{2 \cosh A_i}. \quad (3.14)$$

(c) When either $(u_i = p_i \text{ and } g_i \leq 0)$ or $(u_i = -p_i \text{ and } g_i \geq 0)$, then

$$H_i(\bar{u}, g_i) = \mathbb{1}_{\{g_i \neq 0\}} \frac{|g_i|}{2} \left(\log \left\{ \frac{|g_i| \cosh A_i}{p_i e^{-\text{sgn}(u_i) A_i}} \right\} - 1 \right) + p_i \frac{e^{-\text{sgn}(u_i) A_i}}{2 \cosh A_i}. \quad (3.15)$$

(d) When either $(u_i = p_i \text{ and } g_i > 0)$ or $(u_i = -p_i \text{ and } g_i < 0)$, then $H_i(\bar{u}, g_i) = +\infty$.

Proposition 3.6 For $\bar{\pi} = (\phi_1 \lambda, \phi_2 \lambda)$, $\bar{\phi} = (\phi_1, \phi_2) \in \mathcal{AC}([0, T], B_{1,1})$, we have $I_0(\bar{\pi}) = J_0(\bar{\pi}) = J_1(\bar{\pi})$. Furthermore if $\bar{\phi} \in \mathcal{AC}([0, T], B_{1,1}) \setminus \mathcal{AC}([0, T], B_{p_1, p_2})$, then $I_0(\bar{\phi}) = +\infty$.

Proposition 3.6 shows that the order of supremum and the integrals can be reversed. In particular we can compute the supremum for each point $(t, r) \in [0, T] \times \Lambda$. Next we characterize the finite energy trajectories.

Proposition 3.7 Take $(t, r) \in [0, T] \times \Lambda$.

(a) Let \bar{u} be such that for $i = 1, 2$, $|u_i| < p_i$. There exist positive constants K_1, K_2 and C such that

$$\begin{aligned} & \mathcal{H}(\bar{u}, \bar{g})(t, r) \\ & \leq \sum_{i=1,2} \frac{|g_i|}{2} \left[(\log |g_i|)^+ + \mathbb{1}_{\{g_i > 0\}} \left(\log \frac{1}{p_i - u_i} \right)^+ + \mathbb{1}_{\{g_i < 0\}} \left(\log \frac{1}{p_i + u_i} \right)^+ + K_i \right] (t, r) + C \end{aligned} \quad (3.16)$$

$$\mathcal{H}(\bar{u}, \bar{g})(t, r) \geq \sum_{i=1,2} \frac{|g_i|}{2} [\log |g_i| - K_i](t, r) - C. \quad (3.17)$$

(b) $I_0(\bar{\phi}) < \infty$ if and only if for $i = 1, 2$, $\dot{\phi}_i \log |\dot{\phi}_i|$, $\dot{\phi}_i \log \frac{1}{p_i - \phi_i} \mathbb{1}_{\{\dot{\phi}_i > 0\}}$, $\dot{\phi}_i \log \frac{1}{p_i + \phi_i} \mathbb{1}_{\{\dot{\phi}_i < 0\}}$ belong to $L^1([0, T] \times \Lambda)$.

Proposition 3.8 (1) The functional $I_0(\cdot)$ is lower semicontinuous on $\mathcal{C}([0, T], B_{1,1})$.

(2) The set $D_{L_0} = \{\bar{\pi}; I_0(\bar{\pi}) \leq L_0\}$ is compact in $\mathcal{C}([0, T], B_{1,1})$ for all $L_0 > 0$.

(3) $I_0(\bar{\phi}) \geq 0$, and $I_0(\bar{\phi}) = 0$ if and only if $\bar{\phi}$ is the solution to equation (2.24).

4. Hydrodynamic behavior for the colored system

Throughout the paper, one of the key ingredients to deal with the randomness of the interaction will be the following applications of the ergodic theorem and strong law of large numbers. For all function h on Λ_γ , integer l , we denote by $h^{(l)}$ the averaged function

$$\bar{h}^{(l)}(x) = \frac{1}{(2l+1)^d} \sum_{y \in \Lambda_\gamma, |y-x| \leq l} h(y), \quad x \in \Lambda_\gamma. \quad (4.1)$$

Lemma 4.1 (ergodic theorem for local functions) Let $\Theta(\alpha)$ be a bounded measurable cylinder function on Ω and $G \in \mathcal{C}(\Lambda)$. Then, for almost any disorder configuration α ,

$$\lim_{\gamma \rightarrow 0} \gamma^d \sum_{x \in \Lambda_\gamma} G(\gamma x) \tau_x \Theta(\alpha) = \mathbb{E}[\Theta] \int_{\Lambda} G(r) dr.$$

Proof: Write

$$\gamma^d \sum_{x \in \Lambda_\gamma} G(\gamma x) \tau_x \Theta(\alpha) = \gamma^d \sum_{x \in \Lambda_\gamma} G(\gamma x) [\tau_x \Theta(\alpha) - \mathbb{E}[\Theta]] + \mathbb{E}[\Theta] \gamma^d \sum_{x \in \Lambda_\gamma} G(\gamma x).$$

For any $l \in \mathbb{N}$, by the regularity of G ,

$$|\gamma^d \sum_{x \in \Lambda_\gamma} G(\gamma x) [\tau_x \Theta(\alpha) - \mathbb{E}[\Theta]]| \leq \|G\|_\infty \gamma^d \sum_{x \in \Lambda_\gamma} |(\tau_x \Theta(\alpha))^{(l)}(x) - \mathbb{E}[\Theta]| + \epsilon(\gamma l),$$

where $\lim_{s \rightarrow 0} \epsilon(s) = 0$. Keeping l fixed, by the ergodic theorem,

$$\lim_{\gamma \rightarrow 0} \gamma^d \sum_{x \in \Lambda_\gamma} |(\tau_x \Theta(\alpha))^{(l)}(x) - \mathbb{E}[\Theta]| = \mathbb{E}[|(\tau_x \Theta(\alpha))^{(l)}(x) - \mathbb{E}[\Theta]|].$$

The law of large numbers (letting $l \rightarrow \infty$) gives the result. \square

We introduce (cf. [K]), for $i = 1, 2$, $\delta > 0$,

$$A_{l,\delta}(x, i) = \left\{ \alpha \in \Omega : \left| \alpha_i^{(l)}(x) - \mathbb{E}(\alpha_i(x)) \right| \leq \delta \right\}, \quad x \in \Lambda_\gamma, \quad (4.2)$$

$$\mathcal{E}_i(\delta, l, \gamma, \alpha) = \gamma^d \sum_{x \in \Lambda_\gamma} \mathbb{1}_{A_{l,\delta}^c(x, i)}(\alpha). \quad (4.3)$$

Lemma 4.2 For any $\delta > 0$, for $i = 1, 2$, $\lim_{l \rightarrow \infty} \lim_{\gamma \rightarrow 0} \mathcal{E}_i(\delta, l, \gamma, \alpha) = 0$, \mathbb{P} - a.s.

Proof: Applying Lemma 4.1 to the function $\Theta = \mathbb{1}_{A_{l,\delta}^c(0,i)}$ gives $\lim_{\gamma \rightarrow 0} \mathcal{E}_i(\delta, l, \gamma, \alpha) = \mathbb{P}(A_{l,\delta}^c(0,i))$, \mathbb{P} - a.s. Then by the strong law of large numbers $\lim_{l \rightarrow \infty} \mathbb{P}(A_{l,\delta}^c(0,i)) = 0$. \square

In the following it is convenient to define the random discrete measures $\bar{\lambda}^\gamma(\alpha) = (\lambda_1^\gamma(\alpha), \lambda_2^\gamma(\alpha))$, where

$$\lambda_i^\gamma(\alpha) = \gamma^d \sum_{x \in \Lambda_\gamma} \alpha_i(x) \delta_{\gamma x}, \quad i = 1, 2, \quad \lambda^\gamma = \lambda_1^\gamma(\alpha) + \lambda_2^\gamma(\alpha) = \gamma^d \sum_{x \in \Lambda_\gamma} \delta_{\gamma x}. \quad (4.4)$$

Proof of Theorem 2.5 We follow the general scheme introduced in [KL] chap. 4. We have to show:

- (i) For any α , the sequence $(\bar{Q}_{\sigma^\gamma}^{\gamma,\alpha})_\gamma$ is tight.
- (ii) Any limit point \bar{Q}^α of $(\bar{Q}_{\sigma^\gamma}^{\gamma,\alpha})_\gamma$ is \mathbb{P} -a.s. concentrated on measures $(\bar{\pi}_t)_{t \in [0,T]} \in \mathcal{C}([0,T], \mathcal{M}_{p_1}^{ac} \times \mathcal{M}_{p_2}^{ac})$.
- (iii) For \mathbb{P} -a.s. α , any limit point \bar{Q}^α of $(\bar{Q}_{\sigma^\gamma}^{\gamma,\alpha})_\gamma$ is concentrated on trajectories $(\bar{\pi}_t)_{t \in [0,T]}$ such that $\bar{\pi}_t(dr) = \bar{m}(t,r)dr$, where the density \bar{m} is a weak solution of (2.24).
- (iv) Equation (2.24) has a unique weak solution.

For (ii) we use that the spins are finite-valued (cf. [KL]). Namely, fix $G \in \mathcal{C}(\Lambda)$,

$$\sup_{0 \leq t \leq T} |\langle \pi_{i,t}^\gamma, G \rangle| \leq \gamma^d \sum_{x \in \Lambda_\gamma} |G(\gamma x)| \alpha_i(x), \quad i = 1, 2,$$

because there is at the most one spin per site and $\alpha_i(x) \geq 0$. As in the case without random field, the application that associates to a trajectory $(\pi_{i,t})_{t \in [0,T]}$ the value $\sup_{t \in [0,T]} \langle \pi_{i,t}, G \rangle$ is continuous in the weak topology. Thus by weak convergence and Lemma 4.1 (by the independence of the r.v. α 's) all limits points are concentrated on $(\pi_{i,t})_{t \in [0,T]}$ such that

$$|\langle \pi_{i,t}, G \rangle| \leq \int_\Lambda |G(r)| p_i dr, \quad \mathbb{P} - a.s.$$

Point (iv) is derived similarly to the proof of the Cauchy-Lipschitz theorem. For Points (i) and (iii), let $\bar{G} = (G_1, G_2) \in (\mathcal{C}^{1,0}([0,T] \times \Lambda))^2$. For $\bar{\pi} \in D([0,T], \mathcal{M}_1 \times \mathcal{M}_1)$, let

$$\ell_t(\bar{\pi}, \bar{G}) = \langle \bar{\pi}_t, \bar{G}(t, \cdot) \rangle - \langle \bar{\pi}_0, \bar{G}(0, \cdot) \rangle - \int_0^t \langle \bar{\pi}_s, \partial_s \bar{G}(s, \cdot) \rangle ds. \quad (4.5)$$

We have, for $x \in \Lambda_\gamma$,

$$\begin{aligned} \mathcal{L}^{\gamma,\alpha}(\sigma(x)) &= -\sigma(x) + \sigma(x)(1 - 2c_x^{\gamma,\alpha}(\sigma)) \\ &= -\sigma(x) + \tanh[(J_\gamma \star \sigma)(x) + \theta\alpha(x)]. \end{aligned} \quad (4.6)$$

The $P_{\sigma^\gamma}^{\gamma,\alpha}$ -martingale $\bar{N}_\gamma^{\bar{G}} \equiv (\bar{N}_\gamma^{\bar{G}}(t))_{t \in [0,T]}$ with respect to the natural filtration associated to $(\sigma_t)_{t \in [0,T]}$ (cf. (2.20)) given by

$$\begin{aligned} \bar{N}_\gamma^{\bar{G}}(t) &= \ell_t(\bar{\pi}^\gamma, \bar{G}) - \gamma^d \int_0^t \sum_{i=1,2} \sum_{x \in \Lambda_\gamma} G_i(s, \gamma x) \alpha_i(x) \mathcal{L}^{\gamma,\alpha}(\sigma_s(x)) ds \\ &= \ell_t(\bar{\pi}^\gamma, \bar{G}) + \int_0^t \langle \bar{\pi}_s^\gamma, \bar{G}(s, \cdot) \rangle ds - \sum_{i=1,2} \int_0^t \langle \lambda_i^\gamma(\alpha), G_i(s, \cdot) \tanh[\pi_s^\gamma \star J + a_i \theta] \rangle ds, \end{aligned} \quad (4.7)$$

has quadratic variation

$$\langle \overline{N}_\gamma, \overline{N}_\gamma \rangle (t) = -2\gamma^{2d} \sum_{i=1,2} \sum_{x \in \Lambda_\gamma} \alpha_i(x) G_i^2(s, \gamma x) \int_0^t \{ -1 + \sigma_s(x) \tanh[(J_\gamma \star \sigma_s)(x) + a_i \theta] \} ds. \quad (4.8)$$

Hence, for any $\alpha \in \Omega$, since \tanh is a smooth function and $(J_\gamma \star \sigma)(x) + a_i \theta$ is uniformly bounded in x , σ , by Doob's inequality,

$$\lim_{\gamma \rightarrow 0} P_{\sigma^\gamma}^{\gamma, \alpha} \left(\sup_{t \in [0, T]} |\overline{N}_\gamma(t)| > \delta \right) \leq \lim_{\gamma \rightarrow 0} \frac{1}{\delta^2} E_{\sigma^\gamma}^{\gamma, \alpha} \left(|\overline{N}_\gamma(T)|^2 \right) \leq \lim_{\gamma \rightarrow 0} \frac{1}{\delta^2} C(\overline{G}, T) \gamma^d = 0. \quad (4.9)$$

Bound (4.9) enables to conclude point (i), by Prohorov's criterion. Point (iii) will consist in identification of the limit. To obtain a closed form for the limiting equation, we only need to average over the disorder, that is to replace in the limit $\gamma \rightarrow 0$ the random discrete measures $\lambda_1^\gamma(\alpha)$ and $\lambda_2^\gamma(\alpha)$ by their expectations $p_1 \lambda$ and $p_2 \lambda$ with respect to the environment. Denote

$$\tilde{\ell}_t(\overline{\pi}^\gamma, \overline{G}) = \int_0^t \left\{ \langle \overline{\pi}_s^\gamma, \overline{G}(s, \cdot) \rangle - \sum_{i=1}^2 \langle p_i \lambda^\gamma, G_i(s, \cdot) \tanh(\pi_s^\gamma * J + a_i \theta) \rangle \right\} ds. \quad (4.10)$$

Putting together (4.7), (4.9), and applying Lemma 4.3 (stated below) we get that for all subsequences

$$\liminf_{k \rightarrow +\infty} Q_{\sigma^{\gamma_k}}^{\gamma_k, \alpha} \left(\sup_{t \leq T} \left| \ell_t(\overline{\pi}^{\gamma_k}, \overline{G}) + \tilde{\ell}_t(\overline{\pi}^{\gamma_k}, \overline{G}) \right| > \frac{\delta}{2} \right) = 0. \quad (4.11)$$

Denoting $m = m_1 + m_2$, for $\overline{m} = (m_1, m_2)$, see (2.24), this gives, for almost any α ,

$$\begin{aligned} & Q^\alpha \left(\sup_{t \leq T} \left| \sum_{i=1}^2 \int_\Lambda \left[\{ G_i(t, r) m_i(t, r) - G_i(0, r) m_i(0, r) \} - \int_0^t \partial_s G_i(s, r) m_i(s, r) ds \right] dr \right. \right. \\ & \left. \left. + \sum_{i=1,2} \int_0^t \int_\Lambda \left[G_i(s, r) m_i(s, r) - \mathbb{E}(\alpha_i(0)) \tanh[(J * m)(s, r) + a_i \theta] \right] dr ds \right| > \frac{\delta}{2} \right) \\ & = Q^\alpha \left(\sup_{t \leq T} \left| \ell_t(\overline{m}, \overline{G}) + \tilde{\ell}_t(\overline{m}, \overline{G}) \right| > \frac{\delta}{2} \right) = 0, \end{aligned} \quad (4.12)$$

where we set by an abuse of notation

$$\ell_t(\overline{m}, \overline{G}) = \langle \overline{m}(t, \cdot), \overline{G}(t, \cdot) \rangle - \langle \overline{m}(0, \cdot), \overline{G}(0, \cdot) \rangle - \int_0^t \langle \overline{m}(s, \cdot), \partial_s \overline{G}(s, \cdot) \rangle ds, \quad (4.13)$$

$$\tilde{\ell}_t(\overline{m}, \overline{G}) = \int_0^t \left\{ \langle \overline{m}(s, \cdot), \overline{G}(s, \cdot) \rangle - \sum_{i=1}^2 \langle p_i \lambda, G_i(s, \cdot) \tanh(J * m(s, \cdot) + a_i \theta) \rangle \right\} ds. \quad (4.14)$$

This leads to identification of the limit (iii), that is to equation (2.24). \square

Lemma 4.3 For $i = 1, 2$, $G_i \in C^{1,0}([0, T] \times \Lambda)$, there exists a positive function ϵ on \mathbb{R}_+ with $\lim_{s \rightarrow 0} \epsilon(s) = 0$ such that for all $l \in \mathbb{N} \setminus \{0\}$, $\delta > 0$, and $\mathcal{E}_i(\delta, l, \gamma, \alpha)$ defined in (4.3), the quantity

$$\Delta_i^\gamma(\alpha, \sigma, T) = \int_0^T \left| \langle \lambda_i^\gamma(\alpha) - p_i \lambda, G_i(s, \cdot) \tanh[\pi_s^\gamma * J + a_i \theta] \rangle \right| ds \quad (4.15)$$

satisfies

$$\Delta_i^\gamma(\alpha, \sigma, T) \leq \frac{\delta}{2} T \|G_i(s, \cdot)\|_1 + 2T \|G_i(s, \cdot)\|_\infty \mathcal{E}_i(\delta, l, \gamma, \alpha) + \epsilon(\gamma l)T + \epsilon(\gamma)T, \quad (4.16)$$

$$\lim_{\gamma \rightarrow 0} \Delta_i^\gamma(\alpha, \sigma, T) = 0, \quad \mathbb{P} - a.s. \quad (4.17)$$

Proof: We introduce averages over large microscopic boxes of size l but small w.r.t. the range γ^{-1} of the interaction (l will go to infinity but after the limit $\gamma \rightarrow 0$). To keep notation readable, the function ϵ may vary from one line to another but keeping the same property that $\lim_{s \rightarrow 0} \epsilon(s) = 0$. Since J and \tanh are uniformly Lipschitz, and G_i is uniformly continuous (in space), there are constants $c_1 > 0$, $c_2 > 0$ such that, see (4.1), for $x \in \Lambda_\gamma$,

$$\begin{aligned} \sup_{\sigma \in \mathcal{S}_\gamma} \left| (\pi^\gamma(\sigma) * J)(\gamma x) - ((\pi^\gamma(\sigma) * J)(\gamma \cdot))^{(l)}(x) \right| &\leq c_1 \gamma l, \\ \sup_{\sigma \in \mathcal{S}_\gamma} \left| \tanh[\beta((\pi^\gamma(\sigma) * J)(\gamma x) + a_i \theta)] - (\tanh[(\pi^\gamma(\sigma) * J)(\gamma \cdot) + a_i \theta])^{(l)}(x) \right| &\leq c_2 \gamma l, \\ \sup_{0 \leq s \leq T} \left| G_i(s, \gamma x) - (G_i(s, \gamma \cdot))^{(l)}(x) \right| &\leq \epsilon(\gamma l). \end{aligned}$$

