Typical Gibbs configurations for the 1d Random Field Ising Model with long range interaction. *

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Abstract We study one–dimensional Ising spin systems with ferromagnetic, long–range interactions decaying as $n^{-2+\alpha}$, $\alpha \in [0, \frac{1}{2}]$, in the presence of external random fields. We assume that the random fields are given by a collection of symmetric, independent, identically distributed real random variables, which are gaussian or subgaussian with variance θ. We show that when the temperature and the variance of the randomness are sufficiently small, with overwhelming probability with respect to the random fields, the typical configurations, within intervals centered at the origin whose length grow faster than any power of θ^{-1}, are intervals of $+$ spins followed by intervals of $-$ spins whose typical length is $\simeq \theta^{-\frac{2}{(1-2\alpha)}}$ for $0 \le \alpha < 1/2$ and between $e^{\frac{1}{\theta}}$ and $e^{\frac{1}{\theta^2}}$ for $\alpha = 1/2$.

1 Introduction

We consider a one dimensional ferromagnetic Ising model with a two body interaction $J(n) = n^{-2+\alpha}$ where n denotes the distance of the two spins and $\alpha \in [0, 1/2]$ tunes the decay of the interaction. We add to this term an external random field $h[\omega] = \{h_i[\omega], i \in \mathbb{Z}\}$ given by a collection of independent random variables, with mean zero, symmetrically distributed, with variance θ, gaussian or sub–gaussian defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. We study the magnetization profiles that are typical for the Gibbs measure when θ and the temperature are suitably small. The results hold on a subspace $\Omega_1(\theta) \subset \Omega$ whose probability goes to 1 when $\theta \downarrow 0$.

A systematic and successful analysis of this model for $\theta = 0$ *i.e.* when the magnetic fields are absent has been already accomplished more than twenty years ago [21,10,11,12,13,14,15,1,16]. In particular it has been shown that it exhibits a phase transition only for $\alpha \in [0, 1)$. The presence of external random fields ($\theta \neq 0$) modifies this picture. In [2], it has been proved that for $\alpha \in [0, 1/2]$ there exists a unique infinite volume Gibbs measure *i.e.* there is no phase transition. More recently in [8] it has been proved that when $\alpha \in (1/2, \frac{\log 3}{\log 2} - 1)$ the situation is analogous to the three dimensional short range random field Ising model [4] : for temperature and variance of the randomness small enough, there exist at least two distinct infinite volume Gibbs states, namely the μ^+ and the μ^- Gibbs states. The proof is based on the notion of contours introduced in [14] but using the geometrical description implemented in [5] which is better suited to describe the contribution of the random fields. A Peierls argument is obtained by using a lower bound of the deterministic part of the cost to erase a contour and controlling the stochastic part.

The method used in [2] to prove the uniqueness of the Gibbs measure is very powerful and general but does not provide any insight about the most relevant spin configurations of this measure.

In this paper we show that for temperature and variance of the randomness small enough the typical configurations are intervals of $+$ spins followed by intervals of $-$ spins whose typical length is $\theta^{-\frac{2}{(1-2\alpha)}}$ for $0 \le \alpha < 1/2$ and becomes exponentially larger in term of θ^{-1} for $\alpha = 1/2$. When $\theta > 0$ the Gibbs measures are random valued measures. We need therefore to localize the region in which we inspect the system. All

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our results are given uniformly for an increasing sequence of intervals, centered in one point, with a diameter going to infinity when $\theta \downarrow 0$.

The modification induced by the presence of random fields has been already studied for the one dimensional Kac model with range γ^{-1} [6,7,19]. In this case for θ and γ sufficiently small the typical length is γ^{-2}. The results are consistent if one recalls that the one dimensional random field Kac model exhibits a phase transition for $\gamma \downarrow 0$.

The method applied to derive the upper bound for the length of the intervals having all spins alike, is similar to the one applied for the Kac model [6]. The derivation of lower bound relies on Peierls type arguments. Similar estimates were used in [8], to prove existence of phase transition. In this paper we use them to show that configurations having spins alike for intervals smaller then some value $L_{max}(\alpha)$, see Proposition 4.1, have small Gibbs probability.

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2 Model, notations and main results

2.1. The model

Let $h \equiv \{h_i\}_{i\in\mathbb{Z}}$ be a family of independent, identically distributed, symmetric random variables defined on the probability space $(\Omega, \mathcal{A}, I\!P)$. We assume that each h_i is Bernoulli distributed with $I\!P[h_i = +1] = I\!P[h_i = -1] = 1/2$. By minor modifications, which we will mention in the sequel, we could take h_0 to be a Gaussian random variable with mean 0 and variance 1 or even a subgaussian *i.e.* $I\!E[\exp(th_0)] \le \exp(t^2/2)\,\forall t \in I\!R$, see [17] for basic properties of sub–gaussian random variables.

We denote by $\mathcal{S} \equiv \{-1,+1\}^{\mathbb{Z}}$ the spin configurations space. If $\sigma \in \mathcal{S}$ and $i \in \mathbb{Z}$, σ_i represents the value of the spin at site i. The pair interaction among spins is given by $J(|i-j|)$ defined by

$$J(n) = \begin{cases} J(1) >> 1; \\ \dfrac{1}{n^{2-\alpha}} \quad \text{if} \quad n > 1, \quad \text{with } \alpha \in (-\infty, 1). \end{cases} \tag{2.1}$$

