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A Simple Proof of Stability of Fronts for the Cahn–Hilliard Equation

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Dedicated to Joel Lebowitz on the occasion of his 70th birthday

Abstract: We apply a method developed in our earlier work on a non-local phase kinetics equation to give a simple proof of the non-linear stability of fronts for the Cahn–Hilliard equation.

1. Introduction

In this paper we consider the one dimensional Cahn–Hilliard equation, which is a particularly interesting example of a class of equations for the transport of a conserved order parameter $m(x)$ on $\mathbb{R}$. Such equations generally have the form

$$\frac{\partial}{\partial t} m = \frac{\partial}{\partial x} J, \tag{1.1}$$

where the current J is given in terms of the variation of a free energy functional $\mathcal{F}$ through

$$J = \frac{\partial}{\partial x}\left(\frac{\delta \mathcal{F}}{\delta m}\right). \tag{1.2}$$

In this particular case, the free energy $\mathcal{F}$ is

$$\mathcal{F}(m) = \int_{\mathbb{R}} \left[\frac{1}{2} \left| \frac{\partial}{\partial x} m \right|^2 + \frac{1}{8}(1 - m^2)^2 \right] \mathrm{d}x. \tag{1.3}$$

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The variation in (1.2) is to be computed with respect to the L^2 norm on $\mathbb{R}$, and hence

$$\frac{\delta\mathcal{F}}{\delta m} = -\frac{\partial^2}{\partial x^2}m - \frac{1}{2}m(1-m^2) \tag{1.4}$$

and the equation is

$$\frac{\partial}{\partial t}m = \frac{\partial^2}{\partial x^2}\left(-\frac{\partial^2}{\partial x^2}m - \frac{1}{2}m(1-m^2)\right). \tag{1.5}$$

Clearly the free energy is a decreasing function under this evolution:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(m) = -\int_{\mathbb{R}}\left|\frac{\partial}{\partial x}\frac{\delta\mathcal{F}}{\delta m}(m)\right|^2 \mathrm{d}x, \tag{1.6}$$

and thus our evolution has a Lyapunov functional. We will denote $-\mathrm{d}\mathcal{F}(m)/\mathrm{d}t$ by $\mathcal{I}(m(t))$.

Moreover, the evolution has a conservation law: For all $t > 0$,

$$\int_{\mathbb{R}}(m(x,t) - m(x,0))\mathrm{d}x = 0. \tag{1.7}$$

Replacing derivatives by gradients and divergences in the obvious places, one obtains a two or three dimensional version. In such cases, $m(x)$ represents the order parameter in the model of a binary alloy with a phase transition. The two global equilibrium states correspond to the two minima of the potential $W(m) = (1-m^2)^2/8$. Clearly these are $m = 1$ and $m = -1$. At the boundary between two regions of different phases, there will be a transition from $m = 1$ to $m = -1$. Since the evolution decreases the free energy, we expect that after a short initial time period, these transitions should occur in a way that minimizes the cost in excess free energy. Therefore, in the one dimension across the boundary between two regions of different phase, we expect a "transition profile" that is very close to some translate of $\bar{m}_0$, where

$$\mathcal{F}(\bar{m}_0) = \inf\left\{ \mathcal{F}(m) \,\middle|\, \mathrm{sgn}(x)m(x) \geq 0,\ \lim_{x\to\pm\infty}\mathrm{sgn}(x)m(x) > 0\right\}. \tag{1.8}$$

The minimizer is well known, and easily seen, to be $\bar{m}_0(x) = \tanh(x/2)$. The physical interest in the one dimensional problem is that stability of these minimal free energy transition profiles, which we simply call "fronts" in the rest of the paper, is important for understanding how the boundaries between regions of different phases evolve in higher dimension. Without further mention of the higher dimensional case, we now turn to this stability problem.

The subscript 0 on the minimizer in (1.8) is present because the constraint imposed in (1.8) breaks the translational invariance of the free energy. For any a in $\mathbb{R}$, define

$$\bar{m}_a(x) = \bar{m}_0(x-a). \tag{1.9}$$

These functions $\bar{m}_a$ are the fronts whose stability is to be investigated here. Clearly $\mathcal{F}(\bar{m}_a) = \mathcal{F}(\bar{m}_0)$, so that $\bar{m}_0$ belongs to a one parameter family of minimizers of the free energy. Another family is obtained by reflecting this one because the free energy is also reflection invariant. However, these two families of minimizers separated in all of the relevant metrics, and it suffices to consider just one.

It is easy to guess the result of solving (1.5) for initial data m_0 that is a small perturbation of the front $\bar{m}_0$. The excess free energy should decrease in a way that forces the solution $m(t)$ to tend to the family of fronts, and the conservation law should select $\bar{m}_a$ as the front it should be converging to, so the result should be that, in any reasonable sense, $\lim_{t\to\infty}(m(x,t) - \bar{m}_a(x)) = 0$ with a given in terms of the initial data m_0 through (1.7) in the form

$$\int \big(m(x,0) - \bar{m}_a(x)\big)\mathrm{d}x = 0. \tag{1.10}$$

Our main result is a proof that this is the case. The result has recently been obtained in this case by Bricmont, Kupiainen and Taskinen [2] using renormalization group methods. Their result gives a tighter estimate on the decay rate, but in a weaker norm that does not control the excess free energy. We recently proved such a result for a related equation, the LOP equation, which first appeared in [10] and later rigorously derived from an underlying microscopic model in [7]. The method that we used was developed to deal with the non-local nature of the LOP equation, and the fact that one has no explicit formula for $\bar{m}$ in that case, which precluded the explicit spectral analysis required in the renormalized group method. However, as we show here, the method developed for the LOP equation also applies to the local Cahn–Hilliard equation, and yields a fairly simple proof of the non-local stability. Moreover, this method works directly in physical norms, and it provides an estimate on the rate of decrease of the excess free energy. The result is:

Theorem 1.1. *Consider initial data $m_0(x)$ for the one dimensional Cahn–Hilliard equation* (1.5) *such that*

$$\int x^2(m_0(x) - \bar{m}_0(x))^2\mathrm{d}x \le c_0,$$

where c_0 is any positive constant. Then for any $\epsilon > 0$ there is a strictly positive constant $\delta = \delta(\epsilon, c_0)$ depending only on ϵ and c_0 such that for all inital data with

$$\int (m_0(x) - \bar{m}_0(x))^2\mathrm{d}x \le \delta,$$

the excess free energy $\mathcal{F}(m(t)) - \mathcal{F}(m_0)$ of the corresponding solution $m(t)$ of (1.5) *satisfies*

$$\mathcal{F}(m(t)) - \mathcal{F}(\bar{m}) \le c_2(1 + c_1 t)^{-(9/13-\epsilon)}$$

and

$$\|m(t) - \bar{m}_a\|_1 \le c_2(1 + c_1 t)^{-(5/52-\epsilon)},$$

where c_1 and c_2 are finite constants depending only on ϵ and c_0 and a is given by (1.10).

Since the problem has both a Lyapunov functional and a conservation law, it may appear that it should be a simple matter to prove this result. One reason that it is not so simple is that the decrease of the excess free energy provides only L^2 control, and by itself, only partial control at that. To use the conservation law, one needs L^1 control. Our equation is not dissipative in L^1, a circumstance which is closely related to the lack of a maximum principle. Decrease of free energy can be used to show that the solution $m(x,t)$ approaches some moving front $m_{a(t)}(x)$ in some norm other than L^2. For example, Asselah did this in [1] for the LOP equation studied in [4] and [5], with the approach controlled in the L^∞ norm. But since the free energy is translation invariant, it cannot provide any control over $a(t)$. Moreover, without control on $a(t)$ that prevents it from "running away", it is not at all clear how one can even get L^2 control on the difference between $m(x,t)$ and $m_{a(t)}(x)$, or get a rate estimate. The difficulties in this sort of problem are discussed in more detail in [4]. Here we move directly on to the solution.