Recalling notation (4.4), by summation by parts we get

$$\int_0^T \left| \langle \lambda_i^\gamma(\alpha) - \lambda_i^\gamma(\alpha^{(l)}), G_i(s, \cdot) \tanh[\pi_s^\gamma * J + a_i \theta] \rangle \right| ds \leq \epsilon(\gamma l)T$$

and by uniform continuity or Lipschitz condition

$$\int_0^T \left| \langle p_i \lambda^\gamma - p_i \lambda, G_i(s, \cdot) \tanh[\pi_s^\gamma * J + a_i \theta] \rangle \right| ds \leq \epsilon(\gamma)T.$$

Therefore, we have

$$\Delta_i^\gamma(\alpha, \sigma, T) \leq \epsilon(\gamma l)T + \epsilon(\gamma)T + \int_0^T \left| \langle \lambda_i^\gamma(\alpha^{(l)}) - p_i \lambda^\gamma, G_i(s, \cdot) \tanh[\pi_s^\gamma * J + a_i \theta] \rangle \right| ds.$$

To derive (4.16), we take into account definitions (4.2), (4.3), and that $|\alpha_i^{(l)}(x) - p_i| \leq 2$, to write

$$\begin{aligned} & \left| \langle \lambda_i^\gamma(\alpha^{(l)}) - p_i \lambda^\gamma, G_i(s, \cdot) \tanh[\pi_s^\gamma * J + a_i \theta] \rangle \right| = \\ & \left| \gamma^d \sum_{x \in \Lambda_\gamma} G_i(s, \gamma x) \tanh[(J_\gamma * \sigma_s)(x) + a_i \theta] \left[\mathbb{1}_{A_{l,\delta}(x,i)}(\alpha) + \mathbb{1}_{A_{l,\delta}^c(x,i)}(\alpha) \right] \left(\alpha_i^{(l)}(x) - \mathbb{E}(\alpha_i(x)) \right) \right| \\ & \leq \gamma^d \sum_{x \in \Lambda_\gamma} |G_i(s, \gamma x) \tanh[(J_\gamma * \sigma_s)(x) + a_i \theta]| \left[\delta + 2 \mathbb{1}_{A_{l,\delta}^c(x,i)}(\alpha) \right] \\ & \leq \delta \|G_i(s, \cdot)\|_1 + 2 \|G_i(s, \cdot)\|_\infty \mathcal{E}_i(\delta, l, \gamma, \alpha). \end{aligned} \quad (4.18)$$

Applying Lemma 4.2 to (4.16) we get (4.17). \square

Proof of Corollary 2.7: First notice that applying Lebesgue dominated convergence theorem in the time integral, Theorem 2.5 implies that for any $G_i \in \mathcal{C}^{0,1}([0, T] \times \Lambda)$ we have

$$\lim_{\gamma \rightarrow 0} P_{\sigma_\gamma}^{\gamma, \alpha} \left[\int_0^T \left| \langle \pi_{i,s}^\gamma, G_i(s, \cdot) \rangle - \langle m_i(s, \cdot), G_i(s, \cdot) \rangle \right| ds \geq \delta \right] = 0. \quad (4.19)$$

Now remark that integrating in time (2.24),

$$\begin{aligned} & \langle m_i(t, \cdot), G_i(t, \cdot) \rangle = \langle m_i(0, \cdot), G_i(0, \cdot) \rangle + \int_0^t \langle m_i(s, \cdot), \partial_s G_i(s, \cdot) - G_i(s, \cdot) \rangle ds \\ & + p_i \int_0^t \langle \tanh[(J * m)(s, \cdot) + a_i \theta], G_i(s, \cdot) \rangle ds. \end{aligned}$$

Introducing the martingale \overline{N}_γ^G , see (4.7), and using (4.9) we get

$$P_{\sigma_\gamma}^{\gamma, \alpha} \left[\sup_{t \in [0, T]} |\langle \pi_{i, t}^\gamma, G_i(t, \cdot) \rangle - \langle m_i(t, \cdot), G_i(t, \cdot) \rangle| \geq \delta \right] \leq A_\gamma + B_\gamma + C_\gamma + \epsilon(\gamma),$$

with $\lim_{\gamma \rightarrow 0} \epsilon(\gamma) = 0$ and

$$\begin{aligned} A_\gamma &= P_{\sigma_\gamma}^{\gamma, \alpha} \left[\int_0^T |\langle \pi_{i, s}^\gamma - m_i(s, \cdot), \partial_s G_i(s, \cdot) - G_i(s, \cdot) \rangle| ds \geq \frac{\delta}{4} \right], \\ C_\gamma &= P_{\sigma_\gamma}^{\gamma, \alpha} \left[|\langle \pi_{i, 0}^\gamma - m_i(0, \cdot), G_i(0, \cdot) \rangle| \geq \frac{\delta}{4} \right], \end{aligned}$$

$$\begin{aligned} B_\gamma &= P_{\sigma_\gamma}^{\gamma, \alpha} \left[\int_0^T |\langle \lambda_i^\gamma(\alpha), G_i(s, \cdot) \tanh[\pi_s^\gamma * J + a_i \theta] \rangle - \langle p_i \lambda, G_i(s, \cdot) \tanh[(J * m)(s, \cdot) + a_i \theta] \rangle| ds \geq \frac{\delta}{4} \right] \\ &\leq P_{\sigma_\gamma}^{\gamma, \alpha} \left[\int_0^T |\langle \lambda_i^\gamma(\alpha) - p_i \lambda, G_i(s, \cdot) \tanh[\pi_s^\gamma * J + a_i \theta] \rangle| ds \geq \frac{\delta}{8} \right] \\ &\quad + P_{\sigma_\gamma}^{\gamma, \alpha} \left[\int_0^T |\langle p_i \lambda, G_i(s, \cdot) \tanh[\pi_s^\gamma * J + a_i \theta] - \tanh[(J * m)(s, \cdot) + a_i \theta] \rangle| ds \geq \frac{\delta}{8} \right]. \end{aligned} \tag{4.20}$$

From (2.13) and (4.19), $\lim_{\gamma \rightarrow 0} A_\gamma = \lim_{\gamma \rightarrow 0} C_\gamma = 0$. For B_γ , from Lemma 4.3, the limit when $\gamma \rightarrow 0$ of the first term in the right hand side of (4.20) is equal to zero; the second term vanishes from (4.19) since the function \tanh is Lipschitz continuous. \square

5 The perturbed dynamics and Radon-Nikodym Derivative

The general strategy to derive the large deviations principle prescribes to find a family of mean one positive martingales that can be expressed as functions of the empirical measures. Following [DV], the relevant positive martingales are obtained as Markovian perturbations of the original process. In this section we define a class of time dependent, random external potentials, *the perturbations*, to which we can associate a trajectory $m(t, \cdot)$, $t \in [0, T]$ smooth in time. We show the law of large numbers for the empirical measures of the dynamics associated to these smooth perturbations and derive the Radon-Nikodym derivative of the perturbed process with respect to the unperturbed one.

Given a realization α of the magnetic field, $\overline{V} = (V_1, V_2) \in (\mathcal{C}^{1,0}([0, T], \Lambda))^2$, let

$$V(t, \gamma x, \alpha(x)) = \sum_{i=1,2} \alpha_i(x) V_i(t, \gamma x) \tag{5.1}$$

be the full external random perturbation for the magnetization trajectories $\pi^\gamma(\sigma)$ (not colored). As pointed out in the introduction this perturbation strongly depends on the randomness. It is therefore convenient to consider a Glauber evolution for the colored particle system, associated to the Hamiltonian obtained by summing up (2.4) and

$$H^{\bar{V},\gamma,\alpha}(\sigma) = - \sum_{x \in \Lambda_\gamma} \sum_{i=1,2} \alpha_i(x) V_i(t, \gamma x) \sigma(x). \quad (5.2)$$

To this aim we define time dependent rates, for all $x \in \Lambda_\gamma$, $\sigma \in \mathcal{S}_\gamma$,

$$c_x^{\bar{V},\gamma,\alpha}(\sigma, t) = e^{-\sigma(x)2V(t,\gamma x,\alpha(x))} c_x^{\gamma,\alpha}(\sigma) = \frac{e^{-\sigma(x)[(J_\gamma \star \sigma)(x) + \theta\alpha(x) + 2V(t,\gamma x,\alpha(x))]}{2 \cosh[(J_\gamma \star \sigma)(x) + \theta\alpha(x)]}. \quad (5.3)$$

Assume $(\sigma^\gamma)_\gamma, m_0$ satisfy (2.13). We denote by $P_{\sigma^\gamma}^{\bar{V},\gamma,\alpha}$ the law (and by $E_{\sigma^\gamma}^{\bar{V},\gamma,\alpha}$ the expectation) of the corresponding inhomogeneous Glauber process $(\sigma_t)_{t \in [0, T]}$ on \mathcal{S}_γ , that is the unique probability measure on $D([0, T], \mathcal{S}_\gamma)$ with initial condition σ^γ under which $f(\sigma_t) - f(\sigma_0) - \int_0^t \mathcal{L}_s^{\bar{V},\gamma,\alpha}(f)(\sigma_s) ds$ is a martingale w.r.t. the canonical filtration, for all cylinder function f , where

$$\mathcal{L}_s^{\bar{V},\gamma,\alpha}(f)(\sigma_s) = \sum_{x \in \Lambda_\gamma} c_x^{\bar{V},\gamma,\alpha}(\sigma_s, s) [f(\sigma_s^x) - f(\sigma_s)]. \quad (5.4)$$

Let $\bar{Q}_{\sigma^\gamma}^{\bar{V},\gamma,\alpha}$ be the law of the corresponding empirical measures.

Theorem 5.1 *Assume $(\sigma^\gamma)_\gamma, m_0$ satisfy (2.13). For all $t \in [0, T]$, $\bar{G} = (G_1, G_2) \in (\mathcal{C}^1(\Lambda))^2$, and $\delta > 0$,*

$$\lim_{\gamma \rightarrow 0} \bar{Q}_{\sigma^\gamma}^{\bar{V},\gamma,\alpha} \left[\left| \langle \bar{\pi}_t^\gamma, \bar{G} \rangle - \langle \bar{m}^{\bar{V}}(t, \cdot), \bar{G} \rangle \right| \geq \delta \right] = 0, \quad \mathbb{P} - a.s.,$$

where $\bar{m}^{\bar{V}} = (m_1^{\bar{V}}, m_2^{\bar{V}})$ is the solution to, for $i = 1, 2$,

$$\begin{cases} \partial_t m_i(t, r) = \{-m_i(t, r) + p_i \tanh[(J * m)(t, r) + a_i \theta + 2V_i(t, r)]\} \frac{\cosh[(J * m)(t, r) + a_i \theta + 2V_i(t, r)]}{\cosh[(J * m)(t, r) + a_i \theta]}, \\ m_i(0, \cdot) = p_i m_0(\cdot), \quad m = m_1 + m_2. \end{cases} \quad (5.5)$$

Remark 5.2 . For existence and uniqueness of the solution $\bar{m}^{\bar{V}} \in (\mathcal{C}([0, T], L^\infty(\Lambda)))^2$, we refer to Remark 2.6. Notice that the set $\{\bar{m} \in (L^\infty(\Lambda))^2 : \|m_i\|_\infty \leq p_i\}$ is still left invariant.

Proof: We proceed as for Theorem 2.5. We use

$$1 = \frac{(1 - \sigma_s(x))}{2} + \frac{(1 + \sigma_s(x))}{2} = \mathbb{1}_{\{\sigma_s(x)=-1\}} + \mathbb{1}_{\{\sigma_s(x)=1\}}. \quad (5.6)$$

For $i \in \{1, 2\}$ we have

$$\begin{aligned} \mathcal{L}_s^{\bar{V},\gamma,\alpha}(\alpha_i(x)\sigma(x)) &= -2\alpha_i(x)\sigma(x)c_x^{\bar{V},\gamma,\alpha}(\sigma, s) \\ &= -\alpha_i(x)e^{-\sigma(x)2V_i(s,\gamma x)}2\sigma(x)\frac{e^{-\sigma(x)[(J_\gamma \star \sigma)(x) + a_i \theta]}}{2 \cosh[(J_\gamma \star \sigma)(x) + a_i \theta]} \\ &= -\alpha_i(x) \left[\frac{(\sigma(x) + 1)}{2} \frac{e^{-[2V_i(s,\gamma x) + (J_\gamma \star \sigma)(x) + a_i \theta]}}{\cosh[(J_\gamma \star \sigma)(x) + a_i \theta]} + \frac{(\sigma(x) - 1)}{2} \frac{e^{[2V_i(s,\gamma x) + (J_\gamma \star \sigma)(x) + a_i \theta]}}{\cosh[(J_\gamma \star \sigma)(x) + a_i \theta]} \right] \\ &= -\alpha_i(x)\sigma(x)\frac{\cosh[2V_i(s,\gamma x) + (J_\gamma \star \sigma)(x) + a_i \theta]}{\cosh[(J_\gamma \star \sigma)(x) + a_i \theta]} - \alpha_i(x)\frac{\sinh[2V_i(s,\gamma x) + (J_\gamma \star \sigma)(x) + a_i \theta]}{\cosh[(J_\gamma \star \sigma)(x) + a_i \theta]}. \end{aligned} \quad (5.7)$$

□

We have the analogous result to Corollary 2.7:

Corollary 5.3 For all $\bar{G} = (G_1, G_2) \in (\mathcal{C}^1(\Lambda))^2$, and $\delta > 0$,

$$\lim_{\gamma \rightarrow 0} \bar{Q}_{\sigma^\gamma}^{\bar{V}, \gamma, \alpha} \left[\sup_{t \in [0, T]} \left| \langle \bar{\pi}_t^\gamma, \bar{G} \rangle - \langle \bar{m}^{\bar{V}}(t, \cdot), \bar{G} \rangle \right| \geq \delta \right] = 0. \quad (5.8)$$

Theorem 5.4 Let $\bar{V} = (V_1, V_2) \in (\mathcal{C}^{1,0}([0, T] \times \Lambda))^2$. The Radon-Nikodym derivative is given by

$$\frac{dP_{\sigma^\gamma}^{\bar{V}, \gamma, \alpha}}{dP_{\sigma^\gamma}^{\gamma, \alpha}}(\sigma_{[0, T]}) = \exp \left\{ \gamma^{-d} \left(\ell_T(\bar{\pi}^\gamma(\sigma), \bar{V}) - \frac{1}{2} \int_0^T F_{\bar{V}(s)}(\bar{\lambda}^\gamma(\alpha), \bar{\pi}_s^\gamma) ds \right) \right\}, \quad (5.9)$$

where ℓ_T was defined in (4.5), $\bar{\lambda}^\gamma(\alpha)$ in (4.4), $F_{\bar{V}(s)}(\cdot, \cdot)$ in (3.3), and we have abbreviated $\sigma_{[0, T]} = (\sigma_t)_{t \in [0, T]}$.

Proof: The Radon-Nikodym derivative associated with rates (5.3) is given by (see [HS] or [KL], Appendix 1, Proposition 7.3)

$$\begin{aligned} \frac{dP_{\sigma^\gamma}^{\bar{V}, \gamma, \alpha}}{dP_{\sigma^\gamma}^{\gamma, \alpha}}(\sigma_{[0, t]}) &= \exp \left\{ -H^{\bar{V}, \gamma, \alpha}(\sigma_t) + H^{\bar{V}, \gamma, \alpha}(\sigma_0) \right. \\ &\quad \left. - \int_0^t \exp \left\{ H^{\bar{V}, \gamma, \alpha}(\sigma_s) \right\} (\partial_s + \mathcal{L}_\gamma) \exp \left\{ -H^{\bar{V}, \gamma, \alpha}(\sigma_s) \right\} ds \right\} \\ &= \exp \left\{ \ell_t(\bar{\pi}^\gamma(\sigma), \bar{V}) - \int_0^t \sum_{x \in \Lambda_\gamma} c_x^{\gamma, \alpha}(\sigma_s) \left[e^{-2\sigma_s(x)V(s, \gamma x, \alpha(x))} - 1 \right] ds \right\}, \end{aligned}$$

because of (4.5), (5.2). To get (5.9), we use trigonometric formulas to write (remember (2.8), (2.20), (5.7))

$$\begin{aligned} &2\gamma^d \sum_{x \in \Lambda_\gamma} c_x^{\gamma, \alpha}(\sigma_s) \left[e^{-2\sigma_s(x)V(s, \gamma x, \alpha(x))} - 1 \right] \\ &= 2\gamma^d \sum_{x \in \Lambda_\gamma} \left\{ \frac{(1 - \sigma_s(x))}{2} + \frac{(1 + \sigma_s(x))}{2} \right\} \left\{ \sum_{i=1,2} \alpha_i(x) \right\} c_x^{\gamma, \alpha}(\sigma_s) \left[e^{-2\sigma_s(x)V(s, \gamma x, \alpha(x))} - 1 \right] \\ &= \gamma^d \sum_{i=1,2} \sum_{x \in \Lambda_\gamma} \frac{(1 - \sigma_s(x))}{2} \alpha_i(x) \frac{\exp[(J_\gamma \star \sigma_s)(x) + a_i \theta]}{\cosh[(J_\gamma \star \sigma_s)(x) + a_i \theta]} \left[e^{2V_i(s, \gamma x)} - 1 \right] \\ &\quad + \gamma^d \sum_{i=1,2} \sum_{x \in \Lambda_\gamma} \frac{(1 + \sigma_s(x))}{2} \alpha_i(x) \frac{\exp[-(J_\gamma \star \sigma_s)(x) - a_i \theta]}{\cosh[(J_\gamma \star \sigma_s)(x) + a_i \theta]} \left[e^{-2V_i(s, \gamma x)} - 1 \right] \\ &= \gamma^d \sum_{i=1,2} \sum_{x \in \Lambda_\gamma} \alpha_i(x) \{ \cosh[2V_i(s, \gamma x)] - 1 + \tanh[(J_\gamma \star \sigma_s)(x) + a_i \theta] \sinh[2V_i(s, \gamma x)] \} \\ &\quad - \gamma^d \sum_{i=1,2} \sum_{x \in \Lambda_\gamma} \alpha_i(x) \sigma_s(x) \{ \tanh[(J_\gamma \star \sigma_s)(x) + a_i \theta] (\cosh[2V_i(s, \gamma x)] - 1) + \sinh[2V_i(s, \gamma x)] \} \\ &= F_{\bar{V}(s)}(\bar{\lambda}^\gamma(\alpha), \bar{\pi}_s^\gamma). \end{aligned}$$

□

Note that the Radon-Nikodym derivative depends on the randomness through $\bar{\pi}^\gamma$ and $\bar{\lambda}^\gamma(\alpha)$. By next proposition, we can replace $\bar{\lambda}^\gamma(\alpha)$ in $F_{\bar{V}(s)}(\cdot, \bar{\pi}_s^\gamma)$ with $(p_1\lambda^\gamma, p_2\lambda^\gamma)$, making an error which goes uniformly (for all $\sigma \in \mathcal{S}_\gamma$ and \mathbb{P} -a.s.) to zero as $\gamma \rightarrow 0$.