For $\Lambda \subseteq \mathbb{Z}$ we set $\mathcal{S}_\Lambda = \{-1,+1\}^\Lambda$; its elements are denoted by σ_Λ; also, if $\sigma \in \mathcal{S}$, σ_Λ denotes its restriction to Λ. Given $\Lambda \subset \mathbb{Z}$ finite, define

$$H_0(\sigma_\Lambda) = \frac{1}{2} \sum_{(i,j)\in\Lambda\times\Lambda} J(|i-j|)(1-\sigma_i\sigma_j), \tag{2.2}$$

and for $\omega \in \Omega$

$$G(\sigma_\Lambda)[\omega] = -\theta \sum_{i\in\Lambda} h_i[\omega]\sigma_i.$$

We consider the Hamiltonian given by the random variable on $(\Omega, \mathcal{A}, I\!P)$

$$H(\sigma_\Lambda)[\omega] = \frac{1}{2} \sum_{(i,j)\in\Lambda\times\Lambda} J(|i-j|)(1-\sigma_i\sigma_j) + G(\sigma_\Lambda)[\omega]. \tag{2.3}$$

To take into account the interaction between the spins in Λ and those outside Λ we set for $\eta \in \mathcal{S}$

$$W(\sigma_\Lambda, \eta_{\Lambda^c}) = \sum_{i\in\Lambda}\sum_{j\in\Lambda^c} J(|i-j|)(1-\sigma_i\eta_j) \tag{2.4}$$

and denote

$$H^\eta(\sigma_\Lambda)[\omega] = H(\sigma_\Lambda)[\omega] + W(\sigma_\Lambda, \eta_{\Lambda^c}). \tag{2.5}$$

In the following we drop out the ω from the notation. We denote by

$$\mu^\eta_\Lambda(\sigma_\Lambda) = \frac{1}{Z^\eta_\Lambda}\exp\{-\beta H^\eta(\sigma_\Lambda)\} \qquad \sigma_\Lambda \in \mathcal{S}_\Lambda, \tag{2.6}$$

where Z^η_Λ is the normalization factor, the corresponding *Gibbs measure* on the finite volume Λ, at inverse temperature $\beta > 0$, with boundary condition η. It is a random variable with values on the space of probability measures on $\mathcal{S}_\Lambda$.

When the configuration η is taken so that $\eta_i = \tau$, $\tau \in \{-1,+1\}$, for all $i \in \mathbb{Z}$ we denote the corresponding Gibbs measure by μ^+_Λ when $\tau = 1$ and μ^-_Λ when $\tau = -1$. By FKG inequality the infinite volume limit $\Lambda \uparrow \mathbb{Z}$ of μ^+_Λ and μ^-_Λ exists, say μ^+, μ^-. By the result of Aizenman and Wehr, see [2], *, when $\alpha \in [0, \frac{1}{2}]$ for $\mathbb{P}$–almost all ω, $\mu^+ = \mu^-$ and therefore there is a unique infinite volume Gibbs measure that will be denoted by $\mu = \mu[\omega]$.

2.2. Main result

Any spin configuration $\sigma \in \{-1,+1\}^{\mathbb{Z}}$ can be described in term of runs of spins of the same sign τ, $\tau \in \{-1,+1\}$, *i.e.* sequences of consecutive sites $i_1, i_1+1, i_1+2\ldots, i_1+n \in \mathbb{Z}$, $n = n(\sigma) \in \mathbb{N}$, where $\sigma_k = \tau, \forall k \in \{i_1, \ldots i_1+n\}$, and $\sigma_{i_1-1} = \sigma_{i_1+n+1} = -\tau$. A run could have length 1. To enumerate the runs we do as it follows. Start from the site $i = 0$. Let $\sigma_0 = \tau$, $\tau \in \{-1,+1\}$, call $\mathcal{L}^\tau_1 = \mathcal{L}^\tau_1(\sigma)$ the run containing the origin, $\mathcal{L}^{-\tau}_2$ the run on the right of $\mathcal{L}^\tau_1$ and $\mathcal{L}^{-\tau}_0$ the run on the left of $\mathcal{L}^\tau_1$. In this way to each configuration σ, we assign in a one to one way a sign $\tau = \sigma_0$ and a family of runs $(\mathcal{L}^{(-1)^{j+1}\tau}_j, i \in \mathbb{Z})$. To shorten notation we drop the $(-1)^{j+1}\tau$ and write simply $(\mathcal{L}_j, j \in \mathbb{Z})$.

Given an interval $V \subset \mathbb{Z}$ and a configuration σ_V, let $e_V = e_V(\sigma_V) = \sup(j \in \mathbb{Z} : \mathcal{L}_j \subset V)$ be the index of the rightmost run contained in V and $b_V = b_V(\sigma_V) = \inf(j \in \mathbb{Z} : \mathcal{L}_j \subset V)$ the index of the leftmost run contained in V. We consider the sequences of runs $(\mathcal{L}_j, b_V \le j \le e_V)$ and give upper bounds and lower bounds on their lengths in the regime β large and θ small. More precisely, in Theorem 2.1 we show that in an interval V centered at the origin, longer than any inverse power of θ up to subdominant terms, with $\mathbb{P}$–probability larger than $1 - e^{-g(\theta)}$, where $g(\theta)$ is a function slowly going to infinity as $\theta \downarrow 0$, the typical configurations have runs with length of order $\theta^{-\frac{2}{1-2\alpha}}$ when $0 \le \alpha < 1/2$. When $\alpha = \frac{1}{2}$ we show in Theorem 2.2 that with overwhelming $\mathbb{P}$–probability the typical run that contains the origin is larger than $e^{c/\theta}$ and smaller than e^{c'/θ^2}, where c and c' are suitable positive constants.

