Stability of planar fronts for a non–local phase kinetics equation with a conservation law in $D \leq 3$

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Abstract

We consider, in a $D-$dimensional cylinder, a non–local evolution equation that describes the evolution of the local magnetization in a continuum limit of an Ising spin system with Kawasaki dynamics and Kac potentials. We consider sub–critical temperatures, for which there are two local spatially homogeneous equilibria, and show a local nonlinear stability result for the minimum free energy profiles for the magnetization at the interface between regions of these two different local equilibrium; i.e., the planar fronts: We show that an initial perturbation of a front that is sufficiently small in L^2 norm, and sufficiently localized yields a solution that relaxes to another front, selected by a conservation law, in the L^1 norm at an algebraic rate that we explicitly estimate. We also obtain rates for the relaxation in the L^2 norm and the rate of decrease of the excess free energy.

Key words: phase kinetics, fronts, non-linear stability, non local equation
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Contents

1 Introduction and main results

We consider the nonlocal and nonlinear evolution equation

$$\begin{aligned} \frac{\partial}{\partial t} m(x,t) &= \nabla \cdot \big(\nabla m(x,t) - \beta(1 - m(x,t)^2)(J \star \nabla m)(x,t)\big) \quad x \in \mathbb{R} \times \Lambda, \\ m(0,x) &= m_0(x), \quad m_0(x) \in [-1,1] \end{aligned} \tag{1.1}$$

in the $D-$dimensional cylinder $\mathbb{R} \times \Lambda$ where Λ is a $(D-1)$-dimensional torus of side-length $L > 1$, (equipped with the periodic Euclidean metric), $\beta > 1$, $\star$ denotes convolution, and J is smooth, spherically symmetric probability density on $\mathbb{R}^D$ with compact support. We assume, without loss of generality, that the support of J is contained in a ball of radius 1. In the following we set $d = D - 1$.

This equation first appeared in the literature in a study [12] of the dynamics of Ising systems with a long–range interaction and so–called "Kawasaki" or "exchange" dynamics. In this physical context, $m(x,t)$ is the magnetization density at x at time t, viewed on the length scale of the interaction J, and β is the inverse temperature. The derivation of (1.1) from the underlying stochastic dynamics with x taking values in a torus T^d is done in [9]. Equation (1.1) has been object of several studies that shall be quoted later.

Our investigation in this paper turns on the fact that the equation (1.1) can be written in a gradient flow form: Introduce the **Gates-Penrose-Lebowitz free energy functional** $\mathcal{F}$ defined on all measurable functions from $\mathbb{R} \times \Lambda$ by

$$\mathcal{F}(m) = \int_{\mathbb{R}\times\Lambda} [f(m(x)) - f(m_\beta)] \mathrm{d}x + \frac{1}{4} \int_{\mathbb{R}\times\Lambda} \int_{\mathbb{R}\times\Lambda} J(x-y)[m(x) - m(y)]^2 \mathrm{d}x\mathrm{d}y \tag{1.2}$$

where $f(m)$ is

$$f(m) = -\frac{1}{2}m^2 + \frac{1}{\beta}\left[\left(\frac{1+m}{2}\right)\ln\left(\frac{1+m}{2}\right) + \left(\frac{1-m}{2}\right)\ln\left(\frac{1-m}{2}\right)\right]. \tag{1.3}$$

For $\beta > 1$, this potential function f is a symmetric double well potential on $[-1, 1]$. We denote the positive minimizer of f on $[-1, 1]$ by m_β. It is easy to see that m_β is the positive solution of the equation

$$m_\beta = \tanh(\beta m_\beta).$$

The functional (1.2) is well defined on the set of measurable functions from $\mathbb{R}\times\Lambda$ to $[-1, 1]$, although it might be infinity. The equation (1.1) can be written in the gradient flow form

$$\frac{\partial}{\partial t}m = \nabla \cdot \left[\sigma(m)\nabla\left(\frac{\delta\mathcal{F}}{\delta m}\right)\right] \tag{1.4}$$

where the *mobility* $\sigma(m)$ is given by

$$\sigma(m) = \beta(1 - m^2). \tag{1.5}$$

From this it follows, at least on a formal level, that $\mathcal{F}$ is decreasing along the flow described by (1.1): The formal Frechet derivative of the free energy $\dfrac{\delta\mathcal{F}}{\delta m}$ is

$$\frac{\delta\mathcal{F}}{\delta m} = \frac{1}{\beta}\text{arctanh}(m) - J \star m, \tag{1.6}$$

and thus, one formally derives

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(m(t)) = -\int_{\mathbb{R}\times\Lambda}\left|\nabla\left(\frac{\delta\mathcal{F}}{\delta m}\right)\right|^2\sigma(m(t))\mathrm{d}x =: -\mathcal{I}(m(t)) \ . \tag{1.7}$$

Based on this calculation, one might hope that $\mathcal{F}$ would be a Lyapunov function governing the approach of solutions of (1.1) to a minimizer of $\mathcal{F}$.

The global minimizers of $\mathcal{F}$ are of course the two constant profiles $m(x) = m_\beta$ and $m(x) = -m_\beta$ for all x in the cylinder $\mathbb{R} \times \Lambda$. Here we study a more interesting class of profiles m under the constraint that $m(x)$ is very close to $-m_\beta$ far to the left in the cylinder, and is very close to m_β far to the right in the cylinder.

More precisely, let us write $x = (x_1, x_1^\perp)$ where the first coordinate x_1 runs along the length of the cylinder, and $x^\perp$ along the cross section Λ. Consider the class $\mathcal{C}$ of measurable functions m from $\mathbb{R} \times \Lambda$ to $[-1, 1]$ such that for almost every $x_1^\perp \in \Lambda$,

$$\lim_{x_1\to-\infty} m(x_1, x_1^\perp) = -m_\beta \qquad \text{and} \qquad \lim_{x_1\to\infty} m(x_1, x_1^\perp) = m_\beta \ .$$

The minimizers of $\mathcal{F}$ over $\mathcal{C}$ can be expressed in terms of the minimizers of a simpler functional of one dimensional profiles. More specifically, in [8] is was shown that there exists a unique function $\overline{m}_0(\cdot)$, such that

$$\mathcal{F}_1(\overline{m}_0) = \inf\left\{ \mathcal{F}_1(m) \;\middle|\; \text{sgn}(x_1)m(x_1) \geq 0, \lim_{x_1\to\pm\infty} \text{sgn}(x_1)m(x_1) > 0\right\}, \tag{1.8}$$

where $\mathcal{F}_1$ is is the functional

$$\mathcal{F}_1(m) = \int_{\mathbb{R}} [f(m(x_1)) - f(m_\beta)]\mathrm{d}x_1 + \frac{1}{4}\int_{\mathbb{R}}\int_{\mathbb{R}} \overline{J}(x_1 - y_1)[m(x_1) - m(y_1)]^2 \mathrm{d}x_1 \mathrm{d}y_1,$$

and

$$\overline{J}(x_1) = \int_\Lambda J(x_1, x^\perp)\mathrm{d}x^\perp. \tag{1.9}$$

Furthermore it is shown that $\overline{m}_0$ is an odd, $C^\infty(\mathbb{R})$, increasing function, and

$$\begin{aligned} 0 &< m_\beta^2 - \overline{m}_0^2(x_1) \le Ce^{-\alpha|x_1|} ,\\ 0 &< \overline{m}_0'(x_1) \le Ce^{-\alpha|x_1|} ,\\ 0 &< |\overline{m}_0''(x_1)| \le Ce^{-\alpha|x_1|} , \end{aligned}$$

for positive constants C and α depending on J and β. The first two of these estimates are proved in [8] and the third one in [6]. A review of these and related results can be found in Chapter 8 of the book [13]. The subscript 0 on the minimizer refers to the fact that the constraint imposed in (1.8) breaks the translational invariance of the free energy. For any a in $\mathbb{R}$, define

$$\overline{m}_a(x) = \overline{m}_a(x_1, x_1^\perp) = \overline{m}_a(x_1) = \overline{m}_0(x_1 - a), \quad x \in \mathbb{R}\times\Lambda. \tag{1.10}$$

Clearly

$$\mathcal{F}(\overline{m}_a) = \mathcal{F}(\overline{m}_0), \qquad \frac{\delta\mathcal{F}}{\delta m}(\overline{m}_a) = \frac{1}{\beta}\mathrm{arctanh}(\overline{m}_a) - J\star\overline{m}_a = 0. \tag{1.11}$$

Thus the profiles $\overline{m}_a$ are at least critical points of the free energy $\mathcal{F}$ in the class $\mathcal{C}$. Since they are built out of minimizing one dimensional profiles, it is natural to guess that they are in fact minimizers in $\mathcal{C}$. This has been proved by Alberti and Bellettini [1], who showed moreover that every minimizer of $\mathcal{F}$ in $\mathcal{C}$ is of this form. The functions in this one parameter family of minimizers of the free energy $\overline{m}_a$, $a \in \mathbb{R}$, are the stationary solutions of (1.1) whose stability is to be investigated here.

Because the free energy is reflection invariant, there is also another family, obtained by reflecting the previous one. However, these two families of minimizers are well separated in all of the metrics in which we shall work, and it suffices to consider only one of them.

We shall be concerned here with the evolution of small perturbations of m from $\overline{m}_0$, and their relaxation to $\overline{m}_a$ for some a under the dynamics introduced above. We shall show that if the perturbation is suitably small, then this happens, and moreover, we shall find the value of a, and estimate the rate of convergence.

The equation (1.1) not only has a Lyapunov function; it has a conservation law as well: For times t in any interval on which $m(x,t) - \mathrm{sgn}(x_1)m_\beta$ is integrable,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}\times\Lambda}\big(m(x,t) - \overline{m}_b(x)\big)\mathrm{d}x = 0$$

for any b. Therefore, if one defines a in terms of the initial data m_0 for (1.1) by

$$\int_{\mathbb{R}\times\Lambda}\big(m(x,0) - \overline{m}_a(x)\big)\mathrm{d}x = 0\ , \tag{1.12}$$

one has for the solution

$$\int_{\mathbb{R}\times\Lambda}\big(m(x,t)-\overline{m}_a(x)\big)\mathrm{d}x=0$$

for all t or at least all t such that $m(s,x)-\mathrm{sgn}(x_1)m_\beta$ is integrable for all $s\le t$.

Now, formally invoking the Lyapunov function and the conservation law, it is easy to guess the result of solving (1.1) for initial data m_0 that is a small perturbation of the front $\overline{m}_0$: The decrease of the excess free energy "should" force the solution $m(t)$ to tend to the family of fronts, and the conservation law "should" select $\overline{m}_a$ as the front it should be converging to, so the result "should" be that

$$\lim_{t\to\infty}\int_{\mathbb{R}\times\Lambda}|m(x,t)-\overline{m}_a(x)|\mathrm{d}x=0$$

with a given in terms of the initial data m_0 through (1.12).

There is a fundamental obstacle in the way of this optimistic line of reasoning: Consider a very small $\epsilon>0$ and consider

$$m(x):=\overline{m}_0(x)+\epsilon 1_{[\epsilon^{-3/2},2\epsilon^{-3/2}]}(x_1)\ .$$

Then it is very easy to see that

$$\mathcal{F}(m)-\mathcal{F}(\overline{m}_0)=\mathcal{O}(\epsilon^{1/2})\quad\text{while}\quad \inf_{a\in\mathbb{R}}\|m-\overline{m}_a\|_1=\mathcal{O}(\epsilon^{-1/2})\ .$$

That is, perturbations of a minimizer with extremely small excess free energy can be very far from any minimizer in the L^1 norm: If the only information on the evolution that one had was that the excess free energy was decreasing to zero, one could not rule out the possibility that the L^1 distance to the nearest minimizer might be increasing to infinity. As we shall see, all profiles m with small excess free energy and a large L^1 distance to the nearest minimizer are very delocalized perturbations of minimizers, spread out on a very large scale, as in the example we have given. To rule this out, we have to assume moment conditions that prevent our perturbations from being too delocalized at the beginning, and then we must work to show that this localization does not deteriorate too rapidly. This accounts for the moment conditions in the theorem stated below. These moment conditions are essential for bounding the rate of convergence; a small initial excess free energy is not enough.

In the following, whenever there is no ambiguity, we denote by $\|u\|_p$ the $L^p(\mathbb{R}\times\Lambda)$ of a function u. If $u\in W^{s,2}(\mathbb{R}\times\Lambda)$, $s\in\mathbb{N}$, the space of functions $u\in L^2(\mathbb{R}\times\Lambda)$ whose distributional derivatives of order $\le s$ are in $L^2(\mathbb{R}\times\Lambda)$, we denote by $\|u\|_{W^{s,2}}=\sum_{|\alpha|\le s}\|D^\alpha u\|_2$ its norm.

We have the following main result.

1.1 THEOREM. *Let $m(t)$ be the solution of equation (1.1) in the D-dimensional cylinder $\mathbb{R}\times\Lambda$, $D\le 3$, and with initial data m_0 such that*

$$\int_{\mathbb{R}\times\Lambda}x_1^2(m_0(x)-\overline{m}_0(x))^2\mathrm{d}x\le c_0\ ,$$

where c_0 is any positive constant. Then for any $\delta>0$ there is a strictly positive constant $\epsilon=\epsilon(\delta,c_0,\beta,J,L)$ depending only on δ, c_0, β, J and L such that for all initial data m_0 with $-1\le m_0\le 1$, and with

$$\int_{\mathbb{R}\times\Lambda}(m_0(x)-\overline{m}_0(x))^2\mathrm{d}x\le\epsilon\ ,$$

the excess free energy $\mathcal{F}(m(t)) - \mathcal{F}(m_0)$ of the corresponding solution $m(t)$ of (1.1) satisfies

$$\mathcal{F}(m(t)) - \mathcal{F}(\overline{m}) \leq c_2(1+c_1 t)^{-(9/13-\delta)} \tag{1.13}$$

and

$$\|m(t) - \overline{m}_a\|_1 \leq c_2(1+c_1 t)^{-(5/52-\delta)} \tag{1.14}$$

where c_1 and c_2 are finite constants depending only on δ, c_0, J, β and L and a is given by (1.12).

In $D = 1$ the same stability problem for the equation (1.1) was addressed in the papers [3] and [4]. The strategy used in these papers was applied in [5] to show the local non-linear stability of the interface solution for the Cahn-Hilliard equation, always in $D = 1$.

The method applied in $D = 1$ has been adapted in this paper to show local non-linear stability of the interface solution of (1.1) when dimension $D \geq 2$. To apply the previous strategy in $D \geq 2$ one needs to control the transverse contribution of the perturbation to the planar fronts.

This is done by a suitable splitting of a function in $\mathbb{R} \times \Lambda$ as the sum of two functions, one depending only on $x_1 \in \mathbb{R}$ and the other with mean zero in the direction orthogonal to x_1. This allows us to effectively decouple the problem into transverse and longitudinal parts, and to control the gradient of the function in the transverse direction applying the Poincaré inequality.

The method is robust enough and it should allow to deal with nonlinear local stability problems for other equations of Cahn-Hilliard type.

There are very few results in the literature regarding stability of the planar fronts in infinite domain for equations of Cahn-Hilliard type. The only paper to our knowledge dealing with the stability of the planar front for Cahn-Hilliard equation in $\mathbb{R}^D$, for $D \geq 3$ is the paper by Korvola, Kupianen and Taskinen [11]. They proved that the leading asymptotic of the solution is characterized by a length scale proportional to $t^{\frac{D-1}{3}}$ instead of the usual $t^{\frac{D}{2}}$ typical to parabolic problems. In contrast to the one dimensional and to $D-$ dimensional cylinder setting, considered in this paper, they show that the translation of the front tends to zero as time tends to infinity. This is because a localized perturbation is not able to produce a constant shift in the whole transverse space $\mathbb{R}^{D-1}$. In our case, a perturbation of an equilibrium front need not return asymptotically to the initial front. Indeed, there is no easy argument using only decrease of free energy to show that the perturbation does not cause the front to "run away to infinity". Our method provides a proof, with quantitate estimates, on the size of the shift that can result as the perturbation is dissipated way.

The restriction to $D \leq 3$ is for reasons that are surely technical; the condition $D \leq 3$ is used only in proving certain regularity estimates that are required in our central arguments. Very likely with more labor (and more pages), these could be proved in higher dimension as well.

To implement the heuristics discussed before the theorem is not so simple as one might hope. There are several reasons for this. The first has to do with the relevant norms.

To explain the physical relevance of the L^2 norm, we note that $a \mapsto \|m - \overline{m}_a\|_2^2$ is strictly convex whenever for some $a \in \mathbb{R}$,

$$\|(m - \overline{m}_a)'\|_2 < \|\overline{m}_0'\|_2 \ .$$

Using this we show in Theorem 2.3 that under suitable smallness assumptions on $m - \overline{m}_a$ for some

a, there exists a unique $b \in \mathbb{R}$ so that

$$\|m - \overline{m}_b\|_2 = \inf_{a\in\mathbb{R}}\{\|m - \overline{m}_a\|_2\}\ .$$

Further we show, see Lemma 8.1, that the excess free energy measures the distance to this closest front in the L^2 metric in the sense that

$$\mathcal{F}(m) - \mathcal{F}(\overline{m}_b) \simeq C\|m - \overline{m}_b\|_2^2 \tag{1.15}$$

under suitable smallness assumptions on $m - \overline{m}_b$.

We use the smoothing properties of (1.1) to obtain the condition on the derivative of m for all $t \geq t_0$, for some finite t_0 so that we can apply Lemma 8.1.

On account of (1.15), for any solution $m(t)$ of (1.1), define $a(t)$ to be that value of a such that

$$\|m(t) - \overline{m}_{a(t)}\|_2 = \inf_{a\in\mathbb{R}}\{\|m(t) - \overline{m}_a\|_2\} \tag{1.16}$$

and note that $a(t)$ is a well–defined function as long as $\|m(t) - \overline{m}_{a(t)}\|_2$ stays sufficiently small since then the minimum is uniquely attained. (We shall do all of our analysis in this paper for times t in an interval (t_0, T_0) on which $\|m(t) - \overline{m}_{a(t)}\|_2$ does stay small, and then at the end we shall show that $T_0 = \infty$.) Hence, if one proves that the excess free energy decreases to zero, the best one can obtain from this is that

$$\lim_{t\to\infty} \|m(t) - \overline{m}_{a(t)}\|_2 = 0.$$

However, this does not yield any information on $a(t)$ – and it cannot by the translation invariance of the free energy. The conservation law would give us information on $a(t)$, but to use it we require L^1 control on $m(\cdot,t) - \overline{m}_{a(t)}(\cdot)$. Since

$$\|m(\cdot,t) - \overline{m}_{a(t)}(\cdot)\|_\infty \leq 2$$

a–priori, L^1 control would give us L^2 control through

$$\|m(\cdot,t) - \overline{m}_{a(t)}(\cdot)\|_2^2 \leq 2\|m(\cdot,t) - \overline{m}_{a(t)}(\cdot)\|_1$$

but not *vice–versa*. In order to use the conservation law to show that $\lim_{t\to\infty} a(t) = a$ where a is given by (1.12), we must, and shall, show that

$$\lim_{t\to\infty} \|m(\cdot,t) - \overline{m}_{a(t)}\|_1 = 0\ .$$

Before discussing the L^1 behavior of perturbations of fronts, we make the following convention, to be used throughout the paper, whenever some solution $m(x,t)$ is under discussion:

$$v(x,t) = m(x,t) - \overline{m}_{a(t)}(x) \tag{1.17}$$

where $a(t)$ is given in (1.16), and moreover

$$\overline{m}(x) \qquad \text{denotes} \qquad \overline{m}_{a(t)}(x). \tag{1.18}$$

As explained above, we shall have to look into the details of the free energy dissipation $\mathcal{I}$, see (1.7), in order to understand whatever stability properties our equation may have. To begin this, we write

$$\begin{aligned}\mathcal{I}(m) &= \int_{\mathbb{R}\times\Lambda} \sigma(m(x)) \left[\frac{\partial}{\partial x_1}\left(\frac{1}{\beta}\text{arctanh}m(x) - (J\star m)(x)\right)\right]^2 \mathrm{d}x \\ &+ \int_{\mathbb{R}\times\Lambda} \sigma(m(x)) \left[\nabla^{\perp}\left(\frac{1}{\beta}\text{arctanh}m(x) - (J\star m)(x)\right)\right]^2 \mathrm{d}x, \end{aligned} \tag{1.19}$$

where $\nabla^{\perp}$ is the gradient in the orthogonal direction of x_1.

One result of the paper, Theorem 3.1, gives a lower bound on the rate of dissipation of the excess free energy, whenever the dimension $D \leq 3$. For any $\epsilon > 0$,

$$\begin{aligned}\frac{d}{dt}\big[\mathcal{F}\big(m(t)\big) - \mathcal{F}\big(\overline{m}\big)\big] &= -\mathcal{I}\big(m(t)\big) \\ &\leq -(1-3\epsilon)\sum_{i\geq 1}\int_{\mathbb{R}\times\Lambda} \sigma(\overline{m})\big[(\mathcal{B}v(t))_{x_i}\big]^2 \mathrm{d}x, \end{aligned} \tag{1.20}$$

where $\mathcal{I}\big(m(t)\big)$ is given in (1.19) and, recall (1.18), $\bar{m} := \bar{m}_{a(t)}$. To get this result we need smoothness estimates to hold for the solution $m(t)$, in order to apply Sobolev inequalities. Namely (1.20) holds when the $||v||_{W^{s,2}} \leq \kappa_1(\beta, J, L, \epsilon)$ and $||v||_2 \leq \delta_1(\beta, J, L, \epsilon)$ for some strictly positive constants $\kappa_1(\beta, J, L, \epsilon)$ and $\delta_1(\beta, J, L, \epsilon)$, where $||v||_{W^{s,2}}$ is the Sobolev norm, see in the Appendix, Lemma 8.4 and $s > \frac{D}{2}$. We have quantitative estimates, see Theorem 2.2, of the derivative of all order of $m(t)$ only when $D \leq 3$. This is the rather technical reason, noted above, for which we impose the constraint on the dimension. The $\mathcal{B}$ in (1.20) denotes the second variation of the free energy $\mathcal{F}$ at $\overline{m}$. By our convention, $\overline{m}$ denotes $\overline{m}_{a(t)}$, and while it is occasionally preferable to write $\mathcal{B}_{a(t)}$ to make this explicit, we shall generally simply write $\mathcal{B}$, and leave the dependence on $a(t)$ implicit. However, in recalling the definition, we shall be explicit:

$$\langle u, \mathcal{B}_a u\rangle_{L^2} = \frac{\mathrm{d}^2}{\mathrm{d}s^2}\mathcal{F}(\overline{m}_a + su)\bigg|_{s=0}. \tag{1.21}$$

The properties of $\mathcal{B}$ that we shall use in our analysis are discussed in Section 3. Because of the derivatives, the quadratic form on the right in (1.20) has no spectral gap. If it did, this together with (1.15) would provide an exponential rate of decrease of the excess free energy, and hence of $\|v(t)\|_2$. Since there is no spectral gap here, one needs additional monotonicity, or at least *a–priori* boundedness properties to exploit (1.20), as explained in [3] and [4]. In the study of parabolic equations

$$\frac{\partial u}{\partial t} = \nabla\cdot\big(\mathbf{D}(u)\nabla u\big)\ , \tag{1.22}$$

where $\mathbf{D}(\cdot)$ is the diffusivity matrix for which there is also no spectral gap,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int |u(x,t)|\mathrm{d}x \leq 0 \tag{1.23}$$

which trivially provides the additional monotonicity required to show that

$$\sup_{t\geq 0}\|u(t)\|_1 \leq \|u(0)\|_1\ .$$

Then a standard argument with the Nash inequality allows one to conclude that $\|u(t)\|_2$ decreases to zero at an algebraic rate, at least when the diffusivity $\mathbf{D}(\cdot)$ in (1.22) is bounded from below.

This route is closed to us since the analog of (1.23) does not hold for $v(t)$ when $m(t)$ is a solution to (1.1), see [3] and [4] for further details. Moreover, there are other problematic non-dissipative features, the maximum principle fails to hold for (1.1) and the free energy is *not Frechet differentiable* on the natural set of functions that is invariant under the evolution prescribed by (1.1). Namely, recall (1.6), for any m with $-1 \le m \le 1$, $J \star m$ is bounded, but $\operatorname{arctanh}(m) = \pm\infty$ on $\{x \mid m(x) = \pm 1\}$. This means that some care must be taken with the use of the key dissipativity property (1.7) whose formal derivation depends on this Frechet differentiability. Even worse, however, is that the mobility (1.5) vanishes where $m = \pm 1$, and with it the local contribution to the dissipation in (1.7).

One way to obtain bounds on the decay in the L^1 norm is to apply a strong formulation of the "uncertainty principle"; as done in [4].

We illustrate this in the case of the heat equation in the $D-$ dimensional cylinder $\mathbb{R} \times \Lambda$. Recall that we denote $x = (x_1, x_1^\perp)$, $x_1 \in \mathbb{R}$. Consider a solution u of the heat equation

$$\frac{\partial}{\partial t} u(x,t) = \Delta u(x,t), \quad x \in \mathbb{R} \times \Lambda$$

with integrable initial data u_0, and suppose that

$$\int_{\mathbb{R}\times\Lambda} u_0(x)\mathrm{d}x = 0.$$

We show in Theorem 2.1 of [4] that under the constraint

$$\int \psi(x_1)\mathrm{d}x_1 = 0$$

one has

$$\left(\int x_1^2 |\psi(x_1)|^2 \mathrm{d}x_1\right)\left(\int |\psi'(x_1)|^2 \mathrm{d}x_1\right) \ge \frac{9}{4}\left(\int |\psi(x_1)|^2 \mathrm{d}x_1\right)^2. \tag{1.24}$$

Define

$$f(t) = \int_{\mathbb{R}\times\Lambda} |u(x,t)|^2 \mathrm{d}x \qquad \text{and} \qquad \phi(t) = \int_{\mathbb{R}\times\Lambda} x_1^2 |u(x,t)|^2 \mathrm{d}x + 1. \tag{1.25}$$

One then computes that

$$\frac{\mathrm{d}}{\mathrm{d}t} f(t) = -2\int |\nabla u(x,t)|^2 \mathrm{d}x. \tag{1.26}$$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\phi(t) &= -4\int x_1 e_1 \cdot u(x,t)(\nabla u)(x,t)\mathrm{d}x - 2\int x_1^2 |(\nabla u)(x,t)|^2 \mathrm{d}x \\ &\le 2f(t). \end{aligned} \tag{1.27}$$

We would like, as it will be clear in the following, to write equations (1.26) and (1.27) in a closed form, i.e. to write the right hand side of (1.26) in term of $f(t)$ and $\phi(t)$. Denote

$$J(u(t)) = \int |\nabla u(x,t)|^2 \mathrm{d}x.$$

Since the dependence on t does not play any role in the following calculations, we do not write it explicitly. Set

$$u(x) = v_1(x_1) + w(x)$$

where

$$v_1(x_1) := \frac{1}{L^d}\int_\Lambda u(x_1, x^\perp)dx^\perp, \quad x_1 \in \mathbb{R}.$$

By construction

$$\int_\Lambda w(x_1, x^\perp)dx^\perp = 0, \quad \forall x_1 \in \mathbb{R}. \tag{1.28}$$

We have

$$\begin{aligned} J(u) &= \int |\nabla u(x)|^2 \mathrm{d}x = \int |u_{x_1}(x)|^2 \mathrm{d}x + \int |\nabla^\perp u(x)|^2 \mathrm{d}x \\ &= \int |v_1'(x_1) + w_{x_1}(x)|^2 \mathrm{d}x + \int |\nabla^\perp w(x)|^2 \mathrm{d}x, \end{aligned}$$

where v_1' is the spatial derivative of v_1. Notice that, because of (1.28),

$$\int_{\mathbb{R}\times\Lambda} v_1'(x_1) w_{x_1}(x)\mathrm{d}x = 0.$$

So we get

$$J(u) = \int_{\mathbb{R}\times\Lambda} |v_1'(x_1)|^2 \mathrm{d}x + \int_{\mathbb{R}\times\Lambda} |w_{x_1}(x)|^2 \mathrm{d}x + \int |\nabla^\perp w(x)|^2 \mathrm{d}x. \tag{1.29}$$

Again by (1.28) we can bound from below the last term of (1.29) using the Poincaré inequality in Λ. It states that for $g \in L^2(\Lambda)$ and $\nabla g \in L^2(\Lambda)$

$$c(d)\|g - \overline{g}\|_2^2 \le L^2 \|\nabla g\|_2^2$$

where $\overline{g} = \frac{1}{|\Lambda|}\int_\Lambda g(x)dx$ and $c(d)$ depends on dimensions.

Applying this for each x_1 in (1.29), and using the fact that because of (1.28), $\overline{w}(x_1) = 0$, we obtain

$$\begin{aligned} \|\nabla^\perp w\|_2^2 &= \int_{\mathbb{R}} \mathrm{d}x_1 \left(\sum_{i\ge 2} \int_\Lambda \mathrm{d}x^\perp |(w(x_1, x^\perp))_{x_i}|^2 \right) \\ &\ge \frac{c(d)}{L^2} \int_{\mathbb{R}} \int_\Lambda |w(x_1, x^\perp)|^2 dx. \end{aligned} \tag{1.30}$$

To lower bound the first term of (1.29) we apply the uncertainty principle (1.24) as following:

$$\begin{aligned} \int_{\mathbb{R}\times\Lambda} |v_1'(x_1)|^2 \mathrm{d}x &\ge \frac{9}{4} L^d \frac{\left(\int_{\mathbb{R}} |v_1(x_1)|^2 \mathrm{d}x_1\right)^2}{\left(\int x_1^2 |v_1(x_1)|^2 \mathrm{d}x_1\right)} \\ &= \frac{9}{4} \frac{\left(\int_{\mathbb{R}\times\Lambda} |v_1(x_1)|^2 \mathrm{d}x\right)^2}{\left(\int_{\mathbb{R}\times\Lambda} x_1^2 |v_1(x_1)|^2 \mathrm{d}x\right)}. \end{aligned} \tag{1.31}$$

Therefore, taking into account (1.30) and (1.31) we lower bound (1.29) as following:

$$\begin{aligned} J(u) &\geq \frac{9}{4}\frac{\left(\int_{\mathbb{R}\times\Lambda}|v_1(x_1)|^2\right)^2\mathrm{d}x}{\left(\int_{\mathbb{R}\times\Lambda}x_1^2|v_1(x_1)|^2\mathrm{d}x\right)}+\frac{c(d)}{L^2}\int_{\mathbb{R}\times\Lambda}|w(x)|^2\mathrm{d}x \\ &\geq \frac{9}{4}\frac{1}{\left(\int_{\mathbb{R}\times\Lambda}x_1^2|v_1(x_1)|^2\mathrm{d}x\right)+1}\left\{\left(\int_{\mathbb{R}\times\Lambda}|v_1(x_1)|^2\mathrm{d}x\right)^2+\frac{4}{9}\frac{c(d)}{L^2}\int_{\mathbb{R}\times\Lambda}|w(x)|^2\mathrm{d}x\right\} \end{aligned} \tag{1.32}$$

Notice that we obtained this estimate by dropping the contribution of the second term in (1.29), i.e. $\int_{\mathbb{R}\times\Lambda}|w_{x_1}(x)|^2\mathrm{d}x$. When dealing with equation (1.1) we shall need to keep this term to control the non linearity. By orthogonality

$$\|u\|^2_{L^2(\mathbb{R}\times\Lambda)}=\|v_1\|^2_{L^2(\mathbb{R}\times\Lambda)}+\|w\|^2_{L^2(\mathbb{R}\times\Lambda)}.$$

Therefore

$$\begin{aligned} \|u\|^4_{L^2(\mathbb{R}\times\Lambda)} &= \left(\|v_1\|^2_{L^2(\mathbb{R}\times\Lambda)}+\|w\|^2_{L^2(\mathbb{R}\times\Lambda)}\right)^2 \\ &= \|v_1\|^4_{L^2(\mathbb{R}\times\Lambda)}+\left(\|w\|^2_{L^2(\mathbb{R}\times\Lambda)}+2\|v_1\|^2_{L^2(\mathbb{R}\times\Lambda)}\right)\|w\|^2_{L^2(\mathbb{R}\times\Lambda)}. \end{aligned}$$

Suppose that

$$\left(\|w\|^2_{L^2(\mathbb{R}\times\Lambda)}+2\|v_1\|^2_{L^2(\mathbb{R}\times\Lambda)}\right)\leq\frac{4}{9}\frac{c(d)}{L^2}.$$

Therefore

$$\|u\|^4_{L^2(\mathbb{R}\times\Lambda)}\leq\|v_1\|^4_{L^2(\mathbb{R}\times\Lambda)}+\frac{4}{9}\frac{c(d)}{L^2}\|w\|^2_{L^2(\mathbb{R}\times\Lambda)}. \tag{1.33}$$

Further

$$\begin{aligned} \int_{\mathbb{R}\times\Lambda}x_1^2|v_1(x_1)|^2\mathrm{d}x &\leq \int_{\mathbb{R}\times\Lambda}x_1^2\left[|v_1(x_1)|^2+|w(x)|^2\right]\mathrm{d}x \\ &= \int_{\mathbb{R}\times\Lambda}x_1^2\left[|v_1(x_1)+w(x)|^2\right]\mathrm{d}x=\int_{\mathbb{R}\times\Lambda}x_1^2|u(x)|^2\mathrm{d}x. \end{aligned} \tag{1.34}$$

Taking into account (1.25), (1.32), (1.33), (1.34) we have

$$J(u(t))\geq\frac{9}{4}\frac{f^2(t)}{\phi(t)}.$$

Therefore, from (1.26) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}f(t)\leq-\frac{9}{2}\frac{f^2(t)}{\phi(t)}.$$

Recalling (1.27) we get the system of differential inequalities,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}f(t) &\leq -A\frac{f(t)^2}{\phi(t)} \\ \frac{\mathrm{d}}{\mathrm{d}t}\phi(t) &\leq Bf(t) \end{aligned} \tag{1.35}$$

with $A = 9/2$ and $B = 2$. Theorem 5.1 of [3] says that for any non negative solution of (1.35), the following holds

$$\begin{aligned} f(t) &\leq f(0)^{1-q}\phi(0)^q\left(\frac{\phi(0)}{f(0)} + (A+B)t\right)^{-q} \\ \phi(t) &\leq f(0)^{1-q}\phi(0)^q\left(\frac{\phi(0)}{f(0)} + (A+B)t\right)^{1-q} \end{aligned} \tag{1.36}$$

where

$$q = \frac{A}{A+B} \ .$$

In the case at hand, this is

$$q = \frac{9}{13} \ .$$

Since this value exceeds $1/2$, we get L^1 decay in the following way: We prove in Section 6, Lemma 6.2 that for any function f, so that $\|(1+x_1^2)^{1/2}f\|_{L^2(\mathbb{R}\times\Lambda)}$ is finite and for any $0 < \delta < 1$ we have

$$\|f\|_1 \leq C(\delta, L)\|(1+x_1^2)^{1/2}f\|_2^{(1+\delta)/2}\|f\|_2^{(1-\delta)/2}$$

where $C(\delta, L)$ is given explicitly in the lemma. Since $9/13 > 1/2$ for δ sufficiently small, we have that $\|(1+x_1^2)^{1/2}u(t)\|_2^{(1+\delta)/2}$, see (1.36), increases more slowly than $\|u(t)\|_2^{(1-\delta)/2}$ increases, and so $\|u(t)\|_1$ decreases to zero. In fact, the rate one gets for $\|u(t)\|_1$ is arbitrarily close to $t^{-5/13}$, for δ sufficiently small. Actually, one can do better for the heat equation. One can obtain, as in [3], $\frac{\mathrm{d}}{\mathrm{d}t}\phi(t) \leq \frac{3}{2}f(t)$ (we have 2 in (1.27)). Then $B = \frac{3}{2}$ and the rate one gets for $\|u(t)\|_1$ is arbitrarily close to $t^{-\frac{1}{4}}$, for δ sufficiently small.