Proposition 5.5 *Let $\bar{V} = (V_1, V_2) \in (\mathcal{C}^{1,0}([0, T] \times \Lambda))^2$. There exists a positive function ϵ on \mathbb{R}_+ with $\lim_{s \rightarrow 0} \epsilon(s) = 0$ such that for any $\delta > 0$, $l \in \mathbb{N} \setminus \{0\}$, we have*

$$\left| \int_0^T \left[F_{\bar{V}(s)}(\bar{\lambda}^\gamma(\alpha), \bar{\pi}_s^\gamma) - F_{\bar{V}(s)}((p_1\lambda^\gamma, p_2\lambda^\gamma), \bar{\pi}_s^\gamma) \right] ds \right| \leq \epsilon(\gamma l)T + TC(V_1, V_2)[\delta + \sum_{i=1,2} \mathcal{E}_i(\delta, l, \gamma, \alpha)]$$

where the positive constant $C(V_1, V_2)$ depends on the L^∞ norm of (V_1, V_2) .

Proof: Let $s \in [0, T]$, $C(V_1, V_2) = \sum_{i=1,2} C(V_i) = \sum_{i=1,2} \sup_{s \in [0, T]} \sup_{r \in \Lambda} (\sinh^2[V_i(s, r)] + |\sinh[2V_i(s, r)]|)$, then

$$\left| F_{\bar{V}(s)}(\bar{\lambda}^\gamma(\alpha), \bar{\pi}_s^\gamma) - F_{\bar{V}(s)}((p_1\lambda^\gamma, p_2\lambda^\gamma), \bar{\pi}_s^\gamma) \right| \leq \left| \frac{\gamma^d}{2} \sum_{x \in \Lambda_\gamma} [\alpha_i(x) - p_i] \mathcal{B}_i(x, \sigma, s) \right|,$$

with

$$\begin{aligned} \mathcal{B}_i(x, \sigma, s) &= \cosh[2V_i(s, \gamma x)] - 1 + \tanh[(J_\gamma \star \sigma_s)(x) + a_i \theta] \sinh[2V_i(s, \gamma x)] \\ |\mathcal{B}_i(x, \sigma, s)| &\leq \sinh^2[V_i(s, \gamma x)] + |\sinh[2V_i(s, \gamma x)]| \leq C(V_i). \end{aligned} \quad (5.10)$$

Take $l \in \mathbb{Z}$, $l \neq 0$. Since $\mathbb{E}[\alpha_i(x)] = p_i$ for all $x \in \Lambda_\gamma$,

$$\begin{aligned} \frac{\gamma^d}{2} \sum_{x \in \Lambda_\gamma} [\alpha_i(x) - p_i] \mathcal{B}_i(x, \sigma, s) &= \frac{\gamma^d}{2} \sum_{x \in \Lambda_\gamma} \frac{1}{(2l+1)^d} \sum_{|y| \leq l} [\mathcal{B}_i(x+y, \sigma, s) - \mathcal{B}_i(x, \sigma, s)] \alpha_i(x+y) \\ &\quad - \frac{\gamma^d}{2} \sum_{x \in \Lambda_\gamma} \mathcal{B}_i(x, \sigma, s) [\alpha_i^{(l)}(x) - \mathbb{E}[\alpha_i(x)]]. \end{aligned} \quad (5.11)$$

Using uniform continuity as in the proof of Lemma 4.3, there exists a positive function ϵ on \mathbb{R}_+ with $\lim_{s \rightarrow 0} \epsilon(s) = 0$ (depending only on T , J and \bar{V}) such that the first term on the r.h.s. of (5.11) is bounded uniformly in α and σ ; for the second term, let $\delta > 0$ and $\mathcal{E}_i(\delta, l, \gamma, \alpha)$ defined in (4.3). We conclude by

$$\left| \frac{\gamma^d}{2} \sum_{x \in \Lambda_\gamma} [\alpha_i(x) - p_i] \mathcal{B}_i(x, \sigma, s) \right| \leq \epsilon(\gamma l) + C(V_i) [\delta + \mathcal{E}_i(\delta, l, \gamma, \alpha)].$$

□

6. Upper Bound

By the following exponential tightness, it is enough to derive the upper bound of the large deviations principle for compact subsets.

Proposition 6.1 *For any $\ell \geq 1$, there exists a compact subset $\bar{K}_\ell \subset D([0, T], \mathcal{M}_1 \times \mathcal{M}_1)$ such that for any $\sigma_\gamma \in \mathcal{S}_\gamma$,*

$$\limsup_{\gamma \rightarrow 0} \gamma^d \log \bar{Q}_{\sigma_\gamma}^{\gamma, \alpha}(\bar{K}_\ell^c) \leq -\ell. \quad (6.1)$$

Proof: Consider a sequence of functions $\{H_k\}_{k \geq 1}$ in $\mathcal{C}^2(\Lambda)$ dense in $\mathcal{C}(\Lambda)$ for the uniform topology with $\|H_k\|_\infty \leq 1$. Denote for all integers $m \geq 1$, $\ell \geq 1$, and $\delta > 0$,

$$\mathcal{A}_{m,\delta,\ell} = \{\bar{\pi} \in D([0, T], \mathcal{M}_1 \times \mathcal{M}_1) : \inf_{\{t'_i\}} \max_i \sup_{t'_i \leq s < t'_{i+1}} \sum_{j=1}^m \frac{|\langle \bar{\pi}_t, H_j \rangle - \langle \bar{\pi}_s, H_j \rangle|}{4^j} \leq \frac{\ell + 1}{m}\}, \quad (6.2)$$

where the infimum extends over all positive integers K and all finite sets of points $\{t'_i, 0 \leq i \leq K\}$ satisfying $0 = t'_0 < t'_1 < \dots < t'_K = T$, $t'_{i+1} - t'_i > \delta$. We first show that for $m \geq 1$ and for $\ell \geq 1$ there exists $\delta(m, \ell)$ and $\gamma_0(m, \ell)$ so that for all $\gamma \leq \gamma_0$,

$$\overline{Q}_{\sigma^\gamma}^{\gamma, \alpha}[\bar{\pi} \notin \mathcal{A}_{m,\delta(m,\ell),\ell}] \leq e^{-\frac{\ell+1}{\gamma^\alpha}}. \quad (6.3)$$

This is done taking into account that

$$\begin{aligned} & \left\{ \inf_{\{t'_i\}} \max_i \sup_{t'_i \leq s < t'_{i+1}} |\langle \bar{\pi}_t, H \rangle - \langle \bar{\pi}_s, H \rangle| \geq \frac{\ell + 1}{m} \right\} \\ & \subset \bigcup_{k=0}^{\frac{T}{\delta}} \left\{ \sup_{k\delta \leq t < (k+1)\delta} |\langle \bar{\pi}_t, H \rangle - \langle \bar{\pi}_{k\delta}, H \rangle| \geq \frac{\ell + 1}{4m} \right\}, \end{aligned}$$

and estimating the right hand side as in [KL] (p. 271, after formula (4.6)). Then the construction of the compact \overline{K}_ℓ is obtained by a general procedure as explained in [Bill] and Section 8 of [QRV]. \square

The strategy now follows the martingale approach introduced by [DV]. We need to show that trajectories which are not absolutely continuous with respect to the Lebesgue measure and not absolutely continuous in time can be neglected in the large deviations regime. To exclude these “bad” paths, we introduce an energy functional via an exponential martingale. With this we prove an upper bound with an auxiliary rate functional which is infinite on the set of bad trajectories.

For $\bar{\pi} \in D([0, T], \mathcal{M}_1 \times \mathcal{M}_1)$, $\overline{G} = (G_1, G_2) \in (\mathcal{C}^{1,0}([0, T] \times \Lambda))^2$ denote

$$\begin{aligned} \mathbf{J}_{\overline{G}}(\bar{\pi}) &= \ell_T(\bar{\pi}, \overline{G}) + \tilde{\ell}_T(\bar{\pi}, \overline{G}) \\ &- 2 \sum_{i=1}^2 \int_0^T \left\{ \langle p_i \lambda, G_i^2(s, \cdot) \rangle - \langle \pi_{i,s}, G_i^2(s, \cdot) \tanh(\pi_s * J + a_i \theta) \rangle \right\} ds, \end{aligned} \quad (6.4)$$

where $\ell_T, \tilde{\ell}_T$ are defined in (4.13), (4.14). We define the auxiliary rate functional $\mathcal{J} : D([0, T], \mathcal{M}_1 \times \mathcal{M}_1) \rightarrow \overline{\mathbb{R}}$ as

$$\mathcal{J}(\bar{\pi}) = \begin{cases} \sup_{\overline{G} \in (\mathcal{C}^{1,0}([0, T] \times \Lambda))^2} \left(\mathbf{J}_{\overline{G}}(\bar{\pi}) \right) & \text{if } \bar{\pi} \in D([0, T], \mathcal{M}_1^{ac} \times \mathcal{M}_1^{ac}), \\ +\infty & \text{otherwise.} \end{cases} \quad (6.5)$$

Lemma 6.2 *For all $\bar{\pi} \in D([0, T], \mathcal{M}_1 \times \mathcal{M}_1)$, if $\mathcal{J}(\bar{\pi}) < \infty$, then $\bar{\pi} \in \mathcal{C}([0, T], \mathcal{M}_1^{ac} \times \mathcal{M}_1^{ac})$.*

Proof: Fix $\bar{\pi} \in D([0, T], \mathcal{M}_1 \times \mathcal{M}_1)$ such that $\mathcal{J}(\bar{\pi}) < \infty$. By definition of $\mathcal{J}(\cdot)$, $\bar{\pi} \in D([0, T], \mathcal{M}_1^{ac} \times \mathcal{M}_1^{ac})$. Let $\bar{g} = (g_1, g_2) \in (\mathcal{C}^0(\Lambda))^2$ and $0 \leq s < t < T$. For each $\delta > 0$, let $\psi_{s,t}^\delta : [0, T] \rightarrow \mathbb{R}$ be the function given by

$$\psi_{s,t}^\delta(\tau) = \begin{cases} 0 & \text{if } 0 \leq \tau \leq s \text{ or } t \leq \tau \leq T, \\ \frac{\tau - s}{\delta} & \text{if } s \leq \tau \leq s + \delta, \\ 1 & \text{if } s + \delta \leq \tau \leq t - \delta, \\ \frac{t - \tau}{\delta} & \text{if } t - \delta \leq \tau \leq t. \end{cases} \quad (6.6)$$

Denote $\bar{G}^\delta(\tau, r) = \psi_{s,t}^\delta(\tau)\bar{g}(r)$. Since \bar{G}^δ can be approximated by functions in $(\mathcal{C}^{1,0}([0, T] \times \Lambda))^2$, considering $\frac{\bar{G}^\delta}{\sqrt{t-s}}$ as a test function and performing the limit $\delta \rightarrow 0$, we obtain

$$\begin{aligned} \sqrt{t-s} \lim_{\delta \rightarrow 0} \mathbf{J} \frac{\bar{G}^\delta}{\sqrt{t-s}}(\bar{\pi}) &= \langle \bar{\pi}_t, \bar{g} \rangle - \langle \bar{\pi}_s, \bar{g} \rangle \\ &+ \int_s^t \left\{ \langle \bar{\pi}_\tau, \bar{g} \rangle - \sum_{i=1}^2 \langle p_i \lambda, g_i \tanh(\pi_\tau * J + a_i \theta) \rangle \right\} d\tau \\ &- 2 \frac{1}{\sqrt{t-s}} \sum_{i=1}^2 \int_s^t \left\{ \langle p_i \lambda, g_i^2 \rangle - \langle \pi_{i,\tau}, g_i^2 \tanh(\pi_\tau * J + a_i \theta) \rangle \right\} d\tau. \end{aligned} \quad (6.7)$$

Since $\sqrt{t-s} \lim_{\delta \rightarrow 0} \mathbf{J} \frac{\bar{G}^\delta}{\sqrt{t-s}}(\bar{\pi}) \leq \sqrt{t-s} \mathcal{J}(\bar{\pi})$, we get

$$\begin{aligned} \left| \langle \bar{\pi}_t, \bar{g} \rangle - \langle \bar{\pi}_s, \bar{g} \rangle \right| &\leq C_0(t-s) \sum_{i=1}^2 \left\{ \|g_i\|_1 + \frac{1}{\sqrt{t-s}} \|g_i\|_2^2 \right\} + \sqrt{t-s} \mathcal{J}(\bar{\pi}) \\ &= C_0(t-s) \sum_{i=1}^2 \|g_i\|_1 + \sqrt{t-s} \left\{ C_0 \sum_{i=1}^2 \|g_i\|_2^2 + \mathcal{J}(\bar{\pi}) \right\}, \end{aligned} \quad (6.8)$$

for some positive constant C_0 . This implies that $\bar{\pi} \in \mathcal{C}([0, T], \mathcal{M}_1^{qc} \times \mathcal{M}_1^{qc})$. \square

Lemma 6.3 *Let $\bar{\pi} = (\phi_1(s, r)dr, \phi_2(s, r)dr) \in D([0, T], \mathcal{M}_1^{qc} \times \mathcal{M}_1^{qc})$ such that $\mathcal{J}(\bar{\pi}) < \infty$, then*
 (a) *for $i = 1, 2$, $F_i(s, r) := [p_i - \phi_i(s, r) \tanh(\pi_s(r) * J + a_i \theta)] \geq 0$ for almost all $(s, r) \in [0, T] \times \Lambda$,*
 (b) *$(\phi_1, \phi_2) \in \mathcal{AC}([0, T], B_{1,1})$.*

Proof: (a) Taking as a test function $AG(\cdot, \cdot)$, for all $\bar{G} = (G_1, G_2) \in (\mathcal{C}^{1,0}([0, T] \times \Lambda))^2$ and $A > 0$, we obtain from Definition (6.5) of the rate function \mathcal{J} ,

$$-\sum_{i=1}^2 \int_0^T \int_\Lambda G_i^2(s, r) F_i(s, r) dr ds \leq -\frac{1}{2A} \left\{ \ell_T(\bar{\pi}, \bar{G}) + \tilde{\ell}_T(\bar{\pi}, \bar{G}) \right\} + \frac{1}{2A^2} \mathcal{J}(\bar{\pi}). \quad (6.9)$$

Letting $A \uparrow \infty$, we get

$$\sum_{i=1}^2 \int_0^T \int_\Lambda G_i^2(s, r) F_i(s, r) dr ds \geq 0.$$

Since \bar{G} is arbitrary, we conclude that $F_i(s, r) \geq 0$ for $1 \leq i \leq 2$, almost everywhere.

(b) We show the absolute continuity in time for ϕ_1 . The proof for ϕ_2 is similar. We apply the characterization of $\mathcal{AC}([0, T], B_{1,1})$ given in Proposition 3.2. For all positive integer k let $\{A_i, 1 \leq i \leq k\}$ be rectangles of Λ and $\{(s_i, t_i), 1 \leq i \leq k\}$ be nonempty disjoint intervals of $[0, T]$. For $i = 1, \dots, k$, denote

$$\eta_i := \text{sgn} \left(\int_{A_i} [\phi_1(t_i, r) - \phi_1(s_i, r)] dr \right),$$

where $\text{sgn} : \mathbb{R} \rightarrow \mathbb{R}$ is given by

$$\text{sgn}(x) = \begin{cases} \frac{x}{|x|} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases} \quad (6.10)$$

For each $1 \leq i \leq k$, $0 < \delta < \frac{1}{4} \min_{1 \leq i \leq k} (t_i - s_i)$, we set, see (6.6),

$$V_1(t, r) = \sum_{i=1}^k \eta_i \times \psi_{s_i, t_i}^\delta(t) \times \mathbb{1}_{A_i}(r), \quad V_2(t, r) = 0, \quad \bar{V} = (V_1, V_2). \quad (6.11)$$

Since V_1 can be approximated by functions in $\mathcal{C}^{1,0}([0, T] \times \Lambda)$, proceeding as in (6.7), we obtain for any $b > 0$, see (6.9),

$$\begin{aligned} \sum_{i=1}^k \eta_i \left\{ \int_{A_i} [\phi_1(t_i, r) - \phi_1(s_i, r)] dr \right\} &\leq - \sum_{i=1}^k \int_{s_i}^{t_i} \langle \eta_i \mathbb{1}_{A_i}, [\phi_1(s, \cdot) - p_1 \tanh(\pi_s * J + a_1 \theta)] \rangle ds \\ &\quad + 2b \sum_{i=1}^k \int_{s_i}^{t_i} \int_{\Lambda} |\eta_i| \mathbb{1}_{A_i}(r) F_1(s, r) dr ds + \frac{\mathcal{J}(\bar{\pi})}{b}. \end{aligned}$$

Minimizing over b yields

$$\begin{aligned} \sum_{i=1}^k \left| \int_{A_i} [\phi_1(t_i, r) - \phi_1(s_i, r)] dr \right| &\leq 2 \sum_{i=1}^k (t_i - s_i) \lambda(A_i) \\ &\quad + 2\sqrt{2\mathcal{J}(\bar{\pi})} \left(\sum_{i=1}^k \int_{s_i}^{t_i} \int_{\Lambda} |\eta_i| \mathbb{1}_{A_i}(r) F_1(s, r) dr ds \right)^{1/2} \\ &\leq 2 \sum_{i=1}^k (t_i - s_i) \lambda(A_i) + 4\sqrt{\mathcal{J}(\bar{\pi})} \sqrt{\sum_{i=1}^k (t_i - s_i) \lambda(A_i)}. \end{aligned} \quad (6.12)$$

For all $\varepsilon > 0$ denote $\Delta = \min(\varepsilon/4, \varepsilon^2/(64\mathcal{J}(\bar{\pi})))$. It follows from (6.12) that $\sum_{i=1}^k (t_i - s_i) \lambda(A_i) \leq \Delta$ implies $\sum_{i=1}^k \left| \int_{A_i} [\phi_1(t_i, r) - \phi_1(s_i, r)] dr \right| \leq \varepsilon$. This concludes the proof. \square

For $\bar{\pi} \in D([0, T], \mathcal{M}_1 \times \mathcal{M}_1)$, ℓ_T in (4.13) and $F_{\bar{V}(s)}$ defined in (3.3) denote

$$\widehat{\mathcal{J}}_{\bar{V}}(\bar{\pi}) = \ell_T(\bar{\pi}, \bar{V}) - \frac{1}{2} \int_0^T F_{\bar{V}(s)}((p_1 \lambda, p_2 \lambda), \bar{\pi}_s) ds, \quad (6.13)$$

and

$$\widehat{\mathcal{J}}(\bar{\pi}) = \sup_{\bar{V} \in (\mathcal{C}^{1,0}([0, T] \times \Lambda))^2} \left\{ \widehat{\mathcal{J}}_{\bar{V}}(\bar{\pi}) \right\}. \quad (6.14)$$

Remark that when $\bar{\pi} = (\phi_1 \lambda, \phi_2 \lambda)$, with $\bar{\phi} = (\phi_1, \phi_2) \in \mathcal{AC}([0, T], B_{1,1})$, the functional $\widehat{\mathcal{J}}$ coincides with the functional $I_0 = J_0 = J_1$ (cf. Proposition 3.6). The proof of the upper bound relies on the following proposition.