Theorem 2.1 *For $\alpha \in [0, \frac{1}{2})$ and $\zeta = \zeta(\alpha) = 1 - 2(2^\alpha - 1)$ there exist $\theta_0 = \theta_0(\alpha)$, $\beta_0 = \beta_0(\alpha)$ and constants $c_i(\alpha)$, such that for all $0 < \theta \le \theta_0$, for all β*

$$\beta \ge \frac{\zeta}{2^8\theta^2} \ge \beta_0 \tag{2.7}$$

* A simplified proof of this result which avoids the introduction of metastates, by applying the FKG inequalities, is given by Bovier, see [3], chapter 7. Notice that although we assume that the distribution of the random field has isolated point masses, the result [2] still holds.

if $0 < \alpha < 1/2$, $g(\theta) = (\log \frac{1}{\theta})(\log\log \frac{1}{\theta})$ *and* V *the interval centered at the origin having diameter*

$$\mathrm{diam}(V) = c_0(\alpha) e^{g(\theta)} \left(\frac{1}{\theta}\right)^{\frac{2}{1-2\alpha}} \tag{2.8}$$

then with $I\!P$*-probability larger than* $1 - e^{-g(\theta)}$ *and with* $\mu[\omega]$ *Gibbs measure larger than* $1 - e^{-g(\theta)}$ *the spin configurations are made of runs* $(\mathcal{L}_j, b_V \le j \le e_V)$ *satisfying*

$$c_1(\alpha)\left(\log\frac{1}{\theta}\right)^{-\frac{2}{1-2\alpha}}\left(\log\log\frac{1}{\theta}\right)^{-\frac{1}{1-2\alpha}} \le \theta^{\frac{2}{1-2\alpha}}\left|\mathcal{L}_j\right| \le c_2(\alpha)(\log\frac{1}{\theta})(\log\log\frac{1}{\theta}), \tag{2.9}$$

for all $j \in \{b_V, \ldots e_V\}$.
If $\alpha = 0$, $g(\theta)$ *has to be replaced by* $\hat{g}(\theta) = \log\left(\frac{\log\frac{1}{\theta}}{\theta}\right)$ *and (2.9) becomes*

$$c_1(0) \le \theta^2\left|\mathcal{L}_i\right| \le c_2(0)\left(\log\frac{1}{\theta}\right)^3 \tag{2.10}$$

for all $j \in \{b_{\hat{V}}, \ldots, e_{\hat{V}}\}$ *where* $\hat{V}$ *satisfies*

$$\mathrm{diam}(\hat{V}) = c_0(0) e^{\hat{g}(\theta)}\left(\frac{1}{\theta}\right)^2. \tag{2.11}$$

The proof of Theorem 2.1 follows from Propositions 3.1 and 4.1 and easy estimates.

Theorem 2.2 *For* $\alpha = 1/2$, *there exists* θ_0 *and* β_0 *and constants* c_i, *so that for* $0 < \theta \le \theta_0$ *and* $\beta > \beta_0$ *satisfying (2.7), with* $I\!P$*-probability larger than* $1 - e^{-\frac{c_0}{\theta^2}}$ *and with* $\mu[\omega]$ *Gibbs measure larger than* $1 - e^{-\frac{c_0}{\theta^2}}$ *we have*

$$\frac{c_1}{\theta} \le \log|\mathcal{L}_1| \le \frac{c_2}{\theta^2}, \tag{2.12}$$

where $\mathcal{L}_1$ *is the run containing the origin.*

Remark 2.3 . *The results for* $\alpha = 1/2$ *are less sharp and general than the ones for* $\alpha \in [0, \frac{1}{2})$. *The probability estimates obtained in (4.74) for the lower bound do not allow to get a result uniformly on interval of exponential length. However the estimates for the upper bound are true on a larger scale, see (3.6) and (3.7).*

3 The upper bound

Let $I \subset \mathbb{Z}$ be an interval and denote

$$R^\tau(I) = \{\sigma \in \mathcal{S} : \sigma_i = \tau, \forall i \in I\}, \qquad \tau \in \{-1, +1\}, \tag{3.1}$$

the set of spin configurations equal to τ in the interval I and

$$R(I) = R^+(I) \cup R^-(I). \tag{3.2}$$

Let $L_{\max}$ be a positive integer and $V \subset \mathbb{Z}$ an interval centered at the origin with $\mathrm{diam(V)} > \mathrm{L_{max}}$. Denote

$$\mathcal{R}(V, L_{\max}) = \bigcup_{I \subset V,\ |I| \geq L_{\max}} R(I), \tag{3.3}$$

the set of spin configurations having at least one run of $+1$ or -1 larger than $L_{\max}$ in V. The main result of this section is the following.

Proposition 3.1 *Let $\alpha \in [0, \frac{1}{2}]$, there exist positive constants c_α, c'_α and $\theta_0 = \theta_0(\alpha)$ such that for all $\beta > 0$, for all decreasing real valued function $g_1(\theta) \geq 1$ defined on $\mathbb{R}$ that satisfies $\lim_{\theta \downarrow 0} g_1(\theta) = \infty$ there exist an $\Omega_3(\alpha) \subset \Omega$ with*

$$\mathbb{P}[\Omega_3(\alpha)] \geq \begin{cases} 1 - 2e^{-g_1(\theta)}, & \text{if } 0 \leq \alpha < \frac{1}{2}; \\ 1 - e^{-\frac{1}{2}e^{g_1(\theta)}}, & \text{if } \alpha = \frac{1}{2}, \end{cases} \tag{3.4}$$

$$L_{max}(\alpha) = \begin{cases} c'_\alpha g_1(\theta) \ \left(\frac{1}{\theta^2}\right)^{\frac{1}{1-2\alpha}}, & \text{if } 0 < \alpha < 1/2; \\ c'_0 g_1(\theta) \ \frac{1}{\theta^2}\left(\log \frac{1}{\theta}\right)^2, & \text{if } \alpha = 0; \\ c'_{1/2} e^{g_1(\theta)} \ e^{\frac{3}{2}\frac{8^2}{\theta^2}}(1 + \frac{8}{\theta})^3, & \text{if } \alpha = 1/2, \end{cases} \tag{3.5}$$