Despite what has been said above, understanding the free energy functional $\mathcal{F}$ is still central to understanding the stability. To begin, we introduce the operator $\mathcal{A}$ associated with its second variation at a front $\bar{m}$. First, throughout this paper, we make the following convention: whenever some solution $m(x,t)$ of (1.5) is under discussion, then $v(x,t)$ is defined by

$$v(x,t) = m(x,t) - \bar{m}_{a(t)}(x), \tag{1.11}$$

where $a(t)$ is defined to be that value of c such that

$$\|m(t) - \bar{m}_{a(t)}\|_2 = \inf_{c\in\mathbb{R}}\{\|m(t) - \bar{m}_c\|_2\}. \tag{1.12}$$

It is shown in [4] that $a(t)$ is a well–defined function as long as $\|m(t) - \bar{m}_{a(t)}\|_2$ stays sufficiently small since then the minimum is uniquely attained. Finally, it will be convenient to have the convention that $\bar{m}(x)$ denotes $\bar{m}_{a(t)}(x)$. In the same vein, we shall generally simply write $\mathcal{A}$ in place of $\mathcal{A}_{a(t)}$ for the second variation of $\mathcal{F}$ at $\bar{m}_{a(t)}$, and leave the $a(t)$ implicit. However, in the definition, we shall be explicit:

$$\langle v, \mathcal{A}_a v\rangle_{L^2} = \frac{\mathrm{d}^2}{\mathrm{d}s^2}\mathcal{F}(\bar{m}_a + sv)\bigg|_{s=0}. \tag{1.13}$$

One easily computes that

$$\mathcal{A}v(x) = -v''(x) + V(x)v(x) + v(x), \tag{1.14}$$

where

$$V(x) = \frac{3}{2}\left(\bar{m}^2 - 1\right) = \frac{3}{2}\left(\tanh^2\left(\frac{x}{2}\right) - 1\right). \tag{1.15}$$

The operator $\mathcal{A}$ has a spectral gap:

Lemma 1.2. *In the spectrum of $\mathcal{A}$, 0 is an isolated eigenvalue of multiplicity one. In fact,*

$$\int v(x)\mathcal{A}v(x)\mathrm{d}x \geq \frac{3}{4}\|v\|_2^2$$

for all v with $\int v(x)\bar{m}'(x)\mathrm{d}x = 0$.

Proof. We consider the operator H given by $Hv(x) = -v''(x) + V(x)v(x)$. We know that $\bar{m}'$ is an eigenvector, and that the corresponding eigenvalue is -1. Let $-1 = e_0, e_1, e_2, \dots$ be the negative eigenvalues of H, repeated according to their multiplicity. Then by a bound of Lieb and Thirring [9], one has

$$\sum_j |e_j|^{3/2} \leq \frac{3}{16}\int_{\mathbb{R}} |V(x)|^2 \mathrm{d}x.$$

The integral is easily evaluated and equals 6. Keeping only the first two terms in the sum on the left $1 + |e_1|^{3/2} \leq 18/16$ and this implies that $|e_1| \leq 1/4$. Thus $e_1 \geq -1/4$, and this completes the proof. □

As indicated in Theorem 1.1, we shall start out with $\|v\|_2$ small, and then, because of the smoothing properties of the equation [3, 5], it will be the case that at least a short time later, $\|v\|_2$ is still small, and then $\|v'\|_2$ is small as well. We shall obtain a number of a-priori estimates that hold when $\|v\|_2$ and $\|v'\|_2$ are both small, and shall use them in the final section of the paper to prove that this condition persists indefinitely. The first estimate that we obtain under these conditions shows that the excess free energy of $\bar{m} + v$ is comparable to $\langle v, \mathcal{A}v\rangle$.

Lemma 1.3. *For all $\epsilon > 0$, there are $\delta, \kappa > 0$ so that whenever $\|v\|_2 \leq \delta$ and $\|v'\|_2 \leq \kappa$, then*

$$\frac{1-\epsilon}{2}\langle v, \mathcal{A}v\rangle \leq \mathcal{F}(\bar{m}+v) - \mathcal{F}(\bar{m}) \leq \frac{1+\epsilon}{2}\langle v, \mathcal{A}v\rangle.$$

Proof. One easily computes that

$$\mathcal{F}(\bar{m}+v) - \mathcal{F}(\bar{m}) = \frac{1}{2}\langle v, \mathcal{A}v\rangle + \frac{1}{4}\int \left(2\bar{m}v^3 + v^4\right)\mathrm{d}x.$$

Using the inequality $\|v\|_\infty^2 \leq 2\|v\|_2\|v'\|_2$, one obtains

$$\left|\int \left(2\bar{m}v^3 + \frac{v^4}{2}\right)\mathrm{d}x\right| \leq \left(2\sqrt{2\kappa\delta} + \kappa\delta\right)\|v\|_2^2.$$

By the previous lemma, for κ and δ small enough, $\left(2\sqrt{2\kappa\delta} + \kappa\delta\right)\|v\|_2^2 \leq (\epsilon/2)\langle v, \mathcal{A}v\rangle$, and this completes the proof. □

The first key result is a lower bound on the dissipation in terms of $\mathcal{A}$:

Lemma 1.4. *For any $\epsilon > 0$,*

$$\mathcal{I}(m(t)) = -\frac{d}{dt}\left[\mathcal{F}\big(m(t)\big) - \mathcal{F}\big(\bar{m}\big)\right] \geq (1-\epsilon)\int \left[(\mathcal{A}v)'(x)\right]^2 \mathrm{d}x \tag{1.16}$$

whenever $||v'||_2 \leq \kappa_1(\epsilon)$ and $||v||_2 \leq \delta_1(\epsilon)$ for some strictly positive constants $\kappa_1(\epsilon)$ and $\delta_1(\epsilon)$. Moreover, there exists a constant $\gamma > 0$ so that

$$\int \left[(\mathcal{A}v)'\right]^2 \mathrm{d}x \geq \gamma||v'||_2^2 \tag{1.17}$$

whenever $\int v(x)\bar{m}'(x)dx = 0$.

This theorem is proved in Sect. 2. We use (1.16) only when $\mathcal{I}(m(t)) << \left[\mathcal{F}\big(m(t)\big) - \mathcal{F}\big(\bar{m}\big)\right]$; i.e., when the dissipation is very small compared to the excess free energy. The point is that in this case, v must be very "smooth and spread out" and so both $\langle v', v'\rangle$ and $\langle v, Vv\rangle$ are negligible compared to $\langle v, v\rangle$. That is, $\langle v, \mathcal{A}v\rangle \approx \langle v, v\rangle$, and so the operator $\mathcal{A}$ differs negligibly from the identity. *But if we replace $\mathcal{A}$ by the identity in the linearized evolution equation, it simply becomes the heat equation.* Therefore, when the dissipation is small compared to the excess free energy, we expect heat equation behavior, which we can calculate, to govern this dissipation process.