Proposition 6.4 *Let \bar{K} be a compact set of $D([0, T], \mathcal{M}_1 \times \mathcal{M}_1)$. For any $0 < b < 1$,*

$$\limsup_{\gamma \rightarrow 0} \gamma^d \log \bar{Q}_{\sigma^\gamma}^{\gamma, \alpha}(\bar{K}) \leq -\frac{1}{1+b} \inf_{\bar{\pi} \in \bar{K}} \left[\widehat{\mathcal{J}}(\bar{\pi}) + b\mathcal{J}(\bar{\pi}) \right].$$

Proof: For $\varepsilon > 0$, $\mu \in \mathcal{M}_1$, $g \in \mathcal{C}(\Lambda)$ denote by ι_ε the approximation of the identity

$$\iota_\varepsilon(r) = \frac{1}{(2\varepsilon)^d} \mathbb{I}\{[-\varepsilon, \varepsilon]^d\}(r), \quad r \in \Lambda \quad (6.15)$$

and by $\mu * \iota_\varepsilon$ the measure defined by $\langle \mu * \iota_\varepsilon, g \rangle = \langle \mu, g * \iota_\varepsilon \rangle$. It is absolutely continuous with respect to the Lebesgue measure with density

$$\frac{d(\mu * \iota_\varepsilon)}{d\lambda}(r) = \langle \mu, \iota_\varepsilon(r - \cdot) \rangle, \quad r \in \Lambda.$$

In general, we can only bound this density by $\|\iota_\varepsilon\|_\infty$ which is of order ε^{-d} . Nevertheless, in the case of the empirical measure, we have

$$|\langle \pi_s^\gamma, \iota_\varepsilon(r - \cdot) \rangle| = \left| \frac{\gamma^d}{(2\varepsilon)^d} \sum_{x: \gamma x \in [r-\varepsilon, r+\varepsilon]} \sigma_s(x) \right| \leq 1, \quad \text{for almost all } 0 \leq s \leq T,$$

which means that $\pi^\gamma * \iota_\varepsilon \in \mathcal{M}_1^{ac}$, when $0 < \gamma < \varepsilon$. Furthermore for any $\bar{\pi} \in D([0, T], \mathcal{M}_1 \times \mathcal{M}_1)$, denote by $\bar{\pi}_s * \iota_\varepsilon := (\pi_{1,s} * \iota_\varepsilon, \pi_{2,s} * \iota_\varepsilon)$, $0 \leq s \leq T$ the trajectory in $D([0, T], \mathcal{M}_1^{ac} \times \mathcal{M}_1^{ac})$.

Fix a function $\bar{G} \in (\mathcal{C}^{1,0}([0, T] \times \Lambda))^2$. Consider the mean one exponential martingale $(\bar{\mathcal{Z}}_t^{\bar{G}, \gamma})_{t \geq 0}$

$$\bar{\mathcal{Z}}_t^{\bar{G}, \gamma} = \exp \left\{ \gamma^{-d} \bar{N}_\gamma^{\bar{G}}(t) - \frac{\gamma^{-2d}}{2} \langle \bar{N}_\gamma^{\bar{G}}, \bar{N}_\gamma^{\bar{G}} \rangle(t) \right\}, \quad (6.16)$$

where the martingale $(\bar{N}_\gamma^{\bar{G}}(t))_{t \geq 0}$ and its quadratic variation $(\langle \bar{N}_\gamma^{\bar{G}}, \bar{N}_\gamma^{\bar{G}} \rangle(t))_{t \geq 0}$ were given in (4.7) and (4.8). Using the same arguments as in Proposition 5.5, by smoothness of \bar{G} and $\pi^\gamma * J$, a spatial summation by parts and Taylor expansion permit to rewrite the martingale $\bar{\mathcal{Z}}_t^{\bar{G}, \gamma}$ as

$$\bar{\mathcal{Z}}_t^{\bar{G}, \gamma} = \exp \left\{ \gamma^{-d} \mathbb{J}_{\bar{G}}(\bar{\pi}^\gamma * \iota_\varepsilon) + \gamma^{-d} r(\bar{G}, \gamma, \varepsilon, l, \delta, \alpha) \right\}, \quad (6.17)$$

where $0 < \gamma, \varepsilon, \delta < 1$, l is a positive integer. Here and in the sequel, $r(\bar{G}, \gamma, \varepsilon, l, \delta, \alpha)$ (resp. $r(\bar{G}, \bar{V}, \gamma, \varepsilon, l, \delta, \alpha)$ later on) stands for some random variable satisfying

$$\limsup_{\delta \rightarrow 0} \limsup_{l \rightarrow \infty} \limsup_{\varepsilon \rightarrow 0} \limsup_{\gamma \rightarrow 0} r(\bar{G}, \gamma, \varepsilon, l, \delta, \alpha) = 0, \quad \mathbb{P} - \text{a-e.} \quad (6.18)$$

Let \bar{K} be a compact set of $D([0, T], \mathcal{M}_1 \times \mathcal{M}_1)$. By Hölder inequality,

$$\begin{aligned} \gamma^d \log \bar{Q}_{\sigma^\gamma}^{\gamma, \alpha}(\bar{K}) &= \gamma^d \log E^{Q_{\sigma^\gamma}^{\gamma, \alpha}} \left[\mathbb{I}_{\bar{K}}(\bar{\pi}^\gamma) (\bar{\mathcal{Z}}_t^{\bar{G}, \gamma})^{\frac{b}{1+b}} \times (\bar{\mathcal{Z}}_t^{\bar{G}, \gamma})^{\frac{-b}{1+b}} \right] \\ &\leq \frac{b}{1+b} \gamma^d \log E^{Q_{\sigma^\gamma}^{\gamma, \alpha}} \left[\mathbb{I}_{\bar{K}}(\bar{\pi}^\gamma) \bar{\mathcal{Z}}_t^{\bar{G}, \gamma} \right] + \frac{1}{1+b} \gamma^d \log E^{Q_{\sigma^\gamma}^{\gamma, \alpha}} \left[\mathbb{I}_{\bar{K}}(\bar{\pi}^\gamma) (\bar{\mathcal{Z}}_t^{\bar{G}, \gamma})^{-b} \right] \\ &\leq \frac{1}{1+b} \gamma^d \log E^{Q_{\sigma^\gamma}^{\gamma, \alpha}} \left[\mathbb{I}_{\bar{K}}(\bar{\pi}^\gamma) (\bar{\mathcal{Z}}_t^{\bar{G}, \gamma})^{-b} \right]. \end{aligned} \quad (6.19)$$

To exclude paths whose densities are not absolutely continuous with respect to the Lebesgue measure, we follow arguments used in [FLM]. Fix a sequence $\{F_k : k \geq 1\}$ of smooth nonnegative functions dense in $\mathcal{C}(\Lambda)$ for the uniform topology. For $k \geq 1$, $\varrho > 0$ and $\delta > 0$, let

$$D_{k, \varrho} = \left\{ \bar{\pi} \in D([0, T], \mathcal{M}_1 \times \mathcal{M}_1) : 0 \leq |\pi_{i,t}|, F_k \leq \int_\Lambda F_k(x) dx + C_k \varrho, 0 \leq t \leq T, i = 1, 2 \right\},$$

where $C_k = C(\|\nabla F_k\|_\infty)$ is a constant depending on the gradient ∇F_k of F_k . The sets $D_{k,\varrho}$, $k \geq 1$ are closed subsets of $D([0, T], \mathcal{M}_1 \times \mathcal{M}_1)$, as well as

$$E_{m,\varrho} = \bigcap_{k=1}^m D_{k,\varrho}, \quad m \geq 1.$$

Note that the empirical measure $\bar{\pi}^\gamma$ belongs to $E_{m,\varrho}$ for γ sufficiently small. We have that

$$D([0, T], \mathcal{M}_1^{ac} \times \mathcal{M}_1^{ac}) = \bigcap_{n \geq 1} \bigcap_{m \geq 1} E_{m,1/n}. \quad (6.20)$$

Fix $0 < b < 1$. For $\bar{G}, \bar{V} \in (\mathcal{C}^{1,0}([0, T] \times \Lambda))^2$, $\varepsilon > 0$ and $m, n \in \mathbb{Z}_+$, let $\hat{\mathbf{J}}_{\bar{V}, \bar{G}, \varepsilon}^{b,m,n} : D([0, T], \mathcal{M}_1 \times \mathcal{M}_1) \rightarrow \mathbb{R} \cup \{\infty\}$ be the functional given by

$$\hat{\mathbf{J}}_{\bar{V}, \bar{G}, \varepsilon}^{b,m,n}(\bar{\pi}) = \begin{cases} \hat{J}_{\bar{V}}(\bar{\pi} * \iota_\varepsilon) + b\mathbf{J}_{\bar{G}}(\bar{\pi} * \iota_\varepsilon) & \text{if } \bar{\pi} \in E_{m, \frac{1}{n}}, \\ +\infty & \text{otherwise.} \end{cases} \quad (6.21)$$

It is lower semicontinuous because so is $\bar{\pi} \mapsto \hat{J}_{\bar{V}}(\bar{\pi} * \iota_\varepsilon) + b\mathbf{J}_{\bar{G}}(\bar{\pi} * \iota_\varepsilon)$, and because $E_{m,1/n}$ is closed.

We now return to inequality (6.19). By Proposition 5.5, the exponential martingale $\mathcal{M}_t^{\bar{V}, \gamma}$ defined by the Girsanov formula (5.9) satisfies

$$\mathcal{M}_T^{\bar{V}, \gamma} := \frac{dP_{\sigma^\gamma}^{\bar{V}, \gamma, \alpha}}{dP_{\sigma^\gamma}^{\gamma, \alpha}}(\sigma_{[0, T]}) = \exp \left\{ \gamma^{-d} \hat{J}_{\bar{V}}(\bar{\pi}^\gamma * \iota_\varepsilon) + \gamma^{-d} r(\bar{V}, \gamma, \varepsilon, l, \delta, \alpha) \right\}. \quad (6.22)$$

We rewrite (6.19) as

$$\gamma^d \log \bar{Q}_{\sigma^\gamma}^{\gamma, \alpha}(\bar{K}) \leq \frac{1}{1+b} \gamma^d \log E^{Q_{\sigma^\gamma}^{\gamma, \alpha}} \left[\mathbb{1}_{\bar{K}}(\bar{\pi}^\gamma) \mathcal{M}_T^{\bar{V}, \gamma} \times (\mathcal{M}_T^{\bar{V}, \gamma})^{-1} \times (\bar{Z}_t^{\bar{G}, \gamma})^{-b} \right]. \quad (6.23)$$

Since $\mathcal{M}_T^{\bar{V}, \gamma}$ is a mean one positive martingale, taking into account (6.17) and (6.22) and optimizing over $\bar{\pi}$ in \bar{K} , we obtain, for all positive integers m, n ,

$$\begin{aligned} \limsup_{\gamma \rightarrow 0} \gamma^d \log \bar{Q}_{\sigma^\gamma}^{\gamma, \alpha}(\bar{K}) &\leq \frac{1}{1+b} \sup_{\bar{\pi} \in \bar{K} \cap E_{m, \frac{1}{n}}} \left\{ -\hat{J}_{\bar{V}}(\bar{\pi} * \iota_\varepsilon) - b\mathbf{J}_{\bar{G}}(\bar{\pi} * \iota_\varepsilon) \right\} \\ &\quad + \limsup_{\gamma \rightarrow 0} r(\bar{V}, \bar{G}, \gamma, \varepsilon, l, \delta, \alpha) \\ &= \frac{1}{1+b} \sup_{\bar{\pi} \in \bar{K}} \left\{ -\hat{\mathbf{J}}_{\bar{V}, \bar{G}, \varepsilon}^{b,m,n}(\bar{\pi}) \right\} + \limsup_{\gamma \rightarrow 0} r(\bar{V}, \bar{G}, \gamma, \varepsilon, l, \delta, \alpha). \end{aligned}$$

Optimizing the previous expression with respect to $\bar{V}, \bar{G}, \varepsilon, l, \delta, m, n$, taking into account (6.18), we get

$$\limsup_{\gamma \rightarrow 0} \gamma^d \log \bar{Q}_{\sigma^\gamma}^{\gamma, \alpha}(\bar{K}) \leq \inf_{\bar{V}, \bar{G}, \varepsilon, l, \delta, m, n} \left\{ \frac{1}{1+b} \sup_{\bar{\pi} \in \bar{K}} \left\{ -\hat{\mathbf{J}}_{\bar{V}, \bar{G}, \varepsilon}^{b,m,n}(\bar{\pi}) \right\} \right\}. \quad (6.24)$$

Since \bar{K} is compact and $\bar{\pi} \mapsto \frac{1}{1+b} \sup_{\bar{\pi} \in \bar{K}} \left\{ -\hat{\mathbf{J}}_{\bar{V}, \bar{G}, \varepsilon}^{b,m,n}(\bar{\pi}) \right\}$ is lower semi-continuous for all \bar{V}, \bar{G} and $\varepsilon, l, \delta, m, n$, we may apply the arguments presented in [V], Lemma 11.3 to exchange the supremum with the infimum. In this way we obtain that the right hand side of (6.24) is bounded above by

$$\sup_{\bar{\pi} \in \bar{K}} \inf_{\bar{V}, \bar{G}, \varepsilon, l, \delta, m, n} \left\{ -\frac{1}{1+b} \hat{\mathbf{J}}_{\bar{V}, \bar{G}, \varepsilon}^{b,m,n}(\bar{\pi}) \right\}.$$

By (6.20) we have

$$\limsup_{\varepsilon \rightarrow 0} \limsup_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \widehat{\mathbf{J}}_{\overline{V}, \overline{G}, \varepsilon}^{b, m, n}(\overline{\pi}) := \widehat{\mathbf{J}}_{\overline{V}, \overline{G}}^b(\overline{\pi}) = \begin{cases} \widehat{J}_{\overline{V}}(\overline{\pi}) + b\mathbf{J}_G(\overline{\pi}) & \text{if } \overline{\pi} \in D([0, T], \mathcal{M}_1^{ac} \times \mathcal{M}_1^{ac}), \\ +\infty & \text{otherwise.} \end{cases}$$

By (6.5) and (6.14) we have that $\sup_{\overline{V}, \overline{G}} \left\{ \widehat{\mathbf{J}}_{\overline{V}, \overline{G}}^b(\overline{\pi}) \right\} = \widehat{J}(\overline{\pi}) + b\mathcal{J}(\overline{\pi})$. Therefore,

$$\begin{aligned} \limsup_{\gamma \rightarrow 0} \gamma^d \log \overline{Q}_{\sigma^\gamma}^{\gamma, \alpha}(\overline{K}) &\leq \sup_{\overline{\pi} \in \overline{K}} \inf_{\overline{V}, \overline{G}} \left\{ -\frac{1}{1+b} \widehat{\mathbf{J}}_{\overline{V}, \overline{G}}^b(\overline{\pi}) \right\} \\ &= -\frac{1}{1+b} \inf_{\overline{\pi} \in \overline{K}} \sup_{\overline{V}, \overline{G}} \left\{ \widehat{\mathbf{J}}_{\overline{V}, \overline{G}}^b(\overline{\pi}) \right\} = -\frac{1}{1+b} \inf_{\overline{\pi} \in \overline{K}} \left\{ \widehat{J}(\overline{\pi}) + b\mathcal{J}(\overline{\pi}) \right\}. \end{aligned} \quad (6.25)$$

□

Proof of the upper bound Let \overline{K} be a compact set of $D([0, T], \mathcal{M}_1 \times \mathcal{M}_1)$. By Proposition 6.4, if $\mathcal{J} \equiv +\infty$ on \overline{K} , then the large deviations upper bound is satisfied. Otherwise, there exists $\overline{\pi} \in \overline{K}$ such that $\mathcal{J}(\overline{\pi}) < \infty$. By semicontinuity of the functional $\overline{\pi} \mapsto \mathcal{J}(\overline{\pi})$, we obtain from (6.25) for any $0 < b < 1$,

$$\limsup_{\gamma \rightarrow 0} \gamma^d \log \overline{Q}_{\sigma^\gamma}^{\gamma, \alpha}(\overline{K}) \leq -\frac{1}{1+b} \inf_{\overline{\pi} \in \overline{K}, \mathcal{J}(\overline{\pi}) < \infty} \widehat{J}(\overline{\pi}) - \frac{b}{1+b} \inf_{\overline{\pi} \in \overline{K}} \mathcal{J}(\overline{\pi}).$$

Letting $b \rightarrow 0$, we get

$$\limsup_{\gamma \rightarrow 0} \gamma^d \log \overline{Q}_{\sigma^\gamma}^{\gamma, \alpha}(\overline{K}) \leq -\inf_{\overline{\pi} \in \overline{K}, \mathcal{J}(\overline{\pi}) < \infty} \left\{ \widehat{J}(\overline{\pi}) \right\} \leq -\inf_{\overline{\pi} \in \overline{K}} \left\{ I_0(\overline{\pi}) \right\}.$$

For the last inequality we used Lemma 6.3. By Proposition 6.1 the proof of the upper bound is completed. □

7. Lower Bound

We first get a lower estimate for the probability of a neighborhood of suitable trajectories. We perform the computation with the uniform metric on the time interval $[0, T]$ defined as following: for μ and ν in $D([0, T], \mathcal{M}_1)$ and $\rho(\cdot, \cdot)$ defined in (2.9).

$$\rho_{[0, T]}(\mu, \nu) = \sup_{t \in [0, T]} \rho(\mu_t, \nu_t) \quad \text{and} \quad \rho_{[0, T]}(\overline{\mu}, \overline{\nu}) = \sum_{i=1, 2} \rho_{[0, T]}(\mu_i, \nu_i), \quad (7.1)$$

Taking into account that if $d_{[0, T]}^S(\mu, \nu)$ denotes the Skorohod distance, then

$$d_{[0, T]}^S(\mu, \nu) \leq \rho_{[0, T]}(\mu, \nu) \quad (7.2),$$

the result holds for the Skorohod topology as well, see [Bill].