and an interval $V(\alpha) \subset \mathbb{Z}$ centered at the origin

$$\mathrm{diam(V(\alpha))} = \begin{cases} c'_\alpha e^{g_1(\theta)} \ \left(\frac{1}{\theta^2}\right)^{\frac{1}{1-2\alpha}}, & \text{if } 0 < \alpha < 1/2; \\ c'_0 e^{g_1(\theta)} \ \frac{1}{\theta^2}\left(\log \frac{1}{\theta}\right)^2, & \text{if } \alpha = 0; \\ c'_{1/2} e^{\frac{1}{2}\exp(g_1(\theta))} \ e^{\frac{8^2}{\theta^2}}\left(1 + \frac{8}{\theta}\right)^3, & \text{if } \alpha = 1/2, \end{cases} \tag{3.6}$$

so that on $\Omega_3(\alpha)$, uniformly with respect to $\Lambda \subset \mathbb{Z}$,

$$\sup_\eta \mu^\eta_\Lambda\left(\mathcal{R}(V(\alpha), L_{\max}(\alpha))\right) \leq \begin{cases} 2e^{g_1(\theta)} e^{-\beta c_\alpha \theta^{-\frac{2\alpha}{1-2\alpha}}}, & \text{if } 0 < \alpha < 1/2; \\ 2e^{g_1(\theta)} e^{-\beta c_0 \log\left(\frac{1}{\theta}\log\frac{1}{\theta}\right)}, & \text{if } \alpha = 0; \\ e^{\frac{\exp(g_1(\theta))}{2}} \ \exp(-\beta c_{1/2} e^{\frac{8^2}{2\theta^2}}), & \text{if } \alpha = 1/2. \end{cases} \tag{3.7}$$

Remark:

There are various way to choose $g_1(\theta)$. To get a good probability estimate in (3.4) and to have $L_{max}(\alpha)$ of the order of $\theta^{-\frac{2}{1-2\alpha}}$ when $0 < \alpha < 1/2$, we take $g_1(\theta)$ to be a slowly varying function at zero. Note that $g_1(\theta) = (\log[1/\theta])(\log\log[1/\theta])$ has the following advantages: $e^{-g_1(\theta)}$ decays faster than any inverse power of θ^{-1}, diam(V) increases faster than any polynomial in θ^{-1} and the asymptotic behavior of (3.7) is unaffected.

Proof: Let $I \subset \mathbb{Z}$ be an interval and $R(I)$ defined in (3.2). Since $I' \subset I$ implies $R(I) \subset R(I')$ we have

$$\bigcup_{I \subset V,\ |I| \geq L} R(I) \subset \bigcup_{I \subset V,\ |I| = L} R(I). \tag{3.8}$$

Therefore it is enough to consider the right hand side of (3.8) instead of the left hand side.

Assume that $I = \cup_{\ell=1}^M \Delta(\ell)$ where $\Delta(\ell)$, $\ell \in \{1, \ldots, M\}$, are adjacent intervals of length $|\Delta|$. We denote by Δ a generic interval $\Delta(\ell)$, $\ell \in \{1, \ldots, M\}$. We start estimating $\mu^\eta_\Lambda(R^+(\Delta))$. We bound from below Z^η_Λ

by the sum over configurations constrained to be in $R^-(\Delta)$ and collect the contributions of the magnetic fields in Δ both in the numerator and in the denominator. We obtain:

$$
\begin{aligned}
\mu_\Lambda^\eta(R^+(\Delta)) &\le \frac{\sum_{\sigma_\Lambda} e^{-\beta H^\eta(\sigma_\Lambda)} 1\!\!1_{R^+(\Delta)}}{\sum_{\sigma_\Lambda} e^{-\beta H^\eta(\sigma_\Lambda)} 1\!\!1_{R^-(\Delta)}} \\
&\le e^{2\beta\theta \sum_{i\in\Delta} h_i[\omega]} \sup_{\sigma_{\Lambda\setminus\Delta}} \sup_{\eta_{\Lambda^c}} \frac{e^{-\beta[W(\sigma_\Delta,\sigma_{\Lambda\setminus\Delta})+W(\sigma_\Delta,\eta_\Lambda^c)]} 1\!\!1_{R^+(\Delta)}(\sigma_\Delta)}{e^{-\beta[W(\sigma_\Delta,\sigma_{\Lambda\setminus\Delta})+W(\sigma_\Delta,\eta_\Lambda^c)]} 1\!\!1_{R^-(\Delta)}(\sigma_\Delta} \\
&\le e^{2\beta\theta \sum_{i\in\Delta} h_i[\omega]} e^{2\beta[\sum_{i\in\Delta}\sum_{j\in\Delta^c} J(|i-j|)]} \le e^{2\beta\theta \sum_{i\in\Delta} h_i[\omega]} e^{2\beta E_\alpha(|\Delta|)},
\end{aligned}
\tag{3.9}
$$

where $E_\alpha(|\Delta|)$ is defined by

$$
E_\alpha(|\Delta|) = \begin{cases} 2(J(1)-1) + \frac{2|\Delta|^\alpha}{\alpha(1-\alpha)}, & \text{if } 0<\alpha<1; \\ 2(J(1)-1) + 2\log(|\Delta|) + 4, & \text{if } \alpha = 0. \end{cases} \tag{3.10}
$$