On the other hand, when the dissipation is *not* small compared to the excess free energy, we are in a position to benefit from the plentiful dissipativity to compute bounds on the rate at which it occurs. This is the *dissipation–dichotomy* in our argument.

To make precise use of the dichotomy, introduce a small parameter ϵ_1 to be fixed later, and distinguish between the times t for which

$$\mathcal{I}(m(t)) \le \epsilon_1\left[\mathcal{F}(m(t)) - \mathcal{F}(\bar{m})\right] \tag{1.18}$$

or

$$\mathcal{I}(m(t)) \ge \epsilon_1\left[\mathcal{F}(m(t)) - \mathcal{F}(\bar{m})\right]. \tag{1.19}$$

When (1.19) is true, there is plenty of dissipation, and the excess free energy is decaying at an exponential rate, and it will be relatively simple to exploit this.

We have already explained that condition (1.18) will help us because under this condition, we will be able to show that $\|v\|_1$ dissipates away *as though* it were a solution of the heat equation with $\int_{\mathbb{R}} v(t)\mathrm{d}x = 0$ for all t. We will return to this shortly, but it depends on the fact that when (1.18) holds, v is very "smooth and spread out". This is used several ways in the proof. Indeed, combining (1.18) with the key bound (1.16),

$$\|\big(\mathcal{A}v\big)'\|_2^2 \le \epsilon_1\left[\mathcal{F}(\bar{m}+v) - \mathcal{F}(\bar{m})\right]$$

under appropriate conditions on v. Then since $\langle v, \mathcal{A}v\rangle$ is comparable with the excess free energy of $\bar{m}+v$, when (1.18) holds, one has $\|\big(\mathcal{A}v\big)'\|_2^2 << \langle v, \mathcal{A}v\rangle$. Any function v such that this is the case is so smooth and spread out that

$$\|\mathcal{A}v\|_2^2 \approx \langle v, \mathcal{A}v\rangle \approx 2[\mathcal{F}(m(t)) - \mathcal{F}(\bar{m})]. \tag{1.20}$$

The precise version of this is given in Theorem 2.3, and it is the key inequality behind the *dissipation–dichotomy* argument. It enables us to "drop" extra powers of $\mathcal{A}$ when (1.18) holds.

We shall also need certain moment inequalities, which show that $v(t)$ can't spread out *too* fast.

Theorem 1.5. *Let $m = \bar{m} + v$ be a solution of* (1.5) *and let C be a positive number. Define $\phi(t)$ by*

$$\phi(t) = 1 + \int_{\mathbb{R}} |x\,(\mathcal{A}v)\,|^2\mathrm{d}x + C\left[\mathcal{F}(\bar{m}+v) - \mathcal{F}(\bar{m})\right]. \tag{1.21}$$

Then for any $\epsilon > 0$, there is a choice of $C < \infty$ and an $\epsilon_1 > 0$ so that one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(t) \le 4(1+\epsilon)\left[\mathcal{F}(\bar{m}+v) - \mathcal{F}(\bar{m})\right] \tag{1.22}$$

whenever (1.18) *holds, and* $||v||_2 \leq \delta_1(\epsilon)$, $||v'||_2 \leq \kappa_1(\epsilon)$, *and* $|a(t)| \leq 1$ *for some strictly positive constants* $\kappa_1(\epsilon)$ *and* $\delta_1(\epsilon)$. *Regardless of whether* (1.18) *holds or not, there is a constant* $K < \infty$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(t) \leq K\,[\mathcal{F}(\bar{m}+v) - \mathcal{F}(\bar{m})] \tag{1.23}$$

for as long as $||v'||_2 \leq \kappa_1(\epsilon)$, $||v||_2 \leq \delta_1(\epsilon)$ *and* $|a(t)| \leq 1$.

Theorem 1.5 is proved in Sect. 3. Theorems 1.4 and 1.5 are the main ingredients of our argument specific to the Cahn–Hilliard equation. The other two ingredients are a constrained form of the uncertainty principle inequality and decay estimate for a system of differential inequalities introduced in [5].

We will now explain what these are, and how they work together to provide the proof of Theorem 1.1.

The constrained form of the uncertainty principle inequality [5] is the following: *Under either of the constraints* $\int \psi(x)\mathrm{d}x = 0$ *or* $\psi(0) = 0$, *one has*

$$\left(\int x^2|\psi(x)|^2\mathrm{d}x\right)\left(\int |\psi'(x)|^2\mathrm{d}x\right) \geq \frac{9}{4}\left(\int |\psi(x)|^2\mathrm{d}x\right)^2. \tag{1.24}$$

The difference between (1.24) and the usual uncertainty principle is a factor of 9 in the constant, and, as we showed in [5], this is crucial for L^1 control. We wish to apply this to $\psi = \mathcal{A}v$. It is clear that $\mathcal{A}v$ will have a zero somewhere, a technical argument is needed to control the location. To explain how all of the pieces of the argument fit together, assume for the moment that the initial data is antisymmetric. Then the solution will be antisymmetric for all time and so

$$\mathcal{A}v(0,t) = 0 \tag{1.25}$$

for all t. The technical argument needed to remove the antisymmetry assumption will be given in Sect. 2. However, assuming (1.25) , we have from (1.16) and (1.24) that

$$\frac{d}{dt}\big[\mathcal{F}\big(m(t)\big) - \mathcal{F}\big(\bar{m}\big)\big] \leq -(1-\epsilon)\frac{9}{4}\frac{\|\mathcal{A}v\|_2^4}{\|x\mathcal{A}v\|_2^2}. \tag{1.26}$$

The problem with this inequality is that the right hand side does not directly involve the excess free energy $\mathcal{F}\big(m(t)\big) - \mathcal{F}\big(\bar{m}\big)$. If it did, we could hope to get a Gronwall inequality for the decay of the excess free energy. The problem is thus one of closure: we have to relate the quantity on the right-hand side to the excess free energy.

Now we are ready to put the pieces together. When (1.20) is valid, interpreting the approximation sign appropriately in terms of ϵ, we can rewrite (1.26) as

$$\frac{d}{dt}\big[\mathcal{F}\big(m(t)\big) - \mathcal{F}\big(\bar{m}\big)\big] \leq -9(1-\epsilon)\frac{[\mathcal{F}(m(t)) - \mathcal{F}(\bar{m})]^2}{\|x\mathcal{A}v\|_2^2}. \tag{1.27}$$

Now define

$$f(t) = \mathcal{F}(\bar{m}+v(t)) - \mathcal{F}(\bar{m}) \tag{1.28}$$

and define $\phi(t)$ as in Theorem 1.5. Then (1.27) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathcal{F}\big(m(t)\big) - \mathcal{F}\big(\bar{m}\big)\right] \le -9(1-\epsilon)\frac{\left[\mathcal{F}\big(m(t)\big) - \mathcal{F}\big(\bar{m}\big)\right]^2}{\phi(t)},$$

and from Theorem 1.5 we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(t) \le (1+\epsilon)4\left[\mathcal{F}(\bar{m}+v) - \mathcal{F}(\bar{m})\right].$$

Notice the condition that $|a(t)| \le 1$ in Theorem 1.5, to which we shall return. Thus, when (1.18) holds, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}f(t) \le -\tilde{A}\frac{f(t)^2}{\phi(t)} \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}\phi(t) \le \tilde{B}f(t) \tag{1.29}$$

with the difference between $\tilde{A}/(\tilde{A}+\tilde{B})$ and $9/13$ arbitrarily small for ϵ small enough for all times t such that (1.18) holds, $\|v(t)\|_2$, $\|v'(t)\|_2$ are sufficiently small and $|a(t)| \le 1$.