The previous argument presented for the heat equation can be implemented for equation (1.1). We define

$$f(t) = \mathcal{F}(\overline{m} + v(t)) - \mathcal{F}(\overline{m}) \qquad \text{and} \qquad \phi(t) = L^d + \int \sigma(\overline{m})x_1^2|\mathcal{B}v(x,t)|^2\mathrm{d}x \ ,$$

where v as in (1.17) and $\mathcal{B}$ as in (1.21). We could estimated the time derivatives of these quantities obtaining bounds of the form given in (1.35), but with inexplicit constants A and B.

Now, the rate of decay that one gets by this method depends very much on the ratio of the constants A and B in (1.35). To get L^1 decay, we need this ratio to be fairly close to the ratio $9/13$ obtained for the heat equation.

We do this by exploiting the following alternatives: for any $\epsilon_1 > 0$, at any time t, one has either

$$\mathcal{I}(m(t)) \leq \epsilon_1\big[\mathcal{F}(m(t)) - \mathcal{F}(\overline{m})\big] \ , \tag{1.37}$$

or

$$\mathcal{I}(m(t)) \geq \epsilon_1\big[\mathcal{F}(m(t)) - \mathcal{F}(\overline{m})\big] \ , \tag{1.38}$$

where $\mathcal{I}$ is the dissipation functional (1.7).

We prove in Section 4 that for any $\epsilon > 0$, there are strictly positive constants $\delta_0(\beta, J, L, \epsilon)$, $\kappa_0(\beta, J, L, \epsilon)$ and $\epsilon_1(\beta, J, L, \epsilon)$ depending only on β, J, L and ϵ, such that for all t for which (1.37) is satisfied together with $||v(t)||_{W^{s,2}} < \kappa_0$, $s > \frac{D}{2}$, $||v(t)||_2 < \delta_0$ and $|a(t)| \leq 1$, it is the case that

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[\mathcal{F}(m(t)) - \mathcal{F}(\overline{m})\big] \leq -9(1-\epsilon)(1-\sigma(m_\beta))^2\frac{\big[\mathcal{F}(m(t)) - \mathcal{F}(\overline{m})\big]^2}{\phi(t)} \ .$$

We then show in Section 5 that under the same assumptions of Section 4, it is the case that

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(t) \leq (1+\epsilon)4(1-\sigma(m_\beta))^2\big[\mathcal{F}(\overline{m}+v)-\mathcal{F}(\overline{m})\big]\ .$$

Notice the condition that $|a(t)| \leq 1$, to which we shall return. Thus, when (1.37) holds, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}f(t) &\leq -\tilde{A}\frac{f(t)^2}{\phi(t)} \\ \frac{\mathrm{d}}{\mathrm{d}t}\phi(t) &\leq \tilde{B}f(t) \end{aligned} \tag{1.39}$$

with the difference between $\tilde{A}/\tilde{B}$ and 9/13 arbitrarily small for ϵ small enough for all times t such that $\|v(t)\|_2$, $||v(t)||_{W^{s,2}}$, $s > \frac{D}{2}$, are sufficiently small and $|a(t)| \leq 1$.

On the other hand, when (1.37) is violated and (1.38) holds, the dissipation is large, and this works in our favor. In Section 6, we exploit this alternative to prove Theorem 1.1. The proof is still somewhat intricate, and it would have been simplified had we been able to show the existence of a time $t_\star$ such that (1.37) holds for all $t \geq t_\star$. If this were the case, the constants $\tilde{A}$ and $\tilde{B}$ above would govern the decay, and we would obtain a bound on the excess fee energy of the form

$$\big[\mathcal{F}(m(t))-\mathcal{F}(\overline{m})\big] \leq C(1+A(1-\sigma(m_\beta))^2t)^{-q}$$

where A does not depend on β. Since $(1-\sigma(m_\beta))^2$ vanishes as the critical temperature is approached, this would indicate how the rate of relaxation slows in this limit. In any case, our results do show that it is possible to estimate the *exponent* in the rate of relaxation independently of β.

To explain why (1.37) enables us to obtain what are essentially heat equation constant in (1.39), one has to view it as a smoothness condition. Indeed, it follows from Theorem 3.1 that

$$(1-\epsilon)\sigma(m_\beta)|||\nabla(\mathcal{B}v)|||_2^2 \leq \mathcal{I}(\overline{m}+v)$$

for any ϵ, under appropriate conditions on v. Hence, by Lemma 8.1 which compares $\|\mathcal{B}v\|_2^2$ and the excess free energy of $\overline{m}+v$, when (1.37) holds,

$$\|\nabla\,(\mathcal{B}v)\,\|_2^2 << \|\mathcal{B}v\|_2^2\ .$$

Next, the action of $\mathcal{B}$ on functions w that satisfy

$$\|\nabla w\|_2 << \|w\|_2$$

is particularly simple: As shown in Lemma 8.3 in the appendix,

$$\mathcal{B}w \approx \tilde{\alpha}w$$

where $\tilde{\alpha} = 1/\sigma(m_\beta) - 1$. Once one may replace $\mathcal{B}$ with multiplication by α, the linearized version of (1.1) does become essentially the heat equation. This discussion is heuristic, but in no way misleading, and hopefully motivates the technical preliminaries in Section 2.

Before turning to Section 2 we state the notations which will be used in the following sections. For a function $f(\cdot,t)$, of one spatial variable and time t, we denote by $f'(\cdot,t)$ and $f''(\cdot,t)$ the first and the second derivative with respect to the spatial variable. We will denote by $\dot{f}(\cdot,\cdot)$ the time derivative. Further, we will denote by $C = C(\beta, J, d)$ a positive constant which depends only on these quantities and which might change from one occurrence to one other.

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2 Smoothing estimates and differentiability of the free energy

In this section we state some technical results upon which our analysis in the following sections depends.

It is not hard to show that classical solutions of our equation exist and are unique by adapting to our non-local equation standard fixed-point arguments for semi-linear equations, and it is easy to see from a maximum principle argument that these classical solutions satisfy $|m(x,t)| \leq 1$ for all x and t since the non-local drift term vanishes wherever $m(x,t) = 1$.

The integration by parts leading from (1.4) to (1.7) is problematic if $m(x,t)$ is not bounded away from ± 1. Therefore, fix $0 < \lambda < 1$, which we shall take increasing to 1 shortly, and consider the function $t \mapsto \mathcal{F}_\lambda(\lambda m(t))$ where m solves our equation, and where $\mathcal{F}_\lambda$ differs from $\mathcal{F}$ be having the term $[f(m) - f(m_\beta)]$ in the integrand in (1.2) replaced by $[f(m) - f(\lambda m_\beta)]$, and where f is given by (1.3). With this definition, $\mathcal{F}_\lambda(\lambda m)$ is finite whenever $\mathcal{F}(m)$ is finite.

There is no trouble integrating by parts in $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_\lambda(\lambda m(t))$ for any $t > 0$, as long as ∇m is square integrable, and hence we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_\lambda(\lambda m(t)) = \lambda \int_{\mathbb{R}\times\Lambda} [\frac{1}{\beta}\mathrm{arctanh}(\lambda m) - \lambda J \star m]\nabla \cdot [\nabla m(t) - \sigma(m)\nabla J \star m(t)]\mathrm{d}x$$

$$= -\lambda \int_{\mathbb{R}\times\Lambda} \left[\frac{\lambda}{\beta}\frac{|\nabla m(t)|^2}{1-\lambda^2 m^2(t)} - \lambda\frac{1-m^2(t)}{1-\lambda^2 m^2(t)}\nabla m \cdot \nabla J \star m(t) - \lambda \nabla m \cdot \nabla J \star m + \lambda\sigma(m)|\nabla J \star m(t)|^2\right] \mathrm{d}x$$

$$= -\lambda^2 \int_{\mathbb{R}\times\Lambda} \left[\frac{1}{\beta}\frac{|\nabla m(t)|^2}{1-\lambda^2 m^2(t)} - \frac{1-m^2(t)}{1-\lambda^2 m^2(t)}\nabla m \cdot \nabla J \star m(t) - \nabla m \cdot \nabla J \star m + \sigma(m)|\nabla J \star m(t)|^2\right] \mathrm{d}x.$$

Therefore,

$$\mathcal{F}_\lambda(\lambda m(0)) \geq \mathcal{F}_\lambda(\lambda m(t)) +$$
$$\lambda^2 \int_0^t \int_{\mathbb{R}\times\Lambda} \left[\frac{1}{\beta}\frac{|\nabla m(t)|^2}{1-\lambda^2 m^2(t)} - \frac{1-m^2(t)}{1-\lambda^2 m^2(t)}\nabla m \cdot \nabla J \star m(t) - \nabla m \cdot \nabla J \star m + \sigma(m)|\nabla J \star m(t)|^2\right] \mathrm{d}x\ .$$

Now a simple argument with Fatou's Lemma shows that

$$\mathcal{I}(m(t))$$
$$\leq \liminf_{\lambda\to 1} \int_{\mathbb{R}\times\Lambda} \left[\frac{1}{\beta}\frac{|\nabla m(t)|^2}{1-\lambda^2 m^2(t)} - \frac{1-m^2(t)}{1-\lambda^2 m^2(t)}\nabla m \cdot \nabla J \star m(t) - \nabla m \cdot \nabla J \star m + \sigma(m)|\nabla J \star m(t)|^2\right] \mathrm{d}x\ ,$$

and that

$$\mathcal{F}(m(t)) \leq \liminf_{\lambda\to 1} \mathcal{F}_\lambda(\lambda m(t))\ .$$

On the other hand, for $m(0)$ strictly and uniformly bounded away from ± 1, it is easy to show that

$$\mathcal{F}(m(0)) = \liminf_{\lambda\to 1} \mathcal{F}_\lambda(\lambda m(0))\ .$$

Thus, one obtains for every classical solution such that $|\nabla m(x,t)|^2$ is integrable in x for each t, and which has initial data strictly and uniformly bounded away from ± 1 that

$$\mathcal{F}(m(t)) + \int_0^t \mathcal{I}(m(s))\mathrm{d}s \leq \mathcal{F}(m(0))\ . \tag{2.1}$$

In Section 7 we shall prove that classical solutions whose initial data is a small L^2 perturbation of an instanton do have square integrable gradients for all positive times. Note that even if the gradient of $m(t)$ is square integrable for all t, it is still possible for $\mathcal{I}(m(t))$ to be infinite, though by (2.1), the set of such t must be a null set.

In what follows, we shall need to integrate by parts frequently, and it will be convenient to use the energy dissipation relation with equality in place of the inequality in (2.1). Our main goal in this section is to explain results showing that if the initial data is a small perturbation of an instanton in the L^2 norm, then, after waiting a short time t_0 later, the solution is regular and is a small perturbation of the instanton in Sobolev norms that guarantee that it is also a small perturbation in the L^∞ norm. Thus if we wait a short time, starting from initial data that is a sufficiently small L^2 perturbation of an instanton, the solution $m(t)$ will be strictly bounded away from ± 1, at the very least on an open interval to the right if t_0, and then we no longer need the λ-regularization used above: We can simply integrate by parts to rigorously obtain the *identity* (1.7). Moreover, we shall know that $\mathcal{F}(m(t_0))$ is still bounded by $\mathcal{F}(m(0))$. We shall also show that the moment bounds on the initial data in our main theorem are effectively propagated forward to time t_0.

Therefore, the results in this section permit us throughout the rest of the paper to restrict our attention to the behavior of the free energy functional $\mathcal{F}$ on the set of profiles

$$\mathcal{M} = \{m : \ | \ \|m\|_\infty < 1 \quad \text{and} \quad \|m - \bar{m}\|_2 < \infty\}, \tag{2.2}$$

equipped with the the metric $d(m_1, m_2) = \|m_1 - m_2\|_\infty + \|m_1 - m_2\|_2$. Note that this is not a subset of $L^\infty \cap L^2$ since profiles in $\mathcal{M}$ are never square integrable. However, $\mathcal{M}$ is open in this metric topology, and $\mathcal{F}$ is Frechet differentiable on $\mathcal{M}$, and for any differentiable curve $m(t)$ in $\mathcal{M}$, one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(m(t)) = \int_{\mathbb{R}} \frac{\delta \mathcal{F}}{\delta m}\frac{\partial}{\partial t} m(t)\mathrm{d}x \ ,$$

where

$$\frac{\delta \mathcal{F}}{\delta m} = \frac{1}{\beta}\mathrm{arctanh}(m) - J \star (m).$$

The convolution term satisfies $\|J \star m\|_\infty \leq 1$, but on any set where $m = \pm 1$ we have

$$\mathrm{arctanh} m = \pm\infty \ .$$

The following theorem summarizes our discussion so far in this section, except for the fact that it remains to be proved that, as claimed, the conditions on the initial data do indeed ensure square integrability of the gradient. This is done in Section 7.

2.1 THEOREM. *There is a $\delta > 0$ such that for all initial data m_0 in $\mathcal{M}$ with $\mathcal{F}(m_0) < \infty$ and $\|m_0 - \bar{m}_a\|_2 < \delta$, some $a \in \mathbb{R}$, the corresponding solution $m(x,t)$ of equation (1.1) satisfies*

$$\mathcal{F}(m(0)) \geq \mathcal{F}(m(t)) + \int_0^t \mathcal{I}(m(s))\mathrm{d}s \quad \text{for all} \quad t > 0$$

where

$$\mathcal{I}(m) = \int_{\mathbb{R}\times\Lambda} \sigma(m)\left(\nabla \frac{\delta \mathcal{F}}{\delta m}\right)^2 \mathrm{d}x,$$

is the quantity defined in (1.19). In particular, $\mathcal{F}(m(t))$ is monotonically decreasing.

The proofs of many the results established in this paper depend on certain smoothing properties of the evolution (1.1). The required *a–priori* smoothing estimates are summarized in the following theorem which holds only when $D \le 3$. The constraint on the dimensions depends on the application of Sobolev estimates to control the L_∞ norm of the gradient of v, see Lemma 7.5.

2.2 THEOREM. *Let $D \le 3$ and $m(t)$ be any solution of (1.1). Let $\epsilon > 0$, $t_0 > 0$ and $k \in \mathbb{N}$ be given. Then there is a $\delta = \delta(\epsilon, t_0, k) > 0$ and T_0 such that provided*

$$\|m(t) - \bar{m}_{a(t)}\|_2^2 \le \delta \qquad \text{for all} \qquad t \le T_0 \ ,$$

then

$$\sum_{j=0}^{k} \|(-\Delta)^{j/2}[m(t) - \bar{m}_{a(t)}]\|_2^2 \le \epsilon \qquad \text{for all} \quad t_0 \le t \le T_0$$

and also such that

$$\|x_1(m(t) - \bar{m}_{a(0)})\|_2^2 \le 2\|x_1(m(0) - \bar{m}_{a(0)})\|_2^2 \ .$$

The proof is based on several intermediate results and it is given in Section 7.

Next we prove that $a(\cdot)$ is differentiable and estimate $\dot{a}(\cdot)$.

2.3 THEOREM. *Let m be a solution of (1.1). Then there is a $\delta_0 > 0$ so that whenever*

$$\inf_{a\in\mathbb{R}} \{\|m(t) - \bar{m}_a\|_2\} < \delta_0 \tag{2.3}$$

there is a unique value $a(t)$ at which the infimum in (2.3) is attained. Moreover, for any $\kappa > 0$, there is a $\delta_1(\kappa, \beta, J)$ such that whenever $\|v(t)\|_{W^{s,2}} \le \kappa$ for $s > \frac{D}{2}$ and $\|v(t)\|_2 \le \delta_1$, $a(t)$ is differentiable and

$$|\dot{a}(t)| \le D(\kappa, \beta, J)\|v(t)\|_2$$

where $D(\kappa, \beta, J)$ is a constant depending only on κ, β and J.

Proof: Let $a(t)$ be any minimizer in (2.3). Clearly there is at least one and what we must show is that there is exactly one. Define $d(b) = \|m(t) - \bar{m}_b\|_{L^2(\mathbb{R}\times\Lambda)}^2$. We have

$$d'(b) = 2\int_{\mathbb{R}\times\Lambda} [m(t,x) - \bar{m}_b(x_1)]\bar{m}_b'(x_1)dx, \tag{2.4}$$

and

$$2\int_{\mathbb{R}\times\Lambda} \bar{m}_b(x_1)\bar{m}_b'(x_1)dx = 0.$$

Further deriving (2.4) and recalling that $m(t) = \bar{m}_{a(t)} + v(t)$ we have

$$\begin{aligned} d''(b) &= -2\int_{\mathbb{R}\times\Lambda} m(x,t)\bar{m}_b''(x_1)\mathrm{d}x = -2\int_{\mathbb{R}\times\Lambda} [\bar{m}_{a(t)}(x_1) + v(t,x)]\bar{m}_b''(x_1)\mathrm{d}x \\ &= 2\int_{\mathbb{R}\times\Lambda} \bar{m}_{a(t)}'(x_1)\bar{m}_b'(x_1)\mathrm{d}x - 2\int_{\mathbb{R}\times\Lambda} v(x,t)\bar{m}_b''(x_1)\mathrm{d}x. \end{aligned}$$

Hence,

$$d''(b) \geq 2\int_{\mathbb{R}\times\Lambda} \bar{m}'_{a(t)}(x_1)\bar{m}'_b(x_1)\mathrm{d}x - 2\delta_0\|\bar{m}''_b\|_{L^2(\mathbb{R}\times\Lambda)} \ .$$

But by continuity,

$$\int_{\mathbb{R}\times\Lambda} \bar{m}'_{a(t)}(x_1)\bar{m}'_b(x_1)\mathrm{d}x > \frac{1}{2}\|\bar{m}'_b\|^2_{L^2(\mathbb{R}\times\Lambda)}$$

on some interval $(a(t)-c, a(t)+c)$ for some c depending only on β and J. Therefore, choose

$$\delta_0 \leq \frac{\|\bar{m}'_b\|^2_{L^2(\mathbb{R}\times\Lambda)}}{4\|\bar{m}''_b\|_{L^2(\mathbb{R}\times\Lambda)}}$$

and it follows that $d''(b) > 0$ on $(a(t)-c, a(t)+c)$, and hence there is exactly one critical point of $d(b)$ on $(a(t)-c, a(t)+c)$. However, if b is any value with

$$\|m(t) - \bar{m}_b\|_{L^2(\mathbb{R}\times\Lambda)} = \|m(t) - \bar{m}_{a(t)}\|_{L^2(\mathbb{R}\times\Lambda)}$$

it follows that

$$\|\bar{m}_b - \bar{m}_{a(t)}\|_{L^2(\mathbb{R}\times\Lambda)} \leq 2\|m(t) - \bar{m}_{a(t)}\|_{L^2(\mathbb{R}\times\Lambda)} \leq 2\delta_0 \ .$$

But there is a constant C depending only on β and J so that

$$\|\bar{m}_b - \bar{m}_a\|_{L^2(\mathbb{R})} \geq \frac{(b-a)^2}{C+(b-a)^2}$$

and thus,

$$L^d\frac{(b-a)^2}{C+(b-a)^2} \leq \|\bar{m}_b - \bar{m}_a\|_{L^2(\mathbb{R}\times\Lambda)} \leq 2\delta_0 \ .$$

Decreasing δ_0 if necessary, one can ensure that $|b-a| < c$. Hence any putative second minimum must occur within $(a(t)-c, a(t)+c)$ where there is only the single critical point $a(t)$. Hence there is no other minimum. This proves that $a(t)$ is a well–defined function under the condition (2.3). To show that $a(t)$ is continuously differentiable, we use the Implicit Function Theorem. Define

$$f(a,t) := \int_{\mathbb{R}\times\Lambda} (m(t,x) - \bar{m}_a(x_1))\bar{m}'_a(x_1)\mathrm{d}x \ .$$

This is a C^1 function on $\mathbb{R}^2$, and in fact even C^2. By what we have proved above, for each t, there is exactly on $a(t)$ so that $f(a(t),t) = 0$, and at no such point does the gradient of f vanish, since the a-component of this gradient is non-zero. Hence, by the Implicit Function Theorem, the curve $t \mapsto (a(t),t)$ is continuously differentiable. Moreover, since this curve is the graph of the function $t \mapsto a(t)$, we have that $a(t)$ is continuously differentiable, as claimed.

We now bound $|\dot{a}(t)|$. Differentiating $f(a(t),t) = 0$ in t, one obtains

$$\dot{a}(t)\big(\|\bar{m}'_a\|^2_{L^2(\mathbb{R}\times\Lambda)} - \langle v, \bar{m}''_a\rangle_{L^2(\mathbb{R}\times\Lambda)}\big) = -\int_{\mathbb{R}\times\Lambda} \frac{\partial m}{\partial t}\bar{m}'_{a(t)} \ .$$

Taking into account (1.4) and integrating by part we have

$$\begin{aligned} -\int_{\mathbb{R}\times\Lambda} \frac{\partial m}{\partial t}\bar{m}'_{a(t)} &= \int_{\mathbb{R}\times\Lambda} \sigma(m)\nabla(\frac{\delta\mathcal{F}}{\delta m})e_1\bar{m}''_{a(t)} \\ &= \int_{\mathbb{R}\times\Lambda} \frac{\delta\mathcal{F}}{\delta m}\nabla\cdot[\sigma(m)e_1\bar{m}''_{a(t)}] \\ &= \int_{\mathbb{R}\times\Lambda} \frac{\delta\mathcal{F}}{\delta m}(\bar{m}+v)[\sigma(m)\bar{m}''_{a(t)}]_{x_1}. \end{aligned}$$

Assume that

$$\|v\|_{L^2(\mathbb{R}\times\Lambda)} \le \frac{1}{2}\frac{\|\bar{m}'_a\|^2_{L^2(\mathbb{R}\times\Lambda)}}{\|\bar{m}''_a\|_{L^2(\mathbb{R}\times\Lambda)}}.$$

We thus obtain

$$\begin{aligned} |\dot{a}(t)| &\le \frac{2}{\|\bar{m}'_a\|^2_{L^2(\mathbb{R}\times\Lambda)}}\left|\int_{\mathbb{R}\times\Lambda}\frac{\delta\mathcal{F}}{\delta m}(\bar{m}+v)[\sigma(m)\bar{m}''_{a(t)}]_{x_1}\right| \\ &\le C(\beta,J)\frac{2}{\|\bar{m}'_a\|^2_{L^2(\mathbb{R}\times\Lambda)}}\|[\sigma(m)\bar{m}''_{a(t)}]_{x_1}\|_{L^2(\mathbb{R}\times\Lambda)}\|v\|_{L^2(\mathbb{R}\times\Lambda)}. \end{aligned}$$

Taking into account (1.11), we bounded the L^2 norm of $\delta\mathcal{F}/\delta m(\bar{m}+v)$ by a constant times the L^2 norm of v whenever $\|v(t)\|_{W^{s,2}} \le \kappa$ with $s > \frac{D}{2}$ and κ sufficiently small to guarantee that $\|v\|_\infty \le (1-m_\beta^2)/2$. □

3 Bound on the dissipation rate of the free energy in terms of the dissipation rate for the linearized evolution

In this section we establish a bound on the rate $\mathcal{I}(m(t))$ defined in (1.19) at which the excess free energy $\mathcal{F}(m(t)) - \mathcal{F}(\overline{m})$ is dissipated in terms of the dissipation rate for the *linearized evolution*, see Theorem 3.1.

To state the main result of this section we need the following definitions. Denote by $\mathcal{B}_a$, $a \in \mathbb{R}$ the family of linear operators in $L^2(\mathbb{R}\times\Lambda)$,

$$\mathcal{B}_a v = \left(\frac{v}{\beta(1-\overline{m}_a^2)} - J\star v\right), \tag{3.1}$$

where $\overline{m}_a$, $a\in\mathbb{R}$ is the planar front defined in (1.10). Denote, by an abuse of notation, $\overline{m}'_a(x_1) = \frac{\partial}{\partial x_1}\bar{m}_a(x_1, x_1^\perp)$. It is immediate to verify that

$$(\mathcal{B}_a\overline{m}'_a)(x) = 0, \qquad x\in\mathbb{R}\times\Lambda, \tag{3.2}$$

$\overline{m}'_a$ is therefore the eigenfunction corresponding to the zero eigenvalue. Further $\mathcal{B}_a$ is a selfadojnt operator in $L^2(\mathbb{R}\times\Lambda)$ and Weil's theorem, by the same argument used in [7] for the $d=1$ case, assures the existence of a gap in the spectrum: For $v\in L^2(\mathbb{R}\times\Lambda)$

$$\int_{\mathbb{R}\times\Lambda} v(x)\overline{m}'_a(x_1)\mathrm{d}x = 0,$$

$$\langle\mathcal{B}_a v, v\rangle_{L^2(\mathbb{R}\times\Lambda)} \ge \gamma(L)\|v\|^2_{L^2(\mathbb{R}\times\Lambda)}, \tag{3.3}$$

where $\gamma(L)>0$. A quantitative argument given in [2] proves that $\gamma(L) = \frac{a(\beta,J)}{L^2}$. To our aims the merely existence of a gap is enough. Denote by $\mathcal{A}_a$, $a\in\mathbb{R}$, the family of linear operator in $L^2(\mathbb{R})$

$$\mathcal{A}_a v := \left(\frac{v}{\beta(1-\overline{m}_a^2)} - \overline{J}\star v\right), \tag{3.4}$$

where $\overline{J}$ is defined in (1.9). In [7] was shown that $\mathcal{A}_a$ has a gap γ_0. The eigenfunction corresponding to the zero eigenvalue is $\overline{m}'_a(\cdot)$. Then for all $v \in L^2(\mathbb{R})$ so that

$$\int_{\mathbb{R}} v(x_1)\overline{m}'_a(x_1)\mathrm{d}x_1 = 0$$
$$\langle \mathcal{A}_a v, v\rangle_{L^2(\mathbb{R})} \geq \gamma_0 \|v\|^2_{L^2(\mathbb{R})}.$$

The operator $\mathcal{A}_a$ is defined in term of $\bar{J}$ the one dimensional projection of J and the eigenfunction corresponding to the zero eigenvalue is $\bar{m}'_a(x_1), x_1 \in \mathbb{R}$. The operator $\mathcal{B}_a$ is the multidimensional version of the operator $\mathcal{A}_a$. Notice that the eigenfunction corresponding to the zero eigenvalue of $\mathcal{B}_a$, see (3.2), is the multidimensional version of the one of $\mathcal{A}_a$. In the following we will drop the subscript a if no confusion arises. We have:

3.1 THEOREM. *Let $D \leq 3$, $m(\cdot,t)$ be a solution of (1.1) and $m(\cdot,t) = \overline{m}_{a(t)}(\cdot) + v(\cdot,t)$ where $a(t)$ is chosen so that minimizes $\|m(t) - \overline{m}_a\|^2_{L^2(\mathbb{R}\times\Lambda)}$. For any $\epsilon > 0$ small enough, there is $\delta_1 = \delta_1(\epsilon, \beta, J, L) > 0$ so that at all time t for which $\|v(t)\|_{W^{s,2}} \leq \delta_1$, where $s > \frac{D}{2}$, we have that*

$$\begin{aligned} \frac{d}{dt}\big[\mathcal{F}\big(m(t)\big) - \mathcal{F}\big(\overline{m}\big)\big] &= -\mathcal{I}\big(m(t)\big) \\ &\leq -(1-3\epsilon)\sum_{i\geq 1}\int_{\mathbb{R}\times\Lambda} \sigma(\overline{m})\big[(\mathcal{B}v(t))_{x_i}\big]^2 \mathrm{d}x, \end{aligned}$$

where $\mathcal{I}\big(m(t)\big)$ is given in (1.19).

The proof of Theorem 3.1 is based on several intermediate results. We start proving the following estimate for the mobility $\sigma(m) = \beta(1-m^2)$.

3.2 LEMMA. *Set $m = \overline{m}_a + v$ where a is chosen so that it minimizes $\|m - \overline{m}_a\|^2_{L^2(\mathbb{R}\times\Lambda)}$. For any $\epsilon > 0$ there exists $\delta_1(\epsilon) > 0$ such that*

$$(1-\epsilon)\sigma(\overline{m}) \leq \sigma(m) \leq \sigma(\overline{m})(1+\epsilon)$$

when $\|v\|_{W^{s,2}} \leq \delta_1$ and $s > \frac{D}{2}$.

Proof: Write $\sigma(m) = \sigma(\overline{m})\left[1 + \frac{1}{\sigma(\overline{m})}\beta(2\overline{m}+v)v\right]$. One easily obtains the pointwise bound

$$\left|\frac{1}{\sigma(\overline{m})}\beta(2\overline{m}+v)v\right| \leq 2|v| + v^2 \leq 2C(d,s)\|v\|_{W^{s,2}},$$

where in the last passage we estimated the $\|v\|_\infty$ by Lemma 8.4 for $s > \frac{D}{2}$. Take δ_1 so that $2C(d,s)\delta_1 \leq \epsilon$. □

3.3 LEMMA. *Set $m = \overline{m} + v$, $v \in L^2(\mathbb{R}\times\Lambda)$, $\int \overline{m}'(x_1)v(x)\mathrm{d}x = 0$. For any $\epsilon > 0$, for $s > \frac{D}{2}$ there exists $d_1 := d_1(\epsilon, L, \beta, d)$, defined in condition (3.12), so that if $\|v\|_{W^{s,2}} \leq d_1$, we have*

$$\begin{aligned} \mathcal{I}(m) \geq{} & (1-2\epsilon)\int_{\mathbb{R}\times\Lambda} \sigma(\overline{m})\big[(\mathcal{B}v)_{x_1}\big]^2\mathrm{d}x \\ &+ \epsilon\int_{\mathbb{R}\times\Lambda}\sigma(\overline{m})\big[(\mathcal{B}v)_{x_1}\big]^2\mathrm{d}x - \frac{1}{\epsilon}\int_{\mathbb{R}\times\Lambda}\sigma(\overline{m})[U(v)]^2\mathrm{d}x \\ &+ (1-2\epsilon)\sum_{i\geq 2}\int_{\mathbb{R}\times\Lambda}\sigma(\overline{m})\big[(\mathcal{B}v)_{x_i}\big]^2\mathrm{d}x \end{aligned} \tag{3.5}$$

where $U(v)$ is defined in (3.8).

Proof: Since $\overline{m}$ depends only on x_1 and by assumption $m(x) = \overline{m}(x_1) + v(x)$ we decompose $\mathcal{I}(m)$, see (1.19), as

$$\mathcal{I}(\overline{m} + v) = \mathcal{I}_1(\overline{m} + v) + \mathcal{I}_2(\overline{m} + v)$$

where

$$\mathcal{I}_1(\overline{m} + v) = \int \sigma(m) \left[\left(\frac{m_{x_1}}{\beta(1-m^2)} - J \star m_{x_1} \right) \right]^2 \mathrm{d}x,$$

and

$$\mathcal{I}_2(\overline{m} + v) = \sum_{i \geq 2} \int \sigma(m) \left[\left(\frac{v_{x_i}}{\beta(1-m^2)} - J \star v_{x_i} \right) \right]^2 \mathrm{d}x.$$

We have

$$\left(\frac{m_{x_1}}{\beta(1-m^2)} - J \star m_{x_1} \right) = \left(\frac{m_{x_1}}{\beta(1-m^2)} - \frac{m_{x_1}}{\beta(1-\overline{m}^2)} \right) + \mathcal{B} m_{x_1},$$

where $\mathcal{B}$ is the linear operator defined in (3.1). Denote

$$\tilde{U}(v) := \frac{1}{\beta} \left(\frac{1}{1-m^2} - \frac{1}{1-\overline{m}^2} \right) = \frac{1}{\beta} \frac{2\overline{m}}{(1-\overline{m}^2)^2} v + \frac{1 + 3\overline{m}^2 + 2\overline{m}v}{(1-\overline{m}^2)^2(1-m^2)} v^2. \tag{3.6}$$

Since $\mathcal{B} m_{x_1} = \mathcal{B}\overline{m}' + \mathcal{B} v_{x_1} = \mathcal{B} v_{x_1}$, by (3.2), we have

$$\begin{aligned} &\frac{1}{\beta} \frac{m_{x_1}}{1-m^2} - J \star m_{x_1} = \mathcal{B} v_{x_1} + \tilde{U}(v)(v_{x_1} + \overline{m}') \\ &= \mathcal{B} v_{x_1} + \frac{1}{\beta} \frac{2\overline{m}\overline{m}'}{(1-\overline{m}^2)^2} v + U(v) \end{aligned} \tag{3.7}$$

where

$$U(v) = \frac{1}{\beta} \frac{2\overline{m}}{(1-\overline{m}^2)^2} v v_{x_1} + \frac{1}{\beta} \frac{1 + 3\overline{m}^2 + 2\overline{m}v}{(1-\overline{m}^2)^2(1-m^2)} v^2 (v_{x_1} + \overline{m}'). \tag{3.8}$$

But since

$$\frac{2\bar{m}\bar{m}'}{(1-\overline{m}^2)^2} = \frac{\mathrm{d}}{\mathrm{d}x_1} \left(\frac{1}{1-\overline{m}^2} \right)$$

(3.7) is the same as

$$\frac{1}{\beta} \frac{m_{x_1}}{1-m^2} - J \star m_{x_1} = (\mathcal{B} v)_{x_1} + U(v) \ .$$

Applying Lemma 3.2 we have that

$$\mathcal{I}_1(m + v) \geq (1-\epsilon) \int_{\mathbb{R}\times\Lambda} \sigma(\overline{m}) \big[(\mathcal{B}v)_{x_1} + U(v) \big]^2 \mathrm{d}x \tag{3.9}$$

provided d_1 is less than the δ_1 of Lemma 3.2.