Lemma 7.1 *Assume $(\sigma^\gamma)_\gamma, m_0$ satisfy (2.13). Let $\delta > 0$ and $\overline{\mu} = \overline{m}^{\overline{V}} \lambda$, where $\overline{m}^{\overline{V}}$ is the solution to (5.5) for $\overline{V} = (V_1, V_2) \in (\mathcal{C}^{1,0}([0, T] \times \Lambda))^2$ and $m_i^{\overline{V}}(0, \cdot) = p_i m_0(\cdot)$ for $i = 1, 2$. Then we have, for $\mathcal{V}_\delta(\overline{\mu}) = \{\overline{\mu}' \in D([0, T], \mathcal{M}_1 \times \mathcal{M}_1) : \rho_{[0, T]}(\overline{\mu}, \overline{\mu}') < \delta\}$, and I_{m_0} given in (2.28),*

$$\liminf_{\gamma \rightarrow 0} \gamma^d \log \overline{Q}_{\sigma^\gamma}^{\gamma, \alpha}(\mathcal{V}_\delta(\overline{\mu})) \geq -I_{m_0}(\overline{\mu}), \quad IP - a.s. \quad (7.3)$$

Proof: We introduce the perturbed process. By Jensen inequality we get

$$\log \bar{Q}_{\sigma^\gamma}^{\gamma, \alpha}(\mathcal{V}_\delta(\bar{\mu})) \geq E_{\sigma^\gamma}^{\bar{V}, \gamma, \alpha} \left[\mathbb{1}_{\mathcal{V}_\delta(\bar{\mu})}(\bar{\pi}_{[0, T]}^\gamma) \log \frac{dP_{\sigma^\gamma}^{\gamma, \alpha}}{dP_{\sigma^\gamma}^{\bar{V}, \gamma, \alpha}}(\sigma_{[0, T]}) \right] \left(\bar{Q}_{\sigma^\gamma}^{\bar{V}, \gamma, \alpha}(\mathcal{V}_\delta(\bar{\mu})) \right)^{-1} + \log \bar{Q}_{\sigma^\gamma}^{\bar{V}, \gamma, \alpha}(\mathcal{V}_\delta(\bar{\mu})).$$

By Corollary 5.3, $\lim_{\gamma \rightarrow 0} \bar{Q}_{\sigma^\gamma}^{\bar{V}, \gamma, \alpha}(\mathcal{V}_\delta(\bar{\mu})) = 1$. By Lebesgue dominated convergence theorem,

$$\liminf_{\gamma \rightarrow 0} \gamma^d \log \bar{Q}_{\sigma^\gamma}^{\gamma, \alpha}(\mathcal{V}_\delta(\bar{\mu})) \geq \liminf_{\gamma \rightarrow 0} E_{\sigma^\gamma}^{\bar{V}, \gamma, \alpha} \left[\gamma^d \log \frac{dP_{\sigma^\gamma}^{\gamma, \alpha}}{dP_{\sigma^\gamma}^{\bar{V}, \gamma, \alpha}}(\sigma_{[0, T]}) \right].$$

By Radon-Nikodym formula, see Theorem 5.4, and Proposition 5.5 we have

$$\begin{aligned} \gamma^d \log \frac{dP_{\sigma^\gamma}^{\gamma, \alpha}}{dP_{\sigma^\gamma}^{\bar{V}, \gamma, \alpha}}(\sigma_{[0, T]}) &\geq -\ell_T(\bar{\pi}^\gamma(\sigma), \bar{V}) + \frac{1}{2} \int_0^T \Gamma_{\bar{V}(s, \cdot)}(\bar{\pi}_s^\gamma) ds \\ &\quad - \epsilon(\gamma l)T - TC(V_1, V_2)[\delta + \sum_{i=1,2} \mathcal{E}_i(\delta, l, \gamma, \alpha)]. \end{aligned}$$

From Theorem 5.1, recalling the definition of $K_{\bar{V}}(\cdot)$ given in (3.8), we get that for any l ,

$$\liminf_{\gamma \rightarrow 0} \gamma^d \log \bar{Q}_{\sigma^\gamma}^{\gamma, \alpha}(\mathcal{V}_\delta(\bar{\mu})) \geq -K_{\bar{V}}(\bar{\mu}) - TC(V_1, V_2)[\delta + \lim_{\gamma \rightarrow 0} \sum_{i=1,2} \mathcal{E}_i(\delta, l, \gamma, \alpha)],$$

which yields the result letting $l \rightarrow \infty$ by Lemma 4.2 and Proposition 3.6. \square

To conclude the proof of the lower bound in Theorem 2.9, it remains to show that all $\bar{\pi}$'s such that $I_0(\bar{\pi}) < \infty$ can be approximated by a sequence $(\bar{\pi}_n)_n$ of smooth trajectories, for which Lemma 7.1 holds with $\lim_{n \rightarrow \infty} I_0(\bar{\pi}_n) = I_0(\bar{\pi})$. We first show that any trajectory \bar{m} smooth enough and far away from the boundaries $(\pm p_1, \pm p_2)$ is associated to a function $\bar{V}(\cdot, \cdot)$ in the following sense:

Lemma 7.2 *Given $\bar{m} = (m_1, m_2) \in (\mathcal{C}^{2,0}([0, T] \times \Lambda))^2$, with, for $i = 1, 2$, $|m_i(t, r)| < p_i$, for all $t \in [0, T]$, $r \in \Lambda$, there exists $\bar{V} = (V_1, V_2) \in (\mathcal{C}^{1,0}([0, T] \times \Lambda))^2$ such that $\bar{m} = \bar{m}^{\bar{V}}$ is the solution to (5.5). For $(t, r) \in (0, T] \times \Lambda$,*

$$\begin{aligned} 2V_i(t, r) &= \\ \log \left\{ \partial_t m_i(t, r) \cosh [(J * m)(t, r) + a_i \theta] + \sqrt{(\partial_t m_i(t, r) \cosh [(J * m)(t, r) + a_i \theta])^2 + p_i^2 - m_i^2(t, r)} \right\} &(7.4) \\ - [(J * m)(t, r) + a_i \theta] - \log \{p_i - m_i(t, r)\}, & \end{aligned}$$

and for $t = 0$ we set $\lim_{t \rightarrow 0} V_i(t, r) = V_i(0, r)$.

Proof: By (5.5), for $t \in (0, T]$, we determine $\bar{V}(t, \cdot) = (V_1(t, \cdot), V_2(t, \cdot))$ with $V_i \in \mathcal{C}^{1,0}((0, T] \times \Lambda)$ for $i = 1, 2$, such that $\bar{m} = \bar{m}^{\bar{V}}$. Namely, for $(t, r) \in (0, T] \times \Lambda$, denoting $A_i = (J * m)(t, r) + a_i \theta$, $Y_i = (\cosh A_i) \partial_t m_i(t, r)$, $Z_i = -m_i(t, r)$, (5.5) is written as

$$Y_i = Z_i \cosh [A_i + 2V_i(t, r)] + p_i \sinh [A_i + 2V_i(t, r)].$$

We multiply both sides by $X_i = e^{2V_i(t,r)}$ and obtain

$$e^{A_i}(Z_i + p_i)X_i^2 - 2Y_iX_i + e^{-A_i}(Z_i - p_i) = 0.$$

Its positive solution is

$$X_i = \frac{Y_i + \sqrt{Y_i^2 - (Z_i^2 - p_i^2)}}{e^{A_i}(Z_i + p_i)},$$

which gives (7.4). Note that $\bar{V} = (V_1, V_2)$ has the same spatial regularity as \bar{m} , namely the argument of the square root is always strictly positive. \square

Corollary 7.3 *If \bar{m} is solution of (2.24) then $\bar{V} = 0$ in (7.4).*

Remark 7.4 . Lemma 7.2 could have been stated requiring $\bar{m} \in \mathcal{AC}([0, T], B_{1,1})$. In this case one would get $\bar{V} \in (L^1([0, T], \mathcal{C}(\Lambda)))^2$. We prefer to obtain more regularity in time for \bar{V} , so that uniformity and other technical needs become straightforward.

Given $\bar{m}_0 \in B_{p_1, p_2}$, denote by $\bar{R}(t, \cdot)$, $t \in [0, T]$ the solution of (2.24) with $\bar{R}(0, \cdot) = \bar{m}_0(\cdot)$: for $i = 1, 2$,

$$R_i(t, \cdot) = e^{-t}m_i(0, \cdot) + p_i \int_0^t e^{-(t-s)} \tanh[(J * R)(s, \cdot) + a_i \theta] ds, \quad (7.5)$$

where $R = R_1 + R_2$. It is continuously differentiable in time, actually it is \mathcal{C}^∞ in time for $t \geq t_0 > 0$, and there exists δ_i which depends on T such that $|R_i(t, \cdot)| \leq p_i - \delta_i$ for $t \in [t_0, T]$. Namely, since $|\tanh z| \leq 1 - d$, for $|z| \leq K(\beta, \theta)$ with $1 > d = d(\beta, \theta) > 0$, we have, for $t \in [0, T]$,

$$|R_i(t, \cdot)| \leq e^{-t}m_i(0, \cdot) + p_i(1 - d) \int_0^t e^{-(t-s)} ds \leq p_i[1 - d(1 - e^{-t})]. \quad (7.6)$$

Recall that $I_0(\bar{R}) = 0$, see (3) of Proposition 3.8. Define the sets:

$$\mathcal{C}_0 = \mathcal{C}_0(m_0) = \{\bar{\phi} \in \mathcal{AC}([0, T], B_{p_1, p_2}) : \bar{\phi}(0) = \bar{m}_0, I_0(\bar{\phi}) < \infty\}, \quad (7.7)$$

$$\mathcal{C}_1 = \{\bar{\phi} \in \mathcal{C}_0 : \exists 0 < \eta < T, \bar{\phi}(t) = \bar{R}(t), t \in [0, \eta]\}, \quad (7.8)$$

$$\mathcal{C}_2 = \{\bar{\phi} \in \mathcal{C}_1 : \forall \eta \in (0, T], \exists \delta_i = \delta_i(\bar{\phi}) > 0, i = 1, 2 : \|\phi_i(t)\|_\infty \leq p_i - \delta_i, \quad t \in [\eta, T]\}, \quad (7.9)$$

$$\mathcal{C}_3 = \{\bar{\phi} \in \mathcal{C}_2 : \phi_i \in \mathcal{C}^2((0, T], B_{p_1, p_2}), i = 1, 2, \phi_i(t) \in \mathcal{C}(\Lambda), \forall t \in (0, T]\}. \quad (7.10)$$

By construction $\mathcal{C}_3 \subset \mathcal{C}_2 \subset \mathcal{C}_1 \subset \mathcal{C}_0$. Notice that, by Lemma 7.2, we can associate a function \bar{V} to $\bar{\phi} \in \mathcal{C}_3$. To extend the lower bound, we show that for $i \in \{1, 2, 3\}$, \mathcal{C}_i is $(\rho_{[0, T]}, I_0)$ -dense in \mathcal{C}_{i-1} , that is, for all $\bar{\phi} \in \mathcal{C}_{i-1}$ there exists a sequence $(\bar{\phi}_n)_n \subset \mathcal{C}_i$ such that

$$\lim_{n \rightarrow \infty} \rho_{[0, T]}(\bar{\phi}_n, \bar{\phi}) = 0, \quad \lim_{n \rightarrow \infty} I_0(\bar{\phi}_n) = I_0(\bar{\phi}). \quad (7.11)$$

This method has been inspired by a similar strategy in [QRV].

Lemma 7.5 \mathcal{C}_1 is $(\rho_{[0, T]}, I_0)$ -dense in \mathcal{C}_0 .

Proof: Fix $\bar{m} \in \mathcal{C}_0$. Let $\bar{R}(t, \cdot)$, $t \in [0, T]$, be the solution of (2.24) with initial datum $\bar{R}(0, \cdot) = \bar{m}_0(\cdot)$. For any $\eta \in (0, T)$, define

$$\bar{m}^\eta(t, \cdot) = \begin{cases} \bar{R}(t, \cdot) & \text{for } t \in [0, \eta], \\ \bar{R}(2\eta - t, \cdot) & \text{for } t \in (\eta, 2\eta], \\ \bar{m}(t - 2\eta, \cdot) & \text{for } t \in (2\eta, T]. \end{cases}$$

We have $\bar{m}^\eta \in \mathcal{C}_1$ for any $0 < \eta < T$ and $\lim_{\eta \rightarrow 0} \rho_{[0, T]}(\bar{m}^\eta, \bar{m}) = 0$. Since I_0 is lower semicontinuous it remains to show

$$\lim_{\eta \rightarrow 0} I_0(\bar{m}^\eta) \leq I_0(\bar{m}). \quad (7.12)$$

We split $[0, T]$ into $[0, 2\eta]$ and $[2\eta, T]$ in the integration. We have that

$$\int_{2\eta}^T \int_{\Lambda} \mathcal{H}(\bar{m}^\eta, \dot{\bar{m}}^\eta)(t, r) dr dt = \int_0^{T-2\eta} \int_{\Lambda} \mathcal{H}(\bar{m}, \dot{\bar{m}})(t, r) dr dt \leq I_0(\bar{m}).$$

Next we show that

$$\lim_{\eta \rightarrow 0} \int_0^{2\eta} \int_{\Lambda} \mathcal{H}(\bar{m}^\eta, \dot{\bar{m}}^\eta)(t, r) dr dt = 0.$$

Since $\bar{m}^\eta = \bar{R}$ for $t \in [0, \eta]$ solves (2.24), by (3) of Proposition 3.8,

$$\int_0^\eta \int_{\Lambda} \mathcal{H}(\bar{R}, \dot{\bar{R}})(t, r) dr dt = 0. \quad (7.13)$$

Since the profile \bar{m}^η in $(\eta, 2\eta]$ is the profile in $(0, \eta]$ backwards in time, we have

$$\int_\eta^{2\eta} \int_{\Lambda} \mathcal{H}(\bar{m}^\eta, \dot{\bar{m}}^\eta)(t, r) dr dt = \int_0^\eta \int_{\Lambda} \mathcal{H}(\bar{R}, -\dot{\bar{R}})(t, r) dr dt. \quad (7.14)$$

Because \bar{R} solves (2.24) and for $t > 0$, $|R_i(t, \cdot)| \leq p_i - \delta_i$, for $i = 1, 2$, $\mathcal{H}(\bar{R}, \dot{\bar{R}})$ belongs to $L^1([0, T] \times \Lambda)$, as well as $\mathcal{H}(\bar{R}, -\dot{\bar{R}})$, see explicit formula (3.14). By dominated convergence,

$$\lim_{\eta \rightarrow 0} \int_\eta^{2\eta} \int_{\Lambda} \mathcal{H}(\bar{m}^\eta, \dot{\bar{m}}^\eta)(t, r) dr dt = 0.$$

In this way we prove (7.12). □

Lemma 7.6 \mathcal{C}_2 is $(\rho_{[0, T]}, I_0)$ -dense in \mathcal{C}_1 .

Proof: Let $\bar{m} \in \mathcal{C}_1$ and $\eta \in (0, T)$ so that $\bar{m}(t, \cdot) = \bar{R}(t, \cdot)$ for $t \in [0, \eta]$. By (7.6), $\|m_i(\eta, \cdot)\|_\infty \leq p_i - \delta_i$ for some $\delta_i > 0$ and $i = 1, 2$. Define

$$m_i^n(t, r) = \begin{cases} m_i(t, r) & \text{for } t \in [0, \eta], \\ m_i(\eta, r) + \left(1 - \frac{1}{n}\right) (m_i(t, r) - m_i(\eta, r)) & \text{for } t \in (\eta, T]. \end{cases} \quad (7.15)$$

By construction and from (7.13), $I_0(\bar{m}^n) = \int_\eta^T \int_{\Lambda} \mathcal{H}(\bar{m}^n(t, r), \frac{\partial \bar{m}^n}{\partial t}(t, r)) dr dt$. Moreover, since $I_0(\bar{m}) < \infty$, by Proposition 3.5 we have $\|m_i(t)\|_\infty \leq p_i$ for $t \in [\eta, T]$, then

$$\|m_i^n(t)\|_\infty \leq p_i - \frac{\delta_i}{n}, \quad \forall t \in [\eta, T]. \quad (7.16)$$

Hence $\bar{m}^n \in \mathcal{C}_2$ for all n . Furthermore $\lim_{n \rightarrow \infty} m_i^n(t, r) = m_i(t, r)$ and $\frac{\partial m_i^n}{\partial t}(t, r) = (1 - \frac{1}{n}) \frac{\partial m_i}{\partial t} \rightarrow \frac{\partial m_i}{\partial t}(t, r)$ for almost all $(t, r) \in [\eta, T] \times \Lambda$. Then, by Proposition 3.5, $\mathcal{H}(\bar{m}^n(t, r), \frac{\partial \bar{m}^n}{\partial t}(t, r))$ is given by (3.14), while $\mathcal{H}(\bar{m}(t, r), \frac{\partial \bar{m}}{\partial t}(t, r))$ is given either by (3.14) when $|m_i(t, r)| < p_i$, or, when $|m_i(t, r)| = p_i$, by (3.15), or is infinite. We hence check that pointwise

$$\lim_{n \rightarrow \infty} \mathcal{H}(\bar{m}^n(t, r), \frac{\partial \bar{m}^n}{\partial t}(t, r)) = \mathcal{H}(\bar{m}(t, r), \frac{\partial \bar{m}}{\partial t}(t, r)). \quad (7.17)$$

To apply the Lebesgue dominated convergence Theorem we upper bound, uniformly with respect to n , $|\mathcal{H}(\bar{m}^n, \frac{\partial \bar{m}^n}{\partial t})(t, r)|$ (see also [C] p. 174). For that we combine (3.16) with the facts that, for $i = 1, 2$,

$$\{(t, r) : \frac{\partial m_i^n}{\partial t} > 0\} = \{(t, r) : \frac{\partial m_i}{\partial t} > 0\},$$

and on the set $\{(t, r) : m_i^n(t, r) \geq p_i - \delta_i\}$ we have $m_i(t, r) - m_i(0, r) \geq 0$ and $p_i - m_i^n(t, r) \geq p_i - m_i(t, r)$.