Calling

$$
\Omega_1^-(\Delta) = \big\{\omega : \theta \sum_{i\in\Delta} h_i[\omega] < -2E_\alpha(|\Delta|)\big\}, \tag{3.11}
$$

on $\Omega_1^-(\Delta)$ we have

$$
\sup_{\Lambda\subset\subset\mathbb{Z}} \sup_\eta \mu_\Lambda^\eta(R^+(\Delta)) \le e^{-2\beta E_\alpha(|\Delta|)}. \tag{3.12}
$$

Define

$$
\Omega_2^-(I) = \big\{\omega : \exists \ell_I^* \in \{1,\dots,M\} : \theta \sum_{i\in\Delta(\ell_I^*)} h_i[\omega] < -2E_\alpha(|\Delta|)\big\}. \tag{3.13}
$$

On $\Omega_2^-(I)$ we have

$$
R^+(I) \subset R^+(\Delta(\ell_I^*)), \tag{3.14}
$$

therefore, by (3.12),

$$
\sup_{\Lambda\subset\subset\mathbb{Z}} \sup_\eta \mu_\Lambda^\eta(R^+(I)) \le e^{-2\beta E_\alpha(|\Delta|)}. \tag{3.15}
$$

Assume $V = [-N|\Delta|, N|\Delta|]$. We can, then, cover V with overlapping intervals $I_k = [k|\Delta|, M|\Delta| + k|\Delta|)$ for $k \in \{-N,\dots,(N-M)\}$. It is easy to check that for any interval I of length $M|\Delta|$, $I \subset V$, there exists a unique $k \in \{-N,\dots,(N-M-1)\}$ such that

$$
I \supset I_k \cap I_{k+1}. \tag{3.16}
$$

Therefore one gets

$$
\bigcup_{I\subset V,\,|I|=M|\Delta|} R^+(I) \subset \bigcup_{k=-N}^{N-M-1} \bigcup_{\substack{I: I_k\cap I_{k+1}\subset I\subset V \\ |I|=M|\Delta|}} R^+(I) \subset \bigcup_{k=-N}^{N-M-1} R^+(I_k\cap I_{k+1}). \tag{3.17}
$$

Note that for all k there are $M-1$ consecutive blocks of size $|\Delta|$ in $I_k \cap I_{k+1}$ that will be indexed by $\ell_k \in \{2,\dots,M\}$. Define

$$
\Omega_3^-(V) = \big\{\omega : \forall k \in \{-N,\dots,N-M\}, \exists \ell_k^* \in \{2,\dots,M\} :\ \theta \sum_{i\in\Delta(\ell_k^*)} h_i < -2E_\alpha(|\Delta|)\big\}. \tag{3.18}
$$

If we notice that $R^+(I_k \cap I_{k+1}) \subset R^+(\Delta(\ell_k^*))$, it follows from (3.3), (3.17), and (3.15), that on $\Omega_3^-(V)$, uniformly with respect to $\Lambda \subset \mathbb{Z}$ we have

$$\sup_\eta \mu_\Lambda^\eta(R^+(V, M|\Delta|)) \le (2N+1)e^{-2\beta E_\alpha(|\Delta|)}. \tag{3.19}$$

Next we make a suitable choice of the parameters $|\Delta|, M, N$. Consider first the case $0 < \alpha < 1/2$.
Since the h_i are independent symmetric random variables, we have, see (3.11),

$$\mathbb{P}[\Omega_1^-(\Delta)] = \frac{1}{2}\Big(1 - \mathbb{P}\big[\big|\sum_{i\in\Delta} h_i\big| \le \frac{2E_\alpha(|\Delta|)}{\theta}\big]\Big) \equiv \frac{1}{2}(1-p_1), \tag{3.20}$$

hence, see (3.13),

$$\mathbb{P}[\Omega_2^-(I)] \ge 1 - \big(1 - \mathbb{P}[\Omega_1^-]\big)^M = 1 - \left(\frac{1+p_1}{2}\right)^M, \tag{3.21}$$

and, see (3.18),

$$\mathbb{P}[\Omega_3^-(V)] \ge 1 - (2N+1)\left(\frac{1+p_1}{2}\right)^{M-1}. \tag{3.22}$$

To estimate p_1, we apply Le Cam's inequality, see [18], pg 407, which holds for i.i.d. random variables, symmetric and subgaussian:

$$\sup_{x\in\mathbb{R}} \mathbb{P}\Big[\sum_{i=1}^{|\Delta|} h_i \in [x, x+\tau]\Big] \le \frac{2\sqrt{\pi}}{\sqrt{|\Delta| \mathbb{E}[1 \wedge (h_1/\tau)^2]}}. \tag{3.23}$$

For symmetric Bernoulli random variables, assuming that $\tau \ge 1$, one has

$$\mathbb{E}[(h_1/\tau)^2 1\!\!1_{|h_1|\le\tau}] \ge \tau^{-2},$$

for random variables having different distributions see Remark 3.2. Taking $\tau = 2E_a(|\Delta|)/\theta \ge 1$ and

$$|\Delta| = \left(\frac{32}{B\theta\alpha(1-\alpha)}\right)^{\frac{2}{1-2\alpha}} \tag{3.24}$$

where $0 < B < 1$ we have

$$p_1 \le \frac{8E_\alpha(|\Delta|)\sqrt{\pi}}{\theta\sqrt{|\Delta|}} \le B. \tag{3.25}$$

It is easy to check that there exists $\theta_0 = \theta_0(\alpha, J(1))$, independent on B, such that (3.25) and $\tau \ge 1$ are satisfied for all $0 < \theta \le \theta_0$. Choosing

$$M = \frac{2g_1(\theta)}{\log\frac{2}{1+B}} \tag{3.26}$$

and

$$2N+1 = e^{g_1(\theta)}\frac{1+B}{2} \tag{3.27}$$

with $g_1(\theta)$ so that $\lim_{\theta\downarrow 0} g_1(\theta) = \infty$, (3.4), (3.5), (3.6), and (3.7) are proven for $0 < \alpha < 1/2$. The actual value of B affects only the values of the constants.