We apply to (3.9), inequality (8.12) stated in the appendix with $\lambda = 1 - \epsilon$, where $\epsilon > 0$ is small and arbitrarily chosen. We obtain

$$\begin{aligned} &\int_{\mathbb{R}\times\Lambda} \sigma(\overline{m}) \big[(\mathcal{B}v)_{x_1} + U(v) \big]^2 \mathrm{d}x \\ &\geq (1-2\epsilon) \int_{\mathbb{R}\times\Lambda} \sigma(\overline{m}) \big[(\mathcal{B}v)_{x_1} \big]^2 \mathrm{d}x \\ &+ \epsilon \int_{\mathbb{R}\times\Lambda} \sigma(\overline{m}) \big[(\mathcal{B}v)_{x_1} \big]^2 \mathrm{d}x - \frac{1}{\epsilon} \int_{\mathbb{R}\times\Lambda} \sigma(\overline{m}) [U(v)]^2 \mathrm{d}x. \end{aligned}$$

We proceed similarly for $\mathcal{I}_2(\overline{m}+v)$, taking in account that $\overline{m}$ depends only on x_1. We have

$$\frac{1}{\beta}\frac{v_{x_i}}{1-m^2} - J\star v_{x_i} = \mathcal{B}v_{x_i} + \tilde{U}(v)v_{x_i}, \quad i\geq 2$$

where $\tilde{U}(v)$ is given in (3.6). Then

$$\begin{aligned}\mathcal{I}_2(\overline{m}+v) &\geq (1-2\epsilon)\sum_{i\geq 2}\int_{\mathbb{R}\times\Lambda}\sigma(\overline{m})\left[\mathcal{B}v_{x_i}\right]^2 \mathrm{d}x \\ &+ \epsilon\sum_{i\geq 2}\left(\int_{\mathbb{R}\times\Lambda}\sigma(\overline{m})\left[\mathcal{B}v_{x_i}\right]^2\mathrm{d}x - \frac{1}{\epsilon}\int_{\mathbb{R}\times\Lambda}\sigma(\overline{m})[\tilde{U}(v)v_{x_i}]^2\mathrm{d}x\right). \end{aligned} \tag{3.10}$$

Next we show that the last line of (3.10) is positive when $\|v\|^2_{W^{s,2}}$ is small enough. By periodicity $\int_\Lambda v_{x_j}(x)\mathrm{d}x^\perp = 0$ for all $j\geq 2$. This implies that $\int_{\mathbb{R}\times\Lambda} v_{x_j}(x)\overline{m}'(x_1)\mathrm{d}x = 0$ for $j\geq 2$, therefore by (3.3)

$$\sum_{i\geq 2}\int_{\mathbb{R}\times\Lambda}\sigma(\overline{m})\left[\mathcal{B}v_{x_i}\right]^2\mathrm{d}x \geq \gamma(L)^2\|\nabla^\perp v\|_2^2. \tag{3.11}$$

We have

$$\sum_{i\geq 2}\int_{\mathbb{R}\times\Lambda}\sigma(\overline{m})[\tilde{U}(v(x))v_{x_i}]^2\mathrm{d}x \leq c(\beta)\sup_{x\in\mathbb{R}\times\Lambda}|\tilde{U}(v)|^2\|\nabla^\perp v\|_2^2 \leq C(d,\beta)\|v\|^2_{W^{s,2}}\|\nabla^\perp v\|_2^2,$$

by Lemma 8.4 and $s>\frac{D}{2}$. Then for any given $\epsilon>0$ take $d_1 := d_1(L,\beta,\epsilon)$ so that for $\|v\|^2_{W^{s,2}}\leq d_1$, see (3.10) and (3.11),

$$\epsilon\gamma^2(L) \geq \frac{1}{\epsilon}C(d,\beta)d_1. \tag{3.12}$$

We then obtain (3.5) □

We would like to show that the quantity on the right hand side of (3.5) is strictly positive. There is no hope to show that the second line on the right hand side of (3.5) is positive. We cannot expect to control the nonlinear contribution of the dissipation of the free energy by only the derivative in the x_1 direction. We need to take into account also the gradient in the orthogonal direction of x_1. To this aim we denote, see (3.5),

$$G_\epsilon(v) := \epsilon\left(\sum_{i\geq 1}\int_{\mathbb{R}\times\Lambda}\sigma(\overline{m})\left[(\mathcal{B}v)_{x_i}\right]^2\mathrm{d}x\right) - \frac{1}{\epsilon}\int_{\mathbb{R}\times\Lambda}\sigma(\overline{m})[U(v)]^2\mathrm{d}x. \tag{3.13}$$

In the next proposition we show that $G_\epsilon(v)$ is positive under smoothing assumptions on v.

3.4 PROPOSITION. *Let $s>\frac{D}{2}$, $v\in W^{s+1,2}(\mathbb{R}\times\Lambda)$, $\int_{\mathbb{R}\times\Lambda} v(x)\overline{m}'(x_1)\mathrm{d}x = 0$. For any $\epsilon>0$*

$$G_\epsilon(v)\geq 0, \tag{3.14}$$

provided

$$\|v\|^2_{W^{s+1,2}}\leq\epsilon_0 \tag{3.15}$$

for $\epsilon_0 = \epsilon^{2+r}$ for $r=r(L)>0$.

The proof depends on several intermediate results, and it is given at the end of the section.

In Lemma 3.3 we took advantage by decomposing $m = \overline{m} + v$, $v \in L^2(\mathbb{R} \times \Lambda)$. In the following it is helpful to split $v \in L^2(\mathbb{R} \times \Lambda)$ in the manner:

$$v(x) := v_1(x_1) + w(x), \tag{3.16}$$

where

$$v_1(x_1) := \frac{1}{L^d} \int_\Lambda v(x_1, x^\perp) dx^\perp, \quad x_1 \in \mathbb{R}.$$

By construction,

$$\int_\Lambda w(x_1, x^\perp) dx^\perp = 0 \quad \forall x_1 \in \mathbb{R}. \tag{3.17}$$

Further if $\int_{\mathbb{R}\times\Lambda} v(x)\overline{m}'(x_1)dx = 0$ then

$$\int_{\mathbb{R}\times\Lambda} w(x)\overline{m}'(x_1)dx = \int_{\mathbb{R}} dx_1 \overline{m}'(x_1) \int_\Lambda w(x) dx_1^\perp = 0,$$

$$\int_{\mathbb{R}} v_1(x_1)\overline{m}'(x_1)dx = 0. \tag{3.18}$$

Using decomposition (3.16) we get the following useful result.

3.5 LEMMA. *Let $v \in L^2(\mathbb{R} \times \Lambda)$, $v_{x_1} \in L^2(\mathbb{R} \times \Lambda)$, $v = v_1 + w$ as in (3.16). We have*

$$\begin{aligned}\int_{\mathbb{R}\times\Lambda} \sigma(\overline{m})\left[(\mathcal{B}v)_{x_1}\right]^2 \mathrm{d}x &= \int_{\mathbb{R}\times\Lambda} \sigma(\overline{m})\left[(\mathcal{A}v_1)_{x_1}\right]^2 \mathrm{d}x \\ &+ \int_{\mathbb{R}\times\Lambda} \sigma(\overline{m})\left[(\mathcal{B}w)_{x_1}\right]^2 \mathrm{d}x.\end{aligned}$$

Proof: Take v as in (3.16)

$$\begin{aligned}
&\int_{\mathbb{R}\times\Lambda} \sigma(\overline{m})\left[(\mathcal{B}v)_{x_1}\right]^2 \mathrm{d}x \\
= &\int \sigma(\overline{m}(x_1)) \left[\frac{\partial}{\partial x_1}\left(\frac{v_1}{\beta(1-\overline{m}^2)} - \overline{J}\star v_1\right) + \frac{\partial}{\partial x_1}\left(\frac{w}{\beta(1-\overline{m}_0^2)} - J\star w\right)\right]^2 \mathrm{d}x \\
= &\int \sigma(\overline{m}(x_1)) \left[\frac{\partial}{\partial x_1}\left(\frac{v_1}{\beta(1-\overline{m}^2)} - \overline{J}\star v_1\right)\right]^2 \mathrm{d}x \\
+ &\int \sigma(\overline{m}(x_1)) \left[\frac{\partial}{\partial x_1}\left(\frac{w}{\beta(1-\overline{m}^2)} - J\star w\right)\right]^2 \mathrm{d}x \\
+ \; 2&\int \sigma(\overline{m}(x_1)) \left[\frac{\partial}{\partial x_1}\left(\frac{v_1}{\beta(1-\overline{m}^2)} - \overline{J}\star v_1\right)\right]\left[\frac{\partial}{\partial x_1}\left(\frac{w}{\beta(1-\overline{m}^2)} - J\star w\right)\right] \mathrm{d}x.
\end{aligned} \tag{3.19}$$

Integrating per part with respect to x_1 the last term in (3.19) we have

$$\begin{aligned}
&2\int \sigma(\overline{m}(x_1)) \left[\frac{\partial}{\partial x_1}\left(\frac{v_1}{\beta(1-\overline{m}^2)} - \overline{J}\star v_1\right)\right]\left[\frac{\partial}{\partial x_1}\left(\frac{w}{\beta(1-\overline{m}^2)} - J\star w\right)\right] \mathrm{d}x \\
= &-2\int \frac{\partial}{\partial x_1}\left[\sigma(\overline{m}(x_1))\frac{\partial}{\partial x_1}\left(\frac{v_1}{\beta(1-\overline{m}^2)} - \overline{J}\star v_1\right)\right]\left[\frac{w}{\beta(1-\overline{m}^2)} - J\star w\right] \mathrm{d}x_1 \mathrm{d}x_1^\perp \\
= &-2\int_{\mathbb{R}} \frac{\partial}{\partial x_1}\left[\sigma(\overline{m}(x_1))\frac{\partial}{\partial x_1}\left(\frac{v_1}{\beta(1-\overline{m}^2)} - \overline{J}\star v_1\right)\right]\left(\int_\Lambda \left[\frac{w}{\beta(1-\overline{m}^2)} - J\star w\right] \mathrm{d}x_1^\perp\right) \mathrm{d}x_1 = 0.
\end{aligned}$$

Namely for each x_1 integrating with respect to $x_1^\perp$ we have

$$\int_\Lambda \left[\frac{w}{\beta(1-\overline{m}^2)} - J \star w \right] \mathrm{d}x_1^\perp = 0.$$

□

Taking into account Lemma 3.5 we write, see (3.13),

$$\begin{aligned} G_\epsilon(v) = \epsilon & \left[\int_{\mathbb{R}\times\Lambda} \sigma(\overline{m}) \big[(\mathcal{A}v_1)_{x_1} \big]^2 \mathrm{d}x + \sum_{i\ge 2} \int_{\mathbb{R}\times\Lambda} \sigma(\overline{m}) \big[\mathcal{B}(w_{x_i}) \big]^2 \mathrm{d}x \right] \\ & + \epsilon \int_{\mathbb{R}\times\Lambda} \sigma(\overline{m}) \big[(\mathcal{B}w)_{x_1} \big]^2 \mathrm{d}x - \frac{1}{\epsilon} \int_{\mathbb{R}\times\Lambda} \sigma(\overline{m}) [U(v)]^2 \mathrm{d}x. \end{aligned} \tag{3.20}$$

To show Proposition 3.4 we bound from below the first three terms of (3.20) and from above the last term of (3.20) in term of comparable quantities. We estimate the first and third term of (3.20) in Lemma 3.6 and Lemma 3.7. The lower bound for the second term in (3.20) is easily obtained taking into account that $\int_{\Lambda\times\mathbb{R}} w_{x_i} \overline{m}'(x_1) dx = 0$ and applying (3.3)

$$\sum_{i\ge 2} \int_{\mathbb{R}\times\Lambda} \sigma(\overline{m}) \big[\mathcal{B}(w_{x_i}) \big]^2 \mathrm{d}x \ge \gamma(L) \sigma(m_\beta) \sum_{i\ge 2} \|w_{x_i}\|_2^2. \tag{3.21}$$

Then we estimate from above in term of the same quantities the $U(v)$ term, see Lemma 3.8.

3.6 LEMMA. *Let $v \in L^2(\mathbb{R}\times\Lambda)$, $v_{x_1} \in L^2(\mathbb{R}\times\Lambda)$, $\int_{\mathbb{R}\times\Lambda} v(x)\overline{m}'(x_1)dx = 0$ and $v(x) = v_1(x_1) + w(x)$, see decomposition (3.16), then there exists a positive constant $\gamma_1 := \gamma_1(\beta, J)$, such that*

$$\int_{\mathbb{R}\times\Lambda} \sigma(\overline{m}) \left[\left(\frac{\partial}{\partial x_1} (\mathcal{A}v_1) \right) \right]^2 \mathrm{d}x \ge \sigma(m_\beta) \gamma_1 \|Pv_1'\|^2_{L^2(\mathbb{R}\times\Lambda)}$$

where $\mathcal{A}$ is the linear operator defined in (3.4), P is the orthogonal projection on the orthogonal complement of $\overline{m}''$ of $L^2(\mathbb{R})$.

Proof: We apply Lemma 3.4 of [3]. The assumption needed is $\int_{\mathbb{R}} v_1(x_1)\overline{m}'(x_1)\mathrm{d}x_1 = 0$ which is indeed satisfied; see (3.18).

We then obtain, from Lemma 3.4 of [3], that there exists a positive constant γ_1 depending on β and J so that

$$\begin{aligned} \int_{\mathbb{R}\times\Lambda} \sigma(\overline{m}) \big[(\mathcal{A}v_1)_{x_1} \big]^2 \mathrm{d}x & \ge \sigma(m_\beta)\gamma_1 \int_\Lambda \mathrm{d}x_1^\perp \int_{\mathbb{R}} \mathrm{d}x_1 [(Pv_1')(x_1)]^2 \\ & = \sigma(m_\beta)\gamma_1 \|Pv_1'\|^2_{L^2(\mathbb{R}\times\Lambda)}, \end{aligned}$$

where P is the orthogonal projection on the orthogonal complement of $\overline{m}''$ in $L^2(\mathbb{R})$, i.e.

$$Pv_1' = v_1' - \overline{m}'' \frac{\int_{\mathbb{R}} v_1'(x_1)\overline{m}''(x_1)\mathrm{d}x_1}{\|\overline{m}''\|^2_{L^2(\mathbb{R})}}.$$

□

Next, we estimate from below the term $\int_{\mathbb{R}\times\Lambda} \sigma(\overline{m}) \left[\left(\frac{\partial}{\partial x_1} (\mathcal{B}w) \right) \right]^2 \mathrm{d}x$. When dealing with the heat equation in our heuristic discussion, the corresponding term was simply dropped. Now we need to bound it from below to get some positive contribution that may be used to cancel negative contributions coming from the last term of (3.20). The estimate is obtained by introducing a cut-off function. Without cut-off we could get an estimate of the type:

$$\int_{\mathbb{R}\times\Lambda} \sigma(\overline{m}) \left[\left(\frac{\partial}{\partial x_1} (\mathcal{B}w) \right) \right]^2 \mathrm{d}x \geq \sigma(m_\beta)\gamma^2(L)\|w_{x_1}\|_2^2 - C\|w\|_2^2. \tag{3.22}$$

The main difference between this and (3.25) is that the term $\|w\|_2^2$ in (3.22) is a priori not small and we do not have a way to control it.

Let $N \geq 1$ and $\phi_N^2(x_1)$, $x_1 \in \mathbb{R}$ be a smooth cut-off function so that

$$\phi_N^2(x_1) = \begin{cases} 0 & |x_1| \leq N \\ 1 & |x_1| \geq 2N \end{cases} \tag{3.23}$$

and

$$|\phi_N(x_1)| \leq 1, \quad |\phi_N'(x_1)| \leq \frac{1}{N}, \qquad |\phi_N''(x_1)| \leq \frac{1}{N^2}. \tag{3.24}$$

The choice of cut-off N will depend on L, the linear size of the transversal direction to the front, and it will be chosen as function of ϵ, see proof of Proposition 3.4. We have the following.

3.7 LEMMA. *Take $v \in L^2(\mathbb{R}\times\Lambda)$, $\int_{\mathbb{R}\times\Lambda} v(x)\overline{m}'(x_1)dx = 0$, $v = v_1 + w$, see decomposition (3.16), $w_{x_1} \in L^2(\mathbb{R}\times\Lambda)$ and ϕ_N^2 the cut-off function defined in (3.23). Then for any $N \geq 1$,*

$$\int_{\mathbb{R}\times\Lambda} \sigma(\overline{m}) \left[\left(\frac{\partial}{\partial x_1} (\mathcal{B}w) \right) \right]^2 \mathrm{d}x \geq \frac{1}{4}\sigma(m_\beta)\gamma^2(L)\|\phi_N(w)_{x_1}\|_2^2 - \|w\|_2^2 \frac{1}{N^2} D(\beta, \gamma(L)) \tag{3.25}$$

where $\mathcal{B}$ is the linear operator defined in (3.1) and $D(\beta,\gamma(L))$ is defined in (3.34).

Proof: We have that

$$\int_{\mathbb{R}\times\Lambda} \sigma(\overline{m}) \left[\left(\frac{\partial}{\partial x_1} (\mathcal{B}w) \right) \right]^2 \mathrm{d}x \geq \int_{\mathbb{R}\times\Lambda} \sigma(\overline{m}) \left[\phi_N \left(\frac{\partial}{\partial x_1} (\mathcal{B}w) \right) \right]^2 \mathrm{d}x. \tag{3.26}$$

Using that for smooth integrable functions g and h one has

$$\int_{\mathbb{R}} [(gh)']^2 = \int_{\mathbb{R}} [gh']^2 - \int_{\mathbb{R}} gg''h^2$$

we have

$$\begin{aligned} &\int_{\mathbb{R}\times\Lambda} \sigma(\overline{m}) \left[\phi_N \left(\frac{\partial}{\partial x_1} (\mathcal{B}w) \right) \right]^2 \mathrm{d}x \\ &= \int_{\mathbb{R}\times\Lambda} \sigma(\overline{m}) \left[\left(\frac{\partial}{\partial x_1} \phi_N (\mathcal{B}w) \right) \right]^2 \mathrm{d}x + \int_{\mathbb{R}\times\Lambda} \sigma(\overline{m}) \phi_N \phi_N'' \left[\mathcal{B}w \right]^2 \mathrm{d}x. \end{aligned} \tag{3.27}$$

By the property of ϕ_N, see (3.24), the last term in (3.27) is estimated as following

$$\left| \int_{\mathbb{R}\times\Lambda} \sigma(\overline{m}) \phi_N \phi_N'' \left[\mathcal{B}w \right]^2 \mathrm{d}x \right| \leq \sup |\phi_N''| \|\mathcal{B}w\|_2^2 \leq \frac{1}{N^2} \|\mathcal{B}w\|_2^2 \leq \frac{1}{N^2} C(\beta) \|w\|_2^2,$$

where in the last inequality we used the fact that $\mathcal{B}$ is a bounded operator in L^2. We then obtain that, see (3.26),

$$\int_{\mathbb{R}\times\Lambda} \sigma(\overline{m}) \left[\left(\frac{\partial}{\partial x_1}(\mathcal{B}w)\right)\right]^2 \mathrm{d}x \geq \int_{\mathbb{R}\times\Lambda} \sigma(\overline{m}) \left[\left(\frac{\partial}{\partial x_1}\phi_N(\mathcal{B}w)\right)\right]^2 \mathrm{d}x - \frac{1}{N^2}C(\beta)\|w\|_2^2. \tag{3.28}$$

Next we estimate the first term on the right hand side of (3.28). We write

$$\phi_N \mathcal{B}w = \mathcal{B}(\phi_N w) - \phi_N J \star w + J \star (\phi_N w),$$

and apply the inequality see Lemma 8.5 in the Appendix, writing λ in (8.12) as $\lambda = \frac{1}{2}$,

$$(a+b)^2 \geq \frac{1}{2}a^2 - b^2.$$

We have

$$\begin{aligned}
&\int_{\mathbb{R}\times\Lambda} \sigma(\overline{m}) \left[\frac{\partial}{\partial x_1}(\phi_N \mathcal{B}w)\right]^2 \mathrm{d}x \\
&\geq \frac{1}{2}\int_{\mathbb{R}\times\Lambda} \sigma(\overline{m}) \left[\left(\frac{\partial}{\partial x_1}\mathcal{B}(\phi_N w)\right)\right]^2 \mathrm{d}x \\
&- \int_{\mathbb{R}\times\Lambda} \sigma(\overline{m}) \left[J \star (\phi_N w) - \phi_N J \star w\right]^2 .
\end{aligned} \tag{3.29}$$

Further, we have

$$\begin{aligned}
&\int_{\mathbb{R}\times\Lambda} \sigma(\overline{m}) \left[\left(\frac{\partial}{\partial x_1}\mathcal{B}(\phi_N w)\right)\right]^2 \mathrm{d}x \\
&= \int_{\mathbb{R}\times\Lambda} \sigma(\overline{m}) \left[\mathcal{B}\left(\frac{\partial}{\partial x_1}(w\phi_N)\right) + w(x)\phi_N \frac{\partial}{\partial x_1}\left(\frac{1}{\beta(1-\overline{m}^2)}\right)\right]^2 \mathrm{d}x \\
&\geq \frac{1}{2}\int_{\mathbb{R}\times\Lambda} \sigma(\overline{m}) \left[\mathcal{B}\left(\frac{\partial}{\partial x_1}(w\phi_N)\right)\right]^2 \mathrm{d}x \\
&- \int_{\mathbb{R}\times\Lambda} \left[\phi_N w(x)\frac{\partial}{\partial x_1}\left(\frac{1}{\beta(1-\overline{m}^2)}\right)\right]^2 \mathrm{d}x.
\end{aligned} \tag{3.30}$$

By (3.17),

$$\int_{\Lambda\times\mathbb{R}} \left(\frac{\partial}{\partial x_1}(\phi_N(x_1)w(x))\right) \bar{m}'(x_1)\mathrm{d}x = -\int_{\Lambda\times\mathbb{R}} \phi_N(x_1)w(x)\bar{m}''(x_1)\mathrm{d}x = 0,$$

and therefore, by (3.3)

$$\int_{\mathbb{R}\times\Lambda} \sigma(\overline{m}) \left[\mathcal{B}\left(\frac{\partial}{\partial x_1}(w\phi_N)\right)\right]^2 \mathrm{d}x \geq \sigma(m_\beta)\gamma^2(L)\|(\phi_N w)_{x_1}\|_2^2.$$

Taking into account that

$$\frac{\partial}{\partial x_1}\left(\frac{1}{\beta(1-\overline{m}^2)}\right) = \frac{2\bar{m}\bar{m}'}{\beta(1-\overline{m}^2)^2}$$

and $\bar{m}'$ is exponential decreasing to zero, see (1.10), we have that

$$\int_{\mathbb{R}\times\Lambda} \left[\phi_N w(x)\frac{\partial}{\partial x_1}\left(\frac{1}{\beta(1-\overline{m}^2)}\right)\right]^2 \mathrm{d}x \leq \|w\|_2^2 C(\beta)e^{-\alpha N}.$$

Therefore (3.30) can be estimated as follows:

$$\begin{aligned}&\int_{\mathbb{R}\times\Lambda}\sigma(\overline{m})\left[\frac{\partial}{\partial x_1}\mathcal{B}(\phi_N w)\right]^2 \mathrm{d}x\\&\geq \frac{1}{2}\sigma(m_\beta)\gamma^2(L)\|(\phi_N w)_{x_1}\|_2^2-\|w\|_2^2 C(\beta)e^{-\alpha N}.\end{aligned}\tag{3.31}$$

We have that

$$|(J\star(\phi_N w)-\phi_N J\star w)(x)|\leq \int_{\mathbb{R}\times\Lambda} J(x-y)\,|w(y)[\phi_N(y_1)-\phi_N(x_1)]|\,dy\leq \frac{1}{N}\int_{\mathbb{R}\times\Lambda}J(x-y)|w(y)|dy,$$

namely by the mean value theorem and (3.24) $|\phi_N(y_1)-\phi_N(x_1)|\leq \frac{1}{N}|y_1-x_1|$. Since J has compact support contained in a ball of radius 1, we have for y_1 and x_1 in the support of J, $|\phi_N(y_1)-\phi_N(x_1)|\leq\frac{1}{N}$. The last term of (3.29) is therefore estimated as the following:

$$\int_{\mathbb{R}\times\Lambda}\sigma(\overline{m})\left[J\star(\phi_N w)-\phi_N J\star w\right]^2\leq \frac{1}{N^2}\int_{\mathbb{R}\times\Lambda}(J\star|w|)^2\leq\frac{1}{N^2}\|w\|_2^2.\tag{3.32}$$

Taking into account (3.31) and (3.32) we estimate (3.29) as follows:

$$\begin{aligned}&\int_{\mathbb{R}\times\Lambda}\sigma(\overline{m})\left[\frac{\partial}{\partial x_1}(\phi_N\mathcal{B}w)\right]^2\mathrm{d}x\\&\geq\frac{1}{4}\sigma(m_\beta)\gamma^2(L)\|(\phi_N w)_{x_1}\|_2^2-\|w\|_2^2C(\beta)e^{-\alpha N}-\frac{1}{N^2}\|w\|_2^2\\&\geq\frac{1}{4}\sigma(m_\beta)\gamma^2(L)\|(\phi_N w)_{x_1}\|_2^2-\|w\|_2^2\left[C(\beta)e^{-\alpha N}+\frac{1}{N^2}\right],\end{aligned}\tag{3.33}$$

Further

$$\|(\phi_N w)_{x_1}\|_2^2=\|\phi_N(w)_{x_1}\|^2-\int_{\mathbb{R}\times\Lambda}\phi_N\phi_N'' w^2\geq\|\phi_N(w)_{x_1}\|_2^2-\frac{1}{N^2}\|w\|_2^2.$$

Finally, from (3.28) and (3.33) obtain

$$\begin{aligned}&\int_{\mathbb{R}\times\Lambda}\sigma(\overline{m})\left[\phi_N\left(\frac{\partial}{\partial x_1}(\mathcal{B}w)\right)\right]^2\mathrm{d}x\\ \geq\;&\frac{1}{4}\sigma(m_\beta)\gamma^2(L)\|\phi_N(w)_{x_1}\|_2^2-\|w\|_2^2\left[\frac{1}{4}\sigma(m_\beta)\gamma^2(L)\frac{1}{N^2}+C(\beta)e^{-\alpha N}+\frac{C(\beta)}{N^2}+\frac{1}{N^2}\right]\\ \geq\;&\frac{1}{4}\sigma(m_\beta)\gamma^2(L)\|\phi_N(w)_{x_1}\|_2^2-\|w\|_2^2\frac{1}{N^2}D(\beta,\gamma(L)),\end{aligned}$$

where

$$D(\beta,\gamma(L)):=\left[\frac{1}{4}\sigma(m_\beta)\gamma^2(L)+2C(\beta)\right].\tag{3.34}$$

□

In the next lemma we estimate $\int\sigma(m)\big[U(v)\big]^2\mathrm{d}x$ from above in two different ways which will be used in different regimes.

3.8 LEMMA. *Let $v \in L^2(\mathbb{R}\times\Lambda)$, $\int_{\mathbb{R}\times\Lambda} v(x)\overline{m}'(x_1)\mathrm{d}x = 0$, $v \in W^{s+1,2}(\mathbb{R}\times\Lambda)$, $s > \frac{D}{2}$, $v = v_1 + w$, as in (3.16), and ϕ_N be the cut-off function defined in (3.23). For the non linear operator $U(\cdot)$ defined in (3.8) the following holds*

$$\begin{aligned}\int \sigma(m)\big[U(v)\big]^2\mathrm{d}x \le &||v_1'||^2_{L^2(\mathbb{R}\times\Lambda)} C(\beta,J)\left\{\|v\|^2_{W^{s,2}} + \|w_{x_1}\|_2^2 N\right\}\\ &+\|w\|_2^2 C(\beta,J)\left\{\|v\|^2_{W^{s,2}} + \|v\|^2_{W^{s+1,2}}\right\}\\ &+\|v\|^2_{W^{s,2}} C(\beta,J)\|\phi_N w_{x_1}\|_2^2.\end{aligned} \tag{3.35}$$

Further, assume that $\|v_1\|^2_{L^2(\mathbb{R}\times\Lambda)} \le k^2$ and $\|v_1'\|^2_{L^2(\mathbb{R}\times\Lambda)} \le k^2$. Then for any given $\epsilon_1 > 0$ there exists $\lambda_0 = \lambda_0(\epsilon_1, k)$, see (3.57), so that

$$\begin{aligned}\int \sigma(m)\big[U(v)\big]^2\mathrm{d}x \le &\left[C(\beta,J) + \frac{2}{\lambda_0}\right] ||v_1'||^4_{L^2(\mathbb{R}\times\Lambda)}\\ &+\|w\|^2_{L^2(\mathbb{R}\times\Lambda)} C(\beta,J,d)\left[\|v\|^2_{W^{s,2}} + (\frac{N}{L^d})^2\|v\|^2_{W^{s+1,2}}\right]\\ &+ 8\epsilon_1||Pv_1'||^2_{L^2(\mathbb{R})} + 2\|v\|_{W^{s,2}}\|\phi_N w_{x_1}\|^2,\end{aligned} \tag{3.36}$$

where P is the orthogonal projection on the orthogonal complement of $\overline{m}''$ in $L^2(\mathbb{R})$.

Proof: Observe that for some constant C depending only on β and J

$$\begin{aligned}|U(v)|^2 \le &\,2\left(\frac{1}{\beta}\frac{2\overline{m}}{(1-\overline{m}^2)^2}\right)^2 v^2 v_{x_1}^2 + 2\left(\frac{1}{\beta}\frac{1+3\overline{m}^2+2\overline{m}v}{(1-\overline{m}^2)^2(1-m^2)}\right)^2 v^4 v_{x_1}^2\\ &+2\left(\frac{1}{\beta}\frac{1+3\overline{m}^2+2\overline{m}v}{(1-\overline{m}^2)^2(1-m^2)}\overline{m}'\right)^2 v^4\\ &\le C(\beta,J)\big(R(x_1)|v|^4 + |v|^2|v_{x_1}|^2\big)\end{aligned} \tag{3.37}$$

where $R(\cdot)$ is non-negative, exponentially decreasing to zero as $|x_1| \uparrow \infty$ and $\int_{\mathbb{R}} R(x_1)\mathrm{d}x_1 = 1$. We start deriving (3.35). Splitting $v = v_1 + w$ as in (3.16), we have

$$v^4 = v^2[v_1 + w]^2 \le 2v^2[v_1^2 + w^2].$$

Then since $\|v\|_\infty \le c(d,s)\|v\|_{W^{s,2}}$, for $s > \frac{D}{2}$, see Lemma 8.4,

$$\int_{\mathbb{R}\times\Lambda} R(x_1)|v|^4\mathrm{d}x \le 2c(d,s)\|v\|^2_{W^{s,2}}\int_{\mathbb{R}\times\Lambda} R(x_1)[v_1^2(x_1) + w^2(x)]\mathrm{d}x. \tag{3.38}$$

We may write

$$v_1(x_1) = v_1(y) + \int_y^{x_1} v'(z)\mathrm{d}z.$$

We then multiply both terms by $\overline{m}'(y)$ and integrate on the real line. Since $\int v_1(y)\overline{m}'(y)dy = 0$ we have

$$v_1(x_1) = \frac{1}{2m_\beta}\int_{-\infty}^{\infty}\overline{m}'(y)\left(\int_y^{x_1} v_1'(z)\mathrm{d}z\right)\mathrm{d}y \tag{3.39}$$

and therefore

$$\begin{aligned}|v_1(x_1)| &\le \frac{1}{2m_\beta}\left(\int \overline{m}'(y)|x_1-y|^{\frac12}\mathrm{d}y\right)||v_1'||_{L^2(\mathbb{R})}\\ &\le \left(\frac{1}{2m_\beta}\int \overline{m}'(y)|x_1-y|^2\mathrm{d}y\right)^{\frac14}||v_1'||_{L^2(\mathbb{R})} \le C(\beta,J)[1+|x_1|^2]^{\frac14}||v_1'||_{L^2(\mathbb{R})}.\end{aligned} \tag{3.40}$$

Therefore from (3.38) and (3.40) we have

$$\begin{aligned}
&\int_{\mathbb{R}\times\Lambda} R(x_1)|v|^4 \mathrm{d}x \leq \\
&2c(d,s)\|v\|^2_{W^{s,2}}\left[||v_1'||^2_{L^2(\mathbb{R})} C(\beta,J)\int_{\mathbb{R}\times\Lambda} R(x_1)((1+|x_1|^2)^{\frac{1}{2}}\mathrm{d}x + \int_{\mathbb{R}\times\Lambda} R(x_1)w^2(x)\mathrm{d}x\right] \leq \\
&\|v\|^2_{W^{s,2}} C(\beta,J,s,d)\left[||v_1'||^2_{L^2(\mathbb{R}\times\Lambda)} + \|w\|^2_{L^2(\mathbb{R}\times\Lambda)}\right].
\end{aligned} \tag{3.41}$$

To estimate the contribution from the last term term in (3.37) we split $v_{x_1} = v_1' + w_{x_1}$ where v_1 and w as in (3.16), obtaining

$$\begin{aligned}
\int |v|^2|v_{x_1}|^2\mathrm{d}x &\leq 2\left[\int |v|^2|v_1'|^2\mathrm{d}x + \int |v|^2|w_{x_1}|^2\mathrm{d}x\right] \\
&\leq 2\|v\|^2_\infty\|v_1'\|^2_{L^2(\mathbb{R}\times\Lambda)} + 2\int |v|^2|w_{x_1}|^2\mathrm{d}x.
\end{aligned} \tag{3.42}$$

Splitting again $v = v_1 + w$ as in (3.16) we estimate the last term of (3.42)

$$\begin{aligned}
\int |v|^2|w_{x_1}|^2\mathrm{d}x &\leq 2\left[\int |v_1|^2|w_{x_1}|^2\mathrm{d}x + \int |w|^2|w_{x_1}|^2\mathrm{d}x\right] \\
&\leq 2\left[\int |v_1|^2|w_{x_1}|^2\mathrm{d}x + \|w\|^2_2\|w_{x_1}\|^2_\infty\right]
\end{aligned} \tag{3.43}$$

The first term of (3.43) is estimated by adding and subtracting the cut-off function ϕ_N^2. Taking into account (3.40), we have

$$\begin{aligned}
&\int |v_1|^2|w_{x_1}|^2\mathrm{d}x = \int |v_1|^2|\phi_N w_{x_1}|^2\mathrm{d}x + \int |v_1|^2(1-\phi_N^2)|w_{x_1}|^2\mathrm{d}x \\
&\leq \|v_1\|^2_\infty\|\phi_N w_{x_1}\|^2_2 + c(\beta,J)\|v_1'\|^2_{L^2(\mathbb{R})}\int (1+|x_1|^2)^{\frac{1}{2}}(1-\phi_N^2)|w_{x_1}|^2\mathrm{d}x \\
&\leq \|v_1\|^2_\infty\|\phi_N w_{x_1}\|^2_2 + \frac{c(\beta,J)}{L^d}\|v_1'\|^2_{L^2(R\times\Lambda)}\|w_{x_1}\|^2_{L^2(\mathbb{R}\times\Lambda)}\sqrt{1+N^2}.
\end{aligned} \tag{3.44}$$