To get shorter notation, we denote for $\bar{\phi}, \bar{\psi} \in \mathcal{AC}([0, T], B_{p_1, p_2})$

$$\Upsilon(\phi_i, \psi_i) = \mathbb{I}_{\{\dot{\phi}_i > 0; p_i - \delta_i \leq \psi_i\}} \left(\log \frac{1}{p_i - \psi_i} \right)^+ + \mathbb{I}_{\{\dot{\phi}_i < 0; -p_i + \delta_i \geq \psi_i\}} \left(\log \frac{1}{p_i + \psi_i} \right)^+. \quad i = 1, 2$$

We have

$$\begin{aligned} & 2\mathcal{H}(\bar{m}^n, \frac{\partial \bar{m}^n}{\partial t})(t, r) \\ & \leq \sum_{i=1,2} |\dot{m}_i| \left[(\log |\dot{m}_i|)^+ + \Upsilon(\dot{m}_i, m_i^n) + K_i \right] (t, r) \\ & + \sum_{i=1,2} |\dot{m}_i| \left[\mathbb{I}_{\{\dot{m}_i > 0; m_i^n < p_i - \delta_i\}} \left(\log \frac{1}{p_i - m_i^n} \right)^+ + \mathbb{I}_{\{\dot{m}_i < 0; m_i^n > -p_i + \delta_i\}} \left(\log \frac{1}{p_i + m_i^n} \right)^+ \right] (t, r) + C \\ & \leq \sum_{i=1,2} |\dot{m}_i| \left[(\log |\dot{m}_i|)^+ + \Upsilon(\dot{m}_i, m_i) + K_i \right] (t, r) \\ & + \sum_{i=1,2} |\dot{m}_i| \left[\mathbb{I}_{\{\dot{m}_i > 0; m_i^n < p_i - \delta_i\}} \log \frac{1}{\delta_i} + \mathbb{I}_{\{\dot{m}_i < 0; m_i^n > -p_i + \delta_i\}} \log \frac{1}{\delta_i} \right] (t, r) + C \\ & \leq \sum_{i=1,2} |\dot{m}_i| \left[(\log |\dot{m}_i|)^+ + \mathbb{I}_{\{\dot{m}_i > 0\}} \left(\log \frac{1}{p_i - m_i} \right)^+ + \mathbb{I}_{\{\dot{m}_i < 0\}} \left(\log \frac{1}{p_i + m_i} \right)^+ + K_i + \log \frac{1}{\delta_i} \right] (t, r) + C. \end{aligned} \quad (7.18)$$

Since by assumption $I_0(\bar{m}) < \infty$, by Proposition 3.7, part (b), the above upperbound is integrable. By Lebesgue dominated convergence Theorem we then have

$$\lim_{n \rightarrow \infty} I_0(\bar{m}^n) = I_0(\bar{m}). \quad (7.19)$$

Obviously $\bar{m}^n \rightarrow \bar{m}$ in the metric (3.2). \square

Lemma 7.7 \mathcal{C}_3 is $(\rho_{[0, T]}, I_0)$ -dense in \mathcal{C}_2 .

Proof: Take $\bar{\psi} \in \mathcal{C}_2$. To get more regularity we convolve with a smooth kernel the function both in time and space. To perform the convolution in time we extend the definition of $\bar{\psi}$ to $[T, T+1]$ by setting, for each $s \in [0, 1]$,

$$\bar{\psi}(T+s, r) = \bar{u}(s, r), \quad (7.20)$$

where $\bar{u} = (\bar{u}_1, \bar{u}_2)$ is the solution of equation (2.24) with initial condition $\bar{\psi}(T, \cdot)$. Since $\bar{\psi} \in \mathcal{C}_2$ there exist $\delta_i, i = 1, 2$, such that $|\psi_i(T, r)| \leq p_i - \delta_i$. It follows from (7.5) that $\bar{\psi}_i(T + s, r) \leq p_i - \tilde{\delta}_i$ for all $s \in [0, 1]$, for some $\tilde{\delta}_i$ smaller than δ_i . In the following we will denote it always by δ_i . Denote by $\theta_s \bar{\psi}$ the time translation of $\bar{\psi}$, $(\theta_s \bar{\psi})(t, r) = \bar{\psi}(t + s, r)$ for $(t, r) \in [0, T] \times \Lambda$. Let Φ_{ϵ_1} be a smooth non-negative kernel, $\Phi_{\epsilon_1} \in \mathcal{C}^\infty(\Lambda)$ with support in a ball of radius ϵ_1 and integral one which we use as spatial mollifier. For $\epsilon_0 > 0$, let Ψ_{ϵ_0} be the $\mathcal{C}^\infty(\mathbb{R})$ non-negative temporal mollifier with support $[0, \epsilon_0]$ and integral one. Set $\epsilon \equiv (\epsilon_0, \epsilon_1)$, $\epsilon \downarrow 0$ stands for $\epsilon_0 \downarrow 0$ and $\epsilon_1 \downarrow 0$. Let $\eta > 0$ be such that $\bar{\psi}(t, \cdot) = \bar{R}(t, \cdot)$ for $t \in [0, 3\eta]$. Let $\chi_1(t), \chi_2(t)$ be a \mathcal{C}^2 partition of the unity defined by:

$$\begin{cases} \chi_1(t) = 1 & \text{for } t \in [0, \eta], & \chi_1(t) = 0 & \text{for } t \in [2\eta, T], \\ \chi_2(t) = 0 & \text{for } t \in [0, \eta], & \chi_2(t) = 1 & \text{for } t \in [2\eta, T], \\ \chi_1(t) + \chi_2(t) = 1, & \forall t \in [0, T]. \end{cases}$$

Let

$$\psi_i^\epsilon(t, \cdot) = \chi_1(t)\psi_i(t, \cdot) + \chi_2(t) \int_{\mathbb{R}} \Psi_{\epsilon_0}(s)(\Phi_{\epsilon_1} * \theta_s \psi_i)(t, \cdot) ds. \quad (7.21)$$

By construction $\psi_i(\cdot, \cdot), i = 1, 2$, satisfies all the regularity requirements to be in \mathcal{C}_3 . Furthermore, since $|\psi_i(t, r)| \leq p_i - \delta_i, \delta_i > 0$, for all $\epsilon > 0$ and $t \in [0, T]$, we still have that

$$|\psi_i^\epsilon(t, r)| \leq p_i - \delta_i, \quad i = 1, 2, \quad (7.22)$$

therefore $\bar{\psi}^\epsilon \in \mathcal{C}_3$. Moreover

$$\lim_{\epsilon \rightarrow 0} \rho_{[0, T]}(\bar{\psi}^\epsilon, \bar{\psi}) = 0.$$

Since I_0 is lower semicontinuous, see Proposition 3.8, (1), it is enough to prove

$$\lim_{\epsilon \rightarrow 0} I_0(\bar{\psi}^\epsilon) \leq I_0(\bar{\psi}). \quad (7.23)$$

By using the expression (3.9) of I_0 , see Proposition 3.6, we have

$$I_0(\bar{\psi}^\epsilon) - I_0(\bar{\psi}) = \int_0^T \int_{\Lambda} \left[\mathcal{H}(\bar{\psi}^\epsilon, \frac{\partial \bar{\psi}^\epsilon}{\partial t})(t, x) - \mathcal{H}(\bar{\psi}, \frac{\partial \bar{\psi}}{\partial t})(t, x) \right] dx dt. \quad (7.24)$$

We split the time integral into 3 pieces: (i) a first integral on $[0, \eta]$, which is equal to 0 by definition (7.21) of $\bar{\psi}^\epsilon$; (ii) a second one on $[\eta, 2\eta]$, treated in Lemma 7.9 below; (iii) a third one on $[2\eta, T]$, that we now analyze. Notice that for $t \geq 2\eta$, see (7.21), $\chi_1(t) = 0$ and $\chi_2(t) = 1$, therefore $\psi_i^\epsilon(t, \cdot)$ reduces to a convex combination, and we exploit that $\mathcal{H}(\bar{m}, \bar{a})$ is convex with respect to \bar{a} . Then, for $t \geq 2\eta$, by Jensen inequality we obtain

$$\mathcal{H}(\bar{\psi}^\epsilon, \frac{\partial \bar{\psi}^\epsilon}{\partial t})(t, x) \leq \int_{\mathbb{R}} \Psi_{\epsilon_0}(s) \int_{\Lambda} \Phi_{\epsilon_1}(y) \mathcal{H}(\bar{\psi}^\epsilon(t, x), \frac{\partial \bar{\psi}}{\partial t}(t + s, x - y)) dy ds. \quad (7.25)$$

For all $s \in [0, 1]$, $s < T$, we have

$$\begin{aligned}
& \int_{2\eta+s}^T \int_{\Lambda} \mathcal{H}(\bar{\psi}, \frac{\partial \bar{\psi}}{\partial t})(t, x) dxdt = \int_{2\eta}^T \int_{\Lambda} \mathcal{H}(\bar{\psi}, \frac{\partial \bar{\psi}}{\partial t})(t+s, x) dxdt - \int_{T-s}^T \int_{\Lambda} \mathcal{H}(\bar{\psi}, \frac{\partial \bar{\psi}}{\partial t})(t+s, x) dxdt \\
& = \int_{2\eta}^T \int_{\Lambda} \mathcal{H}(\bar{\psi}, \frac{\partial \bar{\psi}}{\partial t})(t+s, x) dxdt \\
& = \int_{\mathbb{R}} \Psi_{\epsilon_0}(s) \int_{2\eta}^T dt \int_{\Lambda} \mathcal{H}(\bar{\psi}, \frac{\partial \bar{\psi}}{\partial t})(t+s, x) dxds \\
& = \int_{\mathbb{R}} \Psi_{\epsilon_0}(s) \int_{2\eta}^T \int_{\Lambda} dy \Phi_{\epsilon_1}(y) \int_{\Lambda} \mathcal{H}(\bar{\psi}, \frac{\partial \bar{\psi}}{\partial t})(t+s, x-y) dxdt ds,
\end{aligned} \tag{7.26}$$

where the first equality comes from a change of variables, the second from the definition of $\bar{\psi}$ in $[T, T+1]$ (see (7.20)), the third from $\int_{\Lambda} dy \Phi_{\epsilon_1}(y) = 1$ and $\int_{\Lambda} dx \mathcal{H}(\bar{\psi}, \frac{\partial \bar{\psi}}{\partial t})(t, x-y) = \int_{\Lambda} dx \mathcal{H}(\bar{\psi}, \frac{\partial \bar{\psi}}{\partial t})(t, x)$, and the last from $\int_{\mathbb{R}} ds \Psi_{\epsilon_0}(s) = 1$. Therefore

$$\begin{aligned}
& \int_{2\eta}^T \int_{\Lambda} \left[\mathcal{H}(\bar{\psi}^{\epsilon}, \frac{\partial \bar{\psi}^{\epsilon}}{\partial t})(t, x) - \mathcal{H}(\bar{\psi}, \frac{\partial \bar{\psi}}{\partial t})(t, x) \right] dxdt \\
& = \int_{2\eta}^T \int_{\Lambda} \mathcal{H}(\bar{\psi}^{\epsilon}, \frac{\partial \bar{\psi}^{\epsilon}}{\partial t})(t, x) dxdt - \int_{2\eta}^{2\eta+s} \int_{\Lambda} \mathcal{H}(\bar{\psi}, \frac{\partial \bar{\psi}}{\partial t})(t, x) dxdt - \int_{2\eta+s}^T \int_{\Lambda} \mathcal{H}(\bar{\psi}, \frac{\partial \bar{\psi}}{\partial t})(t, x) dxdt \\
& \leq \int_{\mathbb{R}} \Psi_{\epsilon_0}(s) \int_{2\eta}^T \int_{\Lambda} \Phi_{\epsilon_1}(y) \int_{\Lambda} \mathcal{H}(\bar{\psi}^{\epsilon}(t, x), \frac{\partial \bar{\psi}}{\partial t}(t+s, x-y)) dx dy dt ds \\
& \quad - \int_{\mathbb{R}} \Psi_{\epsilon_0}(s) \int_{2\eta}^T \int_{\Lambda} \Phi_{\epsilon_1}(y) \int_{\Lambda} \mathcal{H}(\bar{\psi}, \frac{\partial \bar{\psi}}{\partial t})(t+s, x-y) dx dy dt ds.
\end{aligned} \tag{7.27}$$

The inequality holds by (7.25), and because $\mathcal{H}(\bar{\psi}, \frac{\partial \bar{\psi}}{\partial t})(t, x) \geq 0$. Finally we use (7.26). To estimate the last difference in (7.27) we add and subtract to it the term

$$\int_{\mathbb{R}} ds \Psi_{\epsilon_0}(s) \int_{2\eta}^T dt \int_{\Lambda} \Phi_{\epsilon_1}(y) dy \int_{\Lambda} dx \mathcal{H}(\bar{\psi}(t, x), \frac{\partial \bar{\psi}}{\partial t}(t+s, x-y)),$$

which gives

$$\int_{2\eta}^T \int_{\Lambda} \left[\mathcal{H}(\bar{\psi}^{\epsilon}, \frac{\partial \bar{\psi}^{\epsilon}}{\partial t})(t, x) - \mathcal{H}(\bar{\psi}, \frac{\partial \bar{\psi}}{\partial t})(t, x) \right] dxdt \leq W_1 + W_2,$$

where

$$\begin{aligned}
W_1 &= \int_{\mathbb{R}} \Psi_{\epsilon_0}(s) \int_{2\eta}^T \int_{\Lambda} \Phi_{\epsilon_1}(y) \int_{\Lambda} \left[\mathcal{H}(\bar{\psi}^{\epsilon}(t, x), \frac{\partial \bar{\psi}}{\partial t}(t+s, x-y)) - \mathcal{H}(\bar{\psi}(t, x), \frac{\partial \bar{\psi}}{\partial t}(t+s, x-y)) \right] dx dy dt ds, \\
W_2 &= \int_{\mathbb{R}} \Psi_{\epsilon_0}(s) \int_{2\eta}^T \int_{\Lambda} \Phi_{\epsilon_1}(y) \int_{\Lambda} \left[\mathcal{H}(\bar{\psi}(t, x), \frac{\partial \bar{\psi}}{\partial t}(t+s, x-y)) - \mathcal{H}(\bar{\psi}, \frac{\partial \bar{\psi}}{\partial t})(t+s, x-y) \right] dx dy dt ds.
\end{aligned} \tag{7.28}$$

Taking into account Lemma 7.8 below we get the result. \square

Lemma 7.8

$$\lim_{\epsilon \rightarrow 0} |W_i| = 0, \quad i = 1, 2.$$

Proof: We exploit that $\bar{\psi} \in \mathcal{AC}([0, T], B_{p_1, p_2})$ and $\bar{\psi}$ is differentiable in time in $(T, T+1]$ (see (7.20)), hence $\frac{\partial \psi_i}{\partial t} \in L^1([0, T+1] \times \Lambda)$. Therefore for $A > 0$ and

$$D_A = \{x \in \Lambda : \sup_{t \in [0, T+1]} \sum_{i=1}^2 \left| \frac{\partial \psi_i}{\partial t}(t, x) \right| > A\}. \quad (7.29)$$

we have for all $s \in [0, 1]$,

$$\lim_{A \rightarrow \infty} \sum_{i=1}^2 \int_0^T \int_{\Lambda} \left| \frac{\partial \psi_i}{\partial t}(t+s, x) \right| \mathbb{1}_{D_A}(x) dx dt = 0. \quad (7.30)$$

By (8.19) of Lemma 8.8, we obtain, for $2\nu_0 = \min\{\delta_1, \delta_2\}$, splitting $\Lambda = D_A \cup D_A^c$,

$$\begin{aligned} |W_1| &\leq \int_{\mathbb{R}} \Psi_{\epsilon_0}(s) \int_{\Lambda} \Phi_{\epsilon_1}(y) \int_{2\eta}^T \int_{\Lambda} \sum_{i=1}^2 \left| \frac{\partial \psi_i}{\partial t}(t+s, x-y) \right| (\mathbb{1}_{D_A}(x-y) + \mathbb{1}_{D_A^c}(x-y)) \left(1 + \frac{K}{\nu_0}\right) \\ &\quad \{|\psi_i^\epsilon(t, x) - \psi_i(t, x)| + \|\psi_i^\epsilon(t, \cdot) - \psi_i(t, \cdot)\|_1\} dx dt dy ds \\ &\leq A \left(1 + \frac{K}{\nu_0}\right) \sup_{t \in [0, T]} \sum_{i=1}^2 \left(\int_{\Lambda} |\psi_i^\epsilon(t, x) - \psi_i(t, x)| dx + \|\psi_i^\epsilon(t, \cdot) - \psi_i(t, \cdot)\|_1 \right) \\ &\quad + 4 \sum_{i=1}^2 \int_{2\eta}^T \int_{\Lambda} \left| \frac{\partial \psi_i}{\partial t}(t+s, x) \right| \mathbb{1}_{D_A}(x) dx dt, \end{aligned} \quad (7.31)$$

where we noticed that $\{|\psi_i^\epsilon(t, x) - \psi_i(t, x)| + \|\psi_i^\epsilon(t, \cdot) - \psi_i(t, \cdot)\|_1\} \leq 4$. Letting first $\epsilon \rightarrow 0$ then $A \rightarrow \infty$, we get $\lim_{\epsilon \rightarrow 0} |W_1| = 0$. Next we estimate W_2 . We apply (8.20) of Lemma 8.8. More precisely

$$\begin{aligned} &\left| \mathcal{H}(\bar{\psi}(t+s, x-y), \frac{\partial \bar{\psi}}{\partial t}(t+s, x-y)) - \mathcal{H}(\bar{\psi}(t, x), \frac{\partial \bar{\psi}}{\partial t}(t+s, x-y)) \right| \\ &\leq \sum_{i=1}^2 \left| \frac{\partial \psi_i}{\partial t}(t+s, x-y) \right| \left(1 + \frac{K}{\nu_0}\right) \{(|\psi_i(t+s, x-y) - \psi_i(t, x)|) + |y|\}. \end{aligned}$$

As before take $A > 0$ large enough, split $\Lambda = D_A \cup D_A^c$, to get

$$\begin{aligned} |W_2| &\leq A \left(1 + \frac{K}{\nu_0}\right) \sum_{i=1}^2 \int_{\mathbb{R}} \Psi_{\epsilon_0}(s) \int_{2\eta}^T \int_{\Lambda} \Phi_{\epsilon_1}(y) \int_{\Lambda} \{(|\psi_i(t+s, x-y) - \psi_i(t, x)|) + |y|\} dx dy dt ds \\ &\quad + C \sum_{i=1}^2 \int_{2\eta}^T \int_{\Lambda} \int_{\mathbb{R}} \Psi_{\epsilon_0}(s) \mathbb{1}_{D_A}(x) \left| \frac{\partial \psi_i}{\partial t}(t+s, x) \right| ds dx dt. \end{aligned}$$

Since

$$\lim_{\epsilon_1 \rightarrow 0} \int_{\Lambda} \Phi_{\epsilon_1}(y) |y| dy = 0, \quad \lim_{\epsilon_0 \rightarrow 0} \int_{\mathbb{R}} x \Psi_{\epsilon_0}(s) \int_{\Lambda} |\psi_i(t+s, x-y) - \psi_i(t, x)| dx ds = 0,$$

letting $\epsilon \rightarrow 0$ and then $A \rightarrow \infty$ we obtain $\lim_{\epsilon \rightarrow 0} |W_2| = 0$. \square

Lemma 7.9

$$\lim_{\epsilon \rightarrow 0} \int_{\eta}^{2\eta} \int_{\Lambda} \left[\mathcal{H}(\bar{\psi}^\epsilon, \frac{\partial \bar{\psi}^\epsilon}{\partial t})(t, x) - \mathcal{H}(\bar{\psi}, \frac{\partial \bar{\psi}}{\partial t})(t, x) \right] dx dt = 0.$$