On the other hand, when (1.19) holds, there is plenty of dissipation, and using (1.19) and the second half of Theorem 1.5, we get (1.29) with some *different* constants $\tilde{A}$ and $\tilde{B}$ (in fact, $\tilde{A}$ will be the constant K from Theorem 1.5), but such that the ratio $\tilde{A}/(\tilde{A}+\tilde{B})$ is the same. The upshot is that we always have (1.29), but at two different time scales according to whether (1.19) or (1.18) holds. The heuristic idea that we will make precise in Sect. 4 is that by taking the slower of these two time scales, we bound the decay of our system.

Therefore we consider the system of differential inequalities

$$\frac{\mathrm{d}}{\mathrm{d}t}f(t) \le -A\frac{f(t)^2}{\phi(t)} \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}\phi(t) \le Bf(t) \tag{1.30}$$

with $A = 9$ and $B = 4$. Theorem 5.1 of [4] says that for any solution of (1.30),

$$f(t) \le f(0)^{1-q}\phi(0)^q\left(\frac{\phi(0)}{f(0)} + (A+B)t\right)^{-q},$$

$$\phi(t) \le f(0)^{1-q}\phi(0)^q\left(\frac{\phi(0)}{f(0)} + (A+B)t\right)^{1-q},$$

where $q = A/(A+B)$. In the case at hand, this is $q = 9/13$. Since this value exceeds $1/2$, we get L^1 decay in the following way: By the elementary Lemma 5.2 of [5], for any function w and any $0 < \delta < 1$,

$$\|w\|_1 \le C(\delta)\|(1+x^2)^{1/2}w\|_2^{(1+\delta)/2}\|w\|_2^{(1-\delta)/2}, \tag{1.31}$$

where $C(\delta)$ is a finite constant. (This same method may be applied to solutions u of the heat equation $\partial u/\partial t = u''$ with $\int_{\mathbb{R}} u(t)\mathrm{d}x = 0$ to estimate the rate of L^1 decay, as shown in [5].)

Here, we apply (1.31) with $w = \mathcal{A}v(t)$, so that we obtain

$$\|\mathcal{A}v(t)\|_1^2 \le C(\delta)\phi(t)^{1+\delta}\|\mathcal{A}v(t)\|_2^{1-\delta}.$$

Since $9/13 > 1/2$ for δ sufficiently small, we have that $\phi(t)^{1+\delta}$ increases more slowly than $\|\mathcal{A}v(t)\|_2^{1-\delta}$ decreases, and so $\|\mathcal{A}v(t)\|_1$ decreases to zero. In fact, the rate one gets is arbitrarily close to $t^{-5/26}$, for δ sufficiently small, as in Theorem 1.1.

This leads to

$$\lim_{t\to\infty}\int_{\mathbb{R}} \mathcal{A}v(x,t)\mathrm{d}x = \lim_{t\to\infty}\int_{\mathbb{R}} (V(x)+1)\, v(x,t)\mathrm{d}x = 0.$$

But $\left|\int_{\mathbb{R}} V(x)v(x,t)\mathrm{d}x\right| \leq \|V\|_2\|v(t)\|_2$, and this tends to zero as t tends to infinity by the above, so that finally, $\lim_{t\to\infty}\int_{\mathbb{R}} v(x,t)\mathrm{d}x = 0$. But (1.7) is equivalent to

$$\int_{\mathbb{R}} \left(\bar{m}_{a(t)}(x) - m(x,0)\right)\mathrm{d}x + \int_{\mathbb{R}} v(x,t)\mathrm{d}x = 0,$$

and hence $\lim_{t\to\infty}\int_{\mathbb{R}}\left(\bar{m}_{a(t)}(x) - m(x,0)\right)\mathrm{d}x = 0$ so that $\lim_{t\to\infty} a(t) = a$, where a is determined through (1.10) . Indeed, the map $a \mapsto \int_{\mathbb{R}} (\bar{m}_a(x) - m(x,0))\,\mathrm{d}x$ is linear, and the slope is $-\int_{\mathbb{R}} \bar{m}_a'(x)\mathrm{d}x = -2$, as one sees simply by differentiating. Thus,

$$\left|\int_{\mathbb{R}} \left(\bar{m}_{a(t)}(x) - m(x,0)\right)\mathrm{d}x\right| = 2|a(t) - a|.$$

2. Free Energy Estimates

It follows from (1.6) and the definition of $\mathcal{A}$, one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(m) = -\int_{\mathbb{R}} \left[\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathcal{A}v - \frac{1}{2}\left(3\bar{m}v^2 + v^3\right)\right)\right]^2 \mathrm{d}x. \tag{2.1}$$

For convenience of notation, define

$$U = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\left(3\bar{m}v^2 + v^3\right) = -\frac{3}{2}\left(\bar{m}'v^2 + 2\bar{m}vv' + v^2v'\right). \tag{2.2}$$

Now for any f and g in L^2 and for any $0 < \epsilon < 1$,

$$\|f+g\|_2^2 \geq (1-\epsilon)\|f\|_2^2 - \frac{1}{\epsilon}\|g\|_2^2. \tag{2.3}$$

Combining (2.1), (2.2) and (2.3), we have

$$-\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(m) = \int_{\mathbb{R}} \left|(\mathcal{A}v)' + U\right|^2 \mathrm{d}x \geq (1-\epsilon)\int_{\mathbb{R}} \left|(\mathcal{A}v)'\right|^2 \mathrm{d}x - \frac{1}{\epsilon}\int_{\mathbb{R}} |U|^2\,\mathrm{d}x. \tag{2.4}$$

The following lemma is closely based on lemmas and arguments in Sect. 3 of [4]. We have stated it so that it applied to a general class of potentials because the proof, although somewhat involved, depends only on fairly general properties of $\bar{m}$ and $\mathcal{A}$.

Theorem 2.1. *Let $v \in L^2(\mathbb{R})$, $v' \in L^2(\mathbb{R})$ and $\int v(x)\bar{m}'(x)dx = 0$ then there exists a positive constant γ, such that*

$$\int \left[(\mathcal{A}v)'\right]^2 \mathrm{d}x \geq \gamma||v'||_2^2, \tag{2.5}$$

where $\mathcal{A}$ is the linear operator defined in (1.14) .

Proof. First observe that $(\mathcal{A}v)' = \mathcal{A}v' + V'v$, where V is given in (1.15). Next, $v(x) = v(y) + \int_y^x v'(z)\mathrm{d}z$. Multiply both sides by $\bar{m}'(y)$, and integrate in y. Since $\int v(y)\bar{m}'(y)dy = 0$, and since $\int \bar{m}'(y)dy = 2$,

$$v(x) = \frac{1}{2}\int_{-\infty}^{\infty} \bar{m}'(y)\left(\int_y^x v'(z)\mathrm{d}z\right)\mathrm{d}y. \tag{2.6}$$

Hence

$$(\mathcal{A}v)' = \mathcal{A}v' + Kv', \tag{2.7}$$

where

$$K\phi(x) = V'(x)\frac{1}{2}\int_{-\infty}^{\infty} \bar{m}'(y)\left(\int_y^x \phi(z)\mathrm{d}z\right)\mathrm{d}y.$$

The operator K is compact on L^2. A detailed proof in a closely related case is given in [4]. Now consider the quadratic form $\mathcal{Q}(\phi)$ given by

$$\mathcal{Q}(\phi) = \|\,(\mathcal{A}+K)\,\phi\|_2^2$$

for ϕ in the domain of $\mathcal{A}$.