Summarizing the previous estimate we have that (3.42) is bounded as following

$$\begin{aligned}
&\int |v|^2|v_{x_1}|^2\mathrm{d}x \leq 2\|v\|^2_\infty\|v_1'\|^2_{L^2(R\times\Lambda)} + 4\|w_{x_1}\|^2_\infty\|w\|^2_2 \\
&+ 4\|v\|^2_\infty\|\phi_N w_{x_1}\|^2_2 + \frac{C(\beta,J)}{L^d}\|v_1'\|^2_{L^2(R\times\Lambda)}\||w_{x_1}\|^2_{L^2(\mathbb{R}\times\Lambda)}\sqrt{1+N^2}.
\end{aligned} \tag{3.45}$$

Hence from (3.37), (3.41) and (3.45) we have

$$\begin{aligned}
\int \sigma(m)\left[U(v)\right]^2\mathrm{d}x \quad &\leq \quad \|v\|^2_{W^{s,2}}C(\beta,J)\left[||v_1'||^2_{L^2(\mathbb{R}\times\Lambda)} + \|w\|^2_{L^2(\mathbb{R}\times\Lambda)}\right] \\
&+ \quad \|v\|^2_{W^{s,2}}C(\beta,J)\|\phi_N w_{x_1}\|^2_{L^2(\mathbb{R}\times\Lambda)} + 4\|w_{x_1}\|^2_\infty\|w\|^2_{L^2(R\times\Lambda)} \\
&+ \quad \frac{C(\beta,J)}{L^d}\|v_1'\|^2_{L^2(\mathbb{R}\times\Lambda)}\|w_{x_1}\|^2_{L^2(\mathbb{R}\times\Lambda)}\sqrt{1+N^2}.
\end{aligned} \tag{3.46}$$

We estimate, see Lemma 8.4 in the Appendix,

$$\|v\|_\infty \leq c(d,s)\|v\|_{W^{s,2}}, \quad s > \frac{D}{2}, \quad \text{and} \qquad \||w_{x_1}\|_\infty \leq c(d,s)\|v_{x_1}\|_{W^{s,2}} \leq c(d,s)\|v\|_{W^{s+1,2}}.$$

Hence (3.46) immediately implies (3.35). We next derive (3.36). For $\alpha \in \mathbb{R}$ write

$$v_1' = \alpha \overline{m}'' + g' \tag{3.47}$$

where $\int_{\mathbb{R}} g'(x_1) m''(x_1) \mathrm{d}x_1 = 0$ so that $P v_1' = g'$. Note that, as indicated in our notation, $P v_1'$ is a derivative since v_1' and $\overline{m}''$ are derivatives. Hence, upon integration

$$v_1 = \alpha \overline{m}' + g. \tag{3.48}$$

The fact that $\int_{\mathbb{R}} v_1(x_1) \overline{m}'(x_1) dx_1 = 0$ means that $\|g\|_{L^2(\mathbb{R})}$ cannot be too small. But what we need to know is that $\|g\|_{L^2(\mathbb{R})} = \|P v_1'\|_{L^2(\mathbb{R})}$ is not too small. In general, these are simply two different things. What provides the crucial connection here is that $(2m_\beta)^{-1} \overline{m}'(x_1) \mathrm{d}x_1$ is a probability measure on $\mathbb{R}$, so that

$$\int_{\mathbb{R}} v_1(x_1) \overline{m}'(x_1) dx_1 = 0$$

implies that

$$||g||_\infty \geq \frac{|\alpha|}{2m_\beta} ||\overline{m}'||_{L^2(\mathbb{R})}.$$

Then one may use $\|g\|_\infty^2 \leq 2 \|g'\|_{L^2(\mathbb{R})} \|g\|_{L^2(\mathbb{R})}$ to conclude that

$$|\alpha|^2 \leq \frac{8m_\beta^2}{\|\overline{m}'\|_{L^2(\mathbb{R})}^2} \|g\|_{L^2(\mathbb{R})} \|g'\|_{L^2(\mathbb{R})} = \frac{8m_\beta^2}{\|\overline{m}'\|_{L^2(\mathbb{R}\times\Lambda)}^2} \|g\|_{L^2(\mathbb{R}\times\Lambda)} \|g'\|_{L^2(\mathbb{R}\times\Lambda)}.$$

Since

$$v^4 = (v_1 + w)^4 \leq 4[v_1^2 + w^2]^2 \leq 8[v_1^4 + w^4]$$

we have that

$$\int_{\mathbb{R}\times\Lambda} R(x_1)|v|^4 \mathrm{d}x \leq 8 \int_{\mathbb{R}\times\Lambda} R(x_1) \left[|v_1|^4 + w^4 \right] \mathrm{d}x. \tag{3.49}$$

For the first term of (3.49) we insert the pointwise bound for v_1, see (3.39), obtaining, since $R(x_1)$ is rapidly decreasing, from properties (1.10),

$$\int_{\mathbb{R}\times\Lambda} R(x_1) v_1^4(x_1) \mathrm{d}x \leq ||v_1'||_{L^2(\mathbb{R})}^4 L^d \int_{\mathbb{R}} R(x_1)(1 + |x_1|^2) \mathrm{d}x_1 = \frac{1}{L^d} C(\beta, J) ||v_1'||_{L^2(\mathbb{R}\times\Lambda)}^4.$$

Next,

$$\begin{aligned} \int_{\mathbb{R}\times\Lambda} R(x_1) w^4 \mathrm{d}x &\leq \sup_{x_1 \in \mathbb{R}} R(x_1) \|w\|_{L^\infty(\mathbb{R}\times\Lambda)}^2 \int_{\mathbb{R}\times\Lambda} w^2 \mathrm{d}x \\ &\leq C(\beta) c(d,s) \|w\|_{W^{s,2}}^2 \|w\|_{L^2}^2, \end{aligned}$$

by Lemma 8.4 in the Appendix. For the other term in (3.37) we write

$$\int_{\mathbb{R}\times\Lambda} v^2 (v_{x_1})^2 \mathrm{d}x \leq 2 \int_{\mathbb{R}\times\Lambda} v^2 (v_1')^2 \mathrm{d}x + 2 \int_{\mathbb{R}\times\Lambda} v^2 (w_{x_1})^2 \mathrm{d}x. \tag{3.50}$$

We start estimating the first term of (3.50)

$$\begin{aligned} \int_{\mathbb{R}\times\Lambda} v^2 (v_1')^2 \mathrm{d}x &= \int_{\mathbb{R}\times\Lambda} [\alpha \overline{m}' + g + w]^2 (v_1')^2 \mathrm{d}x \\ &\leq 2 \int_{\mathbb{R}\times\Lambda} \left[\alpha^2 (\overline{m}')^2 + [g + w]^2 \right] (v_1')^2 \mathrm{d}x. \end{aligned} \tag{3.51}$$

Moreover from (3.47) we have

$$\int_{\mathbb{R}\times\Lambda} v_1'\overline{m}''\mathrm{d}x = \alpha\|\overline{m}''\|^2_{L^2(\mathbb{R}\times\Lambda)}$$

and therefore $|\alpha|$ can be estimated as

$$|\alpha| = \frac{\left|\int_{\mathbb{R}\times\Lambda} v_1'\bar{m}''\mathrm{d}x\right|}{\|\overline{m}''\|^2_{L^2(\mathbb{R}\times\Lambda)}} \le \frac{\|v_1'\|_{L^2(\mathbb{R}\times\Lambda)}}{\|\overline{m}''\|_{L^2(\mathbb{R}\times\Lambda)}}. \tag{3.52}$$

We obtain from (3.51)

$$\begin{aligned}
&\int_{\mathbb{R}\times\Lambda} v^2(v_1')^2\mathrm{d}x \le 2\alpha^2\int_{\mathbb{R}\times\Lambda}(\overline{m}')^2(v_1')^2\mathrm{d}x + 2\int_{\mathbb{R}\times\Lambda}[g+w]^2(v_1')^2\mathrm{d}x\\
&\le 2\frac{\|v_1'\|^2_{L^2(\mathbb{R}\times\Lambda)}}{\|\bar{m}''\|^2_{L^2(\mathbb{R}\times\Lambda)}}\|v_1'\|^2_{L^2(\mathbb{R}\times\Lambda)}\|\overline{m}'\|^2_\infty + 4\int_{\mathbb{R}\times\Lambda} g^2(v_1')^2\mathrm{d}x + 4\int_{\mathbb{R}\times\Lambda} w^2(v_1')^2\mathrm{d}x\\
&\le 2\frac{\|v_1'\|^4_{L^2(\mathbb{R}\times\Lambda)}}{\|\bar{m}''\|^2_{L^2(\mathbb{R}\times\Lambda)}}\|\overline{m}'\|^2_\infty + 4\int_{\mathbb{R}\times\Lambda} g^2(v_1')^2\mathrm{d}x + 4\|v_1'\|^2_\infty\|w\|^2_{L^2}.
\end{aligned} \tag{3.53}$$

Since $\|g\|^2_\infty \le 2\|g\|_{L^2(\mathbb{R})}\|g'\|_{L^2(\mathbb{R})}$ we have

$$\begin{aligned}
&\int_{\mathbb{R}\times\Lambda} g^2(v_1')^2\mathrm{d}x \le \|g\|^2_{L^\infty}\int_{\mathbb{R}\times\Lambda}(v_1')^2\mathrm{d}x\\
\le\ & 2||g||_{L^2(\mathbb{R})}||g'||_{L^2(\mathbb{R})}\int_{\mathbb{R}\times\Lambda}(v_1')^2\mathrm{d}x\\
\le\ & 2\lambda\left(||g||^2_{L^2(\mathbb{R})}||g'||^2_{L^2(\mathbb{R})}\right) + \frac{2}{\lambda}\|v_1'\|^4_{L^2(\mathbb{R}\times\Lambda)},
\end{aligned} \tag{3.54}$$

for any $\lambda > 0$. Because of (3.48) we have

$$||g||^2_{L^2(\mathbb{R})} = ||v_1 - \alpha\overline{m}'||^2_{L^2(\mathbb{R})}.$$

Therefore, by (3.52),

$$||g||^2_{L^2(\mathbb{R})} = ||v_1||^2_{L^2(\mathbb{R})} + |\alpha|^2||\overline{m}'||^2_{L^2(\mathbb{R})} \le ||v_1||^2_{L^2(\mathbb{R})} + \left(\frac{||v_1'||_{L^2(\mathbb{R}\times\Lambda)}}{||\overline{m}''||_{L^2(\mathbb{R}\times\Lambda)}}||\overline{m}'||_{L^2(\mathbb{R})}\right)^2. \tag{3.55}$$

Taking in account (3.54), (3.55) from (3.53) one obtains

$$\begin{aligned}
&\int_{\mathbb{R}\times\Lambda} v^2(v_1')^2\mathrm{d}x \le 2\frac{1}{\|m''\|^2_{L^2(\mathbb{R}\times\Lambda)}}\sup_{x_1}(\overline{m}')^2||v_1'||^4_{L^2(\mathbb{R}\times\Lambda)} + \frac{2}{\lambda}||v_1'||^4_{L^2(\mathbb{R}\times\Lambda)}\\
&+ 2\lambda\left(\left[||v_1||^2_{L^2(\mathbb{R})} + \left(\frac{||v_1'||_{L^2(\mathbb{R}\times\Lambda)}}{||\overline{m}''||_{L^2(\mathbb{R}\times\Lambda)}}||\overline{m}'||_{L^2(\mathbb{R})}\right)^2\right]\|g'\|^2_{L^2(\mathbb{R})}\right) + 4\|v_1'\|^2_\infty\|w\|^2_{L^2(\mathbb{R}\times\Lambda)}.
\end{aligned} \tag{3.56}$$

Assume that

$$\|v_1\|^2_{L^2(\mathbb{R}\times\Lambda)} \le k^2, \qquad \|v_1'\|^2_{L^2(\mathbb{R}\times\Lambda)} \le k^2.$$

Then

$$\left[||v_1||^2_{L^2(\mathbb{R})} + \left(\frac{||v_1'||_{L^2(\mathbb{R}\times\Lambda)}}{||\overline{m}''||_{L^2(\mathbb{R}\times\Lambda)}}||\overline{m}'||_{L^2(\mathbb{R})}\right)^2\right]||g'||^2_{L^2(\mathbb{R})}$$
$$\leq \frac{k^2}{L^d}\left[1 + 1\frac{\|\overline{m}'\|^2_{L^2(\mathbb{R})}}{\|\overline{m}''\|^2_{L^2(\mathbb{R})}}\right]\|g'\|^2_{L^2(\mathbb{R})} = \frac{k^2}{L^{2d}}\left[1 + 1\frac{\|\overline{m}'\|^2_{L^2(\mathbb{R})}}{\|\overline{m}''\|^2_{L^2(\mathbb{R})}}\right]\|g'\|^2_{L^2(\mathbb{R}\times\Lambda)}.$$

Take λ in (3.56) so that

$$2\lambda\frac{k^2}{L^{2d}}\left\{1 + \frac{1}{||\overline{m}''||^2_{L^2(\mathbb{R})}}||\overline{m}'||^2_{L^2(\mathbb{R})}\right\} \leq \epsilon_1. \tag{3.57}$$

We denote such λ by $\lambda_0 = \lambda_0(\epsilon_1, k)$. Then we have

$$\begin{aligned}\int_{\mathbb{R}\times\Lambda} v^2(v_1')^2\mathrm{d}x \leq 2||v_1'||^4_{L^2(\mathbb{R}\times\Lambda)}\left[\frac{1}{\|\bar{m}''\|^2_{L^2(\mathbb{R}\times\Lambda)}}\|\bar{m}'\|^2_\infty + \frac{1}{\lambda_0}\right]\\ + \epsilon_1||g'||^2_{L^2(\mathbb{R}\times\Lambda)} + 4\|v_1'\|^2_\infty\|w\|^2_{L^2(\mathbb{R}\times\Lambda)}.\end{aligned} \tag{3.58}$$

Next we need to estimate the second term of (3.50), $\int_{\mathbb{R}\times\Lambda} v^2(w_{x_1})^2\mathrm{d}x$. Splitting $v = v_1 + w$ as in (3.16) we get

$$\begin{aligned}\int_{\mathbb{R}\times\Lambda} v^2(w_{x_1})^2\mathrm{d}x &\leq 2\int_{\mathbb{R}\times\Lambda} w^2(w_{x_1})^2\mathrm{d}x + \int_{\mathbb{R}\times\Lambda} v_1^2(w_{x_1})^2\mathrm{d}x\\ &\leq 2\|w_{x_1}\|^2_\infty\|w\|^2_2 + \int_{\mathbb{R}\times\Lambda} v_1^2(w_{x_1})^2\mathrm{d}x.\end{aligned} \tag{3.59}$$

We split the last term of (3.59) applying the cut-off function ϕ_N^2 as it was done previously in (3.44) but we need to end up with an estimate where the $\|v_1'\|^4_{L^2(\mathbb{R}\times\Lambda)}$ appears. Denote $h_N(x_1) = (1+|x_1|^2)^{\frac{1}{2}}(1-\phi_N^2(x_1))$. We therefore, see (3.44), have

$$\begin{aligned}&\int |v_1|^2|w_{x_1}|^2\mathrm{d}x\\ &\leq \|v_1\|^2_\infty\|\phi_N w_{x_1}\|^2_{L^2(\mathbb{R}\times\Lambda)} + C(\beta, J)\|v_1'\|^2_{L^2(\mathbb{R})}\int h_N(x_1)|w_{x_1}|^2\mathrm{d}x.\end{aligned} \tag{3.60}$$

Note that $h_N(\cdot)$ is smooth and has support in $[-2N, 2N]$. Integrating by part we have

$$\int h_N(x_1)|w_{x_1}|^2\mathrm{d}x = -\int w\left[h_N'(x_1)w_{x_1} + h_N(x_1)w_{x_1x_1}\right]\mathrm{d}x.$$

By Schwartz inequality we then get

$$\int h_N(x_1)(w_{x_1})^2\mathrm{d}x \leq \|w\|_2\left\{\sup|h_N'(x_1)|\|w_{x_1}\|_2 + \sup|h_N(x_1)|\|w_{x_1x_1}\|_2\right\}.$$

We immediately estimate

$$\sup|h_N'(x_1)| \leq C,$$
$$\sup|h_N(x_1)| \leq \sqrt{1+4N^2} \leq 3N.$$

Summarizing from (3.60) we obtain

$$\int |v_1|^2 (w_{x_1})^2 \mathrm{d}x$$
$$\leq \|v_1\|_\infty^2 \|\phi_N w_{x_1}\|_2^2 + C(\beta,J)\frac{N}{L^d}\|v_1'\|^2_{L^2(\mathbb{R}\times\Lambda)}\|w\|_2 \left\{\|w_{x_1}\|_2 + \|w_{x_1x_1}\|_2\right\}.$$

Since $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$ we have

$$\int |v_1|^2 |w_{x_1}|^2 \mathrm{d}x$$
$$\leq \|v_1\|_\infty^2 \|\phi_N w_{x_1}\|_{L^2}^2 + C(\beta,J)\left[\frac{1}{2}\|v_1'\|^4_{L^2(\mathbb{R}\times\Lambda)} + \frac{1}{2}\|w\|_2^2 (\frac{N}{L^d})^2 \left[\|w_{x_1}\|_2 + \|w_{x_1x_1}\|_2\right]^2\right].$$

Summing up all the estimates, (3.44), (3.58) we have

$$\begin{aligned}\int \sigma(m)\big[U(v)\big]^2 \mathrm{d}x \leq{}& \frac{8}{L^d}C(\beta,J)||v_1'||^4_{L^2(\mathbb{R}\times\Lambda)} + 8C(\beta,J,d)\|w\|^2_{W^{s,2}}\|w\|^2\\
&+ 2\frac{1}{\|\bar{m}''\|^2_{L^2(\mathbb{R}\times\Lambda)}}\|\overline{m}'\|_\infty^2 ||v_1'||^4_{L^2(\mathbb{R}\times\Lambda)} + \frac{2}{\lambda_0}||v_1'||^4_{L_2(\mathbb{R}\times\Lambda)}\\
&+ 8\epsilon_1||g'||^2_{L^2(\mathbb{R}\times\Lambda)} + 4\|v_1'\|_\infty^2\|w\|_2^2\\
&+ 2\|w_{x_1}\|_\infty\|w\|_2^2 + 2\|v_1\|^2_\infty\|\phi_N w_{x_1}\|_2^2\\
&+ C(\beta,J)\left[\|v_1'\|^4_{L^2(\mathbb{R}\times\Lambda)} + \|w\|_2^2\left(\frac{N}{L^d}\right)^2\left[\|w_{x_1}\|_2^2 + \|w_{x_1x_1}\|_2^2\right]\right].\end{aligned}$$

Recalling that $g' = Pv'$, see after (3.48), and estimating

$$\|v_1\|_\infty \leq \|v\|_\infty \leq c(d,s)\|v\|_{W^{s,2}}, \qquad \|v'\|_\infty + \|w_{x_1}\|_\infty \leq \|v_{x_1}\|_\infty \leq c(d,s)\|v\|_{W^{s+1,2}}$$

we get (3.36). □

Proof of Proposition 3.4 Writing $G_\epsilon(v)$ as in (3.20), applying Lemma 3.6, Lemma 3.7 and (3.21) we have that

$$\begin{aligned}G_\epsilon(v) \quad\geq\quad & \epsilon\left[\frac{1}{4}\sigma(m_\beta)\gamma^2(L)\|\phi_N(w)_{x_1}\|_2^2 - \|w\|_2^2\frac{1}{N^2}D(\beta,\gamma(L))\right]\\
+\quad & \epsilon\sigma(m_\beta)\gamma_1\|Pv_1'\|^2_{L^2(\mathbb{R}\times\Lambda)} + \epsilon\gamma(L)\sigma(m_\beta)\sum_{i\geq 2}\|w_{x_i}\|_2^2\\
-\quad & \frac{1}{\epsilon}\int\sigma(\overline{m})\big[U(v)\big]^2\mathrm{d}x,\end{aligned}$$

where $U(v)$ is defined in (3.8). Suppose that

$$||Pv_1'||^2_{L^2(\mathbb{R}\times\Lambda)} > \frac{1}{2}||v_1'||^2_{L^2(\mathbb{R}\times\Lambda)} \tag{3.61}$$

and

$$\|v\|^2_{W^{s+1,2}} \leq \epsilon_0, \quad s > \frac{D}{2}. \tag{3.62}$$

From (3.35) of Lemma 3.8 we obtain

$$
\begin{aligned}
G_\epsilon(v) \geq \epsilon & \left[\frac{1}{4}\sigma(m_\beta)\gamma^2(L)\|\phi_N(w)_{x_1}\|_2^2 - \|w\|_2^2 \frac{1}{N^2} D(\beta,\gamma(L))\right] \\
&+ \epsilon\sigma(m_\beta)\gamma_1 \|Pv_1'\|^2_{L^2(\mathbb{R}\times\Lambda)} + \epsilon\gamma(L)\sigma(m_\beta)\sum_{i\geq 2}\|w_{x_i}\|_2^2 \\
&- \frac{1}{\epsilon}\Big\{||v_1'||^2_{L^2(\mathbb{R}\times\Lambda)} C(\beta,J)\left[\|v\|^2_{W^{s,2}} + \||w_{x_1}\|_2^2 N\right] \\
&+ \|w\|_2^2 C(\beta,J)\left[\|v\|^2_{W^{s,2}} + \|v\|^2_{W^{s+1,2}}\right] \\
&+ \|v\|^2_{W^{s,2}} C(\beta,J)\|\phi_N w_{x_1}\|_2^2\Big\}.
\end{aligned}
$$

To show that $G_\epsilon(v) \geq 0$ under assumption (3.61) it is enough to choose our parameters so that the following three inequalities are satisfied:

$$
\left[\epsilon\frac{1}{4}\sigma(m_\beta)\gamma^2(L) - \frac{1}{\epsilon}\|v\|^2_{W^{s,2}}C(\beta,J)\right]\|\phi_N(w)_{x_1}\|_2^2 \geq 0, \tag{3.63}
$$

$$
\epsilon\gamma(L)\sigma(m_\beta)\sum_{i\geq 2}\|w_{x_i}\|^2 - \|w\|^2\left\{\epsilon\frac{1}{N^2}D(\beta,\gamma(L)) + \frac{1}{\epsilon}C(\beta,J)\left[\|v\|^2_{W^{s,2}} + \|v\|^2_{W^{s+1,2}}\right]\right\} \geq 0 \tag{3.64}
$$

$$
\epsilon\sigma(m_\beta)\gamma_1\|Pv_1'\|^2_{L^2(\mathbb{R}\times\Lambda)} - \frac{1}{\epsilon}\|v_1'\|^2_{L^2(\mathbb{R}\times\Lambda)}C(\beta,J)\left\{\|v\|^2_{W^{s,2}} + \|w_{x_1}\|_2^2 N\right\} \geq 0. \tag{3.65}
$$

To satisfy (3.63), under the assumptions (3.62), we need

$$
\gamma^2(L) \geq C\frac{\epsilon_0}{\epsilon^2} \tag{3.66}
$$

for some positive constant C. By the Poincaré inequality (see the similar estimate in (4.18)), we have

$$
\|\nabla^\perp w\|^2 \geq \frac{c(d)}{L^2}\|w\|^2. \tag{3.67}
$$

We can then satisfy (3.64), if

$$
\epsilon\gamma(L)\sigma(m_\beta)\frac{c(d)}{L^2} - \left\{\epsilon\frac{1}{N^2}D(\beta,\gamma(L)) + \frac{1}{\epsilon}C(\beta,J)\left[\|v\|^2_{W^{s,2}} + \|v\|^2_{W^{s+1,2}}\right]\right] \geq 0.
$$

Under the assumptions (3.61) it is enough to require

$$
\gamma(L)\frac{1}{L^2} \geq \frac{1}{N^2}, \qquad \gamma(L)\frac{1}{L^2} \geq C\frac{\epsilon_0}{\epsilon^2}, \tag{3.68}
$$

for some positive constant C. By (3.61) to fulfill (3.65) we need to require

$$
\epsilon\sigma(m_\beta)\gamma_1\frac{1}{2} - \frac{1}{\epsilon}C(\beta,J)\left\{\|v\|^2_{W^{s,2}} + \|w_{x_1}\|^2_{L^2(\mathbb{R}\times\Lambda)}N\right\} \geq 0,
$$

which means

$$
\gamma_1 \geq \frac{\epsilon_0}{\epsilon^2}N. \tag{3.69}
$$

Choose the cut-off $N = \epsilon^{-a}$ with $a = a(L) > 0$ so that the first requirement in (3.68) holds. Choose then $\epsilon_0 = \epsilon^{2+a+b}$ with $b = b(L)$ so that (3.66), the second condition in (3.68) and (3.69) hold. Let $r = r(L) \geq a + b$ and we get (3.14), that is $G_\epsilon(v) \geq 0$, under condition (3.61).

Next, suppose (3.61) is false, i.e. :

$$||Pv_1'||^2_{L^2(\mathbb{R}\times\Lambda)} \le \frac{1}{2}||v_1'||^2_{L^2(\mathbb{R}\times\Lambda)}. \tag{3.70}$$

Then from (3.47)

$$\|v_1' - g'\|_{L^2(\mathbb{R})} \le |\alpha|\|\overline{m}''\|_{L^2(\mathbb{R})}$$

and applying (8.12) with $\lambda = \frac{1}{3}$ and (3.70) one obtains

$$|\alpha|^2||\overline{m}''||^2_{L^2(\mathbb{R})} \ge ||v_1' - g'||^2_{L^2(\mathbb{R})} \ge \frac{1}{3}||v_1'||^2_{L^2(\mathbb{R})} - \frac{1}{2}||g'||^2_{L^2(\mathbb{R})} \ge \frac{1}{12}||v_1'||^2_{L^2(\mathbb{R})}.$$

Therefore

$$||v_1'||^2_{L^2(\mathbb{R})} \le 12||\overline{m}''||^2_{L^2(\mathbb{R})}|\alpha|^2. \tag{3.71}$$

Since v orthogonal to $\overline{m}'$, implies v_1 and w orthogonal to $\overline{m}'$ we have, see (3.48),

$$\frac{\alpha}{2m_\beta} = \frac{1}{2m_\beta}\int_{\mathbb{R}} g\overline{m}'\mathrm{d}x \le ||g||_\infty. \tag{3.72}$$

Inequality $||g||^2_\infty \le 2||g||_2||g'||_2$, (3.71) and (3.72) imply

$$||v_1'||_{L^2(\mathbb{R})} \le C(\beta, J)||g||^{\frac{1}{2}}_{L^2(\mathbb{R})}||g'||^{\frac{1}{2}}_{L^2(\mathbb{R})}. \tag{3.73}$$

Recall, see (3.47), that $\|Pv_1'\|_2 = \|g'\|_2$. We apply estimate (3.36) with $\|v_1\|^2_{L^2(\mathbb{R})} \le k^2$, $\|v_1'\|^2_{L^2(\mathbb{R})} \le k^2$ and for $\epsilon_1 > 0$. The actual values of k and ϵ_1 will be chosen later. Thus it is enough to show

$$\begin{aligned} G_\epsilon(v) \ge& \epsilon\left[\frac{1}{4}\sigma(m_\beta)\gamma^2(L)\|\phi_N(w)_{x_1}\|_2^2 - \|w\|_2^2\frac{1}{N^2}D(\beta,\gamma(L))\right] \\ &+\epsilon\sigma(m_\beta)\gamma_1\|Pv_1'\|^2_{L^2(\mathbb{R}\times\Lambda)} + \epsilon\gamma(L)\sigma(m_\beta)\sum_{i\ge 2}\|w_{x_i}\|_2^2 \\ &-\frac{b_0}{\epsilon}||v_1'||^4_{L^2(\mathbb{R}\times\Lambda)} - \frac{a_0}{\epsilon}\|w\|^2_{L^2(\mathbb{R}\times\Lambda)} \\ &-\frac{1}{\epsilon}8\epsilon_1\|Pv_1'\|^2_{L^2(\mathbb{R}\times\Lambda)} - \frac{1}{\epsilon}2\|v_1\|^2_\infty\|\phi_N w_{x_1}\|_2^2 \ge 0, \end{aligned} \tag{3.74}$$

where we denoted

$$a_0 = \left[C(\beta,J,d)\|w\|^2_{W^{s,2}} + \left[4\|v_1'\|^2_\infty + 2\|w_{x_1}\|_\infty\right] + \frac{1}{2m_\beta^2}(\frac{N}{L^d})^2\left[\|w_{x_1}\|_2 + \|w_{x_1x_1}\|_2\right]^2\right], \tag{3.75}$$

and

$$b_0 = \left[\frac{1}{L^d}C(\beta,J) + 2\frac{1}{\|m''\|^2_{L^2(\mathbb{R}\times\Lambda)}}\sup_{x_1}(\overline{m}')^2 + \frac{2}{\lambda_0(\epsilon_1,k)} + \frac{1}{2m_\beta^2}\right].$$

We estimate

$$\|v_1'\|_\infty \le c(d,s)\|v\|_{W^{s+1,2}}, \qquad \|w_{x_1x_1}\|_2 \le \|v\|_{W^{2,2}} \le \|v\|_{W^{s+1,2}}, \quad \|w_{x_1}\|_\infty \le c(d,s)\|v\|_{W^{s+1,2}}.$$

Therefore, (3.75), by assumptions (3.15), is bounded by

$$a_0 \le CN^2\|v\|_{W^{s+1,2}} \tag{3.76}$$

We need to require, see (3.74), the following three conditions:

$$\left[\epsilon\frac{1}{4}\sigma(m_\beta)\gamma^2(L)-\frac{1}{\epsilon}2\|v_1\|^2_\infty\right]\geq 0, \tag{3.77}$$

$$\begin{aligned}&\epsilon\gamma(L)\sigma(m_\beta)\sum_{i\geq 2}\|w_{x_i}\|^2\\&-\|w\|^2_{L^2(\mathbb{R}\times\Lambda)}\left[\epsilon\frac{1}{4}\frac{1}{N^2}D(\beta,\gamma(L))+\frac{1}{\epsilon}a_0\right]\geq 0,\end{aligned} \tag{3.78}$$

$$\begin{aligned}&\epsilon\sigma(m_\beta)\gamma_1\|Pv_1'\|^2_{L^2(\mathbb{R}\times\Lambda)}\\&-\frac{1}{\epsilon}b_0||v_1'||^4_{L^2(\mathbb{R}\times\Lambda)}-\frac{1}{\epsilon}8\epsilon_1||g'||^2_{L^2(\mathbb{R}\times\Lambda)}\geq 0.\end{aligned} \tag{3.79}$$

To satisfy (3.77) taking into account that

$$\|v_1\|^2_\infty\leq c(d,s)\|v\|^2_{W^{s,2}}\leq\epsilon_0$$

we need to require

$$\left[\epsilon\frac{1}{4}\sigma(m_\beta)\gamma^2(L)-\frac{1}{\epsilon}\epsilon_0\right]\geq 0.$$

To fulfill (3.78), taking into account (3.67) and (3.76), we need to require

$$\epsilon\gamma(L)\sigma(m_\beta)\frac{1}{L^2}c(d)-\left[\epsilon\frac{1}{4}\frac{1}{N^2}D(\beta,\gamma(L))+\frac{1}{\epsilon}CN^2\epsilon_0\right]\geq 0,$$

therefore

$$\gamma(L)\geq C(\beta,J,d,L)\left[\frac{1}{N^2}+\frac{1}{\epsilon^2}CN^2\epsilon_0\right].$$

This forces the choice we have made of ϵ_0 as $\epsilon_0=\epsilon^{2+2a+b}$. To satisfy (3.79), taking into account (3.73) we require

$$\epsilon\sigma(m_\beta)\gamma_1\|Pv_1'\|^2_{L^2(\mathbb{R}\times\Lambda)}-\frac{1}{\epsilon}||g'||^2_{L^2(\mathbb{R}\times\Lambda)}\left\{b_0\|g\|^2_{L^2(\mathbb{R}\times\Lambda)}+8\epsilon_1\right\}\geq 0.$$

Note that

$$\|Pv_1'\|^2_{L^2(\mathbb{R}\times\Lambda)}=\|g'\|^2_{L^2(\mathbb{R}\times\Lambda)}\ .$$

We now seek to bound $||g||^2_{L^2(\mathbb{R}\times\Lambda)}$ as in (3.55). We would then require, in terms of order of magnitude,

$$\gamma_1-b_0\frac{1}{\epsilon^2}\epsilon_0-8\frac{1}{\epsilon^2}\epsilon_1\geq 0.$$

We then choose $\epsilon_1=\epsilon_0$ and $k^2=\epsilon_0$ when applying (3.37) of Lemma 3.8. Recall that $b_0\leq(C+\frac{1}{\lambda_0})$, and $\lambda_0\simeq\frac{\epsilon_1}{\kappa^2}=1$, for the choice done. We therefore get

$$\gamma_1-\frac{C}{\epsilon^2}\epsilon_0-8\frac{1}{\epsilon^2}\epsilon_0\geq 0.$$

Taking $\epsilon_0=\epsilon^{2+r(L)}$ with $r(L)=2a(L)+b(L)$ we get the thesis. □

Proof of Theorem 3.1: Applying Lemma 3.3, see (3.5), we have

$$\begin{aligned}\mathcal{I}\big(m(t)\big) \;&\geq\; (1-2\epsilon)\int_{\mathbb{R}\times\Lambda}\sigma(\overline{m})\big[(\mathcal{B}v)_{x_1}\big]^2\mathrm{d}x \\ &+\; \left[\epsilon\left(\sum_{i\geq 1}\int_{\mathbb{R}\times\Lambda}\sigma(\overline{m})\big[(\mathcal{B}v)_{x_i}\big]^2\mathrm{d}x\right) - \frac{1}{\epsilon}\int_{\mathbb{R}\times\Lambda}\sigma(\overline{m})[U(v)]^2\mathrm{d}x\right] \\ &+\; (1-3\epsilon)\sum_{i\geq 2}\int_{\mathbb{R}\times\Lambda}\sigma(\overline{m})\big[(\mathcal{B}v)_{x_i}\big]^2\mathrm{d}x.\end{aligned}$$

Proposition 3.4 then delivers the thesis. □

4 Bound on the dissipation rate of the free energy in terms of the excess free energy

In this section we establish a bound on the rate $\mathcal{I}(m(t))$ at which the excess free energy $\mathcal{F}(m(t))-\mathcal{F}(\overline{m})$ is dissipated in term of $\mathcal{F}(m(t))-\mathcal{F}(\overline{m})$ itself, working under the hypothesis that

$$\mathcal{I}(m(t)) << \big[\mathcal{F}(m(t))-\mathcal{F}(\overline{m})\big]\ . \tag{4.1}$$

On the other hand, when (4.1) is not satisfied, there is ample dissipation, as explained in the introduction. Denote by

$$\phi(v(t)) = L^d + \int_{\mathbb{R}\times\Lambda} x_1^2(\mathcal{B}_{a(t)}v(t))^2\mathrm{d}x. \tag{4.2}$$

The main result of this section is the following.