When $\alpha = 0$, Le Cam's inequality suggests

$$|\Delta| = \theta^{-2} \left(\frac{64\sqrt{\pi}}{B} \log \theta^{-1} \right)^2 . \tag{3.28}$$

Taking M and N as in (3.26) and (3.27), one gets (3.4), (3.5), (3.6), and (3.7).
When $\alpha = 1/2$ we have

$$\Omega_1(\Delta) = \{\omega : \theta \sum_{i \in \Delta} h_i \leq -8\sqrt{\Delta}\}. \tag{3.29}$$

Le Cam's inequality is useless. We use the Berry-Esseen Theorem, see [9], that gives

$$I\!P[\Omega_1(\Delta)] \geq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\frac{8}{\theta}} e^{-\frac{x^2}{2}}\, dx - \frac{C_{BE}}{\sqrt{\Delta}} \tag{3.30}$$

where $C_{BE} \leq 7.5$ is the Berry-Esseen constant. By the lower bound $\int_{-\infty}^{-y} e^{-\frac{x^2}{2}}\, dx \geq \frac{y}{1+y^2} e^{-\frac{1}{2}y^2}$, we have

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\frac{8}{\theta}} e^{-\frac{x^2}{2}}\, dx \geq \frac{1}{\sqrt{2\pi}} \frac{1}{1+\frac{8}{\theta}} e^{-\frac{8^2}{2\theta^2}}. \tag{3.31}$$

Choosing

$$\Delta = 16^2 (2\pi) \left(1 + \frac{8}{\theta}\right)^2 e^{\frac{8^2}{\theta^2}}, \tag{3.32}$$

so that the right hand side of (3.30) is strictly positive,

$$M = 2\sqrt{2\pi}(1 + \frac{8}{\theta}) e^{\frac{8^2}{2\theta^2}} e^{g_1(\theta)}, \tag{3.33}$$

and

$$2N + 1 = e^{\frac{1}{2} e^{g_1(\theta)}} \tag{3.34}$$

we get (3.4), (3.5), (3.6), and (3.7).
■

Remark 3.2 . *To apply (3.23), one needs a lower bound for the censored variance at* τ *of* h_1 *which is* $I\!E[1 \wedge (h_1/\tau)^2]$. *A simple one is* $I\!E[(h_1/\tau)^2 1\!\!1_{|h_1| \leq \tau}]$ *which is bounded from below by half the variance of* h_1 *times* τ^{-2} *by taking* τ *large enough. However one can also get a more precise bound since the difference between the censored variance and the variance can be estimated by using an exponential Markov's inequality that can be obtained as a consequence of the definition of sub-gaussian. When* $h_i, i \in Z\!\!Z$ *are normal distributed the bound (3.23) can be easily improved to*

$$\sup_{x \in I\!R} I\!P[\sum_{i=1}^{|\Delta|} h_i \in [x, x+\tau]] \leq \frac{\tau}{\sqrt{2\pi |\Delta|}}. \tag{3.35}$$

4 Lower bound

Let $\Delta \subset \mathbb{Z}$ be an interval, $d(i,\Delta) = \inf_{j\in\Delta} |i-j|$, $\partial\Delta = \{i \in \mathbb{Z} : d(i,\Delta) = 1\}$ and $\tau \in \{-1,+1\}$. Let

$$\mathcal{W}(\Delta,\tau) = \{\sigma \in \mathcal{S} : \sigma_i = \tau, \forall i \in \Delta, \sigma_{\partial\Delta} = -\tau\} \tag{4.1}$$

be the event that there is a run of τ in the interval Δ. Let $L_{\min}$ be a positive integer and $V \subset \mathbb{Z}$ be an interval centered at the origin, with $\mathrm{diam}(V) > L_{\min}$. We denote for $i \in V$ and $\tau \in \{-1,+1\}$,

$$\nu_i(L_{\min},\tau) = \bigcup_{\Delta\ni i,\, |\Delta|\le L_{\min}} \mathcal{W}(\Delta,\tau), \tag{4.2}$$

$$\mathcal{V}(V, L_{\min}) = \bigcup_{i\in V} \left[\nu_i(L_{\min},+) \cup \nu_i(L_{\min},-)\right]. \tag{4.3}$$

The main result of this section is the following.

Proposition 4.1 *Let $\alpha \in [0,\frac{1}{2}]$, $\theta > 0$ and $\zeta = \zeta(\alpha) = 1 - 2(2^\alpha - 1)$. There exists $\theta_0 = \theta_0(\alpha)$ and $\beta_0 = \beta_0(\alpha)$ such that for $0 < \theta < \theta_0$ and $\beta > \beta_0$, for $g_2(x) \equiv g_2(x,\alpha)$ a real positive function with $\frac{g_2(x,\alpha)}{x}$ decreasing and $\lim_{x\uparrow\infty} \frac{g_2(x)}{x} = 0$, such that*

$$g_2(x,\alpha) \ge \begin{cases} 1 + \frac{1}{1-2\alpha}\frac{\log x}{4}, & \text{if } 0<\alpha<1/2;\\ 1 + \frac{3}{2}\log\frac{2x}{3}, & \text{if } \alpha = 0; \end{cases} \tag{4.4}$$

if we denote

$$\bar{b} = \min\left(\frac{\beta\zeta}{4}, \frac{\zeta^2}{2^{10}\theta^2}\right) \quad \text{and} \quad \bar{b}_0 = \min\left(\frac{\beta_0\zeta}{4}, \frac{\zeta^2}{2^{10}\theta_0^2}\right) \tag{4.5}$$