We next show that $\mathcal{Q}(\phi) > 0$ for all ϕ in its domain. Suppose on the contrary that $\mathcal{Q}(\phi) = 0$ for some ϕ in the domain of $\mathcal{Q}$, which is the operator domain of $\mathcal{A}$. Define

$$\eta(x) = \int_0^x \phi(y)\mathrm{d}y = \langle 1_{[0,x]}, \phi\rangle.$$

It follows by the Schwarz inequality that

$$|\eta(x)| \le \|\phi\|_2\sqrt{|x|} \qquad \text{for all} \quad x. \tag{2.8}$$

It then follows that $K\phi = V'\eta - \frac{1}{2}V'\langle\bar{m}', \eta\rangle$, where the inner product on the right is well defined because of the exponential decay of $\bar{m}'$ and (2.8). Hence

$$(\mathcal{A}+K)\,\phi = \mathcal{A}\eta' + V'\eta - \frac{1}{2}V'\langle\bar{m}', \eta\rangle = (\mathcal{A}\eta)' - \frac{1}{2}V'\langle\bar{m}', \eta\rangle.$$

Since the right side is a total derivative, we have

$$\mathcal{A}\eta - \frac{1}{2}V\langle\bar{m}', \eta\rangle = C, \tag{2.9}$$

where C is a constant. To determine C, multiply both sides by $\bar{m}'$, and integrate. Note that $\int \bar{m}'\,(\mathcal{A}\eta)\,\mathrm{d}x = 0$, because (2.8) permits the integration by parts. The computation then yields $C = (1/2)\langle\bar{m}', \eta\rangle$. Putting this in (2.9) yields $\mathcal{A}\left(\eta - (1/2)\langle\bar{m}', \eta\rangle\right) = 0$.

Now any solution ψ of $\mathcal{A}\psi = 0$ either decays exponentially or diverges exponentially at infinity, since, due to the rapid decay of $\bar{m}'$, and hence V, $\phi'' \approx \phi$. The only option consistent with (2.8) is exponential decay. Hence we must have that $\eta - (1/2)\langle\bar{m}', \eta\rangle$ is in the L^2 kernel of $\mathcal{A}$. However, we know from Lemma 1.2 that this is spanned by $\bar{m}'$. So we must have $\eta - (1/2)\langle\bar{m}', \eta\rangle = \alpha\bar{m}'$. Integrating both sides against $\bar{m}'$ yields $\alpha = 0$. Hence η is constant, and so $\phi = 0$, as was to be shown.

We will now show that there is a $\gamma > 0$ so that

$$\mathcal{Q}(\phi) \geq \gamma \|\phi\|_2^2 \tag{2.10}$$

for all ϕ. The proof is similar to the proof of Weyl's lemma, though note that $\mathcal{A} + K$ is not self adjoint.

If (2.10) were false, there would exist an infinite orthonormal sequence $\{\phi_n\}$ in L^2 such that $\lim_{n\to\infty} \mathcal{Q}(\phi_n) = 0$. Since the sequence $\{\phi_n\}$ is orthonormal, it converges weakly to zero. Next, let $c_n = \langle \phi_n, \bar{m}' \rangle$ and note that $\lim_{n\to\infty} c_n = 0$. If the c_n are not all zero, let n_0 be such that $|c_{n_0}| \geq |c_n|$ for all n, and define $\tilde{\phi}_n = \phi_n - (c_n/c_{n_0})\phi_{n_0}$. It is clear that the $\tilde{\phi}_n$ are all orthogonal to $\bar{m}'$, and moreover the modified sequence still converges weakly to zero, and still satisfies $\lim_{n\to\infty} \mathcal{Q}(\tilde{\phi}_n) = 0$ and $\lim_{n\to\infty} \|\tilde{\phi}_n\|_2^2 = 1$. (If all of the c_n vanish, we simply take $\tilde{\phi}_n = \phi_n$ for all n.) Moreover, by Lemma 1.2,

$$\|\mathcal{A}\tilde{\phi}_n\|_2^2 \geq \frac{9}{16}\|\tilde{\phi}_n\|_2^2. \tag{2.11}$$

Since the sequence $\{\tilde{\phi}_n\}$ converges weakly to zero,

$$\lim_{n\to\infty} K\tilde{\phi}_n = 0 \tag{2.12}$$

strongly in L^2. Also, it is clear that the operator domain of $\mathcal{A}$ is the form domain of $\mathcal{Q}$ and that $\|\mathcal{A}\phi\|_2^2 \leq 2\left(\mathcal{Q}(\phi) + \|K\phi\|_2^2\right)$ on this domain. Thus,

$$\|\mathcal{A}\tilde{\phi}_n\|_2^2 \leq 2\left(\mathcal{Q}(\tilde{\phi}_n) + \|K\|^2\|\tilde{\phi}_n\|_2^2\right), \tag{2.13}$$

where $\|K\|$ denote the operator norm of K on L^2. In particular, the $\|\mathcal{A}\tilde{\phi}_n\|_2$ are uniformly bounded by a finite constant. Now,

$$\mathcal{Q}(\tilde{\phi}_n) \leq \|\mathcal{A}\tilde{\phi}_n\|_2^2 + \|K\tilde{\phi}_n\|_2^2 + 2\|\mathcal{A}\tilde{\phi}_n\|_2\|K\tilde{\phi}_n\|_2. \tag{2.14}$$

By (2.12) and (2.13), the last two terms on the right in (2.14) tend to zero with n. Hence for any $\epsilon > 0$, we obtain that $\|\mathcal{A}\tilde{\phi}_n\|_2^2 \leq \epsilon\|\tilde{\phi}_n\|_2^2$ for all sufficiently large n, which would contradict (2.11). This proves (2.10). Now by (2.7), when $\langle \bar{m}', v\rangle = 0$, $\|(\mathcal{A}v)'\|_2^2 = \mathcal{Q}(v')$, and hence we have the result. □

Combining this result with (2.4) , we have

$$-\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(m) \geq (1-2\epsilon)\int_{\mathbb{R}} \left|(\mathcal{A}v)'\right|^2 \mathrm{d}x + \epsilon\gamma||v'||_2^2 - \frac{1}{\epsilon}\int_{\mathbb{R}} |U|^2\, \mathrm{d}x. \tag{2.15}$$

We next show that the quantity on the last line is positive whenever δ and κ are small enough. To accomplish this, we use the following lemma:

Lemma 2.2. *Let $v \in L^2(\mathbb{R})$, $v' \in L^2(\mathbb{R})$. For any $\kappa > 0$ and $\epsilon_0 > 0$ small enough, there exists $\delta(\kappa, \epsilon_0) > 0$ such that the following estimate holds:*

$$\int_{\mathbb{R}} \left[U(v)\right]^2 \mathrm{d}x \leq \epsilon_0 \int |v'|^2 \mathrm{d}x, \tag{2.16}$$

provided $\|v\|_2 \leq \delta$, $\|v'\|_2 \leq \kappa$.

Proof. This follows directly from (2.2) and the bound $\|v\|_\infty^2 \le 2\|v\|_2\|v'\|_2$. □

Proof of Theorem 1.4. Now choose κ and δ so that $\epsilon_0 \le \epsilon^2\gamma$, and then from (2.15), we have the inequality of Theorem 2.1. □

We now prove a bound that will enable us to apply the dissipation–dichotomy argument described in the introduction.