4.1 THEOREM. *Let $m(\cdot,t)$ be a solution of (1.1) and set $m(\cdot,t)=\overline{m}_{a(t)}(\cdot)+v(\cdot,t)$ where $a(t)$ is chosen so that minimizes $\|m(t)-\overline{m}_a\|^2_{L^2(\mathbb{R}\times\Lambda)}$. For any $\epsilon>0$ small enough, there is $\delta_1=\delta_1(\epsilon,d,\beta,J,L)>0$ and $\epsilon_1=\epsilon_1(\epsilon,\beta,J)$ so that at all time t for which $\|v(t)\|_{W^{s+1,2}}\leq\delta_1$, $|a(t)|\leq 1$, where $s>\frac{D}{2}$,*

$$C(\beta)\|v(t)\|^2 \leq \frac{4}{9L^2}, \tag{4.3}$$

and

$$\mathcal{I}(m(t)) \leq \epsilon_1\big[\mathcal{F}\big(m(t)\big)-\mathcal{F}\big(\overline{m}\big)\big] \tag{4.4}$$

we have that

$$\frac{d}{dt}\big[\mathcal{F}\big(m(t)\big)-\mathcal{F}\big(\overline{m}\big)\big] \leq -9(1-\sigma(m_\beta))^2(1+\epsilon)\frac{\big[\mathcal{F}\big(m(t)\big)-\mathcal{F}\big(\overline{m}\big)\big]^2}{\phi(t)}. \tag{4.5}$$

The proof of Theorem 4.1, given at the end of this section, is based on Theorem 3.1 and an application of the following constrained version of Weyl's uncertainty principle proved in Section 2 of [4].

4.2 THEOREM. *Let $\psi(x)$ be a function on the real line such that*

$$\int|\psi'(x)|^2\mathrm{d}x<\infty \qquad \text{and} \qquad \int|x\psi(x)|^2\mathrm{d}x<\infty \tag{4.6}$$

and such that either

$$\psi(0) = 0 \tag{4.7}$$

or

$$\int \psi(x)\mathrm{d}x = 0\ . \tag{4.8}$$

Then

$$\Big(\int |\psi'(x)|^2\mathrm{d}x\Big)\Big(\int |x\psi(x)|^2\mathrm{d}x\Big) \geq \frac{9}{4}\Big(\int |\psi(x)|^2\mathrm{d}x\Big)^2 . \tag{4.9}$$

Notice that under (4.6), ψ is integrable and well–defined at 0, so (4.7) and (4.8) make sense.

Recall that $m(t) = \bar{m}_{a(t)} + v(t)$ and $v = v_1 + w$ as in (3.16). We will apply Theorem 4.2 to v_1, but the argument used in the one dimensional setting, see [4], does not suffice. Namely we get an extra term, see the last term of (4.11), due to the multidimensionality of the problem.

4.3 LEMMA. *Let* $m(t) = \bar{m}_{a(t)} + v(t)$, $v = v_1 + w$ *as in* (3.16) *and* $|a(t)| \leq 1$,

$$\int_R (x_1 v_1(x_1))^2\, dx_1 < \infty,$$

$$\int_R v_1(x_1)(\overline{m})'_{a(t)}(x_1)\mathrm{d}x_1 = 0. \tag{4.10}$$

For any $\epsilon > 0$, *there exists* $\delta_1 = \delta_1(\epsilon)$ *and* $\epsilon_1 = \epsilon_1(\epsilon, \delta_1)$ *so that when* $\|v\|_{W^{s,2}} \leq \delta_1$ *and*

$$\mathcal{I}(m) \leq \epsilon_1\big[\mathcal{F}(m) - \mathcal{F}(\overline{m})\big],$$

we have

$$\begin{aligned}\int_{\mathbb{R}} \big[(\mathcal{A}v_1)_{x_1}\big]^2\mathrm{d}x_1 \quad &\geq \quad \frac{(1-\epsilon)^3}{(1+\epsilon)^2}\frac{9}{4}\frac{\big(\int_{\mathbb{R}}(\mathcal{A}v_1)^2\mathrm{d}x_1\big)^2}{\int_{\mathbb{R}} x_1^2(\mathcal{A}v_1)^2\mathrm{d}x_1 + 1} \\ &- \quad \epsilon_1^2\frac{9}{4}\frac{1}{\epsilon^3 L^{2d}}\frac{\big(C(\beta,J)\,\big[\mathcal{F}\big(m(t)\big) - \mathcal{F}(\overline{m})\big]\big)^2}{\int_{\mathbb{R}} x_1^2(\mathcal{A}v_1)^2\mathrm{d}x_1 + 1}.\end{aligned} \tag{4.11}$$

Proof: The proof of the lemma when m is antisymmetric in the x_1 variable is a straightforward application of (4.9). In such a case, $a(t) = 0$ for all $t \geq 0$ and $(\mathcal{A}v_1)(0) = 0$. By Theorem 4.2 one gets

$$\int_{\mathbb{R}} \big[(\mathcal{A}v_1)_{x_1}\big]^2\mathrm{d}x_1 \geq \frac{9}{4}\frac{\big(\int_{\mathbb{R}}(\mathcal{A}v_1)^2\mathrm{d}x_1\big)^2}{\int_{\mathbb{R}} x_1^2(\mathcal{A}v_1)^2\mathrm{d}x_1}$$

and (4.11) holds for $\epsilon_1 = 0$ and $\epsilon = 0$. Without this symmetry condition, the proof is more involved. The argument used in this case in the one dimensional setting, see [4], requires further elaboration: We get an extra term, see the last term of (4.11), due to the multidimensionality of the problem.

We introduce the smearing operator

$$\mathcal{S}v_1(x_1) = \frac{1}{2m_\beta}\overline{m}' \star v_1(x_1)\ .$$

Notice that $\mathcal{S}$ is a contraction on $L^2(\mathbb{R})$, and it commutes with differentiation. Hence,

$$\|(\mathcal{A}v_1)_{x_1}\|_{L^2(\mathbb{R})} \geq \|\mathcal{S}\big(\mathcal{A}v_1\big)_{x_1}\|_{L^2(\mathbb{R})} = \|\big(\mathcal{S}\mathcal{A}v_1\big)_{x_1}\|_{L^2(\mathbb{R})}\ . \tag{4.12}$$

Further, note that

$$\mathcal{S}(\mathcal{A}v_1)(a(t)) = \frac{1}{2m_\beta}(\overline{m}' \star \mathcal{A}v_1)(a(t)) = \frac{1}{2m_\beta}\int_{\mathbb{R}} (\overline{m})'_{a(t)}(x_1)(\mathcal{A}v_1)(x_1)dx_1 = 0$$

by (4.10). Hence the constrained uncertainty principle applies with the result that, see (4.12),

$$\| \ (\mathcal{A}v_1)_{x_1}\|^2_{L^2(\mathbb{R})} \geq \|\big(\mathcal{S}\mathcal{A}v_1\big)_{x_1}\|^2_{L^2(\mathbb{R})} \geq \frac{9}{4}\frac{\|\mathcal{S}\mathcal{A}v_1\|^4_{L^2(\mathbb{R})}}{\|(x_1 - a(t))\mathcal{S}\mathcal{A}v_1\|^2_{L^2(\mathbb{R})}} \ . \tag{4.13}$$

We now need to remove $\mathcal{S}$. In the numerator we have for all $\epsilon > 0$,

$$\|\mathcal{S}\mathcal{A}v_1\|^2_{L^2(\mathbb{R})} = \|\mathcal{A}v_1 + (\mathcal{S}\mathcal{A}v_1 - \mathcal{A}v_1)\|^2_{L^2(\mathbb{R})} \geq (1-\epsilon)\|\mathcal{A}v_1\|^2_{L^2(\mathbb{R})} - \frac{1}{\epsilon}\|(\mathcal{S}\mathcal{A}v_1 - \mathcal{A}v_1)\|^2_{L^2(\mathbb{R})}.$$

Applying Lemma 8.2 one can show that

$$\|(\mathcal{S}\mathcal{A}v_1 - \mathcal{A}v_1)\|^2_{L^2(\mathbb{R})} \leq C(\beta, J)\|(\mathcal{A}v_1)_{x_1}\|^2_{L^2(\mathbb{R})}.$$

Applying Theorem 3.1, we have in particular that

$$\|(\mathcal{A}v_1)_{x_1}\|^2_{L^2(\mathbb{R})} \leq \frac{1}{L^d}C(\beta, J)\mathcal{I}(m(t)$$

and by (4.4) we have

$$\|(\mathcal{A}v_1)_{x_1}\|^2_{L^2(\mathbb{R})} \leq \epsilon_1\frac{1}{L^d}C(\beta, J)\big[\mathcal{F}\big(m(t)\big) - \mathcal{F}\big(\overline{m}\big)\big].$$

Therefore

$$\|\mathcal{S}\mathcal{A}v_1\|^2_{L^2(\mathbb{R})} \geq (1-\epsilon)\|\mathcal{A}v_1\|^2_2 - \frac{1}{\epsilon}\epsilon_1\frac{1}{L^d}C(\beta, J)\big[\mathcal{F}\big(m(t)\big) - \mathcal{F}\big(\overline{m}\big)\big].$$

Applying inequality (8.12) with $\lambda = 1 - \epsilon$ we have

$$\|\mathcal{S}\mathcal{A}v_1\|^4_{L^2(\mathbb{R})} \geq (1-\epsilon)^3\|\mathcal{A}v_1\|^4_2 - \frac{1}{\epsilon^3}\left(\epsilon_1\frac{1}{L^d}C(\beta, J)\big[\mathcal{F}\big(m(t)\big) - \mathcal{F}\big(\overline{m}\big)\big]\right)^2 . \tag{4.14}$$

To remove $\mathcal{S}$ from the denominator, write

$$\int (x_1 - a(t))^2\big(\mathcal{S}\mathcal{A}v_1\big)^2 \mathrm{d}x_1 \leq (1+\epsilon)\int x_1^2\big(\mathcal{S}\mathcal{A}v_1\big)^2 \mathrm{d}x_1 + \left(\frac{1+\epsilon}{\epsilon}\right)a(t)^2\|\mathcal{S}\mathcal{A}v_1\|^2_{L^2(\mathbb{R})}. \tag{4.15}$$

By Minkowski's inequality and commuting convolution with multiplication by x_1, one has

$$\|x_1\mathcal{S}\mathcal{A}v_1\|_{L^2(\mathbb{R})} \leq \|\mathcal{S}x_1\mathcal{A}v\|_{L^2(\mathbb{R})} + \|\tilde{\mathcal{S}}\mathcal{A}v_1\|_{L^2(\mathbb{R})}$$

where $\tilde{\mathcal{S}}$ denotes convolution by $(2m_\beta)^{-1}x_1\overline{m}'(x_1)$. Clearly $\tilde{\mathcal{S}}$ is bounded on L^2 with norm no greater than $(2m_\beta)^{-1}\|x_1\overline{m}'\|_1$. And since $\mathcal{S}$ is a contraction on L^2, one has

$$\|x_1\mathcal{S}\mathcal{A}v_1\|_2 \leq \|x_1\mathcal{A}v\|_2 + (2m_\beta)^{-1}\|x_1\overline{m}'\|_1\|\mathcal{A}v_1\|_2 \ .$$

Thus, for all $\epsilon > 0$,

$$\|x_1\mathcal{S}\mathcal{A}v_1\|^2_{L^2(\mathbb{R})} \leq (1+\epsilon)\|x_1\mathcal{A}v_1\|^2_{L^2(\mathbb{R})} + \left(\frac{1+\epsilon}{\epsilon}\right)(2m_\beta)^{-2}\|x_1\overline{m}'\|^2_1\|\mathcal{A}v_1\|^2_{L^2(\mathbb{R})} \ . \tag{4.16}$$

Combining (4.16) and (4.15), recalling the hypothesis that $|a(t)| \le 1$, and $\mathcal{A}$ is a bounded operator one has

$$\int_{\mathbb{R}} (x_1 - a(t))^2 \big(\mathcal{S}\mathcal{A}v_1\big)^2 \mathrm{d}x_1 \le (1+\epsilon)^2 \|x_1 \mathcal{A} v_1\|^2_{L^2(\mathbb{R})} + \frac{1}{2} \tag{4.17}$$

when $||v_1||_{L^2(\mathbb{R})}$ is sufficiently small. Combining (4.13), (4.14) and (4.17), we obtain the final result. □

Proof of Theorem 4.1: Since $\mathcal{F}$ is decreasing along the solution of (1.1), see Theorem 2.1, we have

$$\frac{d\mathcal{F}}{dt}\big(m(t)\big) = -\mathcal{I}\big(m(t)\big),$$

where $\mathcal{I}\big(m\big)$ is defined in (1.19). Applying Theorem 3.1, denoting $m = \overline{m} + v$, splitting v as in (3.16) we have

$$\begin{aligned} \mathcal{I}\big(\overline{m} + v\big) \quad &\ge \quad (1-3\epsilon)\left[\int_{\mathbb{R}\times\Lambda} \sigma(\overline{m})\big[(\mathcal{A}v_1)_{x_1}\big]^2 \mathrm{d}x + \int_{\mathbb{R}\times\Lambda} \sigma(\overline{m})\big[(\mathcal{B}w)_{x_1}\big]^2 \mathrm{d}x\right] \\ &+ \quad (1-3\epsilon)\sum_{i\ge 2}\int_{\mathbb{R}\times\Lambda} \sigma(\overline{m})\big[(\mathcal{B}w)_{x_i}\big]^2 \mathrm{d}x. \end{aligned}$$

In particular, since $\sigma(\overline{m}) \ge \sigma(m_\beta)$

$$\begin{aligned} \mathcal{I}\big(\overline{m} + v\big) \quad &\ge \quad (1-3\epsilon)\sigma(m_\beta)\left[\int_{\mathbb{R}\times\Lambda} \big[(\mathcal{A}v_1)_{x_1}\big]^2 \mathrm{d}x\right] \\ &+ \quad (1-3\epsilon)\sigma(m_\beta)\sum_{i\ge 2}\int_{\mathbb{R}\times\Lambda} \big[(\mathcal{B}w)_{x_i}\big]^2 \mathrm{d}x. \end{aligned}$$

Taking into account that for each fixed $x_1 \in \mathbb{R}$, $\int_\Lambda (\mathcal{B}w)(x_1, x^\perp) dx^\perp = 0$ we apply to the last term the Poincaré inequality, see (1.30), obtaining

$$\begin{aligned} \|\nabla^\perp \mathcal{B}w\|^2 &= \int_{\mathbb{R}} \mathrm{d}x_1 \left(\sum_{i\ge 2}\int_\Lambda \mathrm{d}x^\perp |(\mathcal{B}w(x_1, x^\perp))_{x_i}|^2\right) \\ &\ge \frac{c(d)}{L^2}\int_{\mathbb{R}}\int_\Lambda |\mathcal{B}w(x_1,x^\perp)|^2 dx = \frac{c(d)}{L^2}\|\mathcal{B}w\|^2. \end{aligned} \tag{4.18}$$

Then

$$\mathcal{I}\big(\overline{m} + v\big) \ge (1-3\epsilon)\sigma(m_\beta)\left[\int_{\mathbb{R}\times\Lambda} \big[(\mathcal{A}v_1)_{x_1}\big]^2 \mathrm{d}x + \frac{c(d)}{L^2}\|\mathcal{B}w\|_2^2\right].$$

Applying Lemma 4.3, we get

$$\begin{aligned} \mathcal{I}\big(\overline{m} + v\big) &\ge (1-3\epsilon)\sigma(m_\beta)\left[\frac{9}{4}\frac{(1-\epsilon)^3}{(1+\epsilon)^2}\frac{\Big(\int_{\mathbb{R}\times\Lambda}(\mathcal{A}v_1)^2\Big)^2}{\int_{\mathbb{R}\times\Lambda} x_1^2 (\mathcal{A}v_1)^2 \mathrm{d}x + L^d} + \frac{c(d)}{L^2}\|\mathcal{B}w\|_2^2\right] - \epsilon_1^2 \mathcal{R} \\ &\ge (1-3\epsilon)\sigma(m_\beta)\frac{9}{4}\frac{(1-\epsilon)^3}{(1+\epsilon)^2}\frac{1}{\int_{\mathbb{R}\times\Lambda} x_1^2 (\mathcal{A}v_1)^2 \mathrm{d}x + L^d}\left[\left(\int_{\mathbb{R}\times\Lambda}(\mathcal{A}v_1)^2\right)^2 + \frac{4}{9L^2}\|\mathcal{B}w\|_2^2\right] - \epsilon_1^2 \mathcal{R} \end{aligned}$$

where

$$\mathcal{R} = \frac{9}{4}\frac{C(\beta, J)}{\epsilon^3}\frac{\big(\big[\mathcal{F}\big(m(t)\big) - \mathcal{F}\big(\overline{m}\big)\big]\big)^2}{\int_{\mathbb{R}\times\Lambda} x_1^2 (\mathcal{A}v_1)^2 \mathrm{d}x + L^d}. \tag{4.19}$$

Our aim is to prove that

$$\left[\left(\int_{\mathbb{R}\times\Lambda}(\mathcal{A}v_1)^2\right)^2+\frac{4}{9L^2}\|\mathcal{B}w\|_2^2\right]\geq\Big[\ \big[\mathcal{F}\big(m(t)\big)-\mathcal{F}\big(\overline{m}\big)\big]^2\Big].$$

By orthogonality

$$\|\mathcal{A}v_1\|_2^2+\|\mathcal{B}w\|_2^2=\|\mathcal{B}v\|_2^2.$$

Then

$$\|\mathcal{B}v\|_2^4=\big(\|\mathcal{A}v_1\|_2^2+\|\mathcal{B}w\|_2^2\big)^2=\|\mathcal{A}v_1\|_2^4+(\|\mathcal{B}w\|_2^2+2\|\mathcal{A}v_1\|_2^2)\|\mathcal{B}w\|_2^2.$$

Suppose that

$$\|\mathcal{B}w\|_2^2+2\|\mathcal{A}v_1\|_2^2\leq\frac{4}{9L^2}\tag{4.20}$$

then

$$\|\mathcal{A}v_1\|_2^4+\frac{4}{9L^2}\|\mathcal{B}w\|_2^2\geq\|\mathcal{B}v\|_2^4.$$

This implies that

$$\mathcal{I}\big(\overline{m}+v\big)\geq(1-3\epsilon)\sigma(m_\beta)\frac{9}{4}\frac{(1-\epsilon)^3}{(1+\epsilon)^2}\frac{\|\mathcal{B}v\|_2^4}{\int_{\mathbb{R}\times\Lambda}x_1^2(\mathcal{A}v_1)^2\mathrm{d}x+L^d}-\epsilon_1^2\mathcal{R}.$$

To compare $\|\mathcal{B}v\|_2^2$ with $[\mathcal{F}\big(m(t)\big)-\mathcal{F}\big(\overline{m}\big)]$ under assumption (4.4) we apply Lemma 4.4 stated and proven below and we obtain

$$\|\mathcal{B}v\|_2^2\geq2\tilde{\alpha}(1-2\epsilon)[\mathcal{F}\big(m(t)\big)-\mathcal{F}\big(\overline{m}\big)],\tag{4.21}$$

where

$$\tilde{\alpha}=\frac{1}{\beta(1-m_\beta^2)}-1=\frac{1-\sigma(m_\beta)}{\sigma(m_\beta)}.\tag{4.22}$$

Taking into account (4.21) we have

$$\mathcal{I}\big(m(t)\big)\geq(1-\sigma(m_\beta))^2 9\frac{1}{\sigma(m_\beta)}\frac{(1-\epsilon)^3}{(1+\epsilon)^2}\frac{\big[\mathcal{F}\big(m(t)\big)-\mathcal{F}\big(\overline{m}\big)\big]^2}{\int_{\mathbb{R}\times\Lambda}x_1^2(\mathcal{A}v_1)^2\mathrm{d}x+L^d}-\epsilon_1^2\mathcal{R}.$$

Recalling the definition of $\mathcal{R}$, see (4.19), choosing ϵ_1 small enough so that

$$\frac{9}{4}\frac{\epsilon_1}{L^d\epsilon^3}C(\beta,J)\leq\epsilon$$

we get

$$\mathcal{I}\big(m(t)\big)\geq(1-\sigma(m_\beta))^2 9\frac{1}{\sigma(m_\beta)}\frac{(1-2\epsilon)^3}{(1+\epsilon)^2}\frac{\Big[\big[\mathcal{F}\big(m(t)\big)-\mathcal{F}\big(\overline{m}\big)\big]^2\Big]}{\int_{\mathbb{R}\times\Lambda}x_1^2(\mathcal{A}v_1)^2\mathrm{d}x+L^d}.\tag{4.23}$$

By Lemma 3.5 we have that

$$\int_{\mathbb{R}\times\Lambda}x_1^2(\mathcal{A}v_1)^2\mathrm{d}x\leq\int_{\mathbb{R}\times\Lambda}x_1^2(\mathcal{B}v)^2\mathrm{d}x,$$

from (4.23), taking into account (4.2) we get (4.5). Next we verify that requirement (4.20) is indeed satisfied under assumptions (4.3). Namely $\|\mathcal{B}w\|_2^2 + 2\|\mathcal{A}v_1\|_2^2 = \|\mathcal{B}v\|_2^2 + \|\mathcal{A}v_1\|_2^2$ and

$$\|\mathcal{A}v_1\|_2^2 = \|\mathcal{B}v_1\|_2^2 = \int_{\mathbb{R}} \mathrm{d}x_1 \left(\frac{1}{L^d}\int_\Lambda \mathcal{B}v(x_1, x_1^\perp)\mathrm{d}x_1^\perp\right)^2 \le$$
$$\left(\frac{1}{L^d}\right)^2 \int_R \mathrm{d}x_1 \left(\left(\int_\Lambda (\mathcal{B}v)^2(x_1,x_1^\perp)\mathrm{d}x_1^\perp\right)^{\frac{1}{2}} L^{\frac{d}{2}}\right)^2 = \frac{1}{L^d}\int_{\Lambda\times\mathbb{R}} (\mathcal{B}v)^2(x)\mathrm{d}x.$$

Then

$$\|\mathcal{B}w\|_2^2 + 2\|\mathcal{A}v_1\|_2^2 \le \|\mathcal{B}v\|_2^2[1+\frac{1}{L^d}] \le C(\beta)[1+\frac{1}{L^d}]\|v\|_2^2,$$

since $\|\mathcal{B}v\|_2^2 \le C(\beta)\|v\|_2^2$. We get (4.20). □

Next we compare $\|\mathcal{B}v\|_2^2$ with $[\mathcal{F}(m(t)) - \mathcal{F}(\overline{m})]$.

4.4 LEMMA. *Take $v \in W^{1,2}(\mathbb{R}\times\Lambda)$, $\int_{\mathbb{R}\times\Lambda} v\overline{m}' = 0$ and $m = \bar m + v$. For any $\epsilon > 0$ there exists $\epsilon_1 = \epsilon_1(\epsilon, L, \beta, J)$ so that for*

$$\mathcal{I}(m(t)) \le \epsilon_1\big[\mathcal{F}(m(t)) - \mathcal{F}(\overline{m})\big], \tag{4.24}$$

$$\|\mathcal{B}v\|^2_{L^2(\mathbb{R}\times\Lambda)} \ge 2\tilde\alpha(1-2\epsilon)[\mathcal{F}(\bar m + v) - \mathcal{F}(\overline{m})] \tag{4.25}$$

and

$$\|\mathcal{B}v\|^2_{L^2(\mathbb{R}\times\Lambda)} \le 2\tilde\alpha(1+2\epsilon)[\mathcal{F}(\bar m + v) - \mathcal{F}(\overline{m})], \tag{4.26}$$

where $\tilde\alpha$ is defined in (4.22).

Proof: We have that

$$\|\mathcal{B}v\|_2^2 = \langle \mathcal{B}v, \mathcal{B}v\rangle = \tilde\alpha\langle v, \mathcal{B}v\rangle + \langle(\mathcal{B}-\tilde\alpha)v, \mathcal{B}v >= \tilde\alpha\langle v,\mathcal{B}v\rangle + \langle v, (\mathcal{B}-\tilde\alpha)(\mathcal{B}v)\rangle.$$

Therefore

$$\|\mathcal{B}v\|^2 \ge \tilde\alpha\langle v,\mathcal{B}v\rangle - |\langle v, (\mathcal{B}-\tilde\alpha)(\mathcal{B}v)\rangle|.$$

By Lemma 8.3 we have

$$|\langle v, (\mathcal{B}-\tilde\alpha)(\mathcal{B}v)\rangle| \le \|v\|_{L^2(\mathbb{R}\times\Lambda)}\|(\mathcal{B}-\tilde\alpha)(\mathcal{B}v)\|_{L^2(\mathbb{R}\times\Lambda)} \le \|v\|_{L^2(\mathbb{R}\times\Lambda)}\|\|\nabla(\mathcal{B}v)\|_{L^2(\mathbb{R}\times\Lambda)}.$$

By Theorem 3.1 and assumption (4.24) we have

$$C(\beta)\|\nabla(\mathcal{B}v)\|^2_{L^2(\mathbb{R}\times\Lambda)} \le \mathcal{I}(v) \le \epsilon_1[\mathcal{F}(m(t)) - \mathcal{F}(\overline{m})\big].$$

Under this condition

$$|\langle v, (\mathcal{B}-\tilde\alpha)(\mathcal{B}v)\rangle| \le \|v\|_{L^2(\mathbb{R}\times\Lambda)}[\epsilon_1 K(\beta J)\mathcal{F}(m(t)) - \mathcal{F}(\overline{m})\big]^{\frac{1}{2}}.$$

Further, by Lemma 8.1

$$\|v\|^2_{L^2(\mathbb{R}\times\Lambda)} \le \frac{4}{\gamma(L)}[\mathcal{F}(m(t)) - \mathcal{F}(\overline{m})].$$

Therefore

$$|\langle v,(\mathcal{B}-\tilde{\alpha})(\mathcal{B}v)\rangle| \le [\epsilon_1 K(\beta J)\frac{4}{\gamma(L)}]^{\frac{1}{2}}[\mathcal{F}\big(m(t)\big)-\mathcal{F}\big(\overline{m}\big)].$$

Take ϵ_1 small enough so that

$$[\epsilon_1 K(\beta J)\frac{4}{\gamma(L)}]^{\frac{1}{2}} \le \epsilon$$

Then we get (4.25). Arguing in a similar way we get (4.26). □

5 Moment estimates

In this section we control the evolution of

$$\phi(t) = L^d + \int_{\mathbb{R}\times\Lambda} \sigma(\overline{m}_{a(t)})|x_1\left(\mathcal{B}_{a(t)}v(t)\right)|^2 \mathrm{d}x \tag{5.1}$$

in term of the free energy functional $\mathcal{F}$. As we discussed in the introduction it is important to have the right constant multiplying the free energy. In the next theorem we show two estimates. The first estimate (5.2) does not quantify the constant multiplying the free energy and holds under less restrictive assumptions. To show the second estimate, see (5.3), we need that the dissipation $\mathcal{I}(m(t))$ is small compared to the excess free energy, see (5.3). For proving the main result we need both of them.

5.1 THEOREM. *Let $m(\cdot,t)$ be a solution of (1.1). For any $\epsilon > 0$, $L > 0$ there are constants $\kappa_0(\beta,J,\epsilon,L)$, $\delta_0(\beta,J,\epsilon,L)$ and $\epsilon_1(\beta,J,\epsilon,L)$ such that for all t with $\|v(t)\|_{W^{s,2}} \le \kappa_0$, $s > \frac{D}{2}$, $||v(t)||_2 < \delta_0$ and $|a(t)| \le 1$ there exists a positive constant $B = B(\kappa_0,L,d,\beta,J)$*

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(t) \le B\big[\mathcal{F}(m(t)) - \mathcal{F}(\overline{m})\big] \ . \tag{5.2}$$

Further if

$$\mathcal{I}(m(t)) \le \epsilon_1\big[\mathcal{F}(m(t)) - \mathcal{F}(\overline{m})\big] \tag{5.3}$$

then

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(t) \le (1+\epsilon)4(1-\sigma(m_\beta))^2\big[\mathcal{F}(m(t)) - \mathcal{F}(\overline{m})\big] \ . \tag{5.4}$$

The proof of Theorem 5.1 is based on several intermediate results. We start deriving the full non–linear evolution for v inserting $m(t) = \overline{m}_{a(t)} + v(t)$ into (1.1). Taking into account that $\overline{m}$ is a stationary solution of (1.1), i.e

$$\nabla\overline{m} - \beta(1-\overline{m}^2)(J\star\nabla\overline{m}) = 0,$$

we obtain:

$$\begin{aligned}
\frac{\partial v}{\partial t} &= \nabla\cdot\left(\nabla v - \beta(1-\overline{m}^2)J\star\nabla v\right)\\
&+ \beta\nabla\cdot\left(v(v+2\overline{m})J\star(\nabla v+\nabla\overline{m})\right) + \dot{a}(t)\overline{m}'\\
&= \nabla\cdot(\sigma(\overline{m})\nabla(\mathcal{B}v))\\
&+ \beta\nabla\cdot\left(v^2 J\star\nabla\overline{m}\right) + \beta\nabla\cdot((v(v+2\overline{m})J\star\nabla v)\\
&+ \dot{a}(t)\overline{m}'.
\end{aligned} \tag{5.5}$$

Differentiating (5.1) produces terms involving $\dot{a}(t)$. We estimate these by applying Theorem 2.3.

5.2 LEMMA. *Let v be a solution of (5.5). Then for any $\epsilon > 0$ there are constant $\delta = \delta(\epsilon, \beta, J) > 0$ and $\kappa = \kappa(\delta, \epsilon, \beta, J) > 0$, such that for all t with $\|v(t)\|_2 \le \delta$ and $\|v(t)\|_{W^{s+1,2}} \le \kappa$ for $s > \frac{D}{2}$*

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(t) \le 2\int \mathcal{B}\left(\sigma(\overline{m})x_1^2\mathcal{B}v\right)\frac{\partial v}{\partial t}\mathrm{d}x + \epsilon\big[\mathcal{F}(\overline{m}+v) - \mathcal{F}(\overline{m})\big] \ . \tag{5.6}$$

Proof: Since $\mathcal{B}$ is self adjoint,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int \sigma(\overline{m})|x_1\mathcal{B}v|^2\mathrm{d}x &= 2\int \mathcal{B}\big(\sigma(\overline{m})x_1^2\mathcal{B}v\big)\frac{\partial v}{\partial t}\mathrm{d}x \\ &+ \dot{a}(t)2\bigg(\beta\int \overline{m}\overline{m}'x_1^2|\mathcal{B}v|^2\mathrm{d}x + \int \sigma(\overline{m})x_1^2(\mathcal{B}v)\frac{2\overline{m}\overline{m}'}{\beta(1-\overline{m}^2)^2}v\mathrm{d}x\bigg). \end{aligned}\tag{5.7}$$

By the exponential decays properties of $\overline{m}$, see (1.10), the boundedness of $\mathcal{B}$ on $L^2(\mathbb{R}\times\Lambda)$ and Theorem 2.3, which says that $|\dot{a}(t)| \le D(\kappa, \beta, J)\|v(t)\|_2$, one clearly has

$$\begin{aligned}\dot{a}(t)2\bigg(\beta\int \overline{m}\overline{m}'x_1^2|\mathcal{B}v|^2\mathrm{d}x &+ \int \sigma(\overline{m})x_1^2(\mathcal{B}v)\frac{2\overline{m}\overline{m}'}{\beta(1-\overline{m}^2)^2}v\mathrm{d}x\bigg) \\ &\le |\dot{a}(t)|C\left[\|\overline{m}\overline{m}'x_1^2\|_\infty\int|\mathcal{B}v|^2\mathrm{d}x + \|\sigma(\overline{m})x_1^2\frac{\overline{m}\overline{m}'}{\beta(1-\overline{m}^2)^2}\|_\infty\int(\mathcal{B}v)v\mathrm{d}x\right] \\ &\le C(\kappa,\beta,J)\|v(t)\|_2\big[\mathcal{F}(\overline{m}+v) - \mathcal{F}(\overline{m})\big]\end{aligned}$$

where C is a constant depending only on β, J and κ that changes from line to line. In the last inequality we applied Lemma 4.4 and Lemma 8.1given the Appendix. □

We will separately estimate the linear and nonlinear contributions from (5.5) to (5.6). Since $\mathcal{B}\overline{m}' = 0$, the term containing $\dot{a}$ in (5.5) makes no contribution to (5.7).

The basic manipulation, to be done repeatedly in the rest of the proof, is to commute differentiation and multiplication by x_1 with $\mathcal{B}$. Therefore we define

$$g(x_1) = \Big(\frac{1}{\sigma(\overline{m}(x_1))}\Big)' = \frac{2\bar{m}(x_1)\bar{m}'(x_1)}{\beta(1-\overline{m}^2(x_1))^2}, \qquad x_1 \in \mathbb{R}$$

and observe that

$$\nabla\left(\mathcal{B}w\right) = e_1 g w + \mathcal{B}(\nabla w), \tag{5.8}$$

where e_1 is the D unit vector in the x_1- direction, $e_1 = (1, 0, \dots 0)$. In (5.8) and in the following we denote by $\mathcal{B}(\nabla w)$ the D vector with components $\mathcal{B}(w_{x_j})$, $j = 1, \dots, D$ and by $\|\nabla w\|_2^2 = \sum_{i=1}^D \|\frac{\partial w}{\partial x_i}\|_2^2$. Furthermore, define the convolution operator $\mathcal{C}$ by

$$\mathcal{C}w(x) = \int_{\mathbb{R}\times\Lambda} J(y)y_1 w(x-y)\mathrm{d}y.$$

Observe that for any function w,

$$x_1\big(\mathcal{B}w\big) = \mathcal{B}(x_1 w) + \mathcal{C}w \tag{5.9}$$

where $x_1 w$ denotes the function with values $x_1 w(x_1, x_1^\perp)$. By Young's inequality $\mathcal{C}$ is bounded on all L^p with operator norm

$$\|\mathcal{C}\| \le \int_{\mathbb{R}\times\Lambda}|x_1 J(x)|\mathrm{d}x \ . \tag{5.10}$$

We need the following technical lemma.