then for all D such that $D_0 < D \le \frac{\bar{b}_0}{g_2(\bar{b}_0,\alpha)}$ with $D_0 = \max(8, \frac{1}{4}\log\frac{2}{d_0})$ for some absolute constant d_0, there exists $\Omega_5(\alpha) \subset \Omega$ with

$$\mathbb{P}[\Omega_5(\alpha)] \ge \begin{cases} 1 - 6e^{-(2D-5)g_2(\bar{b},\alpha)}, & \text{if } 0<\alpha<1/2;\\ 1 - 6e^6 e^{-(2D-4)g_2(\bar{b},0)} & \text{if } \alpha=0;\\ 1 - e^{-\sqrt{\bar{b}}\frac{(2D-1)}{\sqrt{2D}}}, & \text{if } \alpha = 1/2. \end{cases} \tag{4.6}$$

Then on $\Omega_5(\alpha)$, for

$$L_{min}(\alpha) = \begin{cases} \left(\frac{\bar{b}}{Dg_2(\bar{b},\alpha)}\right)^{\frac{1}{1-2\alpha}} \left(4 + \frac{1}{1-2\alpha}\log\left[\frac{\bar{b}}{Dg_2(\bar{b},\alpha)}\right]\right)^{-\frac{1}{1-2\alpha}}, & \text{if } 0<\alpha<1/2;\\ \frac{\bar{b}}{Dg_2(\bar{b},0)}\left(4 + \log\left[\frac{\bar{b}}{Dg_2(\bar{b},0)}\right]\right), & \text{if } \alpha=0;\\ e^{-4+\sqrt{\frac{\bar{b}}{2D}}}, & \text{if } \alpha=1/2, \end{cases} \tag{4.7}$$

and

$$\mathrm{diam}(V_{min}(\alpha)) = \begin{cases} e^{g_2(\bar{b},\alpha)}(\bar{b})^{\frac{1}{1-2\alpha}}, & \text{if } 0<\alpha<1/2;\\ e^{g_2(\bar{b},0)}\frac{\bar{b}}{Dg_2(\bar{b},0)}\left(4 + \log\left[\frac{\bar{b}}{Dg_2(\bar{b},0)}\right]\right), & \text{if } \alpha=0;\\ e^{\frac{\bar{b}}{4}}, & \text{if } \alpha=1/2, \end{cases} \tag{4.8}$$

for all $\Lambda \subset \mathbb{Z}$ large enough,

$$\mu^+_\Lambda\left(\mathcal{V}(V_{min}(\alpha), L_{min}(\alpha))\right) \le \begin{cases} 6e^{-(2D-5)g_2(\bar{b},\alpha)}, & \text{if } 0<\alpha<1/2;\\ 6e^6 e^{-(2D-4)g_2(\bar{b},0)}, & \text{if } \alpha=0;\\ e^{-\sqrt{\bar{b}}\frac{(2D-1)}{\sqrt{2D}}}, & \text{if } \alpha=1/2. \end{cases} \tag{4.9}$$

Remark 4.2 . The estimate (4.9) are uniform in Λ, therefore by the uniqueness of the infinite volume Gibbs measure, [2], Proposition 4.1 holds for the infinite volume Gibbs measure $\mu[\omega]$.

Proof: Since the boundary conditions are homogeneous equal to + we apply the geometrical description of the spin configurations presented in [5]. In the following we will assume that the notions of triangles, contours and their properties are known to the reader. In Section 5 we summarize definitions and main properties used in the proof. Let $\mathcal{T} = \{\underline{T}\}$ be the set of families of triangles compatible with the chosen + boundary conditions on Λ. Let denote by $|T|$ the *mass* of the triangle T, *i.e.* the cardinality of $T \cap \mathbb{Z}$, see (5.1). It is convenient to identify in $\underline{T} \in \mathcal{T}$ families of triangles having the same mass,

$$\underline{T} = \{\underline{T}^{(1)}, \ldots, \underline{T}^{(k_{\underline{T}})}\}, \tag{4.10}$$

rearranged in increasing order, where $k_{\underline{T}} = \sup\{|T| : T \in \underline{T}\} \in \mathbb{N}$ and for $\ell \in \{1, \ldots, k_{\underline{T}}\}$, $\underline{T}^{(\ell)}$ is the family of $n_\ell \equiv n_\ell(\underline{T}) \in \mathbb{N}$ triangles in $\underline{T}$, all having mass ℓ. By convention $n_\ell(\underline{T}) = 0$ when there is no triangle of mass ℓ in $\underline{T}$. We denote

$$|\underline{T}|^x = \sum_{\ell=1}^{k_{\underline{T}}} n_\ell(\underline{T})\, \ell^x, \; x \in \mathbb{R}, \; x \neq 0, \tag{4.11}$$

and

$$\log |\underline{T}| = \sum_{\ell=1}^{k_{\underline{T}}} n_\ell(\underline{T})(4 + \log \ell). \tag{4.12}$$

Let $\Lambda \subset \mathbb{Z}$ be an interval large enough, $V \subset \Lambda$ and L an integer, $L \leq |V|$. We study $\mu^+_\Lambda(\cup_{i\in V}\nu_i(L,-))$, the case of $\mu^+_\Lambda(\cup_{i\in V}\nu_i(L,+))$ can be treated along the same lines. Since $\mu^+_\Lambda(\cup_{i\in V}\nu_i(L,-)) \leq \sum_{i\in V} \mu^+_\Lambda(\nu_i(L,-))$, it is enough to estimate $\mu^+_\Lambda(\nu_i(L,-))$ for a given $i \in V$. Applying (4.2) one has

$$\mu^+_\Lambda(\nu_i(L,-)) \leq \sum_{\ell_0=1}^{L} \sum_{\Delta:\Delta\ni i, |\Delta|=\ell_0} \mu^+_\Lambda(\mathcal{W}(\Delta,-)). \tag{4.13}$$