Theorem 2.3. *For all $\epsilon > 0$, there is an $\epsilon_0 > 0$ such that for or all v orthogonal to $\bar{m}'$ with*

$$\mathcal{I}(\bar{m} + v) = \|(\mathcal{A}v)'\|_2^2 \le \epsilon_0^2\langle v, \mathcal{A}v\rangle \tag{2.17}$$

one has

$$(1-\epsilon)\|\mathcal{A}v\|_2^2 \le \langle v, \mathcal{A}v\rangle \le (1+\epsilon)\|\mathcal{A}v\|_2^2. \tag{2.18}$$

Proof. First, by Lemma 1.2, inserting $\mathcal{A}^{1/2}v$ in place of v,

$$\langle v, \mathcal{A}v\rangle \le \frac{4}{3}\|\mathcal{A}v\|_2^2 \tag{2.19}$$

so we have that $\|(\mathcal{A}v)'\|_2^2 \le (4\epsilon_0^2/3)\|v\|_2^2$. Then, using the notation of Lemma 1.2,

$$\left|\|\mathcal{A}v\|_2^2 - \langle v, \mathcal{A}v\rangle\right| = \left|\langle v'', \mathcal{A}v\rangle + \langle Vv, \mathcal{A}v\rangle\right| \le \left|\langle v', (\mathcal{A}v)'\rangle\right| + |\langle Vv, \mathcal{A}v\rangle|\,.$$

Now $|\langle Vv, \mathcal{A}v\rangle| \le \|v\|_2\|V\|_2\|\mathcal{A}v\|_\infty$ and by (2.17) and (2.19),

$$\|\mathcal{A}v\|_\infty^2 \le 2\|\mathcal{A}v\|_2\|(\mathcal{A}v)'\|_2 \le \frac{8\epsilon_0}{3}\|\mathcal{A}v\|_2^2.$$

Then, by Lemma 1.2 and Schwarz's inequality, $\|v\|_2 \le (4/3)\|\mathcal{A}v\|_2$, so that, recalling from the proof of Lemma 1.2 that $\|V\|_2^2 = 6$,

$$|\langle Vv, \mathcal{A}v\rangle| \le 8\sqrt{\frac{\epsilon_0}{3}}\|\mathcal{A}v\|_2^2. \tag{2.20}$$

Next we bound $\left|\langle v', (\mathcal{A}v)'\rangle\right|$. First, an easy application of (2.17) and (2.19) yields

$$\left|\langle v', (\mathcal{A}v)'\rangle\right| \le \|v'\|_2\|(\mathcal{A}v)'\|_2 \le \epsilon_0\sqrt{\frac{4}{3}}\|v'\|_2\|\mathcal{A}v\|_2. \tag{2.21}$$

By Theorem 2.1, $\|v'\|_2 \le (1/\sqrt{\gamma})\|\,(\mathcal{A}v)'\,\|_2^2$; hence aplying (2.17) and (2.19) again,

$$\left|\langle v', (\mathcal{A}v)'\rangle\right| \le \epsilon_0^2\frac{4}{3\sqrt{\gamma}}\|\mathcal{A}v\|_2^2. \tag{2.22}$$

Combining (2.20) and (2.22), we have the result. □

3. Moment Estimates

In this section we prove Theorem 1.5 which bounds the growth of

$$\phi(t) = 1 + \int_{\mathbb{R}} |x\,(\mathcal{A}v)\,|^2 \mathrm{d}x + C\,[\mathcal{F}(\bar{m}+v) - \mathcal{F}(\bar{m})]\,, \tag{3.1}$$

where C is a positive constant to be specified. Actually,

$$1 + C\,[\mathcal{F}(\bar{m}+v) - \mathcal{F}(\bar{m})] \tag{3.2}$$

is non-negative and monotone decreasing, so as far as growth is concerned, the quantity of real interest is

$$\psi(t) = \int_{\mathbb{R}} |x\,(\mathcal{A}v)\,|^2 \mathrm{d}x. \tag{3.3}$$

However, (3.2) contributes negative terms to the time derivative of $\phi(t)$ that serve to absorb certain terms that cannot be controlled in terms of the excess free energy, due to the unboundedness of the operator $\mathcal{A}$.

Recall that $\mathcal{A}$ means $\mathcal{A}_{a(t)}$, where the solution $m(x,t)$ has the form $m(x,t) = v(x,t) + \bar{m}_{a(t)}(x)$, and $a(t)$ minimizes $\|m(t) - \bar{m}_a\|_2^2$. Therefore, it follows from (1.14) that

$$\frac{\partial}{\partial t}\left(\mathcal{A}_{a(t)}v(t)\right) = \mathcal{A}_{a(t)}\left(\frac{\partial}{\partial t}v(t)\right) - 3\bar{m}'_{a(t)}\dot{a}(t), \tag{3.4}$$

where $\dot{a}(t)$ denotes the derivative of $a(t)$. We can also rewrite the evolution equation (1.5) in terms of $v(t) = m(t) - \bar{m}_{a(t)}$, and doing so we obtain

$$\mathcal{A}_{a(t)}\left(\frac{\partial}{\partial t}v(t)\right) = \mathcal{A}_{a(t)}\left[\left(\mathcal{A}_{a(t)}v(t)\right) + \frac{1}{2}\left(3\bar{m}_{a(t)}v^2(t) + v^3(t)\right)''\right]. \tag{3.5}$$

(This time there is no contribution involving $\dot{a}(t)$ since $\bar{m}'_{a(t)}$ is annihilated by $\mathcal{A}_{a(t)}$.) Note that the first term on the right is linear in v, and the second term is higher order. The main contribution will come from the linear term, and it is this that we must work hardest to control.

To control the term involving $\dot{a}(t)$, first note that

$$\int (m(t) - \bar{m}_{a(t)})\bar{m}'_{a(t)} \mathrm{d}x = 0$$

which holds for all t. Differentiating this equation in t, one obtains $\dot{a}(t)\big(\|\bar{m}'_a\|_2^2 - \langle v, \bar{m}''_a\rangle\big) = -\int(\partial m/\partial t)\bar{m}'_a$. Thus, we have

$$|\dot{a}(t)| \le 2\left|\int\left(\frac{\delta\mathcal{F}}{\delta m}\right)'\bar{m}''_a \mathrm{d}x\right| \le 2\sqrt{\mathcal{I}(m(t))}\|\bar{m}''\|_2, \tag{3.6}$$

as long as $\|v\|_2$ is sufficiently small that $\left(\|\bar{m}'_a\|_2^2 - \langle v, \bar{m}''_a\rangle\right) > 1/2$. Since $\bar{m}'$ has exponential decay, this gives us the bounds we will need to control the effects of the terms involving $\dot{a}(t)$, as we will see below. The non-linear terms are easily handled without any preparatory analysis.

We now turn to the linear part, which will provide all of the most important terms. Consider the growth of $\psi(t)$ when v evolves according to the linearized equation

$$\frac{\partial}{\partial t} v = (\mathcal{A}v)'' . \tag{3.7}$$

The computations that follow can be more clearly and compactly represented if we introduce the notation

$$\xi = x\,(\mathcal{A}v)' \qquad \text{and} \qquad \eta = \mathcal{A}v. \tag{3.8}$$

Lemma 3.1. *Let $v(x,t)$ solve* (3.7)*, and let $\psi(t)$ be defined in terms of v through* (3.3)*. Then for any $\alpha > 0$,*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\psi(t) = &\left(12 + \frac{1 + 4\|V\|_1}{2\alpha}\right)\langle \eta', \eta'\rangle \\ &+ \left(2 + \frac{\alpha}{2}\|(x^2 V')\|_\infty^2 + 2\alpha\|V\|_1\right)\langle \eta, \eta\rangle,\end{aligned} \tag{3.9}$$

where $\eta = \mathcal{A}v$.