5.3 LEMMA. *For $w \in L^2(\mathbb{R}\times\Lambda)$, $w_{x_i} \in L^2(\mathbb{R}\times\Lambda)$, $i = 1,\dots,D$ we have*

$$\|\sigma(\overline{m})x_1\nabla\big(\mathcal{B}w\big)\|_2 \le \frac{\|\big(\sigma(\overline{m})x_1\overline{m}'\big)'\|_{L^2(\mathbb{R}\times\Lambda)}}{\|\overline{m}'\|_{L^2(\mathbb{R}\times\Lambda)}}\|w\|_2 + \gamma(L)^{-1/2}\|\mathcal{B}^{1/2}\big(\sigma(\overline{m})x_1\nabla\big(\mathcal{B}w\big)\big)\|_2 \tag{5.11}$$

where $\gamma(L)$ is the spectral gap (3.3) of $\mathcal{B}$. Further there is a finite constant $C > 0$ depending only on β and J such that whenever $|a(t)| \le 1$,

$$\|J\star(x_1\nabla w)\|_2 \le C\left(\|w\|_2 + \|\mathcal{B}^{1/2}(\sigma(\overline{m})x_1\nabla\big(\mathcal{B}w\big)\|_2\right). \tag{5.12}$$

Proof: Let P denote the orthogonal projection onto the span of $\overline{m}'$; i.e., the null space of $\mathcal{B}$. Then

$$\begin{aligned} P\big(\sigma(\overline{m})x_1\big(\mathcal{B}w\big)_{x_1}\big) &= \frac{1}{\|\overline{m}'\|_2^2}\langle\overline{m}',\sigma(\overline{m})x_1\big(\mathcal{B}w\big)_{x_1}\rangle_{L^2}\overline{m}' \\ &= -\frac{1}{\|\overline{m}'\|_2^2}\langle\big(\sigma(\overline{m})x_1\overline{m}'\big)',\big(\mathcal{B}w\big)\rangle_{L^2}\overline{m}', \end{aligned}$$

and

$$P\big(\sigma(\overline{m})x_1\big(\mathcal{B}w\big)_{x_j}\big) = \frac{1}{\|\overline{m}'\|_2^2}\langle\overline{m}',\sigma(\overline{m})x_1\big(\mathcal{B}w\big)_{x_j}\rangle_{L^2}\overline{m}' = 0, \quad j \ne 1.$$

Hence, by the Schwarz inequality and the fact that $\mathcal{B}$ is bounded, we get

$$\|P\big(\sigma(\overline{m})x_1\big(\mathcal{B}w\big)_{x_1}\big)\|_2 \le \frac{\|\big(\sigma(\overline{m})x_1\overline{m}'\big)'\|_2}{\|\overline{m}'\|_2}\|w\|_2\ . \tag{5.13}$$

Next,

$$\begin{aligned} \|P^\perp\big(\sigma(\overline{m})x_1\nabla\big(\mathcal{B}w\big)\big)\|_2 &= \|\mathcal{B}^{-1/2}\mathcal{B}^{1/2}P^\perp\big(\sigma(\overline{m})x_1\nabla\big(\mathcal{B}w\big)\big)\|_2 \\ &\le \gamma(L)^{-1/2}\|\mathcal{B}^{1/2}P^\perp\big(\sigma(\overline{m})x_1\nabla\big(\mathcal{B}w\big)\big)\|_2 \\ &= \gamma(L)^{-1/2}\|P^\perp\mathcal{B}^{1/2}\big(\sigma(\overline{m})x_1\nabla\big(\mathcal{B}w\big)\big)\|_2 \\ &\le \gamma(L)^{-1/2}\|\mathcal{B}^{1/2}\big(\sigma(\overline{m})x_1\nabla\big(\mathcal{B}w\big)\big)\|_2\ . \end{aligned}$$

Hence, the Minkowski inequality and (5.13) yield (5.11). To prove (5.12) we define the operator $\mathcal{D}$ by

$$\mathcal{D}w = \frac{1}{\beta(1-m_\beta^2)}w - J\star w\ . \tag{5.14}$$

Fourier transforming, one sees that $\mathcal{D}$ is bounded with a bounded inverse since $\beta(1-m_\beta^2) < 1$. Note that

$$\mathcal{D}w = \mathcal{B}w - \tilde{g}w, \tag{5.15}$$

where

$$\tilde{g}(x_1) = \frac{1}{\beta(1-\overline{m}^2(x_1))} - \frac{1}{\beta(1-m_\beta^2)}, \quad x_1 \in \mathbb{R}. \tag{5.16}$$

Also, $\mathcal{D}$ commutes with convolution by J, and

$$x_1\mathcal{D}w = \mathcal{D}(x_1w) + \mathcal{C}w, \tag{5.17}$$

as with $\mathcal{B}$ in (5.9). Hence,

$$J \star x_1 \nabla w = \mathcal{D}^{-1} J \star \big(\mathcal{D}(x_1 \nabla w)\big) = \mathcal{D}^{-1} J \star \big(x_1 \mathcal{D}(\nabla w) - \mathcal{C}(\nabla w)\big) = \\ - \mathcal{D}^{-1}\mathcal{C}\big(\nabla J \star w\big) + \mathcal{D}^{-1} J \star \big(x_1 \mathcal{B}\nabla w - x_1 \tilde{g} \nabla w\big),$$

where we have used that convolution with J commutes with $\mathcal{C}$ and that $J \star \nabla w = \nabla J \star w$. Next, applying (5.8),

$$\mathcal{D}^{-1} J \star \big(x_1 \mathcal{B}(\nabla w) - x_1 \tilde{g} \nabla w\big) = \mathcal{D}^{-1} J \star \big(x_1 \nabla\big(\mathcal{B} w\big)\big) - \mathcal{D}^{-1} J \star \big(x_1 g w e_1 + x_1 \tilde{g} \nabla w\big).$$

We have that

$$\|\mathcal{D}^{-1} J \star (x_1 g w e_1 + x_1 \tilde{g} \nabla w)\|_2 \leq C(\beta, J)\|w\|_2,$$

where we used the rapidly decay of g and $\tilde{g}$ and that $J \star (x_1 \tilde{g} \nabla w) = \nabla J \star (x_1 \tilde{g} w) + J \star \nabla(x_1 \tilde{g}) w$. Thus,

$$\begin{aligned} \|J \star (x_1 \nabla w)\|_2 &\leq \|\mathcal{D}^{-1}\mathcal{C}\left(\nabla J \star w\right)\|_2 \\ &+ \|\mathcal{D}^{-1} J \star \big(x_1 \nabla\big(\mathcal{B} w\big)\big)\|_2 \\ &+ \|\mathcal{D}^{-1} J \star (x_1 g w e_1 + x_1 \tilde{g} \nabla w)\|_2 \\ &\leq C(\beta, J)\big(\|w\|_2 e_1 + \|\sigma(\overline{m}) x_1 \nabla\big(\mathcal{B} w\big)\|_2\ . \end{aligned}$$

Now application of (5.11) yields (5.12). □

Next we estimate the nonlinear contribution from (5.5) to (5.6).

5.4 LEMMA. *Let v be a solution of (5.5). Then for any $\epsilon > 0$ there are constants $\delta = \delta(\beta, J, \epsilon, L) > 0$ and $\kappa = \kappa(\beta, J, \epsilon, L) > 0$ such that for all t with $\|v(t)\|_2 \leq \delta$, $\|v(t)\|_{W^{s,2}} \leq \kappa$, and $|a(t)| \leq 1$,*

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\phi(t) &\leq 2\int \mathcal{B}\big(\sigma(\overline{m}) x_1^2 \mathcal{B} v\big)\nabla \cdot (\sigma(\overline{m})\nabla(\mathcal{B} v))\,\mathrm{d}x \\ &+ \epsilon\big[\mathcal{F}(\overline{m} + v) - \mathcal{F}(\overline{m})\big] + \epsilon\|\mathcal{B}^{1/2}\big(\sigma(\overline{m}) x_1 \nabla\big(\mathcal{B} v\big)\|_2^2. \end{aligned}$$

Proof: We separately estimate the contribution of the two nonlinear terms in (7.1) to (5.6), beginning with the more difficult of the two:

$$2\int \mathcal{B}\big(\sigma(\overline{m}) x_1^2 \mathcal{B} v\big)\beta\nabla \cdot \big(v(v + 2\overline{m}) J \star \nabla v\big)\mathrm{d}x. \tag{5.18}$$

Now integrating by parts and applying (5.8) to (5.18) yields

$$\int \mathcal{B}\big(\sigma(\overline{m}) x_1^2 \mathcal{B} v\big)\beta\nabla \cdot \big(v(v + 2\overline{m}) J \star \nabla v\big)\mathrm{d}x = A_1 + A_2$$

with

$$\begin{aligned} A_1 &= -2\int g\big(\sigma(\overline{m}) x_1^2 \mathcal{B} v\big) e_1 \beta\big(v(v + 2\overline{m})\nabla J \star v\big)\mathrm{d}x \\ &- 2\int \mathcal{B}\big(\sigma(\overline{m})' x_1^2 \mathcal{B} v\big)\beta\big(v(v + 2\overline{m}) e_1 \nabla J * v\big)\mathrm{d}x, \end{aligned}$$

$$
\begin{aligned}
A_2 &= -4\int \mathcal{B}\big(\sigma(\overline{m})x_1e_1\mathcal{B}v\big)\beta\big(v(v+2\overline{m})J*\nabla v\big)\mathrm{d}x\\
&- 2\int \mathcal{B}\big(\sigma(\overline{m})x_1^2\nabla\left(\mathcal{B}v\right)\big)\beta\big(v(v+2\overline{m})J*\nabla v\big)\mathrm{d}x
\end{aligned}
$$

where the first term in A_1 comes from the first term of (5.8), and the remaining term of A_1 together with the term A_2 from differentiating the product $\sigma(\overline{m})x_1^2\mathcal{B}v$. We have also used the fact that $J\star\nabla v=\nabla J\star v$. Because of (1.10), by Lemma 8.1 in the appendix and the inequality $\|v\|_\infty\le c(d,s)\|v\|_{W^{s,2}}$, for $s>\frac{D}{2}$

$$
|A_1|\le C\|v\|_\infty\|v\|_2^2\le Cc(d,s)\|v\|_{W^{s,2}}\|v\|_2^2\le Cc(d,s)\|v\|_{W^{s,2}}\frac{1}{\gamma(L)}\big[\mathcal{F}(\overline{m}+v)-\mathcal{F}(\overline{m})\big] \tag{5.19}
$$

where C is a constant depending only on β and J. Then for any $\epsilon>0$ there are constants $\delta>$ and $\kappa>0$ such that for all t the quantity in (5.19) is no greater than

$$
|A_1|\le\frac{\epsilon}{3}\big[\mathcal{F}(\overline{m}+v)-\mathcal{F}(\overline{m})\big]\ . \tag{5.20}
$$

To estimate A_2 we need to commute an x_1 past $\mathcal{B}$. Applying (5.9), these become

$$
\begin{aligned}
A_2 &= 4\int \mathcal{C}\big(\sigma(\overline{m})\mathcal{B}v\big)e_1\beta\big(v(v+2\overline{m})J*\nabla v\big)\mathrm{d}x+2\int \mathcal{C}\big(\sigma(\overline{m})x_1\nabla\big(\mathcal{B}v\big)\big)\beta\big(v(v+2\overline{m})J*\nabla v\big)\mathrm{d}x\\
&- 4\int \mathcal{B}\big(\sigma(\overline{m})\mathcal{B}v\big)e_1\beta\big(v(v+2\overline{m})x_1J*\nabla v\big)\mathrm{d}x-2\int \mathcal{B}\big(\sigma(\overline{m})x_1\nabla\big(\mathcal{B}v\big)\big)\beta\big(v(v+2\overline{m})x_1J*\nabla v\big)\mathrm{d}x.
\end{aligned}
$$

Now, it is exactly the convolution by J in $\mathcal{B}$ that doesn't commute with multiplication by x_1 so that

$$
x_1J\star w=J\star(x_1w)+\mathcal{C}w
$$

so that the integrals above can be partially rewritten as

$$
\begin{aligned}
A_2 &= 4\int \mathcal{C}\big(\sigma(\overline{m})\mathcal{B}v\big)\beta\big(v(v+2\overline{m})e_1\nabla J*v\big)\mathrm{d}x+2\int \mathcal{C}\big(\sigma(\overline{m})x_1\nabla\big(\mathcal{B}v\big)\big)\beta\big(v(v+2\overline{m})J*\nabla v\big)\mathrm{d}x\\
&- 4\int \mathcal{B}\big(\sigma(\overline{m})\mathcal{B}v\big)\beta\big(v(v+2\overline{m})e_1\mathcal{C}(\nabla v)\big)\mathrm{d}x-2\int \mathcal{B}\big(\sigma(\overline{m})x_1\nabla\big(\mathcal{B}v\big)\big)\beta\big(v(v+2\overline{m})\mathcal{C}(\nabla v)\big)\mathrm{d}x\\
&- 4\int \mathcal{B}\big(\sigma(\overline{m})\mathcal{B}v\big)\beta\big(v(v+2\overline{m})e_1J*(x_1\nabla v)\big)\mathrm{d}x\\
&- 2\int \mathcal{B}\big(\sigma(\overline{m})x_1\nabla\big(\mathcal{B}v\big)\big)\beta\big(v(v+2\overline{m})J*(x_1\nabla v)\big)\mathrm{d}x.
\end{aligned}
$$

Clearly there is a constant C depending only on β and J so that

$$
\|\mathcal{C}(\nabla v)\|_2\le C\|v\|_2,
$$

and hence the four terms containing $\mathcal{C}$ may be estimated, as in (5.19) by

$$
\|v\|_\infty\|v\|_2^2\le c(d,s)\|v\|_{W^{s,2}}\frac{1}{\gamma(L)}\big[\mathcal{F}(\overline{m}+v)-\mathcal{F}(\overline{m})\big].
$$

Hence there are constants κ and δ so that

$$
Cc(d,s)\|v\|_{W^{s,2}}\frac{1}{\gamma(L)}\big[\mathcal{F}(\overline{m}+v)-\mathcal{F}(\overline{m})\big]\le\frac{\epsilon}{3}\big[\mathcal{F}(\overline{m}+v)-\mathcal{F}(\overline{m})\big] \tag{5.21}
$$

for all t with $\|v(t)\|_2 \leq \delta$, $\|v\|_{W^{s,2}} \leq \kappa$ and $|a(t)| \leq 1$.

Next, by the Schwarz inequality, and then (5.12) of Lemma 5.3,

$$\begin{aligned} -4\int \mathcal{B}\big(\sigma(\overline{m})\mathcal{B}v\big)\beta\big(v(v+2\overline{m})e_1 J*(x_1\nabla v)\big)\mathrm{d}x &\leq C\|v\|_\infty\|v\|_2\|J*(x_1\nabla v)\|_2 \\ &\leq C\|v\|_\infty\|v\|_2 \\ &\times \Big[\|v\|_2 + \|\mathcal{B}^{1/2}\sigma(\overline{m})x_1\nabla\big(\mathcal{B}v\big)\|_2\Big] \end{aligned} \tag{5.22}$$

and

$$\begin{aligned} -2\int \mathcal{B}\big(\sigma(\overline{m})x_1\nabla\big(\mathcal{B}v\big)\big)\beta\big(v(v+2\overline{m})J*(x_1\nabla v)\big)\mathrm{d}x &\leq C\|v\|_\infty\|\mathcal{B}^{1/2}\sigma(\overline{m})x_1\nabla\big(\mathcal{B}v\big)\|_2\|J*(x_1\nabla v)\|_2 \\ &\leq C\|v\|_\infty\|\mathcal{B}^{1/2}\sigma(\overline{m})x_1\nabla\big(\mathcal{B}v\big)\|_2 \\ &\times \Big[\|v\|_2 + \|\mathcal{B}^{1/2}\sigma(\overline{m})x_1\nabla\big(\mathcal{B}v\big)\|_2\Big]. \end{aligned} \tag{5.23}$$

Hence the sum of the two terms in (5.22) and (5.23) is no greater than

$$C\|v\|_\infty \Big[\|v\|_2^2 + \|\mathcal{B}^{1/2}\sigma(\overline{m})x_1\nabla\big(\mathcal{B}v\big)\|_2^2\Big]$$

and now decreasing δ and κ as necessary, we obtain as before from $\|v\|_\infty \leq c(d,s)\|v\|_{W^{s,2}}$ and Lemma 8.1 in the Appendix that this is no greater than

$$|A_2| \leq \frac{\epsilon}{3}\big[\mathcal{F}(\overline{m}+v) - \mathcal{F}(\overline{m})\big] + \epsilon\|\mathcal{B}^{1/2}\sigma(\overline{m})x_1\nabla\big(\mathcal{B}v\big)\|_2^2 \tag{5.24}$$

for all t with $\|v(t)\|_2 \leq \delta$, $\|v\|_{W^{s,2}} \leq \kappa$ and $|a(t)| \leq 1$. Thus the estimate on (5.18) follows from (5.20), (5.21) and (5.24).

It remains to estimate the contributions to (5.6) from the other of the two non-linear terms in (5.5), namely

$$-2\int \nabla\left(\mathcal{B}\left(\sigma(\overline{m})x_1^2\mathcal{B}v\right)\right)\left(v^2 J\star\nabla\overline{m}\right)\mathrm{d}x.$$

Proceeding as above, though with with much less effort, one obtains that this term is bounded by

$$\|v\|_\infty\big(C\|v\|_2^2 + \|\mathcal{B}^{1/2}\sigma(\overline{m})x_1\nabla\big(\mathcal{B}v\big)\|_2^2\big)$$

where the extra factor of $\|v\|_\infty$ comes from the nonlinearity. Using once more the inequality $\|v\|_\infty \leq c(d,s)\|v\|_{W^{s,2}}$, one sees that for δ sufficiently small, one can combine the above estimates, once more using Lemma 8.1 in the Appendix, to obtain the proof of the lemma. □

5.5 THEOREM. *Let v be a solution of (5.5). For any $\epsilon > 0$ there are constants $\delta = \delta(\beta, J, \epsilon, L) > 0$ and $\kappa = \kappa(\beta, J, \epsilon, L) > 0$ such that for all t with $\|v(t)\|_2 \leq \delta$, $\|v\|_{W^{s,2}} \leq \kappa$, for $s > \frac{D}{2}$ and $|a(t)| \leq 1$,*

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\phi(t) &\leq -4\int \mathcal{B}\big(\sigma(\overline{m})\mathcal{B}v\big)x_1\sigma(\overline{m})e_1\cdot\nabla(\mathcal{B}v)\mathrm{d}x - 2\|\mathcal{B}^{1/2}\left(\sigma(\overline{m})x_1\nabla\big(\mathcal{B}v\big)\right)\|_2^2 \\ &+ I_1 + I_2 + I_3 + I_4 \\ &+ \epsilon\big[\mathcal{F}(\overline{m}+v) - \mathcal{F}(\overline{m})\big] + \epsilon\|\mathcal{B}^{1/2}\left(\sigma(\overline{m})x_1\nabla\big(\mathcal{B}v\big)\right)\|_2^2 \end{aligned} \tag{5.25}$$

where

$$\begin{aligned}
I_1 &= -2\int g(x_1)\big(\sigma(\overline{m})x_1^2\mathcal{B}v\big)e_1\sigma(\overline{m})\nabla(\mathcal{B}v)\mathrm{d}x,\\
I_2 &= -2\int \mathcal{B}\left(\sigma(\overline{m})' x_1^2\mathcal{B}v\right)e_1\sigma(\overline{m})\nabla(\mathcal{B}v)\mathrm{d}x,\\
I_3 &= -4\int \mathcal{C}\big(\sigma(\overline{m})\mathcal{B}v\big)e_1\sigma(\overline{m})\nabla(\mathcal{B}v)\mathrm{d}x,\\
I_4 &= -2\int \mathcal{C}\big(\sigma(\overline{m})x_1\nabla(\mathcal{B}v)\big)\sigma(\overline{m})\nabla(\mathcal{B}v)\mathrm{d}x.
\end{aligned}$$

Proof: Denote by A

$$A = 2\int \mathcal{B}\big(\sigma(\overline{m})x_1^2\mathcal{B}v\big)\nabla\cdot(\sigma(\overline{m})\nabla(\mathcal{B}v)) = -2\int \nabla\left(\mathcal{B}\big(\sigma(\overline{m})x_1^2\mathcal{B}v\big)\right)\sigma(\overline{m})\nabla(\mathcal{B}v).$$

By Lemma 5.4 the only term to take care to get (5.25) is A. Now applying (5.8) yields

$$A = -2\int g(x_1)\big(\sigma(\overline{m})x_1^2\mathcal{B}v\big)e_1\sigma(\overline{m})\nabla(\mathcal{B}v)\mathrm{d}x - 2\int \mathcal{B}\left(\nabla\left(\sigma(\overline{m})x_1^2\mathcal{B}v\right)\right)\sigma(\overline{m})\nabla(\mathcal{B}v)\mathrm{d}x. \tag{5.26}$$

Further differentiating the product $\sigma(\overline{m})x_1^2\mathcal{B}v$ we have

$$\begin{aligned}
-2\int \mathcal{B}\left(\nabla\left(\sigma(\overline{m})x_1^2\mathcal{B}v\right)\right)\sigma(\overline{m})\nabla(\mathcal{B}v)\mathrm{d}x &= -2\int \mathcal{B}\left((\sigma(\overline{m}))' x_1^2\mathcal{B}v\right)e_1\sigma(\overline{m})(\nabla(\mathcal{B}v)\mathrm{d}x\\
&- 4\int \mathcal{B}\big(\sigma(\overline{m})x_1\mathcal{B}v\big)e_1\sigma(\overline{m})\nabla(\mathcal{B}v)\mathrm{d}x\\
&- 2\int \mathcal{B}\left(\sigma(\overline{m})x_1^2\nabla(\mathcal{B}v)\right)\big(\sigma(\overline{m})(\nabla(\mathcal{B}v)\big)\mathrm{d}x.
\end{aligned}$$

Denote

$$I_1 = -2\int g(x_1)\big(\sigma(\overline{m})x_1^2\mathcal{B}v\big)e_1\sigma(\overline{m})\nabla(\mathcal{B}v)\mathrm{d}x,$$

$$I_2 = -2\int \mathcal{B}\left((\sigma(\overline{m}))' x_1^2\mathcal{B}v\right)e_1\sigma(\overline{m})\nabla(\mathcal{B}v)\mathrm{d}x.$$

We obtain, see (5.26),

$$\begin{aligned}
A &= I_1 + I_2\\
&- 4\int \mathcal{B}\big(\sigma(\overline{m})x_1\mathcal{B}v\big)e_1\sigma(\overline{m})\nabla(\mathcal{B}v)\mathrm{d}x\\
&- 2\int \mathcal{B}\left(\sigma(\overline{m})x_1^2\nabla(\mathcal{B}v)\right)\big(\sigma(\overline{m})(\nabla(\mathcal{B}v)\big)\mathrm{d}x\ .
\end{aligned}$$

Next, to exploit the positivity of $\mathcal{B}$, we need to distribute the factors of x_1 symmetrically in the last integral. To do this, apply (5.9) to account for commuting multiplication by x_1 with $\mathcal{B}$. We also do this in the other integral, so that the same function $\sigma(\overline{m})x_1\nabla(\mathcal{B}v)$ is produced there as well. The result is

$$A = I_1 + I_2$$

$$
\begin{aligned}
&- \ 4\int \mathcal{C}\big(\sigma(\overline{m})\mathcal{B}v\big)e_1\sigma(\overline{m})\nabla(\mathcal{B}v)\mathrm{d}x\\
&- \ 2\int \mathcal{C}\big(\sigma(\overline{m})x_1\nabla(\mathcal{B}v)\big)\sigma(\overline{m})\nabla(\mathcal{B}v)\mathrm{d}x\\
&- \ 4\int \mathcal{B}\big(\sigma(\overline{m})\mathcal{B}v\big)x_1e_1\sigma(\overline{m})\nabla(\mathcal{B}v)\mathrm{d}x\\
&- \ 2\int \mathcal{B}\big(\sigma(\overline{m})x_1\nabla(\mathcal{B}v)\big)\sigma(\overline{m})x_1\nabla(\mathcal{B}v)\mathrm{d}x \ .
\end{aligned}
$$

Now denote the first two terms after I_1 and I_2; i.e., those containing $\mathcal{C}$, by I_3 and I_4 respectively. Then, by Lemma 4.4, the result is proved. □

Proof of Theorem 5.1: The starting point for proving both (5.2) and (5.4) is Theorem 5.5. We start proving (5.2). The first two terms in (5.25) are the key to the analysis. They correspond to the two terms produced in (1.27) when similar estimates were performed on the heat equation as an illustration of the method. To see this more easily, introduce the following notations:

$$f = e_1\sigma(\overline{m})\mathcal{B}v \tag{5.27}$$

and

$$h = \sigma(\overline{m})x_1\nabla(\mathcal{B}v) \ , \tag{5.28}$$

$$\langle f, h\rangle = \sum_{i=1}^{D}\int_{\mathbb{R}\times\Lambda} f_i(x)h_i(x)dx.$$

Notice that $f_i = 0$ for all $i \geq 2$. These first two terms in (5.25) can be written as following:

$$
\begin{aligned}
-4\langle f, \mathcal{B}h\rangle - 2\langle h, \mathcal{B}h\rangle &= -\langle h+2f, \mathcal{B}(h+2f)\rangle - \langle h, \mathcal{B}h\rangle + 4\langle f, \mathcal{B}f\rangle\\
&\leq -\langle h, \mathcal{B}h\rangle + 4\langle f, \mathcal{B}f\rangle.
\end{aligned} \tag{5.29}
$$

The next step is to estimate each of the I_j appearing in (5.25) in terms of $\|v\|_2^2$, using the negative term in (5.29) to absorb contributions from ∇v.

First, using the Schwarz inequality, and then the arithmetic–geometric mean inequality,

$$I_1 \leq 2\|g\sigma(\overline{m})x_1^2\mathcal{B}v\|_2\|\sigma(\overline{m})\nabla(\mathcal{B}v)\|_2 \leq \lambda\|g\sigma(\overline{m})x_1^2\mathcal{B}v\|_2^2 + \frac{1}{\lambda}\|\sigma(\overline{m})\nabla(\mathcal{B}v)\|_2^2$$

for any $\lambda > 0$. Now choose λ so large that the estimate (5.11) of Lemma 5.3 gives

$$\frac{1}{\lambda}\|\sigma(\overline{m})\nabla(\mathcal{B}v)\|_2^2 \leq \frac{1}{4}\left(\|v\|_2^2 + \langle h, \mathcal{B}h\rangle\right)$$

where h is given in (5.28). The choice of λ depends also on L. One obtains a constant C depending on β, J and L such that

$$I_1 \leq \frac{1}{4}\langle h, \mathcal{B}h\rangle + C\|v\|_2^2 \ . \tag{5.30}$$

It is easier to deal with I_2. Schwarz and (1.10) suffice to establish that there is a constant C depending only on β, and certain finite moments of $\overline{m}'$ so that

$$I_2 \leq \frac{1}{4}\langle h, \mathcal{B}h\rangle + C\|v\|_2^2 \ . \tag{5.31}$$

To bound I_3, we will integrate by parts. Note that using (5.27)

$$I_3 = -4\int (\mathcal{C}f)\sigma(\overline{m})\nabla(\mathcal{B}v)\mathrm{d}x = 4\int \nabla(\mathcal{C}f)\sigma(\overline{m})\big(\mathcal{B}v\big)\mathrm{d}x + 4\int (\mathcal{C}f)\sigma(\overline{m})'\big(\mathcal{B}v\big)\mathrm{d}x.$$

Using this, (5.10) and the rapid decay of $\sigma(\overline{m})'$ coming from (1.10), there is clearly a constant C depending only on β and J so that

$$I_3 \le C\|\sigma(\overline{m})\mathcal{B}v\|_2^2\ . \tag{5.32}$$

Finally, to bound I_4, we use (5.28) and again integrate by parts:

$$I_4 = -2\int (\mathcal{C}h)\sigma(\overline{m})\nabla\big(\mathcal{B}v\big)\mathrm{d}x = 2\int \nabla\big(\mathcal{C}h\big)\sigma(\overline{m})\big(\mathcal{B}v\big)\mathrm{d}x + 2\int \big(\mathcal{C}h\big)\sigma(\overline{m})'\big(\mathcal{B}v\big)\mathrm{d}x.$$

Now proceeding as with I_3, one obtains a constant C depending only on β and J so that

$$I_4 \le C\|\sigma(\overline{m})\mathcal{B}v\|_2\langle h, \mathcal{B}h\rangle^{1/2} \le \frac{1}{4}\langle h, \mathcal{B}h\rangle + 4C^2\|\sigma(\overline{m})\mathcal{B}v\|_2^2\ . \tag{5.33}$$

Then, from (5.30), (5.31), (5.32) and (5.33) we have

$$I_1 + I_2 + I_3 + I_4 \le \frac{3}{4}\langle h, \mathcal{B}h\rangle + C\|v\|_2^2. \tag{5.34}$$

¿From Theorem 5.5, taking into account (5.29) and (5.34) we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\phi(t) \le& -\langle h, \mathcal{B}h\rangle + 4\langle f, \mathcal{B}f\rangle + \frac{3}{4}\langle h, \mathcal{B}h\rangle + C\|v\|_2^2 \\ &+ \epsilon\big[\mathcal{F}(\overline{m}+v) - \mathcal{F}(\overline{m})\big] + \epsilon\|\mathcal{B}^{1/2}\big(\sigma(\overline{m})x_1\nabla\big(\mathcal{B}v\big)\|_2^2.\end{aligned}$$

Take $\epsilon < \frac{1}{4}$, and using the fact that $\mathcal{B}$ is bounded, with a bound depending only on β J and L, Lemma 8.1, we have (5.2). To get (5.4) we estimate I_1 through I_4 under the assumption (5.3). We have

$$\begin{aligned}I_1 &= -2\langle g\sigma^2(\overline{m})x_1^2\mathcal{B}v, (\mathcal{B}v)_{x_1}\rangle_{L^2} \\ &\le 2\|g\sigma^2(\overline{m})x_1^2\|_\infty\|\mathcal{B}v\|_2\|(\mathcal{B}v)_{x_1}\|_2 \\ &\le 2\|g\sigma^2(\overline{m})x_1^2\|_\infty\left(\frac{2\tilde{\alpha}\epsilon_1(1+2\epsilon)}{(1-3\epsilon)\sigma(m_\beta)}\right)^{1/2}\big[\mathcal{F}(\overline{m}+v) - \mathcal{F}(\overline{m})\big]\end{aligned}$$

where we used (4.26) ($\|\mathcal{B}v\|^2_{L^2(\mathbb{R}\times\Lambda)} \le 2\tilde{\alpha}(1+2\epsilon)[\mathcal{F}\big(m(t)\big) - \mathcal{F}\big(\overline{m}\big)]$) of Lemma 4.4, (3.5) of Theorem 3.1 in the last step, together with the assumption (5.3). Note that $\|g\sigma(\overline{m})^2x_1^2\|_\infty$ is bounded by a constant depending only on β and J by (1.10) and the hypothesis that $|a(t)| \le 1$, since (1.10) implies that g is a rapidly decaying bump function centered on $a(t)$. Other L^∞ estimates involving x_1 will be treated in the same way without further mention. This is the only use made of $|a(t)| \le 1$. Similarly,

$$\begin{aligned}I_2 &= -2\langle\sigma(\overline{m})\mathcal{B}\big(\sigma(\overline{m})'x_1^2\mathcal{B}v\big)e_1, \nabla\big(\mathcal{B}v\big)\rangle_{L^2} \\ &\le 2\|\sigma(\overline{m})\|_\infty\|\|\mathcal{B}\|\|\sigma(\overline{m})'x_1^2\|_\infty\|\mathcal{B}v\|_2\|\nabla\big(\mathcal{B}v\big)\|_2 \\ &\le 2\|\sigma(\overline{m})\|_\infty\|\|\mathcal{B}\|\|\sigma(\overline{m})'x_1^2\|_\infty\left(\frac{2\tilde{\alpha}\epsilon_1(1+2\epsilon)}{(1-3\epsilon)\sigma(m_\beta)}\right)^{1/2}\big[\mathcal{F}(\overline{m}+v) - \mathcal{F}(\overline{m})\big]\end{aligned}$$

In the same way, one obtains similar bounds for I_3 and I_4 and then since all of the $\|\cdot\|_\infty$ terms are bounded *a–priori* in terms of β and J, there is a constant C depending only on β and J such that

$$I_1 + I_2 + I_3 + I_4 \le C\epsilon_1\big[\mathcal{F}(\overline{m}+v) - \mathcal{F}(\overline{m})\big]\ .$$

Choosing $\epsilon_1 = \epsilon/C$, one has

$$I_1 + I_2 + I_3 + I_4 \le \epsilon\big[\mathcal{F}(\overline{m}+v) - \mathcal{F}(\overline{m})\big]\ . \tag{5.35}$$

Recalling notations (5.27) and (5.28), Theorem 5.5 and (5.35) one has

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\phi(t) &\le -4\langle f, \mathcal{B}h\rangle_{L^2} - (2-\epsilon)\langle h, \mathcal{B}h\rangle_{L^2}\\ &+ 2\epsilon\big[\mathcal{F}(\overline{m}+v) - \mathcal{F}(\overline{m})\big]\ .\end{aligned} \tag{5.36}$$

Now, since $\mathcal{B}$ is non negative,

$$\begin{aligned}-4\langle f, \mathcal{B}h\rangle_{L^2} - (2-\epsilon)\langle h, \mathcal{B}h\rangle_{L^2} &= -\langle (2-\epsilon)^{1/2}h + 2(2-\epsilon)^{-1/2}f, \mathcal{B}\big((2-\epsilon)^{1/2}h + 2(2-\epsilon)^{-1/2}f\big)\rangle_{L^2}\\ &+ 4(2-\epsilon)^{-1}\langle f, \mathcal{B}f\rangle_{L^2}\\ &\le 4(2-\epsilon)^{-1}\langle f, \mathcal{B}f\rangle_{L^2}.\end{aligned} \tag{5.37}$$

To bound this in terms of the excess free energy, one makes repeated use of (8.10) of Lemma 8.2 together with the self adjointness and boundedness of $\mathcal{B}$, to replace factors of $\sigma(\overline{m})$ with factors of $\sigma(m_\beta)$:

$$\begin{aligned}\langle f, \mathcal{B}f\rangle_{L^2} &= \langle \sigma(\overline{m})\mathcal{B}v, \mathcal{B}\sigma(\overline{m})\mathcal{B}v\rangle_{L^2}\\ &= \langle \sigma(m_\beta)\mathcal{B}v, \mathcal{B}\sigma(\overline{m})\mathcal{B}v\rangle_{L^2} + \langle [\sigma(\overline{m})\mathcal{B}v - \sigma(m_\beta)\mathcal{B}v], \mathcal{B}\sigma(\overline{m})\mathcal{B}v\rangle_{L^2}\\ &\le \sigma(m_\beta)\langle \mathcal{B}v, \mathcal{B}\sigma(\overline{m})\mathcal{B}v\rangle_{L^2} + C\|\nabla\big(\mathcal{B}v\big)\|_2\|\mathcal{B}v\|_2\\ &= \sigma(m_\beta)\langle \mathcal{B}^2v, \sigma(\overline{m})\mathcal{B}v\rangle_{L^2} + C\|\nabla\big(\mathcal{B}v\big)\|_2\|\mathcal{B}v\|_2\\ &\le \sigma(m_\beta)^2\langle \mathcal{B}^2v, \mathcal{B}v\rangle_{L^2} + 2C\|\nabla\big(\mathcal{B}v\big)\|_2\|\mathcal{B}v\|_2\\ &\le \sigma(m_\beta)^2\tilde{\alpha}\langle \mathcal{B}v, \mathcal{B}v\rangle_{L^2} + 3C\|\nabla\big(\mathcal{B}v\big)\|_2\|\mathcal{B}v\|_2\\ &= \sigma(m_\beta)^2\tilde{\alpha}^2\langle v, \mathcal{B}v\rangle_{L^2} + 4C\|\nabla\big(\mathcal{B}v\big)\|_2\|\mathcal{B}v\|_2,\end{aligned} \tag{5.38}$$

where C is constant derived from those in the cited lemmas. Combining estimates (5.36), (5.37) and (5.38) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(t) \le -\sigma(m_\beta)^2\tilde{\alpha}^2 4(2-\epsilon)^{-1}\langle v, \mathcal{B}v\rangle_{L^2} + 4C\|\nabla\big(\mathcal{B}v\big)\|_2\|\mathcal{B}v\|_2 + 2\epsilon\big[\mathcal{F}(\overline{m}+v) - \mathcal{F}(\overline{m})\big]\ .$$

By the hypotheses (5.3), (4.26) of Lemma 4.4 , (3.5) of and Theorem 3.1 we have

$$4C\|\nabla\big(\mathcal{B}v\big)\|_2\|\mathcal{B}v\|_2 \le \epsilon\big[\mathcal{F}(\overline{m}+v) - \mathcal{F}(\overline{m})\big]$$

for ϵ_1 sufficiently small. By Lemma 8.1 of the Appendix $\langle v, \mathcal{B}v\rangle_{L^2} \le \frac{2}{(1-\epsilon)}\big[\mathcal{F}(\overline{m}+v) - \mathcal{F}(\overline{m})\big]$ for δ and κ sufficiently small. Redefining ϵ, one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(t) \le (1+\epsilon)4\tilde{\alpha}^2\sigma(m_\beta)^2\big[\mathcal{F}(\overline{m}+v) - \mathcal{F}(\overline{m})\big]\ .$$

which is the desired result since $\tilde{\alpha}^2\sigma^2(m_\beta) = (1-\sigma(m_\beta))^2$.