Proof: We have

$$\begin{aligned}
& \int_{\eta}^{2\eta} \int_{\Lambda} \left[\mathcal{H}(\bar{\psi}^{\epsilon}, \frac{\partial \bar{\psi}^{\epsilon}}{\partial t})(t, x) - \mathcal{H}(\bar{\psi}, \frac{\partial \bar{\psi}}{\partial t})(t, x) \right] dx dt \\
&= \int_{\eta}^{2\eta} \int_{\Lambda} \left[\mathcal{H}(\bar{\psi}^{\epsilon}, \frac{\partial \bar{\psi}^{\epsilon}}{\partial t})(t, x) - \mathcal{H}(\bar{\psi}, \frac{\partial \bar{\psi}^{\epsilon}}{\partial t})(t, x) \right] dx dt \\
&+ \int_{\eta}^{2\eta} \int_{\Lambda} \left[\mathcal{H}(\bar{\psi}, \frac{\partial \bar{\psi}^{\epsilon}}{\partial t})(t, x) - \mathcal{H}(\bar{\psi}, \frac{\partial \bar{\psi}}{\partial t})(t, x) \right] dx dt.
\end{aligned} \tag{7.32}$$

The first term is estimated by applying (8.19) of Lemma 8.8. We have

$$\begin{aligned}
& \left| \int_{\eta}^{2\eta} \int_{\Lambda} \left[\mathcal{H}(\bar{\psi}^{\epsilon}, \frac{\partial \bar{\psi}^{\epsilon}}{\partial t})(t, x) - \mathcal{H}(\bar{\psi}, \frac{\partial \bar{\psi}^{\epsilon}}{\partial t})(t, x) \right] dx dt \right| \\
&\leq \sum_{i=1,2} \int_{\eta}^{2\eta} \int_{\Lambda} \left(1 + \frac{K}{\nu_0} \right) \left| \frac{\partial \psi_i^{\epsilon}(t, x)}{\partial t} \right| \left(\|\bar{\psi}^{\epsilon}(t, \cdot) - \bar{\psi}(t, \cdot)\|_1 + |\psi_i(t, x) - \psi_i^{\epsilon}(t, x)| \right) dx dt,
\end{aligned} \tag{7.33}$$

where $\nu_0 = \min\{\delta_1, \delta_2\}$. Note that

$$\frac{\partial \psi_i^{\epsilon}}{\partial t}(t, \cdot) = \chi_1(t) \frac{\partial \psi_i}{\partial t}(t, \cdot) + \chi_2(t) \int_{\mathbb{R}} \Psi_{\epsilon_0}(s) (\Phi_{\epsilon_1} * (\theta_s \frac{\partial \psi_i}{\partial t}))(t, \cdot) + \chi_1'(t) [\psi_i(t, \cdot) - \psi_i^{\epsilon}(t, \cdot)] ds, \quad t \in [0, T], \tag{7.34}$$

where we denote $\chi_i'(t) = \frac{d}{dt} \chi_i(t)$, for $i = 1, 2$, and we use that $\chi_2'(t) = -\chi_1'(t)$. Since for $t \in (0, 3\eta)$, $\bar{\psi}(t) = \bar{R}(t)$ solves (2.24) we have

$$\sup_{t \in [\eta, 2\eta]} \sup_{x \in \Lambda} \left| \frac{\partial \psi_i^{\epsilon}}{\partial t}(t, \cdot) \right| \leq \frac{6}{\eta}. \tag{7.35}$$

Therefore for all η letting $\epsilon \rightarrow 0$ the term in the right hand side of (7.33) goes to zero. For the second term in the r.h.s. of (7.32), applying (8.21) in Lemma 8.8 and taking into account (7.35), we get the result. \square

8 Appendix

We prove in this section the properties of the rate functional given in Section 3.

Proof of Lemma 3.3 The differentiability of $\Gamma_{\bar{V}}(\bar{u})$ in $(L^{\infty}(\Lambda))^2$ is easily verified. For the convexity we compute first the Hessian of $\Gamma_{\bar{V}}(\bar{u})$ with respect to V_1 and V_2 . Since the Hessian is a diagonal matrix, it is enough to study separately the convexity with respect to V_1 and V_2 , we do it for V_1 . For $r \in \Lambda$, we set $V_1(r) = x, u_1(r) = m$ and denote by

$$f_1(x) = (p_1 \tanh \vartheta - m) \sinh(2x) + (p_1 - m \tanh \vartheta) (\cosh(2x) - 1)$$

the integrand term in $\Gamma_{\bar{V}}(\bar{u})$ which depends only on \bar{V} with ϑ varying in some bounded interval of \mathbb{R} , $x \in \mathbb{R}$, $|m| \leq p_1$. We then study the sign of the second derivative of f_1 .

$$\frac{1}{4} f_1''(x) = \cosh(2x) [p_1 - m \tanh \vartheta] + \sinh(2x) [p_1 \tanh \vartheta - m].$$

Notice that $p_1 - m \tanh \vartheta \geq 0$, and $p_1 - m \tanh \vartheta \geq p_1 \tanh \vartheta - m \geq -(p_1 - m \tanh \vartheta)$. Since $\cosh(2x) > |\sinh(2x)|$ and $|\tanh \vartheta| < 1$ when ϑ varies in a bounded interval we obtain that f_1 is convex. \square

Proof of Proposition 3.5 Recalling (3.12), for $i = 1, 2$, denote

$$\begin{aligned} F_i(v_i) &= g_i v_i - \frac{1}{2} B_i(\bar{u}, v_i) \\ &= g_i v_i - (p_i - u_i) \frac{e^{A_i}}{4 \cosh A_i} [e^{2v_i} - 1] - (p_i + u_i) \frac{e^{-A_i}}{4 \cosh A_i} [e^{-2v_i} - 1]. \end{aligned} \quad (8.8)$$

Hence

$$\frac{\partial F_i}{\partial v_i} = g_i - (p_i - u_i) \frac{e^{A_i}}{2 \cosh A_i} e^{2v_i} + (p_i + u_i) \frac{e^{-A_i}}{2 \cosh A_i} e^{-2v_i}. \quad (8.9)$$

First assume that $u_i > p_i$. By (8.8), since $-(p_i - u_i) > 0$, we have $\lim_{v_i \rightarrow +\infty} F_i(v_i) = +\infty$. In the same way, we get $\lim_{v_i \rightarrow -\infty} F_i(v_i) = +\infty$ if $u_i < -p_i$ because then $-(p_i + u_i) > 0$. Therefore, (a) holds.

For the remaining cases, we exploit that for $\bar{u} \in B_{p_1, p_2}$, the function $v_i \mapsto B_i(\bar{u}, v_i)$ is convex differentiable on \mathbb{R} .

(b) To compute the Legendre transform of $B_i(\bar{u}, v_i)$, when $|u_i| < p_i$, by (8.9), the maximum in (3.11) is obtained for (remember (3.13))

$$e^{2v_i} = e^{-A_i} \frac{D_i}{p_i - u_i}, \quad \text{hence } v_i = \frac{1}{2} \left(\log \frac{D_i}{p_i - u_i} - A_i \right). \quad (8.10)$$

Inserting (8.10) in (8.8) we have

$$H_i(\bar{u}, g_i) = \frac{g_i}{2} \left[\log \frac{D_i}{p_i - u_i} - A_i \right] + p_i \frac{e^{A_i} + e^{-A_i}}{4 \cosh A_i} - u_i \frac{e^{A_i} - e^{-A_i}}{4 \cosh A_i} - \frac{1}{4 \cosh A_i} \left[D_i + \frac{p_i^2 - u_i^2}{D_i} \right], \quad (8.11)$$

which yields (3.14) since, using (3.13), we write

$$D_i + \frac{p_i^2 - u_i^2}{D_i} = g_i \cosh A_i + R_i + \frac{(p_i^2 - u_i^2)(g_i \cosh A_i - R_i)}{(g_i \cosh A_i)^2 - (g_i \cosh A_i)^2 - p_i^2 + u_i^2} = 2R_i.$$

(c) When $u_i = p_i$ (resp. $u_i = -p_i$),

$$\frac{\partial F_i}{\partial v_i} = g_i + p_i \frac{e^{-\operatorname{sgn}(u_i)A_i}}{\cosh A_i} e^{-2\operatorname{sgn}(u_i)v_i}$$

and to solve $\frac{\partial F_i}{\partial v_i} = 0$ (that is to find a finite extremum) we need $g_i < 0$ (resp. $g_i > 0$), namely

$$g_i = -p_i \frac{e^{-\operatorname{sgn}(u_i)A_i}}{\cosh A_i} e^{-2\operatorname{sgn}(u_i)v_i}.$$

Inserting this value in (8.8) we get (3.15) when $g_i \neq 0$.

When $u_i = p_i$ (resp. $u_i = -p_i$) and $g_i = 0$, (8.8) becomes

$$F_i(v_i) = p_i \frac{e^{-\operatorname{sgn}(u_i)A_i}}{2 \cosh A_i} [1 - e^{-2\operatorname{sgn}(u_i)v_i}].$$

It is an increasing (resp. decreasing) function with a finite maximal limit:

$$\lim_{v_i \rightarrow +\infty} F_i(v_i) = p_i \frac{e^{-\operatorname{sgn}(u_i)A_i}}{2 \cosh A_i} = H_i(\bar{u}, g_i), \quad \text{resp.} \quad \lim_{v_i \rightarrow -\infty} F_i(v_i) = p_i \frac{e^{-\operatorname{sgn}(u_i)A_i}}{2 \cosh A_i} = H_i(\bar{u}, g_i).$$

(d) When $u_i = p_i$ and $g_i > 0$ (resp. $u_i = -p_i$ and $g_i < 0$), (8.8) becomes

$$F_i(v_i) = g_i v_i + p_i \frac{e^{-\operatorname{sgn}(u_i)A_i}}{2 \cosh A_i} [1 - e^{-2\operatorname{sgn}(u_i)v_i}].$$

Hence

$$\lim_{v_i \rightarrow +\infty} F_i(v_i) = +\infty = H_i(\bar{u}, g_i), \quad \text{resp.} \quad \lim_{v_i \rightarrow -\infty} F_i(v_i) = +\infty = H_i(\bar{u}, g_i).$$

□

Proof of Proposition 3.6 This follows and extends the proof in [C], p. 171, Properties III(a). By their respective Definitions (3.6), (3.8), (3.9) (see also (3.4), (3.7), (3.10)), we have $J_0(\bar{\pi}) \leq I_0(\bar{\pi}) \leq J_1(\bar{\pi})$. We now prove that we have equalities. In all cases, for $i = 1, 2$, we denote by ϑ_i the value of v_i that realizes the extremum of $H_i(\bar{u}, g_i)$. From Proposition 3.5, ϑ_i belongs to $\mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$. Therefore we introduce $\vartheta_i^m = \operatorname{sgn}(\vartheta_i) \times [|\vartheta_i| \wedge m]$ and b_i^m the corresponding (finite) value of $H_i(\bar{u}, g_i)$. Then as $m \rightarrow \infty$, $\vartheta_i^m \rightarrow \vartheta_i$ and $b_i^m \rightarrow H_i(\bar{u}, g_i) \in \mathbb{R}^+ \cup \{\infty\}$. According to the case we consider, either a_i and/or b_i are finite, and there is no problem, or $b_i = +\infty$ thus $b_i^m > 0$ for m large enough, or, when $\bar{u} \in B_{p_1, p_2}$, b_i^m is non-negative because ϑ_i^m is between 0 and ϑ_i , and $v_i \mapsto B_i(\bar{u}, v_i)$ is a convex function. Therefore in all cases we apply Fatou's Lemma to get

$$J_1(\bar{\pi}) \leq \liminf_{m \rightarrow \infty} \int_0^T \int_{\Lambda} \sum_{i=1,2} b_i^m(t, r) dr dt$$

which is smaller than $J_0(\bar{\pi})$, giving Proposition 3.6. Notice that this implies that $I_0(\bar{\pi})$ is infinite when $\bar{u} \notin B_{p_1, p_2}$. □

Proof of Proposition 3.7 We use the explicit representation of $\mathcal{H}(\cdot, \cdot)$ given in Proposition 3.5.

(a) We upper bound expression (3.14). The difficulty comes from the term

$$F(\bar{u}, g_i, \theta) = g_i \log \frac{D_i(\bar{u}, g_i, \theta)}{p_i - u_i}, \quad (8.12)$$

where $D_i(\bar{u}, g_i, \theta)$ is defined in (3.13). Let $-\bar{u} = (-u_1, -u_2)$. We have

$$\begin{aligned} F(-\bar{u}, -g_i, \theta) &= g_i \log \frac{(p_i + u_i) \left\{ g_i \cosh[(J * u) - a_i \theta] + \sqrt{(g_i \cosh[(J * u) - a_i \theta])^2 + p_i^2 - u_i^2} \right\}}{- (g_i \cosh[(J * u) - a_i \theta])^2 + (g_i \cosh[(J * u) - a_i \theta])^2 + p_i^2 - u_i^2}} \\ &= g_i \log \frac{g_i \cosh[(J * u) - a_i \theta] + \sqrt{(g_i \cosh[(J * u) - a_i \theta])^2 + p_i^2 - u_i^2}}{p_i - u_i} \\ &= F(\bar{u}, g_i, -\theta). \end{aligned} \quad (8.13)$$

We write

$$F(\bar{u}, g_i, \theta) = F(\bar{u}, g_i, \theta) \mathbb{1}_{\{g_i \geq 0\}} + F(-\bar{u}, -g_i, -\theta) \mathbb{1}_{\{g_i < 0\}}. \quad (8.14)$$

Hence it suffices to estimate $F(\bar{u}, g_i, \theta)$ for $g_i > 0$ and $\theta \in \mathbb{R}$. We get

$$F(\bar{u}, g_i, \theta) \leq |g_i| \left\{ \log D_i(\bar{u}, g_i, \theta) + \left(\log \frac{1}{p_i - u_i} \right)^+ \mathbb{1}_{\{g_i > 0\}} + \left(\log \frac{1}{p_i + u_i} \right)^+ \mathbb{1}_{\{g_i < 0\}} \right\}. \quad (8.15)$$

We obtain (3.16) by the upper bound $D_i(\bar{u}, g_i, \theta) \leq 2|g_i| + 1$. The lower bound (3.17) is obtained as in [C], p. 171. We rely on formulas (3.11), (3.12). Since $e^{\beta a} \leq e^{\beta|a|}$, there exists a constant C such that

$$B_i(\bar{u}, v_i) \leq 2C[e^{2|v_i|} - 1]$$

$$2p_i \geq 2C \geq \max \left\{ (p_i - u_i) \frac{e^{A_i}}{2 \cosh A_i}, (p_i + u_i) \frac{e^{-A_i}}{2 \cosh A_i} \right\}. \quad (8.16)$$

Then

$$H_i(\bar{u}, g_i) \geq \sup_{v_i \in \mathbb{R}} \left\{ g_i v_i - C [e^{2|v_i|} - 1] \right\} = \max \left\{ \frac{|g_i|}{2} \left[\log \frac{|g_i|}{2C} - 1 \right] + C, 0 \right\}.$$

(b) If (3.16) holds then $I_0(\bar{\phi}) < \infty$. For the converse, by (3.17), it is necessary to have $\dot{\phi}_i \log |\dot{\phi}_i| \in L^1([0, T] \times \Lambda)$. To conclude, notice that when $g_i > 0$, uniformly in $\theta \in \mathbb{R}$,

$$F(\bar{u}, g_i, \theta) - g_i \log \{g_i \cosh A_i\} \geq 2g_i \mathbb{I}_{\{g_i > 0\}} \log \frac{1}{p_i - u_i}.$$

□

Proof of Proposition 3.8 For (1), (2) we refer to the similar proof of [C], Theorem III.4, p. 148. To show the first part of (3), notice that for $\bar{V} = 0$ the r.h.s. of the argument of the sup in (3.4) is equal to zero. This implies that for $s \in [0, T]$, $\mathcal{H}^*(\bar{\phi}(s, \cdot), \dot{\bar{\phi}}(s, \cdot)) \geq 0$ in (3.6), therefore $I_0(\bar{\pi}) \geq 0$. For the second half of (3), we start by proving that if $I_0(\bar{\pi}) = 0$, then $\bar{\pi} = (\phi_1 \lambda, \phi_2 \lambda)$ with $\bar{\phi} = (\phi_1, \phi_2) \in \mathcal{AC}([0, T], B_{1,1})$ is the solution to equation (2.24). From Proposition 3.6, we know that $J_0(\bar{\pi}) = 0$ (see (3.7), (3.8)), that is, for any $\bar{V} = (V_1, V_2) \in (L^\infty([0, T] \times \Lambda))^2$, we have

$$\int_0^T \langle \bar{V}(s, \cdot), \dot{\bar{\phi}}(s, \cdot) \rangle ds \leq \frac{1}{2} \int_0^T \Gamma_{\bar{V}(s, \cdot)}(\bar{\phi}(s, \cdot)) ds.$$

Now take $V_2 = 0$ and ηV_1 instead of V_1 , where $\eta > 0$. Denote $\phi = \phi_1 + \phi_2$, then recalling definitions (3.3) and (3.5), we get

$$\begin{aligned} & 2\eta \int_0^T \langle V_1(s, \cdot), \dot{\phi}_1(s, \cdot) \rangle ds \\ & \leq p_1 \int_0^T \langle \tanh(\phi(s, \cdot)) * J + a_1 \theta \sinh(2\eta V_1(s, \cdot)) + \cosh(2\eta V_1(s, \cdot)) - 1 \rangle ds \\ & \quad - \int_0^T \langle \phi_1(s, \cdot) \left(\tanh(\phi(s, \cdot)) * J + a_1 \theta [\cosh(2\eta V_1(s, \cdot)) - 1] + \sinh(2\eta V_1(s, \cdot)) \right) \rangle ds. \end{aligned}$$

Using Taylor expansion in η when $\eta \rightarrow 0$, dividing by η and letting $\eta \rightarrow 0$, we obtain

$$\int_0^T \langle V_1(s, \cdot), \dot{\phi}_1(s, \cdot) \rangle ds \leq p_1 \int_0^T \langle \tanh(\phi(s, \cdot)) * J + a_1 \theta V_1(s, \cdot) \rangle ds - \int_0^T \langle \phi_1(s, \cdot) V_1(s, \cdot) \rangle ds.$$

Since all terms in the previous expression are linear in V_1 , we may change V_1 into $-V_1$ to obtain the converse inequality. Then, exchanging the roles of indices 1 and 2, we have, for $i = 1, 2$,

$$\int_0^T \langle V_i(s, \cdot), \dot{\phi}_i(s, \cdot) \rangle ds = p_i \int_0^T \langle \tanh(\phi(s, \cdot)) * J + a_i \theta V_i(s, \cdot) \rangle ds - \int_0^T \langle \phi_i(s, \cdot) V_i(s, \cdot) \rangle ds.$$

This means that $\bar{\phi}$ is the (unique) weak solution of (2.24), since by definition (2.28) of the rate functional the initial condition is fulfilled.