Proof. Let V be the potential defined in (1.15). Then one easily computes the commutators

$$\left[\frac{\partial}{\partial x}, \mathcal{A}\right] = V' \quad \text{and} \quad [x, \mathcal{A}] = 2\frac{\partial}{\partial x}. \tag{3.10}$$

Clearly,

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi(t) = 2\int_{\mathbb{R}} x^2\,(\mathcal{A}v)\,\mathcal{A}\,(\mathcal{A}v)''\,\mathrm{d}x.$$

Now one commutes derivatives and multiples of x past $\mathcal{A}$ and integrates by parts to obtain a dissipative term of the form $-\int_{\mathbb{R}} x\,(\mathcal{A}v)'\,\mathcal{A}\left(x\,(\mathcal{A}v)'\right)\mathrm{d}x$ into which positive terms can be absorbed.

The result, in the notation (3.8) , is that

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi(t) = -2\langle \xi, \mathcal{A}\xi\rangle - 4\langle \xi, \eta''\rangle - 4\langle x\eta, \mathcal{A}\eta'\rangle - 2\langle \eta', x^2 V'\eta\rangle.$$

The last three terms require further manipulation. First:

$$\langle \xi, \eta''\rangle = -\langle \xi', \eta'\rangle = -\langle x\eta'', \eta'\rangle - \langle \eta', \eta'\rangle = -\frac{1}{2}\langle \eta', \eta'\rangle.$$

This term is controlled by the derivative of the excess free energy. Second, one has, using (3.10)

$$\langle x\eta, \mathcal{A}\eta'\rangle = \langle \eta, \mathcal{A}\xi\rangle + 2\langle \eta, \eta''\rangle = \langle \eta, \mathcal{A}\xi\rangle - 2\langle \eta', \eta'\rangle.$$

Finally, for any $\alpha > 0$,

$$\langle \eta', x^2 V'\eta\rangle \le \langle \eta', \eta'\rangle^{1/2}\langle (x^2V')\eta, (x^2V')\eta\rangle^{1/2} \le \frac{1}{2\alpha}\langle \eta', \eta'\rangle + \frac{\alpha}{2}\|(x^2V')\|_\infty^2\langle \eta, \eta\rangle.$$

Putting everything together, one obtains:

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi(t) \le -2\langle \xi, \mathcal{A}\xi\rangle - 4\langle \xi, \mathcal{A}\eta\rangle + \left(10 + \frac{1}{2\alpha}\right)\langle \eta', \eta'\rangle + \frac{\alpha}{2}\|(x^2V')\|_\infty^2\langle \eta, \eta\rangle.$$

Now one uses that

$$-2\langle \xi, \mathcal{A}\xi\rangle - 4\langle \xi, \mathcal{A}\eta\rangle = -2\langle (\xi+\eta), \mathcal{A}(\xi+\eta)\rangle + 2\langle \eta, \mathcal{A}\eta\rangle. \tag{3.11}$$

But $\langle \eta, \mathcal{A}\eta\rangle = \langle \eta', \eta'\rangle + \langle \eta, V\eta\rangle + \langle \eta, \eta\rangle \le \langle \eta', \eta'\rangle + \|V\|_1\|\eta\|_\infty^2 + \langle \eta, \eta\rangle$, and

$$\|\eta\|_\infty^2 \le 2\|\eta'\|_2\|\eta\|_2 \le \frac{1}{\alpha}\langle \eta', \eta'\rangle + \alpha\langle \eta, \eta\rangle.$$

Altogether

$$\langle \eta, \mathcal{A}\eta\rangle \le \left(1 + \frac{\|V\|_1}{\alpha}\right)\langle \eta', \eta'\rangle + (1 + \alpha\|V\|_1)\langle \eta, \eta\rangle. \tag{3.12}$$

Putting (3.12) into (3.11) gives the result. □

Lemma 3.2.

$$\langle \eta, \eta\rangle \le \langle \eta', \eta'\rangle + \langle v, \mathcal{A}v\rangle.$$

Proof. By Schwarz, for any $\alpha > 0$,

$$\langle \eta, \eta\rangle = \langle \mathcal{A}^{1/2}\eta, \mathcal{A}^{1/2}v\rangle \le \langle \eta, \mathcal{A}\eta\rangle^{1/2}\langle v, \mathcal{A}v\rangle^{1/2} \le \frac{\alpha}{2}\langle \eta, \mathcal{A}\eta\rangle + \frac{1}{2\alpha}\langle v, \mathcal{A}v\rangle,$$

and $\langle \eta, \mathcal{A}\eta\rangle \le \langle \eta', \eta'\rangle + \|V+1\|_\infty\langle \eta, \eta\rangle$. Since $\|V+1\|_\infty = 1$, we can choose $\alpha = 1$ and combine the above to obtain the result. □

Proof of Theorem 1.5. First, we deal with the inhomogenous terms involving $\dot{a}(t)$ on the right in (3.4) , as they contribute to

$$\left|\int_{\mathbb{R}} x^2 \left(\mathcal{A}_{a(t)}v\right) \frac{\partial}{\partial t}\left(\mathcal{A}_{a(t)}v\right) \mathrm{d}x\right|.$$

By symmetry and the Schwarz inequality, we have that

$$\left|3\int_{\mathbb{R}} \left(\mathcal{A}_{a(t)}v\right) x^2\bar{m}'_{a(t)}\mathrm{d}x\right| |\dot{a}(t)| \le 3\|\mathcal{A}_{a(t)}x^2\bar{m}'_{a(t)}\|_2\|v\|_2|\dot{a}(t)|.$$

Now applying (3.6) , the contribution of the term involving $\dot{a}(t)$ is bounded above by

$$6\|\mathcal{A}_{a(t)}x^2\bar{m}_{a(t)}\|_2\|v\|_2\|\bar{m}''_{a(t)}\|_2\sqrt{\mathcal{I}(m(t))}.$$

It is here that we begin using the hypothesis that $|a(t)| \le 1$. The exponential decay of $\bar{m}'_{a(t)}$ would not give a bound on $\|\mathcal{A}_{a(t)}x^2\bar{m}'_{a(t)}\|_2$ that is uniform in t if $|a(t)|$ gets large. Since this is precluded by the hypotheses, for any $\alpha > 0$, there is a universal constant K_α so that

$$\left|3\int_{\mathbb{R}} x^2 \left(\mathcal{A}_{a(t)}v\right) \bar{m}'_{a(t)}\mathrm{d}x\right| |\dot{a}(t)| \le \frac{K_\alpha}{\alpha}\mathcal{I}(m(t)) + \alpha\|v\|_2^2. \tag{3.13}$$

Note that the first term on the right in (3.13) can be absorbed into the negative contribution from the inclusion of the multiple C of the excess free energy in ϕ, at least if C is chosen appropriately large. Therefore, since we can take α arbitrarily small, and can bound of $\|v\|_2^2$ in terms of the excess free energy by Lemma 1.3, this term is under control.