6 Proof of the main results

We begin this section by proving several lemmas concerning the L^1 norm. The first of these will be used in the proof of Theorem 1.1 to control $|a - a(t)|$ when a is given by (1.12) and $a(t)$ denotes the shift from the origin of the closest front to $m(t)$, see (1.16).

6.1 LEMMA. *Let w be a function such that $w - \mathrm{sgn}(\mathrm{x}_1)\mathrm{m}_\beta$, $x_1 \in \mathbb{R}$, is integrable and b be fixed by the condition*

$$\int_{\mathbb{R}\times\Lambda} \big(w(x) - \bar{m}_b(x)\big)\mathrm{d}x = 0. \tag{6.1}$$

Then for any c

$$|b - c| \le \frac{1}{2m_\beta}\frac{1}{L^d}\int_{\mathbb{R}\times\Lambda} \big|w(x) - \bar{m}_c(x)\big|\mathrm{d}x. \tag{6.2}$$

In particular, for any solution $m(t)$ of (1.1) and any t such that $m(t) - \mathrm{sgn}(\mathrm{x}_1)\mathrm{m}_\beta$ is integrable,

$$|a(t) - a| \le \frac{1}{2m_\beta}\frac{1}{L^d}\int_{\mathbb{R}\times\Lambda} \big|m(x,t) - \bar{m}_{a(t)}(x)\big|\mathrm{d}x \tag{6.3}$$

where a is fixed by the condition that

$$\int \big(m(x,0) - \bar{m}_a(x)\big)\mathrm{d}x = 0.$$

Proof: First, since $\int_{\mathbb{R}} \bar{m}_0'(x_1)\mathrm{d}x_1 = 2m_\beta > 0$, there is exactly one b such that (6.1) holds. Next, adding and subtracting $\bar{m}_c$, one sees

$$\int_{\mathbb{R}\times\Lambda} \big(w(x) - \bar{m}_c(x)\big)\mathrm{d}x = \int_{\mathbb{R}\times\Lambda} \big(\bar{m}_b(x) - \bar{m}_c(x)\big)\mathrm{d}x.$$

Also, it is clear that

$$\int_{\mathbb{R}\times\Lambda} \big(\bar{m}_b(x) - \bar{m}_c(x)\big)\mathrm{d}x = 2m_\beta(b-c)L^d$$

and (6.2) easily follows. □

6.2 LEMMA. *Let w be any function such that*

$$\int_{\mathbb{R}\times\Lambda} |w(x)|^2(1 + x_1^2)\mathrm{d}x < \infty\ .$$

For any $0 < \delta < 1$ so that

$$C(\delta, L) = \left(\int_{\mathbb{R}\times\Lambda} (1 + x_1^2)^{-(1+\delta)/2}\mathrm{d}x\right)^{1/2} < \infty$$

we have

$$\|w\|_1 \le C(\delta, L)\|(1 + x_1^2)^{1/2} w\|_2^{(1+\delta)/2}\|w\|_2^{(1-\delta)/2}.$$

Proof: Let $p = (1+\delta)/2$ and observe that

$$\begin{aligned}\int_{\mathbb{R}\times\Lambda} |w(x)|\mathrm{d}x &= \int_{\mathbb{R}\times\Lambda} (1+x_1^2)^{-p/2}(1+x_1^2)^{p/2}|w(x)|\mathrm{d}x \\ &\leq \Big(\int_{\mathbb{R}\times\Lambda} (1+x_1^2)^{-p}\mathrm{d}x\Big)^{1/2}\Big(\int_{\mathbb{R}\times\Lambda} (1+x_1^2)^{p}|w(x)|^2\mathrm{d}x\Big)^{1/2}.\end{aligned} \tag{6.4}$$

Jensen's inequality, for $p < 1$, implies

$$\frac{1}{\|w\|_2^2}\int_{\mathbb{R}\times\Lambda} (1+x_1^2)^p|w(x)|^2\mathrm{d}x \leq \left(\frac{1}{\|w\|_2^2}\int_{\mathbb{R}\times\Lambda}(1+x_1^2)|w(x)|^2\mathrm{d}x\right)^p. \tag{6.5}$$

The result easily follows from (6.4) and (6.5). □

Since $\phi(t)$ is defined in term of moments of $\mathcal{B}v$ instead of v, see (5.1), we need one more lemma to apply the previous one.

6.3 LEMMA. *There is a finite constant C depending only on β and J so that for all t such that $|a(t)| \leq 1$ and $\|v(t)\|_2 \leq 1$,*

$$\|(1+x_1^2)^{1/2}v(t)\|_2^2 \leq C(\beta, J)\phi(t) \tag{6.6}$$

and

$$\big(\phi(t) - L^d\big) \leq C(\beta, J)\|x_1 v(t)\|_2^2. \tag{6.7}$$

Proof: Let $\mathcal{D}$ be the operator defined in (5.14)

$$\mathcal{D}w = \frac{1}{\beta(1-m_\beta^2)}w - J\star w\ .$$

As we have pointed out in Section 5 this operator is bounded and has a bounded inverse on $L^2(\mathbb{R}\times\Lambda)$. Then, using once more the rules for commuting convolution and multiplication by x_1, see (5.17), we have

$$\begin{aligned}\|x_1 v\|_2 \leq& \|\mathcal{D}^{-1}\|\|\mathcal{D}x_1 v\|_2 \\ =& \|\mathcal{D}^{-1}\|\|x_1\mathcal{D}v - (x_1 J)\star v\|_2 \\ \leq& \|\mathcal{D}^{-1}\|\big(\|x_1\mathcal{D}v\|_2 + \|(x_1 J)\|_1\|v\|_2\big) \\ \leq& C\big(\|x_1\mathcal{D}v\|_2 + \|v\|_2\big)\end{aligned}$$

for some constant C depending only on β and J. Next, taking into account (5.15)

$$\begin{aligned}\|x_1\mathcal{D}v\|_2 =& \|x_1\mathcal{B}v - x_1\tilde{g}v\|_2 \\ \leq& \|x_1\mathcal{B}v\|_2 + \|x_1\tilde{g}\|_\infty\|v\|_2,\end{aligned}$$

where, recall (5.16),

$$\tilde{g}(x_1) = \sigma(\bar{m}(x_1))^{-1} - \sigma(m_\beta)^{-1}, \qquad x_1 \in \mathbb{R}.$$

Since the hypothesis $|a(t)| \leq 1$, $\|x_1\tilde{g}\|_\infty \leq C$ for some constant C depending only on β and J. Thus we have

$$\|x_1\mathcal{D}v\|_2 \leq C\big(\|x_1\mathcal{B}v\|_2 + \|v\|_2\big).$$

Finally,

$$\sigma(m_\beta)\|x_1\mathcal{B}v\|_2^2 = \int_{\mathbb{R}\times\Lambda} x_1^2(\sigma(m_\beta)-\sigma(\bar m))\big(\mathcal{B}v\big)^2 \mathrm{d}x + \int_{\mathbb{R}\times\Lambda} x_1^2\sigma(\bar m)\big(\mathcal{B}v\big)^2\mathrm{d}x$$
$$\le \|x_1^2(\sigma(m_\beta)-\sigma(\bar m))\|_\infty \|\mathcal{B}v\|_2^2 + \int_{\mathbb{R}\times\Lambda} x_1^2\sigma(\bar m)\big(\mathcal{B}v\big)^2\mathrm{d}x$$

and the sup norm is again bounded by some constant $C(\beta, J)$ depending only on β and J by (1.10) and the hypothesis $|a(t)| \le 1$. Combining these estimates, one easily obtains

$$\int_{\mathbb{R}\times\Lambda}(1+x_1^2)v^2 dx \le C\left(\int_{\mathbb{R}\times\Lambda} x_1^2\sigma(\bar m)\big(\mathcal{B}v\big)^2\mathrm{d}x + \|v\|_2^2\right)$$

which yields (6.6) since $\|v(t)\|_2 \le 1$ by hypothesis. The proof of (6.7) simply reverses the above steps. With C changing from line to line, one easily obtains

$$\int_{\mathbb{R}\times\Lambda} x_1^2\sigma(\bar m)\big(\mathcal{B}v\big)^2\mathrm{d}x \le C\big(\|x_1\mathcal{B}v\|_2^2 + \|v\|_2^2\big)$$
$$\le C\big(\|x\mathcal{D}v\|_2^2 + \|v\|_2^2\big) \le C\big(\|\mathcal{D}x_1 v\|_2^2 + \|v\|_2^2\big)$$
$$\le C\big(\|x_1 v\|_2^2 + \|v\|_2^2\big)$$

and this complete the proof. □

Proof of Theorem 1.1 : First, fix $\epsilon > 0$, and then choose δ_1, κ_1 and ϵ_1 small enough so that both the following three estimates hold under the condition that

$$\mathcal{I}(m(t)) \le \epsilon_1\big[\mathcal{F}(m(t)) - \mathcal{F}(\overline{m})\big] \tag{6.8}$$

for all t such that $|a(t)| \le 1$, $\|v(t)\|_2 \le \delta_1$ and $\|v(t)\|_{W^{s+1,2}} \le \kappa_1$, $s > \frac{D}{2}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[\mathcal{F}\big(m(t)\big) - \mathcal{F}\big(\bar m\big)\big] \le -9(1-\epsilon)(1-\sigma(m_\beta))^2\frac{\big[\mathcal{F}\big(m(t)\big) - \mathcal{F}\big(\bar m\big)\big]^2}{\phi(t)} \tag{6.9}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(t) \le (1+\epsilon)4(1-\sigma(m_\beta))^2\big[\mathcal{F}(\bar m + v) - \mathcal{F}(\bar m)\big]. \tag{6.10}$$

This is possible by Theorems 4.1 and 5.1. By (5.2) of Theorem 5.1 and Lemma 8.1 in the appendix decreasing $\delta_1 > 0$ and $\kappa_1 > 0$ if need be, we have for a finite constant B and $c(\kappa_1)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(t) \le B\big[\mathcal{F}(\bar m + v) - \mathcal{F}(\bar m)\big] \tag{6.11}$$

and

$$\frac{1}{4}\gamma(L)\|v\|_2^2 \le \big[\mathcal{F}(\bar m + v) - \mathcal{F}(\bar m)\big] \le c(\kappa_1)\|v\|_2^2. \tag{6.12}$$

Next define δ_0 by

$$\delta_0 = \frac{\delta_1\sqrt{\gamma(L)}}{4(c(\kappa_1)+1)} \tag{6.13}$$

where $\gamma(L)$ and $c(\kappa_1)$ are the constants in (6.12). Theorem 2.2 applied with the values of δ_0, δ_1 and κ_1 fixed above, guarantees the existence of an $\epsilon_0 > 0$ and a t_0 so that when the initial data satisfies $\|m_0 - \bar{m}\|_2 \le \epsilon_0$, the solution to (1.1) satisfies

$$\|v(t_0)\|_2 \le \delta_0 \tag{6.14}$$

and

$$\|v(t)\|_{W^{s,2}} \le \kappa_1$$

for all $t \ge t_0$ such that $\|v(t)\|_2 \le \delta_1$. We have from Theorem 2.2 that

$$\int_{\mathbb{R}\times\Lambda} \big(x_1(m(x,t_0) - \bar{m}_0(x)\big)^2 \mathrm{d}x \le 2c_0$$

where c_0 is the constant specified in the hypotheses of Theorem 1.1. Clearly then,

$$\|x_1 v(t_0)\|_2^2 \le 2\big(\|x_1(m(t_0) - \bar{m}_0)\|_2^2 + 4m_\beta a(t_0)L^d\big).$$

By Theorem 2.3 we may suppose, further decreasing δ_1 if need be, that $4m_\beta a(t_0)L^d \le c_0$. Then

$$\|x_1 v(t_0)\|_2^2 \le 5c_0$$

and hence, by (6.7) of Lemma 6.3,

$$\phi(t_0) \le \tilde{c}_0 \tag{6.15}$$

where $\tilde{c}_0$ is a finite constant depending only on c_0, β, J and L. Hence, writing $f(t) = \big[\mathcal{F}(m(t)) - \mathcal{F}(\overline{m})\big]$, we have to control on the values of both $f(t_0)$ and $\phi(t_0)$ through (6.14) and (6.15).

The time t_0 is the time we have to wait for the smoothing properties of the equation (1.1) to regularize our data enough that the estimates above all hold, and it fixes the left end of the interval on which we shall work. To fix the right end, which we shall eventually show to be $+\infty$, define

$$T_0 = \min\{ \ \inf\{ \ t > t_0 \ | \ \|v(t)\|_2 \ge \delta_1/2 \ \} \ , \ \inf\{ \ t > t_0 \ | \ |a(t)| \ge 1 \ \} \ \} \ .$$

Then, uniformly on the interval (t_0, T_0), both of the estimates (6.11) and (6.12) holds. Moreover for those t in (t_0, T_0) such that (6.8) holds, one also has (6.9) and (6.10). Hence we have the following alternative:

One the one hand, in case

$$\begin{aligned} \mathcal{I}(m(t)) &\le \epsilon_1\big[\mathcal{F}(m(t)) - \mathcal{F}(\overline{m})\big] \\ \frac{\mathrm{d}}{\mathrm{d}t} f(t) &\le -\tilde{A}\frac{f(t)^2}{\phi(t)} \\ \frac{\mathrm{d}}{\mathrm{d}t}\phi(t) &\le \tilde{B} f(t) \end{aligned} \tag{6.16}$$

where $\tilde{A}$ and $\tilde{B}$ by

$$\begin{aligned} \tilde{A} &= 9(1-\epsilon)(1-\sigma(m_\beta))^2 \\ \tilde{B} &= 4(1+\epsilon)(1-\sigma(m_\beta))^2. \end{aligned}$$

On the other hand, in case

$$\mathcal{I}(m(t)) \geq \frac{\epsilon_1}{2}\big[\mathcal{F}(m(t)) - \mathcal{F}(\overline{m})\big],$$
$$\frac{\mathrm{d}}{\mathrm{d}t}f(t) \leq -\frac{\epsilon_1}{2}f(t)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(t) \leq Bf(t).$$

In the application of the system of differential inequalities (1.35), it is the *ratio* of the constants A and B that determines the exponent q, see Theorem 5.1 of [3]. Indeed,

$$q = \frac{(A/B)}{(A/B)+1}.$$

The values of A and B themselves can be changed, keeping this ratio fixed, simply by rescaling the time t. Therefore we define

$$A = \frac{\tilde{A}}{\tilde{B}}B$$

and observe that

$$\frac{\epsilon_1}{2}f(t) = \frac{\epsilon_1}{2Af(t)}Af(t)^2 \geq \frac{\epsilon_1}{2Af(t)}A\frac{f(t)^2}{\phi}$$

since $\phi(t) \geq 1$ by definition. Now, by (6.12) we may further decrease $\delta_1 > 0$ if need be to ensure that

$$\frac{\epsilon_1}{2Af(t)} \geq 1\ .$$

Doing so, we have that in case

$$\begin{aligned}
&\mathcal{I}(m(t)) \geq \frac{\epsilon_1}{2}\big[\mathcal{F}(m(t)) - \mathcal{F}(\overline{m})\big]\\
&\frac{\mathrm{d}}{\mathrm{d}t}f(t) \leq -A\frac{f(t)^2}{\phi(t)}\\
&\frac{\mathrm{d}}{\mathrm{d}t}\phi(t) \leq Bf(t)
\end{aligned} \tag{6.17}$$

where

$$\frac{A}{B} = \frac{\tilde{A}}{\tilde{B}}\ .$$

Now suppose that at t_0,

$$\mathcal{I}(m(t_0)) > \frac{\epsilon_1}{2}\big[\mathcal{F}(m(t_0)) - \mathcal{F}(\overline{m})\big]\ .$$

Define

$$t_1 = \inf\{\ t > t_0 \mid \mathcal{I}(m(t)) \leq \frac{\epsilon_1}{2}\big[\mathcal{F}(m(t)) - \mathcal{F}(\overline{m})\big]\}\ ,$$
$$t_2 = \inf\{\ t > t_1 \mid \mathcal{I}(m(t)) \geq \epsilon_1\big[\mathcal{F}(m(t)) - \mathcal{F}(\overline{m})\big]\}\ ,$$
$$t_3 = \inf\{\ t > t_2 \mid \mathcal{I}(m(t)) \leq \frac{\epsilon_1}{2}\big[\mathcal{F}(m(t)) - \mathcal{F}(\overline{m})\big]\}\ ,$$

and so forth. We follow the usual convention that if there is no $t < T_0$ satisfying the condition, the infimum is set to be T_0. Notice that since $\mathcal{I}(m(t))$ and $\mathcal{F}(m(t))$ are continuous function of t, $t_3 > t_2 > t_1 > t_0$. The sequence of times t_j can have no limit point except possibly T_0, since at such a limit point, the continuous function $\mathcal{I}(m(t))$ would take on two values.

If at t_0,

$$\mathcal{I}(m(t_0)) \le \frac{\epsilon_1}{2}\left[\mathcal{F}(m(t_0)) - \mathcal{F}(\overline{m})\right] ,$$

one would define

$$t_1 = \inf\{\ t > t_0 \mid \mathcal{I}(m(t)) \ge \epsilon_1\left[\mathcal{F}(m(t)) - \mathcal{F}(\overline{m})\right]\} ,$$

and then proceed as above with the opposite alternation.

In either case, one produces a sequence of intervals $[t_j, t_{j+1}]$ on which (6.16) and (6.17) hold in successive alternation. On each of these intervals, we may apply Theorem 5.1 of [3]. To put all of these estimates together in a transparent way, we rescale the intervals on which (6.17) holds. Supposing that (6.17) holds on $[t_0, t_1]$, define

$$s(t) = \frac{A}{\tilde{A}}(t - t_0) \qquad \text{and} \qquad s_1 = \frac{A}{\tilde{A}}(t_1 - t_0)$$

for $t_0 < t < t_1$,

$$s(t) = s_1 + (t - t_1) \qquad \text{and} \qquad s_2 = s_1 + (t_2 - t_1)$$

for $t_1 < t < t_2$,

$$s(t) = s_2 + \frac{A}{\tilde{A}}(t - t_2) \qquad \text{and} \qquad s_3 = \frac{A}{\tilde{A}}(t_3 - t_2)$$

for $t_2 < t < t_3$, and so forth in alternation. It follows that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s} f(s) &\le -\tilde{A}\frac{f(s)^2}{\phi(s)} \\ \frac{\mathrm{d}}{\mathrm{d}s} \phi(s) &\le \tilde{B} f(s) \end{aligned}$$

for all s with $0 \le s \le s(T_0)$. By Theorem 5.1 of [3],

$$\begin{aligned} f(s) &\le f(0)^{1-q}\phi(0)^q\left(\frac{\phi(0)}{f(0)} + (\tilde{A}+\tilde{B})s\right)^{-q} \\ \phi(s) &\le f(0)^{1-q}\phi(0)^q\left(\frac{\phi(0)}{f(0)} + (\tilde{A}+\tilde{B})s\right)^{1-q} \end{aligned}$$

where

$$q = \frac{\tilde{A}}{\tilde{A}+\tilde{B}}$$

and where $f(0)$ and $\phi(0)$ are bounded by (6.14) and (6.15). Now, for any $\delta > 0$, we can choose ϵ so that

$$\frac{\tilde{A}}{\tilde{A}+\tilde{B}} = \frac{9}{13} - \delta .$$

We shall now show that for δ small enough, $|a(t)| \le 1/2$ for all $t \le T_0$. Then by Lemmas 6.2, 6.3 and estimate (6.12),

$$\begin{aligned} \|v(s)\|_1 \le & C(\delta, L)\|(1 + x_1^2)^{1/2}v(s)\|_2^{(1+\delta)/2}\|v(s)\|_2^{(1-\delta)/2} \\ \le & C(\delta, L)C(\beta, J)^{(1+\delta)/4}\left(\frac{4}{\gamma(L)}\right)^{(1-\delta)/4}\phi(s)^{(1+\delta)/4}f(s)^{(1-\delta)/4} \\ \le & C(\delta, L)C(\beta, J)^{(1+\delta)/4}\left(\frac{4}{\gamma(L)}\right)^{(1-\delta)/4}f(0)^{(1-q)/4}\phi(0)^{q/4}\left(\frac{\phi(0)}{f(0)} + (\tilde{A}+\tilde{B})s\right)^{(1-2q+\delta)/4} . \end{aligned} \tag{6.18}$$

The right hand side is decreasing for $\delta < 5/26$, and we now choose δ to be at least this small. Moreover, the value at $s = 0$ can be made arbitrarily small by decreasing δ_1. We now do so, if need be, to ensure that

$$\|v(s)\|_1 \leq m_\beta/2$$

for all $s \leq s(T_0)$. Hence, for a, as in (1.12), by Lemma 6.1, we have

$$|a(t) - a| \leq 1/4\frac{1}{L^d}$$

for all $t \leq T_0$. But then by Lemma 6.1 again, this implies that $|a(t)| < \frac{1}{2L^d}$ for all $t \leq T_0$. Hence if $T_0 < \infty$, it is because $\|v(T_0)\|_2 = \delta_1/2$. But since (6.12) is still valid with the same constants on the closed interval $[t_0, T_0]$, and since the excess free energy is monotone decreasing, we have

$$\begin{aligned}
\frac{\delta_1^2}{4} &= \|v(T_0)\|_2^2 \\
&\leq \frac{4}{\gamma(L)}\big(\mathcal{F}(\bar{m} + v(T_0)) - \mathcal{F}(\bar{m})\big) \leq \frac{4}{\gamma(L)}\big(\mathcal{F}(\bar{m} + v(t_0)) - \mathcal{F}(\bar{m})\big) \\
&\leq c(\kappa_1)\frac{4}{\gamma(L)}\|v(t_0)\|_2^2 \leq c(\kappa_1)\frac{4}{\gamma(L)}\delta_0^2.
\end{aligned}$$

This contradicts (6.13), and hence $T_0 < \infty$ is not possible. We now clearly have (1.13) since

$$s(t) \geq \min\left\{\frac{A}{\tilde{A}}\ ,\ 1\right\}(t - t_0)\ .$$

Also from this and (6.18), we have

$$\|m(t) - \bar{m}_{a(t)}\|_1 \leq c_2(1 + c_1 t)^{-(5/52-\delta)}\ .$$

But

$$\begin{aligned}
\|m(t) - \bar{m}_a\|_1 \leq& \|m(t) - \bar{m}_{a(t)}\|_1 + \|\bar{m}_a - \bar{m}_{a(t)}\|_1 \\
=& \|m(t) - \bar{m}_{a(t)}\|_1 + 2m_\beta L^d|a - a(t)| \\
\leq& 2\|m(t) - \bar{m}_{a(t)}\|_1
\end{aligned}$$

by (6.3). Hence (1.14) follows as well. □

7 Proof of smoothing estimates

The main goal of this section is to deduce the regularity properties stated in Theorem 2.2 for the derivatives of the solution $m(t)$ of equation (1.1) that starts sufficiently close in the L^2 norm to $\bar{m}_b$ for some b:

The proof depends on several intermediate results concerning the evolution of $m(t) - \bar{m}_b$ and its derivatives for *fixed* b. To simplify the notation, we will write $\bar{m}$ instead of $\bar{m}_b$, so one should keep in mind that in this section $\bar{m}$ is not necessarily the antisymmetric, increasing instanton.

Let $v = m - \bar{m}$. Notice we are *not* assuming that $\bar{m} = \bar{m}_{a(t)}$, so this definition of $v(t)$ differs slightly from the one used in the rest of the paper. However, for most of this section, it is the most convenient notation. It saves us from keeping $\dot{a}(t)$ terms throughout the many calculations that follow. From the evolution equation

$$\frac{\partial m}{\partial t} = \nabla \cdot \left(\nabla m - \beta(1 - m^2) J * \nabla m \right)$$

and the eigenvalue equation

$$\beta(1 - \bar{m}^2) J * \bar{m}' = \bar{m}'$$

we deduce the following evolution equation for v

$$\begin{aligned}\frac{\partial v}{\partial t} &= \nabla \cdot \left(\nabla v - \beta(1 - \bar{m}^2) \nabla J * v \right) \\ &\quad + \beta \nabla \cdot e_1 \left(v(v + 2\bar{m}) J * \bar{m}' \right) + \beta \nabla \cdot \left(v(v + 2\bar{m}) \nabla J * v \right) .\end{aligned} \tag{7.1}$$

Here and in what follows, e_1 denote the unit vector in the x_1 direction. Define

$$\Psi := \beta \left(e_1 J \star \bar{m}' + \nabla J \star v \right) \qquad \text{and} \qquad \Phi := 2\bar{m}\Psi \ .$$

We can write (7.1) as

$$\frac{\partial v}{\partial t} = \nabla \cdot \left(\nabla v - \beta(1 - \bar{m}^2) \nabla J * v \right) + \nabla \cdot \left(v^2 \Psi \right) + \nabla \cdot (v\Phi) \ . \tag{7.2}$$

Since Ψ and hence Φ depend on v, both the second and the third terms on the right in (7.2) are nonlinear in v. However, because of the convolution with J, the dependence on v that enters through these terms is harmless as far as smoothness of v is concerned. For any multindex α, denote by D^α the corresponding differentiation operator. Since both $\|m\|_\infty \le 1$ and $\|\bar{m}\|_\infty \le m_\beta \le 1$, $\|v\|_\infty \le 2$, we have

$$\|D^\alpha (J * v)\|_\infty = \|(D^\alpha J) * v\|_\infty \le 2\|(D^\alpha J)\|_\infty \ ,$$

independent of v. Then, since $\bar{m}$ is smooth, there exist finite constants C_α depending only on J and α so that

$$\|D^\alpha \Psi\|_\infty \le C_\alpha \qquad \text{and} \qquad \|D^\alpha \Phi\|_\infty \le C_\alpha \ . \tag{7.3}$$

Our first goal is to study the smoothing properties of (7.2). We shall show that on any interval of time on which $\|v(t)\|_2$ stays bounded, solutions immediately develop derivatives of all orders even if the initial data is not smooth. To use this, we need to know that $\|v(t)\|_2$ stays bounded in some interval of the origin. Later of course we shall see that if $\|v(0)\|_2$ is small enough, this holds globally in time. In the next Lemma and in what follows, C will denote a constant depending only on J and β but otherwise changing from line to line.

7.1 LEMMA. *Let v be a solution of* (7.1). *Then there is a finite constant C depending only on J and β so that for all $t > 0$,*

$$\|v(t)\|_2^2 \le e^{Ct} \|v(0)\|_2^2 \ .$$

Proof: From (7.2) we have

$$\begin{aligned}
&\frac{d}{dt}\|v\|_2^2 = 2\int v\frac{\partial v}{\partial t}\mathrm{d}x = \\
&-2\|\nabla v\|_2^2 + 2\beta\int \nabla v\cdot\big((1-\bar{m}^2)\nabla J * v\big)\,\mathrm{d}x \\
&-2\int \nabla v\cdot\big(v^2\Psi + v\Phi\big)\,\mathrm{d}x \\
&\le -2\|\nabla v\|_2^2 + C\|\nabla v\|_2\|v\|_2
\end{aligned}$$

where in the last line we have used the bound $\|v\|_\infty \le 2$. Completing the square leads to $\frac{d}{dt}\|v\|_2^2 \le C\|v\|_2^2$, and the result follows directly. □

7.2 LEMMA. *Let v be a solution of* (7.1) *and suppose that for some finite δ and positive T_δ,*

$$\|v(t)\|_2^2 \le \delta \qquad \text{for all} \quad t \le T_\delta\ .$$

Then

$$\|\nabla v(t)\|_2^2 \le \frac{\delta}{2t} + C\delta \qquad \text{for all} \quad t \le T_\delta$$

where C is a constant depending only on J and β.

Proof: We begin with the L^2 norm of the first derivatives.

$$\begin{aligned}
&\frac{d}{dt}\|\nabla v\|_2^2 = -2\int \Delta v\frac{\partial v}{\partial t}\mathrm{d}x \\
&= -2\|\Delta v\|_2^2 + 2\beta\int \Delta v\nabla\cdot\big((1-\bar{m}^2)\nabla J * v\big)\,\mathrm{d}x \\
&+ 2\int \Delta v\nabla\cdot\big(v^2\Psi + v\Phi\big)\,\mathrm{d}x.
\end{aligned} \tag{7.4}$$

By the Schwarz inequality, this is no more than

$$-2\|\Delta v\|_2^2 + 2\|\Delta v\|_2\left[\beta\|\nabla\cdot(1-\bar{m}^2)\nabla J * v)\|_2 + \|\nabla\cdot v^2\Psi\|_2 + \|\nabla\cdot(v\Phi)\|_2\right]$$

Now,

$$\|\nabla\cdot\big((1-\bar{m}^2)\nabla J * v\big)\,\|_2 \le \|2\bar{m}'\nabla J * v\|_2 + \|(\Delta J) * v\|_2 \le C\|v\|_2. \tag{7.5}$$

Then by (7.3) and the *a-priori* estimate $\|v\|_\infty \le 2$,

$$\|\nabla\cdot v^2\Psi\|_2 \le C(\|\nabla v\|_2 + \|v\|_2) \qquad \text{and} \qquad \|\nabla\cdot(v\Phi)\|_2 \le C(\|\nabla v\|_2 + \|v\|_2)\ .$$

Combining this with (7.5), we obtan

$$\frac{d}{dt}\|\nabla v\|_2^2 \le -2\|\Delta v\|_2^2 + \|\Delta v\|_2 C(\|\nabla v\|_2 + \|v\|_2).$$

We use half of our dissipative term $-2\|\Delta v\|_2^2$ to eliminate reference to ∇v and Δv in the positive part of this bound. To do so, note that by the Schwarz inequality

$$\|\nabla v\|_2^2 = -\int(\Delta v)v\mathrm{d}x \le \|\Delta v\|_2\|v\|_2\ .$$

Therefore, $\|\Delta v\|_2\|\nabla v\|_2 \le \|\Delta v\|_2^{3/2}\|v\|_2^{1/2}$. By the arithmetic–geometric mean inequality,

$$\|\Delta v\|_2^{3/2}\|v\|_2^{1/2} \le \frac{3\epsilon}{4}\|\Delta v\|_2^2 + \frac{1}{4\epsilon^3}\|v\|_2^2$$

for any $\epsilon > 0$. Even more simply $\|\Delta v\|_2\|v\|_2 \le \frac{\epsilon}{2}\|\Delta v\|_2^2 + \frac{1}{2\epsilon}\|v\|_2^2$, and thus,

$$\frac{d}{dt}\|\nabla v\|_2^2 \le \left(-2 + \frac{\epsilon 5C}{4}\right)\|\Delta v\|_2^2 + \frac{3C}{4\epsilon}\|v\|_2^2.$$

Again by the Schwarz inequality

$$\|\Delta v\|_2^2 \ge \frac{\|\nabla v\|_2^4}{\|v\|_2^2} .$$

Using this, and choosing ϵ so that $5C\epsilon \le 4$, one finally obtains

$$\frac{d}{dt}\|\nabla v\|_2^2 \le -\frac{\|\nabla v\|_2^4}{\|v\|_2^2} + C\|v\|_2^2 .$$

Now by hypothesis, for all times t under consideration, we have the bound $\|v\|_2^2 \le \delta$. Letting $x(t)$ denote the value of $\|\nabla v\|_2^2$ at time t, we then have the differential inequality

$$\frac{d}{dt}x \le -\frac{x^2}{\delta} + C\delta .$$

Introducing $y = 1/x$, one obtains a differential inequality of the form

$$\frac{d}{dt}y \ge \frac{1}{\delta} - C\delta y^2.$$

Now let y_*

$$y_* = \frac{1}{\sqrt{2C\delta}} ,$$

so that for $0 \le y \le y_*$, $y' \ge \frac{1}{2\delta}$, and for $y_* \le y \le \sqrt{2}y_*$, we have at least that $y' \ge 0$. This means that y increases with rate at least $1/2\delta$ until y_* is reached. At this point it is still increasing, and it continues to increase until $\sqrt{2}y_*$, and it never again passes below this value, and hence never again below y_* either. Therefore

$$y(t) \ge \min\{t/2\delta\ ,\ y_*\} \qquad \text{for all} \quad t > 0 ,$$

and hence

$$x(t) \le \max\{2\delta/t\ ,\ \sqrt{2C}\delta\} \le 2\delta/t + \sqrt{2C}\delta \qquad \text{for all} \quad t > 0 .$$

This proves the stated assertion about $\|\nabla v\|_2^2$. □

We next consider the second derivatives, where a new feature emerges.