For the reverse, we prove that if $\bar{\phi} \in \mathcal{AC}([0, T], B_{1,1})$ is the solution of equation (2.24), then $\bar{\pi} = (\phi_1 \lambda, \phi_2 \lambda)$ is such that $J_1(\bar{\pi}) = 0$; hence, by Proposition 3.6, $I_0(\bar{\pi}) = 0$. We insert equation (2.24) into the explicit representation (3.14). Namely if $\bar{\pi}$ solves (2.24) then, by Corollary 7.3,

$$\log \frac{D_i}{p_i - \phi_i} - A_i = 0, \quad (8.17)$$

$$R_i = D_i - (\partial_t \phi_i) \cosh A_i = e^{A_i} (p_i - \phi_i) + (\phi_i - p_i \tanh A_i) \cosh A_i.$$

Hence

$$p_i - \phi_i \tanh A_i - \frac{R_i}{\cosh A_i} = (p_i - \phi_i) \left(1 + \tanh A_i - \frac{e^{A_i}}{\cosh A_i}\right) = 0. \quad (8.18)$$

By (8.17), (8.18), the right hand side of (3.14) is equal to zero, which completes the proof of (3). \square

Next lemma states some technical results needed in the extension of the lower bound.

Lemma 8.8 *For $(t, t') \in [0, T]^2$, $(x, y) \in \Lambda^2$, $\delta_0 > 0$, let $\tau \in \{t, t'\}$, $\zeta \in \{x, y\}$. We assume $(\bar{u}(\tau, \cdot), \bar{v}(\tau, \cdot)) \in (B_{p_1 - \delta_0, p_2 - \delta_0})^2$, $g_i(\tau, \zeta) \in \mathbb{R}$ and $h_i(\tau, \zeta) \in \mathbb{R}$, for $i = 1, 2$; we have*

$$|H_i(\bar{u}(t, x), g_i(t', y)) - H_i(\bar{v}(t, x), g_i(t', y))| \leq |g_i(t', y)| \left(1 + \frac{K}{\delta_0}\right) (\|v(t, \cdot) - u(t, \cdot)\|_1 + |u_i(t, x) - v_i(t, x)|), \quad (8.19)$$

$$|H_i(\bar{v}(t, x), g_i(t', y)) - H_i(\bar{v}(t', y), g_i(t', y))| \leq |g_i(t', y)| \left(1 + \frac{K}{\delta_0}\right) (|x - y| + |v_i(t, x) - v_i(t', y)|), \quad (8.20)$$

$$|H_i(\bar{u}(t, x), g_i(t, x)) - H_i(\bar{u}(t, x), h_i(t, x))| \leq \left(K + \log \frac{1}{\delta_0} + K|g_i(t, x)|\right) |g_i(t, x) - h_i(t, x)|, \quad (8.21)$$

where the constant $K = K(J, \theta)$ may change from one occurrence to the other.

Proof: The assumptions enable to prove (8.19)–(8.21) by writing formula (3.14) for H_i , using (3.13) for A_i, D_i, R_i . The latter depend on \bar{u} or \bar{v} , g_i or h_i , (t, x) or (t', y) . In each computation, we stress the dependence on the involved quantities, writing e.g. $A_i(u)$ for (8.19), $R_i(g_i)$ for (8.21). Notice that, unlike in (3.14), those functions depend not only on (t, x) , but on (t, x) and (t', y) ; this does not change the expression of H_i , since (3.14) was established pointwise in the proof of Proposition 3.5. In the intermediate computations, we omit to write $(t, x), (t', y)$.

We begin with auxiliary estimates. For (8.21), notice that $|(J * u)(t, x)| \leq 1$ (since $\int J(r) dr = 1$), and

$$|A_i(u)| \leq K(J, \theta). \quad (8.22)$$

When $|v_i| \leq p_i - \delta_0$ and $g_i > 0$, we have $p_i^2 - v_i^2 = (p_i - |v_i|)(p_i + |v_i|) \geq p_i \delta_0$, hence

$$g_i K(J, \theta) + 1 \geq R_i(\bar{u}, g_i) \geq \sqrt{g_i^2 + p_i \delta_0} \geq \max(g_i, \sqrt{p_i \delta_0}), \quad (8.23)$$

$$g_i K(J, \theta) + 1 \geq D_i(\bar{v}, g_i) \geq g_i + \max(g_i, \sqrt{p_i \delta_0}). \quad (8.24)$$

For (8.19), we need

$$|(J * u)(t, x) - (J * v)(t, x)| \leq \|J\|_\infty \|u(t, \cdot) - v(t, \cdot)\|_1 \quad (8.25)$$

and its consequences

$$\begin{aligned} & |v_i(t, x) \tanh[A_i(v)(t, x)] - u_i(t, x) \tanh[A_i(u)(t, x)]| \\ & \leq |v_i(t, x) - u_i(t, x)| |\tanh[A_i(v)(t, x)]| + |u_i(t, x)| |\tanh[A_i(v)(t, x)] - \tanh[A_i(u)(t, x)]| \\ & \leq |v_i(t, x) - u_i(t, x)| + \|J\|_\infty \|u(t, \cdot) - v(t, \cdot)\|_1. \end{aligned} \quad (8.26)$$

$$|\cosh[A_i(u)] - \cosh[A_i(v)]| \leq K(J, \theta) \|u(t, \cdot) - v(t, \cdot)\|_1. \quad (8.27)$$

Respectively for (8.20), we need, since $|v(\cdot, \cdot)| \leq 1$,

$$\begin{aligned} & |(J * v)(t, x) - (J * v)(t', y)| \leq |(J * v)(t, x) - (J * v)(t, y)| + |(J * v)(t, y) - (J * v)(t', y)| \\ & \leq \|J'\|_\infty |x - y| + \left| \int_\Lambda J(y - z) [v(t', z) - v(t, z)] dz \right| \\ & \leq \|J'\|_\infty (|x - y| + \|v(t', \cdot) - v(t, \cdot)\|_1). \end{aligned} \quad (8.28)$$

as well as its consequence analogous to (8.26). The proofs of (8.19)–(8.20) go along the same scheme. Namely all the estimates are done pointwise, they rely respectively on (8.24) to (8.28), the other changes being straightforward starting from expressions analogous to (8.29) below. Hence we detail only the proof of (8.19). We have

$$\begin{aligned} & 2 [H_i(\bar{u}(t, x), g_i(t', y)) - H_i(\bar{v}(t, x), g_i(t', y))] \\ & = g_i(t', y) [(J * v)(t, x) - (J * u)(t, x)] + g_i(t', y) \left[\log \frac{D_i(\bar{u})(t, x)}{p_i - u_i(t, x)} - \log \frac{D_i(\bar{v})(t, x)}{p_i - v_i(t, x)} \right] \\ & + v_i(t, x) \tanh[A_i(v)(t, x)] - u_i(t, x) \tanh[A_i(u)(t, x)] + \frac{R_i(\bar{v})(t, x)}{\cosh[A_i(v)(t, x)]} - \frac{R_i(\bar{u})(t, x)}{\cosh[A_i(u)(t, x)]}. \end{aligned} \quad (8.29)$$

Next we show

$$\left| g_i(t', y) \left[\log \frac{D_i(\bar{u})(t, x)}{p_i - u_i(t, x)} - \log \frac{D_i(\bar{v})(t, x)}{p_i - v_i(t, x)} \right] \right| \leq \frac{K}{\delta_0} |u_i(t, x) - v_i(t, x)| (1 + |g_i(t', y)|) + |g_i(t', y)| \|u(t, \cdot) - v(t, \cdot)\|_1. \quad (8.30)$$

To this aim, see (8.12)–(8.14), it is enough to estimate, when $g_i(t', y) > 0$ and uniformly for $\theta \in \mathbb{R}$,

$$\begin{aligned} & |F(\bar{u}(t, x), g_i(t', y), \theta) - F(\bar{v}(t, x), g_i(t', y), \theta)| = \left| g_i(t', y) \left[\log \frac{D_i(\bar{u})(t, x)}{p_i - u_i(t, x)} - \log \frac{D_i(\bar{v})(t, x)}{p_i - v_i(t, x)} \right] \right| \\ & \leq g_i \left[\left| \log \frac{p_i - v_i}{p_i - u_i} \right| + \left| \log \frac{D_i(\bar{u})}{D_i(\bar{v})} \right| \right] \\ & \leq g_i \left| \frac{u_i - v_i}{p_i - u_i} \right| + g_i \left| \frac{D_i(\bar{u}) - D_i(\bar{v})}{D_i(\bar{v})} \right|, \end{aligned} \quad (8.31)$$

because $|\log(1 + a)| \leq \log(1 + |a|) \leq |a|$. By (8.24), $g_i \leq D_i(\bar{v})$. Using also (8.27) we get

$$\begin{aligned} & \left| g_i \frac{D_i(\bar{u}) - D_i(\bar{v})}{D_i(\bar{v})} \right| \leq \left| \frac{g_i^2 (\cosh[A_i(u)] - \cosh[A_i(v)])}{D_i(\bar{v})} \right| + \left| g_i \frac{R_i(\bar{u}) - R_i(\bar{v})}{D_i(\bar{v})} \right| \\ & \leq g_i K(J, \theta) \|u(t, \cdot) - v(t, \cdot)\|_1 + |R_i(\bar{u}) - R_i(\bar{v})|. \end{aligned} \quad (8.32)$$

To estimate the second term on the right hand side of (8.32), we apply (8.22), (8.23), (8.27) and obtain

$$\begin{aligned}
|R_i(\bar{u}) - R_i(\bar{v})| &= \left| \frac{[R_i(\bar{u})]^2 - [R_i(\bar{v})]^2}{R_i(\bar{u}) + R_i(\bar{v})} \right| \\
&\leq \left| \frac{g_i^2 (\cosh^2[A_i(u)] - \cosh^2[A_i(v)])}{R_i(\bar{u}) + R_i(\bar{v})} \right| + \left| \frac{v_i^2 - u_i^2}{R_i(\bar{u}) + R_i(\bar{v})} \right| \\
&\leq \frac{g_i^2 K(J, \theta) \|u(t, \cdot) - v(t, \cdot)\|_1}{R_i(\bar{u}) + R_i(\bar{v})} + \left| \frac{v_i^2 - u_i^2}{R_i(\bar{u}) + R_i(\bar{v})} \right| \\
&\leq |g_i| K(J, \theta) \|u(t, \cdot) - v(t, \cdot)\|_1 + \frac{p_i}{\sqrt{p_i} \delta_0} |u_i - v_i|.
\end{aligned} \tag{8.33}$$

Combining (8.31), (8.33) we obtain (8.30). Next we estimate the last term of (8.29). Taking into account (8.27) and (8.33) we have

$$\begin{aligned}
\left| \frac{R_i(\bar{v})}{\cosh[A_i(v)]} - \frac{R_i(\bar{u})}{\cosh[A_i(u)]} \right| &= \left| \frac{R_i(\bar{v}) - R_i(\bar{u})}{\cosh[A_i(v)]} + R_i(\bar{u}) \frac{\cosh[A_i(u)] - \cosh[A_i(v)]}{\cosh[A_i(v)] \cosh[A_i(u)]} \right| \\
&\leq |R_i(\bar{v}) - R_i(\bar{u})| + R_i(\bar{u}) |\cosh[A_i(u)] - \cosh[A_i(v)]| \\
&\leq \sqrt{\frac{p_i}{\delta_0}} |u_i - v_i| + (K(J, \theta) |g_i| + K(J, \theta) [|g_i| + 1]) \|v(t, \cdot) - u(t, \cdot)\|_1.
\end{aligned} \tag{8.34}$$

Finally, combining (8.25), (8.26), (8.30), (8.34) yields (8.19).

We now derive (8.21) in a similar way:

$$\begin{aligned}
&2 [H_i(\bar{u}(t, x), g_i(t, x)) - H_i(\bar{u}(t, x), h_i(t, x))] \\
&= [h_i(t, x) - g_i(t, x)] \left(A_i(u)(t, x) + \log \frac{1}{p_i - u_i(t, x)} \right) + g_i(t, x) \log D_i(\bar{u}, g_i)(t, x) - h_i(t, x) \log D_i(\bar{u}, h_i)(t, x) \\
&+ \frac{R_i(\bar{u}, h_i)(t, x) - R_i(\bar{u}, g_i)(t, x)}{\cosh[A_i(u)(t, x)]}.
\end{aligned} \tag{8.35}$$

We have, restricting ourselves to $g_i > 0, h_i > 0$, see (8.12)–(8.14), and using $|\log(1 + a)| \leq |a|$ as in (8.31), then (8.24),

$$\begin{aligned}
|g_i \log D_i(g_i) - h_i \log D_i(h_i)| &\leq |g_i - h_i| |\log D_i(g_i)| + h_i |\log D_i(g_i) - \log D_i(h_i)| \\
&\leq |g_i - h_i| |\log D_i(g_i)| + h_i \left| \frac{D_i(g_i) - D_i(h_i)}{D_i(h_i)} \right| \\
&\leq |g_i - h_i| \left(|\log D_i(g_i)| + \frac{h_i \cosh[A_i(u)]}{D_i(h_i)} \right) + h_i \left| \frac{R_i(g_i) - R_i(h_i)}{D_i(h_i)} \right| \\
&\leq |g_i - h_i| (g_i + 1) K(J, \theta) + |R_i(g_i) - R_i(h_i)|.
\end{aligned}$$

Then, as in (8.33),

$$R_i(g_i) - R_i(h_i) = \frac{R_i^2(g_i) - R_i^2(h_i)}{R_i(g_i) + R_i(h_i)} = \frac{(g_i^2 - h_i^2) (\cosh[A_i(u)])^2}{R_i(g_i) + R_i(h_i)}.$$

Therefore, using that $|u_i| \leq p_i - \delta_0$, and (8.23),

$$2 |H_i(\bar{u}, g_i) - H_i(\bar{u}, h_i)| \leq |h_i - g_i| \left(K(J, \theta) + \log \frac{1}{\delta_0} + (g_i + 1)K(J, \theta) + 2[K(J, \theta)]^2 \right).$$

□

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References

- [BBI] Bianchi, A., Bovier, A., Ioffe, D. Sharp Asymptotics for metastability in the random field Curie-Weiss model mean-field models, *Electron. J. Probab.* **14**(2009), 1541–1603.
- [Bill] Billingsley, P. *Convergence of probability measures*. Wiley and Sons, New York, 1968.
- [Bo] Bovier, A. *Statistical Mechanics of Disordered Systems. A Mathematical Perspective*. Cambridge University Press, 2006.
- [BEGK] Bovier, A., Eckhoff, M., Gaynard, V., Klein, M. Metastability in stochastic dynamics of disordered mean-field models, *Probab. Theory Relat. Fields* **119**(2001), 99–161.
- [CDS] Collet, F., Dai Pra, P., Sartori, E. A simple mean field model for social interactions: dynamics, fluctuations, criticality. *J. Stat. Phys.* **139**(2010), 820-858.
- [COP] Cassandro, M., Orlandi, E., Picco, P. Typical configurations for one dimensional random field Kac model. *Ann. Probab.*, **27**, no. 3(1999), 1414–1467.
- [COPV] Cassandro, M., Orlandi, E., Picco, P., Vares, M. E. Typical configurations for one dimensional random field Kac model: localization of the phases. *Electron. J. Probab.*, **20**(2005), 786–864.
- [COP4] Cassandro, M., Orlandi, E., Picco, P. The optimal interface profile for a non-local model of phase separation. *Nonlinearity* **15**(2002), 1621–1651.
- [C] Comets, F. Nucleation for a long range magnetic model. *Ann. Inst. Henri Poincaré, Probab. Statist.* **23**, no. 2(1987), 135–178.
- [CE] Comets, F., Eisele, T. Asymptotic dynamics, non-critical and critical fluctuations for a geometric long-range interacting model. *Commun. Math. Phys.* **118**(1988), 531–567.
- [DD] Dai Pra, P., den Hollander, F. McKean-Vlasov limit for interacting random processes in random media. *J. Stat. Phys.* **84**(1996), 735–771.
- [DOPT] De Masi, A., Orlandi, E., Presutti, E. and Triolo, L. Glauber evolution with Kac potentials I. Mesoscopic and macroscopic limits, interface dynamics. *Nonlinearity*, **7**(1996), 287–301.
Glauber evolution with Kac potentials II. Fluctuation. *Nonlinearity*, **9**(1996), 27–51.
Glauber evolution with Kac potentials. III. Spinodal decomposition. *Nonlinearity*, **9**(1996), 53–114.
- [DV] Donsker, M.D., Varadhan S.R.S. Large deviations from an hydrodynamical limit. *Comm. Pure Appl. Math.* **42**(1989), 243–270.
- [DS] Dunford, N., Schwartz, J.T. *Linear operators*. Wiley and Sons, New York, 1963.
- [FLM] Farfan, J.S. Landim, C. and Mourragui, M. Hydrostatics and dynamical large deviations of boundary driven gradient symmetric exclusion processes. *Stoch. Proc. and their Appl.* **121**(2011), 725–758.

- [FMP] Fontes, L.R., Mathieu, P., Picco, P. On the averaged dynamics of the random field Curie-Weiss model. *Ann. Appl. Probab.* **10**(2000), 1212–1245.
- [HS] Holley, R.A., Stroock, D.W. A martingale approach to infinite systems of interacting processes. *Ann. Probab.*, **4**, vol. 2(1976), 196–223.
- [K] Koukkous A.: Hydrodynamic behavior of symmetric zero-range processes with random rates. *Stoch. Proc. Appl.* **84**(1999), 297–312.
- [KL] Kipnis, C., Landim, C. *Hydrodynamic limit of interacting particle systems*. Springer-Verlag, 1999.
- [KUH] Kac, M., Uhlenbeck, G., and Hemmer, P.C. On the van der Waals theory of vapour-liquid equilibrium. I. Discussion of a one-dimensional model. *J. Math. Phys.* **4**(1963), 216–228.
- [LP] Lebowitz J. and Penrose O. Rigorous treatment of the Van der Waals Maxwell theory of the liquid-vapour transition. *J. Math. Phys.* **7**(1966), 98–113.
- [MP] Mathieu, P., Picco, P. Metastability and convergence to equilibrium for the random field Curie-Weiss model, *J. Stat. Phys.* **91**(1998), 679–732.
- [MO] Mourragui, M., Orlandi, E. Large deviations from a macroscopic scaling limit for particle systems with Kac interactions and random potential. *Ann. Inst. Henri Poincaré, Probab. Statist.* **43**(2007), 677–715.
- [OP] Orlandi, E., Picco, P. Weak large deviations principle for one dimensional random field Kac model. *Electron. J. Probab.*, **14**(2009), 1372–1416.
- [QRV] Quastel, J., Rezakhanlou, F. and Varadhan, S.R.S. Large deviations for the symmetric simple exclusion process in dimension $d \geq 3$. *Probab. Theory Relat. Fields*, **113**(1999), 1–84.
- [P] Presutti, E. *Scaling Limits in Statistical Mechanics and Microstructures in Continuum Mechanics*. Springer Berlin Heidelberg, Series: Theoretical and Mathematical Physics, 2009.
- [Sp] Spohn, H. *Large scale dynamics of interacting particles*. Springer, Berlin, 1991.
- [V] Varadhan, S.R.S. *Large deviations and Applications* CBMS-NSF regional conference Series in Applied Mathematics, **46**(1984), SIAM, Philadelphia.