One even more easily handles the contributions of the nonlinear terms in (3.5) using the bound $\|v\|_\infty^2 \leq 2\|v\|_2\|v'\|_2$. We do not give the details here, but turn to the application of the lemmas from this section to control the contribution from the linear terms.

To apply Lemma 3.1, choose α so that

$$\left(2 + \frac{\alpha}{2}\|(x^2V')\|_\infty^2 + 2\alpha\|V\|_1\right) \leq 2\left(1 + \frac{\epsilon}{4}\right).$$

Then, for this choice of α, and using the notation from (3.8) ,

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi(t) = \left(12 + \frac{1 + 4\|V\|_1}{2\alpha}\right)\langle\eta', \eta'\rangle + 2\left(1 + \frac{\epsilon}{4}\right)\langle\eta, \eta\rangle. \tag{3.14}$$

Next, by Theorem 1.4,

$$\frac{\mathrm{d}}{\mathrm{d}t}C\left[\mathcal{F}(\bar{m} + v) - \mathcal{F}(\bar{m})\right] \leq -C(1 - \epsilon)\langle\eta', \eta'\rangle.$$

Therefore, if we choose C so that $C(1 - \epsilon) \geq (12 + (1 + 4\|V\|_1)/(2\alpha))$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(t) \leq 2\left(1 + \frac{\epsilon}{4}\right)\langle\eta, \eta\rangle.$$

It remains to bound $\|\eta\|_2^2$. There are two cases. First suppose that the dissipation is small compared to the excess free energy so that (1.18) holds. Then by Theorem 2.3, $\|\eta\|_2^2 \leq (1 + \epsilon)\langle v, \mathcal{A}v\rangle$, and then by Lemma 1.3, $\|\eta\|_2^2 \leq (1 + \epsilon)^3[\mathcal{F}(m(t)) - \mathcal{F}(\bar{m})]$, for δ and κ sufficiently small. Redefining ϵ, we have proved (1.22) under the hypothesis (1.18).

If we don't assume (1.18), we use

$$\|\eta\|_2^2 = \langle v', \eta'\rangle + \langle(V + 1)v, \eta\rangle \leq \|v'\|_2\sqrt{\mathcal{I}(m(t))} + \|v\|_2\|\eta\|_2$$

since $\|v+1\|_\infty = 1$. This leads to $\|\eta\|_2^2 \leq (2/\gamma)\mathcal{I}(m(t)) + 4\|v\|_2^2$, where γ is the constant in Theorem 1.4. Again, the term involving $\mathcal{I}(m(t))$ can be absorbed by an appropriate choice of C. The remaining term is easily handled by Lemma 1.2 and Lemma 1.3, and so (1.23) is established. □

4. Proof of the Main Theorem

We will be brief in the presentation of this proof since from this point on, it is very close to the one we have given for the LOP equation in Sect. 4 of [5].

Let $m(t)$ be a solution of (1.5) with initial data as specified in Theorem 1.1, where the size of δ is to be specified in the course of the proof. The first step is to wait a bit to acquire some smoothness. For any fixed $\kappa > 0$, if initially $\|v\|_2 \leq \delta/4$, where δ is sufficiently small, we will have that $\|v(1)\|_2 \leq \delta/2$ and $\|v'(1)\|_2 \leq \kappa/2$, and moreover $|a(1)|$ will be small. Regularity theory for $m(t)$ can be found in [3]. Also, the production of smoothness estimates in Sect. 2 of [5] are easily adapted to this case to see the validity of the above assertion.

We now begin the analysis from this starting point. All of the lemmas and theorems that required $\|v(1)\|_2 \leq \delta$, $\|v'(1)\|_2 \leq \kappa$, and $|a(1)| < 1$ can be used until time T, which is the first time that any of them is violated. Of course, we have to show that such a time T never occurs.

Let $f(t)$ and $\phi(t)$ be given in terms of $m(t)$ as in the introduction.

We begin by assuming that at time t, (1.18) holds. Then by Theorem 1.4,

$$\frac{\mathrm{d}}{\mathrm{d}t} f(t) \leq -(1-\epsilon)\|(\mathcal{A}v)'\|_2^2.$$

By convexity $\|(\mathcal{A}v)'\|_2^2 \geq \|(\mathcal{A}\rho * v)'\|_2^2$, where $\rho = (1/2)\bar{m}'$, which is a probability density. Because v is orthogonal to $\bar{m}'$, $\rho * v(a(t)) = 0$. Therefore, by the constrained uncertainty principle (1.24) ,

$$\|(\mathcal{A}v)'\|_2^2 \geq \|(\mathcal{A}\rho * v)'\|_2^2 \geq \frac{9}{4} \frac{\|\mathcal{A}\rho * v\|_2^4}{\|(x-a(t))\,(\mathcal{A}\rho * v)\,\|_2^2}.$$

Now under the condition (1.18) , v is so smooth and spread out that $\rho * v \approx= v$, and we do not lose much in passing from v to $\rho * v$. The estimates are straightforward, making use of (3.10) , and are exactly like those applied on pp. 868–869 of [5]. Without repeating the details, the result is that $\|(\mathcal{A}v)'\|_2^2 \geq (9/4)(1-\epsilon)^2 \left(\|\mathcal{A} * v\|_2^4\right) / \left(\|(x-a(t))\,(\mathcal{A}v)\,\|_2^2\right)$ and hence that, with ϵ redefined, and making use of Lemma 1.3,

$$\frac{\mathrm{d}}{\mathrm{d}t} f(t) \leq -9(1-\epsilon)\frac{f^2(t)}{\phi(t)},$$

where we have used the fact that $|a(t)| < 1$ to absorb the effects of $a(t)$ into the constant term.

By Theorem 1.5, we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(t) \leq 4(1+\epsilon)f(t).$$

Hence for such t, we have (1.30) satisfied with $A/(A+B)$ arbitrarily close to $9/13$.

Now suppose that (1.19) holds. Then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(t) \leq \tilde{B} f(t)$$

from the second half of Theorem 1.5, where $\tilde{B}$ is the constant K given there. From (1.19)

$$\frac{\mathrm{d}}{\mathrm{d}t} f(t) \leq -\epsilon_1 f(t) \leq \tilde{A}\frac{f^2(t)}{\phi(t)}, \tag{4.1}$$

where $\tilde{A}$ can be chosen as large as we like provided $f(t)$ is sufficiently small. Thus with δ chosen sufficiently small, as long as $f(t) < \delta$ holds, we have (**??**) and can arrange for it to hold with a value of $\tilde{A}$ so that $\tilde{A}/(\tilde{A}+\tilde{B}) = A/(A+B)$. Thus, by rescaling the time in those time intervals in which (1.19) holds; i.e., possibly using a slower clock there, we have a system holding for all t. The details of this argument are exactly as in Sect. 5 of [5].

One now concludes that as long as $|a(t)| < 1$, $\|v(t)\|_2 \leq \delta$ and $\|v'(t)\|_2 \leq \kappa$, $f(t)$ decays at a rate close to $t^{-9/13}$ (using the slower of the two time scales). Therefore, as in [5], $|a(t)| < 1$, $\|v(t)\|_2 \leq \delta$ and $\|v'(t)\|_2 \leq \kappa$ hold for all t, and so $f(t)$ decays all the way to zero at a rate close to $t^{-9/13}$, as in Theorem 1.1. As explained at the end of Sect. 1 of this paper, this means that $\|\mathcal{A}v(t)\|_1$ decays to zero at an algebraic rate, and that this forces $\lim_{t\to\infty} a(t) = a$, where a is given by the conservation law.

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