7.3 LEMMA. *Let v be a solution of* (7.1) *and suppose that for some $\delta > 0$ and $T_\delta > 0$,*

$$\|v(t)\|_2^2 + \|\nabla v(t)\|_2^2 \le \delta \qquad \text{for all} \quad t \le T_\delta .$$

Then

$$\|\Delta v(t)\|_2^2 \le \frac{\delta}{2t} + C\delta \qquad \text{for all} \quad t \le T_\delta$$

where C is a constant depending only on J and β.

Proof:

$$\frac{d}{dt}\|\Delta v\|_2^2 = 2\int [(-\Delta)^2 v]\frac{\partial v}{\partial t}\mathrm{d}x = -2\|\nabla\Delta v\|_2^2 + 2\beta\int [\nabla\Delta v]\cdot\nabla\mathrm{div}\left((1-\bar{m}^2)\nabla J * v\right)\mathrm{d}x$$
$$+ 2\int [\nabla\Delta v]\cdot\nabla\mathrm{div}\left[\left(v^2\Psi\right) + (v\Phi)\right]\mathrm{d}x. \tag{7.6}$$

By the Schwarz inequality, this is no more than

$$-2\|(-\Delta)^{3/2}v\|_2^2 + C\|(-\Delta)^{3/2}v\|_2\left[\|\Delta((1-\bar{m}^2)J*\nabla v)\|_2 + \|\Delta\left(v^2\Psi\right)\|_2 + \|\Delta\left(v\Phi\right)\|_2\right]. \tag{7.7}$$

The estimation of this proceeds as before, but with one new feature: Now there is a contribution of the form

$$\Delta v^2 = 2v\Delta v + 2|\nabla v|^2\ .$$

As before, we can use the bound $\|v\|_\infty \le 2$ to conclude that $\|v\Delta v\|_2 \le 2\|\Delta v\|_2$. However

$$\||\nabla v|^2\|_2 = \|\nabla v\|_4^2\ .$$

Thus, using the elementary estimate $\|\nabla v\|_2 \le \|v\|_2 + \|\Delta v\|_2$, we bound the quantity in (7.7) by

$$-2\|(-\Delta)^{3/2}v\|_2^2 + C\|(-\Delta)^{3/2}v\|_2\left[\|\Delta((1-\bar{m}^2)J*\nabla v)\|_2 + \|\Delta\left(v^2\Psi\right)\|_2 + \|\Delta\left(v\Phi\right)\|_2\right] \le$$
$$-2\|(-\Delta)^{3/2}v\|_2^2 + C\|(-\Delta)^{3/2}v\|_2\left(\|v\|_2 + \|\Delta v\|_2 + \||\nabla v|^2\|_2\right)\ . \tag{7.8}$$

To handle $\||\nabla v|^2\|_2$, we compute

$$\begin{aligned}\int|\nabla v|^4\mathrm{d}x &= \int\nabla v\cdot\nabla v|\nabla v|^2\mathrm{d}x\\ &= -\int v[(\Delta v)|\nabla v|^2 - 2D^2v(\nabla v,\nabla v)]\mathrm{d}x\\ &\le C\|D^2v\|_2\||\nabla v|^2\|_2 \le C\|\Delta v\|_2\left(\int|\nabla v|^4\mathrm{d}x\right)^{1/2}\ .\end{aligned}$$

That is,

$$\|\nabla v\|_4^2 \le C\|\Delta v\|_2\ . \tag{7.9}$$

Using this in (7.8), our estimate for right hand side of (7.6) becomes

$$\frac{d}{dt}\|\Delta v\|_2^2 \le -2\|(-\Delta)^{3/2}v\|_2^2 + C\|(-\Delta)^{3/2}v\|_2(\|v\|_2 + \|\Delta v\|_2).$$

Now by Schwarz,

$$\|\Delta v\|_2^2 = \langle(-\Delta)^{3/2}v, (-\Delta)^{1/2}v\rangle \le \|(-\Delta)^{3/2}v\|_2\|\nabla v\|_2. \tag{7.10}$$

Therefore

$$\frac{d}{dt}\|\Delta v\|_2^2 \le -2\|(-\Delta)^{3/2}v\|_2 + \|(-\Delta)^{3/2}v\|_2 C\left(\|(-\Delta)^{3/2}v\|_2^{1/2}\|\nabla v\|_2^{1/2} + \|v\|_2\right)\ .$$

By the same type of arithmetic-geometric mean argument we made earlier, we obtain

$$\frac{d}{dt}\|\Delta v\|_2^2 \le -\|(-\Delta)^{3/2}v\|_2^2 + C\left(\|\nabla v\|_2^2 + \|v\|_2^2\right)\ ,$$

and then by (7.10),

$$\frac{d}{dt}\|\Delta v\|_2^2 \le -\frac{\|\Delta v\|_2^4}{\|\nabla v\|_2^2} + C\left(\|\nabla v\|_2^2 + \|v\|_2^2\right)$$
$$\le -\frac{\|\Delta v\|_2^4}{\delta} + C\delta.$$

The analysis of this differential inequality proceeds exactly as with (7.4) in the previous lemma. □

Up to this point, our analysis had not depended in any significant way on the dimension. To proceed to higher smoothness estimates, we need to take the dimension into account: One last new feature enters in adapting our strategy for proving smoothness to higher derivatives.

7.4 LEMMA. *For dimension $D \le 3$, let v be a solution of* (7.1) *and suppose that for some $\delta > 0$ and $T_\delta > 0$,*

$$\|v(t)\|_2^2 + \|\nabla v(t)\|_2^2 + \|\Delta v(t)\|_2^2 \le \delta \qquad \text{for all} \quad t \le T_\delta \ .$$

Then

$$\|\nabla\Delta v(t)\|_2^2 \le \frac{\delta}{2t} + C\delta \qquad \text{for all} \quad t \le T_\delta$$

where C is a constant depending on J, β and the dimension.

Proof:

$$\frac{d}{dt}\|\nabla\Delta v\|_2^2 = -2\int(-\Delta)^3 v\frac{\partial v}{\partial t}\mathrm{d}x = -2\|(-\Delta)^2 v\|_2^2$$
$$+ 2\beta\int[(-\Delta)^2 v]\Delta\mathrm{div}\left((1-\bar{m}^2)\nabla J * v\right)\mathrm{d}x$$
$$+ 2\int[(-\Delta)^2 v]\left[\Delta\mathrm{div}\left(v^2\Psi\right) + \Delta\mathrm{div}\left(v\Phi\right)\right]\mathrm{d}x$$

and again by the Schwarz inequality, this is no more than

$$-2\|(-\Delta)^2 v\|_2^2 + 2\beta\|(-\Delta)^2 v\|_2\left[\|\Delta\mathrm{div}((1-\bar{m}^2)J * \nabla v)\|_2 + \|\Delta\mathrm{div}\left(v^2\Psi\right)\|_2 + \|\Delta\mathrm{div}\left(v\Phi\right)\|_2\right].$$

As before, in estimating $\|\Delta\mathrm{div}((1-\bar{m}^2)J * \nabla v)\|_2$, we may let all derivatives fall on J to obtain the bound $C\|v\|_2$. Also, by (7.3), we have that

$$\|\Delta\mathrm{div}\left(v\Phi\right)\|_2 \le C\left[\|v\|_2 + \|\nabla v\|_2 + \|\Delta v\|_2 + \|\nabla\Delta v\|_2\right] \ .$$

However, to estimate $\|\Delta\mathrm{div}\left(v^2\Psi\right)\|_2$, we need a bound on

$$\|v^2\|_2 + \|v|\nabla v|\|_2 + \||\nabla v|^2\|_2 + \|v|\nabla\Delta v|\|_2 + \||\nabla v||\Delta v|\|_2$$

for the first, second and fourth terms we may use the *a-priori* bound $\|v\|_\infty \le 2$. We have already estimated the third term in (7.9). The term that forces us to make dimension dependent estimates is $\||\nabla v||\Delta v|\|_2$. The strategy that led to (7.9)does not work here. Instead, we use the Sobolev embedding estimate

$$\|\nabla v\|_\infty \le C(\|\nabla\Delta v\|_2 + \|\nabla v\|_2) \ ,$$

valid in dimensions 2 and 3. We then use the bound $\|\Delta v\|_2 \le \delta \le 1$ to obtain

$$\|\Delta \mathrm{div}\left(v^2\Psi\right)\|_2 \le C\left[\|v\|_2 + \|\nabla v\|_2 + \|\Delta v\|_2 + \|\nabla\Delta v\|_2\right] \ .$$

Finally, we estimate

$$\|\nabla\Delta v\|_2^2 = -\int(-\Delta)^2 v\Delta v dx \le \|(-\Delta)^2 v\|_2\|\Delta v\|_2 \le \delta\|\Delta^2 v\|_2$$

to obtain

$$\frac{d}{dt}\|\nabla\Delta v\|_2^2 \le -2\|(-\Delta)^2 v\|_2^2 + C[\delta + \|(-\Delta)^2 v\|_2] \ .$$

The analysis of this differential inequality proceeds as before. □

We now come to the general case.

7.5 LEMMA. *For dimension $D \le 3$, let v be a solution of* (7.1) *and suppose that for some $\delta > 0$ and $T_\delta > 0$, and $k \in \mathbb{N}$*

$$\sum_{j=0}^{k} \|(-\Delta)^{j/2} v(t)\|_2^2 \le \delta \qquad \text{for all} \quad t \le T_\delta \ .$$

Then

$$\|(-\Delta)^{(k+1)/2} v(t)\|_2^2 \le \frac{\delta}{2t} + C\delta \qquad \text{for all} \quad t \le T_\delta$$

where C is a constant depending on J, β and the dimension.

Proof:

$$\frac{d}{dt}\|(-\Delta)^{(k+1)/2} v\|_2^2 = -2\int(-\Delta)^{k+1} v\frac{\partial v}{\partial t}\mathrm{d}x = -2\|(-\Delta)^{(k+2)/2} v\|_2^2$$
$$+ 2\beta\int[(-\Delta)^{(k+2)/2} v](-\Delta)^{k/2}\mathrm{div}\left((1-\bar{m}^2)\nabla J * v\right)\mathrm{d}x$$
$$+ 2\int[(-\Delta)^{(k+2)/2} v]\left[(-\Delta)^{k/2}\mathrm{div}\left(v^2\Psi\right) + (-\Delta)^{k/2}\mathrm{div}\left(v\Phi\right)\right]\mathrm{d}x.$$

As above, we have

$$\|(-\Delta)^{k/2}\mathrm{div}\left((1-\bar{m}^2)\nabla J * v\right)\|_2 \le C\delta$$

and

$$\|(-\Delta)^{k/2}\mathrm{div}\left(v\Phi\right)\|_2 \le C\delta \ .$$

Also,

$$\|(-\Delta)^{k/2}\mathrm{div}\left(v^2\Psi\right)\|_2 \le C\sum_{j+\ell\le k+1 \ ,j,k \ \ge 0} \||(-\Delta)^{j/2} v||(-\Delta)^{\ell/2} v|\|_2 \ .$$

Since we have already proved the result for $k \le 3$, we may suppose that $k+1 \ge 4$. For $k+1 \ge 4$, whenever two non-negative integers j and ℓ satisfy $j + \ell \le k+1$, at least one of the integers is no greater than $k-1$. Let us suppose that $j \le k-1$. Then we have the sobolev embedding inequality

$$\|(-\Delta)^{j/2} v\|_\infty \le C\left(\|(-\Delta)^{(k+1)/2} v\|_2 + \|(-\Delta)^{j/2} v\|_2\right) \ ,$$

valid in dimensions 2 and 3, Thus, using the fact that $\delta \le 1$, we have

$$\|(-\Delta)^{k/2}\mathrm{div}\left(v^2\Psi\right)\|_2 \le C\left(\delta + \|(-\Delta)^{(k+1)/2}v\|_2\right) \ .$$

Next, by Schwarz

$$\|\|(-\Delta)^{(k+1)/2}v\|_2^2 \le \|\|(-\Delta)^{(k+2)/2}v\|_2\|\|(-\Delta)^{k/2}v\|_2^2 \ ,$$

and so

$$\begin{aligned}
&\frac{d}{dt}\|(-\Delta)^{(k+1)/2}v\|_2^2 \le -2\|(-\Delta)^{(k+2)/2}v\|_2^2 + C\left(\delta + \|(-\Delta)^{(k+2)/2}v\|_2\right)\\
&\le -\|(-\Delta)^{(k+2)/2}v\|_2^2 + C\delta \le -\frac{\|(-\Delta)^{(k+1)/2}v\|_2^4}{\|(-\Delta)^{k/2}v\|_2^2} + C\delta\\
&\le -\frac{\|(-\Delta)^{(k+1)/2}v\|_2^4}{\delta} + C\delta.
\end{aligned}$$

Thus, $x(t) := \|(-\Delta)^{(k+1)/2}v\|_2^2$ satisfies the differential inequality (7.4), and the result now follows as in the proof of Lemma 7.2. □

We are now ready to prove the Theorem 2.2.

Proof of Theorem 2.2: We proceed by induction on k. We shall first show that with b kept fixed, for any $\epsilon > 0$, if δ is sufficiently small, then for any $t_0 > 0$, and any $k \in \mathbb{N}$, there exists T_0 so that

$$\|(-\Delta)^{k/2)}v(t)\|_2^2 \le \epsilon \quad \text{for all} \quad t_0 \le t \le T_0 \ .$$

For $k = 1$, this result follows from Lemma 7.2. Suppose that $k \ge 2$, and the result has been proved for $k-1$. Then by Lemma 7.5 we have this result for k as well. Now, recall that $a(t)$ is defined by

$$\|m(t) - \bar{m}_{a(t)}\|_2 \le \|m(t) - \bar{m}_b\|_2 \quad \text{for all} \quad b \in \mathbb{R} \ .$$

As we have shown, for δ small enough, this uniquely determines $a(t)$. Moreover, as long as $\|m(t) - \bar{m}_{a(0)}\|_2$ is small, so is $a(t) - a(0)$. Then, in the notation of this section,

$$m(t) - \bar{m}_{a(t)} = [m(t) - \bar{m}_{a(0)}] + [\bar{m}_{a(0)} - \bar{m}_{a(t)}] = v(t) + [\bar{m}_{a(0)} - \bar{m}_{a(t)}] \ .$$

Thus

$$\|(-\Delta)^{k/2)}(m(t) - \bar{m}_{a(t)})\|_2 \le \|(-\Delta)^{k/2)}v(t)\|_2^2 + \|(-\Delta)^{k/2)}[\bar{m}_{a(t)} - \bar{m}_{a(0)}]\|_2^2 \ .$$

Next note that

$$\|(-\Delta)^{k/2)}[\bar{m}_{a(t)} - \bar{m}_{a(0)}]\|_2^2 \le C_k|a(t) - a(0)| \ .$$

(Note: The constant C_k contains a multiple of L^d, so the constant also depend on L, which is fixed. This is the first place L enters.) By Theorem 2.3, $t \mapsto a(t)$ is Lipschitz, and so for any $\epsilon > 0$, there is an s_ϵ so that $C_k|a(t) - a(0)| \le \epsilon/2$ provided $t \le s_\epsilon$. Then, by what has been proved above, for any $t_0 < s_\epsilon/2$, there is a $\delta > 0$ so that if $\|v(t)\|_2 \le \delta$ for all $0 \le t \le T_0$, then $\|(-\Delta)^{k/2)}v(t)\|_2^2 \le \epsilon/2$. Combining results, we have that

$$\|(-\Delta)^{k/2)}[m(t) - \bar{m}_{a(t)}]\|_2^2 \le \epsilon$$

for all $t_0 \le t \le s_\epsilon$. The same analysis shows that

$$\|(-\Delta)^{k/2}[m(t) - \bar{m}_{a(t)}]\|_2^2 \le \epsilon$$

for all $s_{\epsilon/2} \leq t \leq \min\{3s_{\epsilon/2}, T_0\}$, and inductively,

$$\|(-\Delta)^{k/2}[m(t) - \bar{m}_{a(t)}]\|_2^2 \leq \epsilon$$

for all $js_{\epsilon/2} \leq t \leq \min\{(j+2)s_{\epsilon/2}, T_0\}$. That is, in steps of fixed length s_ϵ we cover the interval (t_0, T_0).

Finally, we prove the assertion in the Theorem concerning moments. Namely, in the notation being used here, it suffices to show that that for some constant C, which may be made small by choosing $\|v_0\|_2$ small,

$$\int_{\mathbb{R}\times\Lambda} x_1^2 v^2(t)\mathrm{d}x \leq e^{Ct} \int_{\mathbb{R}\times\Lambda} x_1^2 v^2(0)\mathrm{d}x \ .$$

Differentiating the left side, we find

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}\times\Lambda} x_1^2 v^2(t)\mathrm{d}x &= 2\int_{\mathbb{R}\times\Lambda} x_1^2 v(t)\frac{\partial}{\partial t}v(t)\mathrm{d}x \\
&= 2\int_{\mathbb{R}\times\Lambda} x_1^2 v\nabla\cdot\left[\left(\nabla v - \beta(1-\bar{m}^2)\nabla J * v\right) + v^2\Psi + v\Phi\right]\mathrm{d}x \\
&= -2\int_{\mathbb{R}\times\Lambda} x_1^2 v^2 \mathrm{d}x \qquad (7.11)\\
&+ 2\int_{\mathbb{R}\times\Lambda} x_1^2\left[\nabla v\cdot\beta\left(1-\bar{m}^2\right)\nabla J * v\right) - v^2\nabla v\cdot\Psi - v\nabla v\cdot\Phi\right]\mathrm{d}x \qquad (7.12)\\
&- 4\int_{\mathbb{R}\times\Lambda} x_1 v\left[\left(\nabla_1 v - \beta(1-\bar{m}^2)\nabla_1 J * v\right) + v^2\Psi_1 + v\Phi_1\right]\mathrm{d}x\ . \qquad (7.13)
\end{aligned}$$

The term in (7.11) has a sign that allows us to ignore it. The next simplest term is the integral in (7.13). Using the Scwarz inequality, we may bound it in magnitude by

$$\left(\int_{\mathbb{R}\times\Lambda} x_1^2 v^2\right)^{1/2}\left(\int_{\mathbb{R}\times\Lambda}\left[\left(\nabla_1 v - \beta(1-\bar{m}^2)\nabla_1 J * v\right) + v^2\Psi_1 + v\Phi_1\right]^2\mathrm{d}x\right)^{1/2} .$$

By what we have proved above, the second square root on the right is bounded uniformly (and small) on the interval under consideration.

After one more integration by parts in the variable x_1, the contribution in (7.12) is handled the same way. □

8 Appendix

8.1 LEMMA. *Let $m \in \mathcal{M}$, see (2.2), $m = \bar{m} + v$, where $\bar{m}$ is the closest instanton to m in $L^2(\mathbb{R}\times\Lambda)$. There exists $\kappa > 0$, $\delta > 0$, $c = c(\kappa) > 0$ so that for and $\|v\|_{W^{s,2}} \leq \kappa$, where $s > \frac{D}{2}$, we have*

$$\frac{1}{4}\gamma(L)||m - \bar{m}||_2^2 \leq \mathcal{F}\big(m\big) - \mathcal{F}\big(\bar{m}\big) \leq c||m - \bar{m}||_2^2. \qquad (8.1)$$

Moreover for any $\epsilon > 0$ there is a $\tilde{\kappa}(\epsilon, L, \beta)$ so that

$$\frac{1-\epsilon}{2}\langle v, \mathcal{B}v\rangle_{L^2(\mathbb{R}\times\Lambda)} \leq \mathcal{F}\big(m\big) - \mathcal{F}\big(\bar{m}\big) \leq \frac{1+\epsilon}{2}\langle v, \mathcal{B}v\rangle_{L^2(\mathbb{R}\times\Lambda)} \qquad (8.2)$$

provided $\|v\|_{W^{s,2}} \leq \tilde{\kappa}$ for $s > \frac{D}{2}$.

Proof: Denote by $\mathcal{F}'\big(m\big)(v)$ and $\mathcal{F}''\big(m\big)(v,w)$ respectively the first and the second Frechet derivative of $\mathcal{F}(\cdot)$ computed at $m\in\mathcal{M}$ in the directions v and w in $L^2(\mathbb{R}\times\Lambda)$. It is easy to see that for $\|m\|_\infty\le c<1$ repeated Frechet derivatives exist with

$$\mathcal{F}'\big(m\big)(v)=\int_{\mathbb{R}\times\Lambda}\left[\frac{1}{\beta}\text{arctanh}m(x)-J\star m(x)\right]v(x)\mathrm{d}x,$$

and

$$\mathcal{F}''\big(m\big)(v,w)=\int_{\mathbb{R}\times\Lambda}\left[\frac{1}{\beta}\frac{v(x)}{(1-m^2(x))}-J\star v(x)\right]w(x)\mathrm{d}x.$$

When $m=\bar m$ then

$$\mathcal{F}''\big(\bar m\big)(v,w)=\int_{\mathbb{R}\times\Lambda}\left[\frac{1}{\beta}\frac{v(x)}{(1-\bar m^2(x))}-J\star v(x)\right]w(x)\mathrm{d}x=\langle\mathcal{B}v,w\rangle,$$

where $\mathcal{B}$ is the operator defined in (3.1).

Writing $m=\bar m+v$, we can represent

$$\mathcal{F}\big(\bar m+v\big)-\mathcal{F}\big(\bar m\big)=\int_0^1 d\tau\mathcal{F}'\big(\overline{m}+\tau v\big)(v)=\int_0^1 d\tau\int_0^\tau ds\mathcal{F}''\big(\bar m+sv\big)(v,v).$$

In order to get a lower bound for the last term above we expand $\mathcal{F}''\big(\bar m+sv\big)(v,v)$ around $s=0$ obtaining

$$\mathcal{F}''\big(\bar m+s(m-\overline{m})\big)(v,v)=\mathcal{F}''\big(\bar m\big)(v,v)+\mathcal{F}'''\big(\tilde m\big)(v,v,v)$$

where $\tilde m=\bar m+s_0v$ for some s_0 between 0 and 1 by the mean value theorem. Therefore

$$\mathcal{F}\big(m\big)-\mathcal{F}\big(\bar m\big)=\frac{1}{2}\langle\mathcal{B}v,v\rangle+\int_0^1 d\tau\int_0^\tau ds\mathcal{F}'''\big(\tilde m\big)(v,v,v). \tag{8.3}$$

Since $\bar m$ is the closest instanton to m in $L^2(\mathbb{R}\times\Lambda)$, $\int\bar m'(x)v(x)\mathrm{d}x=0$. Therefore by (3.3)

$$\langle\mathcal{B}v,v\rangle\ge\gamma(L)\|v\|_2^2. \tag{8.4}$$

We then need a lower bound on the term involving the third derivative of the free energy. By direct computation,

$$\big|\mathcal{F}'''\big(\tilde m\big)(v,v,v)\big|=\frac{2}{\beta}\big|\int_{\mathbb{R}\times\Lambda}\frac{\tilde m}{(1-\tilde m^2)^2}\big(v(x)\big)^3\mathrm{d}x\big|. \tag{8.5}$$

Take $\|v\|_{W^{s,2}}\le\delta_1$, so that $||\tilde m||_\infty\le 1-\delta_0$, with $\delta_0>0$, see Lemma 8.4. With this choice of δ_1 we have

$$\big|\mathcal{F}'''\big(\tilde m\big)(v,v,v)\big|\le c(\beta,\delta_1)\int_{\mathbb{R}\times\Lambda}|v(x)|^3\mathrm{d}x$$

for some constant $c(\beta,\delta_1)$ depending on β and δ_1. We have that

$$\int_{\mathbb{R}\times\Lambda}|v(x)|^3\mathrm{d}x\le\sup|v(x)|\int_{\mathbb{R}\times\Lambda}|v(x)|^2$$

and, see Lemma (8.4), for $s>\frac{D}{2}$

$$\sup|v(x)|\le C(d,s)\|v\|_{W^{s,2}}.$$

Therefore

$$\big|\mathcal{F}'''\big(\tilde{m}\big) < v, v, v > \big| \le c(\beta, \delta_1, d)\|v\|_{W^{s,2}}\big(\|v\|_2\big)^2 \tag{8.6}$$

and, see (8.3),

$$\mathcal{F}\big(m\big) - \mathcal{F}\big(\bar{m}\big) \ge \|v\|_2^2\left[\frac{1}{2}\gamma(L) - c(\beta, \delta_1, d)\|v\|_{W^{s,2}}\right].$$

Taking $\delta := \min\{\delta_1, \delta(\beta, L)$ so that

$$\frac{1}{4}\gamma(L) - c(\beta, \delta_1, d)\|v\|_{W^{s,2}} \ge 0$$

we have

$$\mathcal{F}\big(m\big) - \mathcal{F}\big(\bar{m}\big) \ge \frac{1}{4}\gamma(L)\|v\|_2^2.$$

Thus, we have established a lower bound for (8.1). The upper bound follows from the boundedness of $\mathcal{B}$ and estimate (8.5) of $\mathcal{F}'''\big(\tilde{m}\big)(v, v, v)$. Note that one needs always a bound on $\|v\|_{W^{s,2}}$ to get $\|\tilde{m}\| \le 1 - \delta_0$. In this way we proved (8.1).

In a similar way the inequalities (8.2)follows. Namely, from (8.3), for any positive ϵ

$$\mathcal{F}\big(m\big) - \mathcal{F}\big(\bar{m}\big) = \frac{1}{2}(1-\epsilon)\langle\mathcal{B}v, v\rangle + \left[\frac{1}{2}\epsilon\langle\mathcal{B}v, v\rangle + \int_0^1 d\tau \int_0^\tau ds\mathcal{F}'''\big(\tilde{m}\big)(v, v, v)\right]. \tag{8.7}$$

From (3.3) and (8.6), denoting $\tilde{\delta}$ the δ appearing in the formula, the last term in (8.7) is bigger or equal to

$$\frac{1}{2}\epsilon\gamma(L)||v||_2^2 - c(\beta, \tilde{\delta})C(d, s)\|v\|_{W^{s,2}}\big(\|v\|_2\big)^2 = ||v||_2^2\left[\frac{1}{2}\epsilon\gamma(L) - c(\beta, \tilde{\delta})C(d, s)\|v\|_{W^{s,2}}\right]. \tag{8.8}$$

Choosing $\tilde{\delta}$ so that the term in (8.8) is strictly positive we get the lower bound (8.2). The upper bound (8.2) follows immediately. □

8.2 LEMMA. *Let $\rho(x)$ be a probability density with*

$$\int |x|\rho(x)\mathrm{d}x < \infty\ .$$

Then for $v \in W^{1,2}(\mathbb{R} \times \Lambda)$

$$\|v - \rho \star v\|_2 \le \|\nabla v\|_2 \int |x|\rho(x)\mathrm{d}x.$$

Proof: We have

$$\begin{aligned} &\|v - \rho \star v\|_2^2 = \\ &\int_{\mathbb{R}} dx \left(\int_{\mathbb{R}} \rho(x-y)[v(y) - v(x)]dy\right)^2 \\ &\le \left(\int |x|\rho(x)\mathrm{d}x\right)^2 \|\nabla v\|_2^2. \end{aligned}$$

□

The next lemma shows that for any function v that is orthogonal to $\bar{m}'$, whenever $\|\nabla v\|_{L^2(\mathbb{R}\times\Lambda)}$ is small compared to $\|v\|_{L^2(\mathbb{R}\times\Lambda)}$, then $\mathcal{B}v$ is very close to being a constant multiple of v, $\tilde{\alpha}v$ where $\tilde{\alpha}$ is defined by

$$\tilde{\alpha} = \frac{1}{\beta(1-m_\beta^2)} - 1$$

and it is strictly positive for $\beta > 1$. The lemma also shows that under the same condition, $\sigma(\bar{m})v$ is very close to $\sigma(m_\beta)v$.

8.3 LEMMA. *Let $v \in W^{1,2}(\mathbb{R}\times\Lambda)$, $\langle v, \bar{m}'\rangle_{L^2} = 0$. There is a finite positive constant $K(\beta, J, L, d)$ so that*

$$\|\mathcal{B}v - \tilde{\alpha}v\|_{L^2(\mathbb{R}\times\Lambda)} \le K(\beta, J, L, d)\|\nabla v\|_{L^2(\mathbb{R}\times\Lambda)}, \tag{8.9}$$

and

$$\|\sigma(\bar{m})v - \sigma(m_\beta)v\|_{L^2(\mathbb{R}\times\Lambda)} \le K(\beta, J, L, d)\|\nabla v\|_{L^2(\mathbb{R}\times\Lambda)}. \tag{8.10}$$

Proof: Clearly,

$$\mathcal{B}v - \tilde{\alpha}v = \frac{1}{\beta}\left(\frac{\bar{m}^2 - m_\beta^2}{(1-\bar{m}^2)(1-m_\beta^2)}\right)v + (v - J\star v).$$

We will estimate these two terms separately. For the second term we apply Lemma 8.2. For the first term split, as done in (3.16),

$$v(x_1, x^\perp) = v_1(x_1) + w(x_1, x^\perp).$$

Then

$$\frac{1}{\beta}\left(\frac{\bar{m}^2 - m_\beta^2}{(1-\bar{m}^2)(1-m_\beta^2)}\right)v = \frac{1}{\beta}\left(\frac{\bar{m}^2 - m_\beta^2}{(1-\bar{m}^2)(1-m_\beta^2)}\right)[v_1 + w]$$

We estimate

$$\frac{1}{\beta}\left(\frac{\bar{m}^2 - m_\beta^2}{(1-\bar{m}^2)(1-m_\beta^2)}\right)v_1$$

as in [4], being one dimensional. First, for any y_1 and x_1 we have

$$v_1(x_1) = v_1(y_1) + \int_{y_1}^{x_1} v_1'(z)\mathrm{d}z\ .$$

Now multiply both sides by $\bar{m}'(y_1)$ and integrate in y_1. By the orthogonality of $\bar{m}'$ and v_1, we have

$$2m_\beta v_1(x_1) = \int_{-\infty}^{\infty} \bar{m}'(y)\left(\int_{y_1}^{x_1} v_1'(z)\mathrm{d}z\right)\mathrm{d}y\ .$$

But $|\int_{y_1}^{x_1} v'(z)\mathrm{d}z| \le |x_1 - y_1|^{1/2}\|v'\|_2$ so that

$$|v_1(x_1)| \le \frac{1}{2m_\beta}\left(\int \bar{m}'(y)|x_1 - y_1|^{1/2}\mathrm{d}y\right)\|v_1'\|_{L^2(\mathbb{R})}\ ,$$

and clearly there is a finite constant $K(\beta, J)$ depending only on β and J so that

$$\frac{1}{2m_\beta}\int \bar{m}'(y)|x_1 - y_1|^{1/2}\mathrm{d}y \le K(\beta, J)(1 + |x_1|),$$

and hence

$$|v_1(x_1)| \leq K(\beta, J)(1 + |x_1|)\|v_1'\|_{L^2(\mathbb{R})}. \tag{8.11}$$

Next, using the pointwise bounds (8.11) established above,

$$\begin{aligned}
&\|(\bar{m}^2 - m_\beta^2)\big((1-\bar{m}^2)(1-m_\beta^2)\big)^{-1} v_1\|^2_{L^2(\mathbb{R}\times\Lambda)} \\
\leq &\|v_1'\|^2_{L^2(\mathbb{R}\times\Lambda)} K(\beta, J) \int (1+|x_1|)^2 (\bar{m}^2 - m_\beta^2)^2 \big((1-\bar{m}^2)(1-m_\beta^2)\big)^{-2} \mathrm{d}x_1 \\
\leq &\tilde{K}(\beta, J)\|v_1'\|^2_{L^2(\mathbb{R}\times\Lambda)},
\end{aligned}$$

where $\tilde{K}(\beta, J)$ is finite by the rapid decay of $(\bar{m}^2 - m_\beta^2)^2$. Further

$$\|\frac{1}{\beta}\left(\frac{\bar{m}^2 - m_\beta^2}{(1-\bar{m}^2)(1-m_\beta^2)}\right) w\|^2_{L^2(\mathbb{R}\times\Lambda)} \leq K(\beta, J)\|w\|^2_{L^2(\mathbb{R}\times\Lambda)}.$$

Applying the Poincaré inequality as in (4.18) we have

$$\|w\|^2_{L^2(\mathbb{R}\times\Lambda)} \leq L^2 c(d)\|\nabla^\perp w\|^2_{L^2(\mathbb{R}\times\Lambda)}.$$

Then

$$\begin{aligned}
&\|\frac{1}{\beta}\left(\frac{\bar{m}^2 - m_\beta^2}{(1-\bar{m}^2)(1-m_\beta^2)}\right) v\|^2_{L^2(\mathbb{R}\times\Lambda)} \leq K(\beta, J)\left[\|v_1'\|^2_{L^2(\mathbb{R}\times\Lambda)} + L^2 c(d)\|\nabla^\perp w\|^2_{L^2(\mathbb{R}\times\Lambda)}\right] \\
&\leq K(\beta, J, d, L)\|\nabla v\|^2_{L^2(\mathbb{R}\times\Lambda)}.
\end{aligned}$$

The proof of (8.10) is very similar to the proof of (8.9). □

For function $v \in W^{s,2}(\mathbb{R}\times\Lambda)$ we have the following result which can be proven by Fourier analysis, see [10].

8.4 LEMMA. *For $v \in W^{s,2}(\mathbb{R}\times\Lambda)$, if $s > \frac{D}{2}$, we have*

$$\|v\|_\infty \leq C(d, s)\|v\|_{W^{s,2}}.$$

8.5 LEMMA. *For any real number a and b and for any λ, $0 < \lambda < 1$,*

$$(a+b)^2 \geq \lambda a^2 - \left(\frac{1}{1-\lambda} - 1\right) b^2.$$

Proof: The proof is immediate:

$$\begin{aligned}
(a+b)^2 &\geq a^2 + b^2 - 2ab \\
&= \lambda a^2 + \Big((1-\lambda)a^2 + b^2 - 2ab\Big) \\
&\geq \lambda a^2 - \left(\frac{1}{1-\lambda} - 1\right) b^2.
\end{aligned} \tag{8.12}$$

The last inequality is obtained adding and subtracting $\frac{1}{1+\lambda} b^2$.

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