It remains to estimate $\mu^+_\Lambda(\mathcal{W}(\Delta,-))$, for a given $i \in V$, $\Delta \ni i$ and $|\Delta| = \ell_0$. We denote by

$$\mathcal{C} = \mathcal{C}(\Delta,-) = \{\underline{T} \in \mathcal{T} \text{compatible with } \mathcal{W}(\Delta,-)\}. \tag{4.14}$$

A family $\underline{T}$ is said compatible with the event $\mathcal{W}(\Delta,-)$ if $\underline{T}$ corresponds to a spin configuration where the event $\mathcal{W}(\Delta,-)$ occurs. By construction the families of triangles in $\mathcal{C}$ satisfy only one of the two following conditions:

- there exists $T_0 \in \mathcal{C}$ so that $\Delta = supp(T_0)$
- there exist two triangles $T_{right} = T_{right}(\Delta)$ and $T_{left} = T_{left}(\Delta)$ one on the right and one on the left of Δ that are adjacent * to Δ.

The fact that T_{left} (resp. T_{right}) is on the left (resp. right) of Δ and is adjacent to it will be denoted by $T_{left} \triangleleft \Delta$, (resp $T_{right} \triangleright \Delta$). By (5.2) $\ell_0 = \text{dist}(T_{left}, T_{right}) \geq |T_{right}| \wedge |T_{left}|$, *i.e.* at least one of the two triangles (T_{left}, T_{right}) has support smaller or equal than ℓ_0. We write

$$\mathcal{C} \subseteq \cup_{j=1}^3 \mathcal{A}_j \tag{4.15}$$

* We say that T is adjacent to an interval Δ if $0 < d(supp(T), \Delta) < 1$. *i.e.* $\Delta \cap supp(T) = \emptyset$ and T is the first triangle on the right or the left of Δ having the support at distance from Δ smaller than 1.

where $\mathcal{A}_j = \mathcal{A}_j(\Delta, i)$ are defined by:

$$\mathcal{A}_1 = \{\underline{T} \in \mathcal{C} : \exists T_0 \in \underline{T}, supp(T_0) = \Delta\}; \tag{4.16}$$

$$\mathcal{A}_2 = \cup_{\ell=1}^{\ell_0} \mathcal{A}_2(\ell) \text{ with } \mathcal{A}_2(\ell) = \{\underline{T} \in \mathcal{C} : \exists T_{left} \in \underline{T},\, T_{left} \triangleleft \Delta,\, |T_{left}| = \ell\}; \tag{4.17}$$

$$\mathcal{A}_3 = \cup_{\ell=1}^{\ell_0} \mathcal{A}_3(\ell) \text{ with } \mathcal{A}_3(\ell) = \{\underline{T} \in \mathcal{C} \setminus \mathcal{A}_2 : \exists T_{right} \in \underline{T},\, T_{right} \triangleright \Delta,\, |T_{right}| = \ell\}. \tag{4.18}$$

We have

$$\mu_\Lambda^+(\mathcal{W}(\Delta, -)) \le \mu_\Lambda^+(\mathcal{A}_1) + \mu_\Lambda^+(\mathcal{A}_2) + \mu_\Lambda^+(\mathcal{A}_3). \tag{4.19}$$

Any family of triangles in $\mathcal{A}_1$ can be written as $\underline{T} \cup T_0 \in \mathcal{A}_1$ where $T_0 \notin \underline{T}$. We denote by $\mathcal{A}_1 \setminus T_0$ the set all these $\underline{T}$ such that $T_0 \cup \underline{T} \in \mathcal{A}_1$, with the same meaning we denote $\mathcal{A}_2(\ell) \setminus T_{left}$ and $\mathcal{A}_3(\ell) \setminus T_{right}$.

We start analyzing the first term on the right hand side of (4.19). Given $\underline{T} = \underline{T}' \cup T_0 \in \mathcal{A}_1$, call $J(T_0, \underline{T})$ the maximum interval with respect to inclusion, containing $\mathrm{supp}(T_0)$ with the property that all the other triangles $S \in \underline{T}'$ with $\mathrm{supp}(S) \subset J(T_0, \underline{T})$ have mass $|S| < |T_0|$. If all the triangles $S \in \underline{T}'$ have mass $|S| < |T_0|$ then $J(T_0, \underline{T}) = \Lambda$, otherwise either $J(T_0, \underline{T})$ is the base of a triangle containing T_0 or is adjacent to at least one triangle with mass larger or equal to $|T_0|$.

For $\underline{T} \in \mathcal{A}_1$ consider the set

$$\mathcal{I}(T_0, \underline{T}) = \{S \in \underline{T} : \mathrm{supp}(S) \subset J(T_0, \underline{T})\},$$

of triangles $S \in \underline{T}$ with $\mathrm{supp}(S) \subset J(T_0, \underline{T})$ and partition them in contours, with a constant $C = |T_0|$ (*c.f.* (5.5) for the definition of C) disregarding any other triangle not in $\mathcal{I}(T_0, \underline{T})$.

These contours have the following properties :

(1) each contour is composed of triangles having mass smaller or equal to $|T_0|$;
(2) the distance between two such contours is larger or equal to $|T_0|$;
(3) all contours are mutually external, *i.e.* there are no contours nested inside other contours;
(4) for all T with $\mathrm{supp}(T) \subset J(T_0, \underline{T})$, $\mathrm{dist}(T, J^c(T_0, \underline{T})) \ge |T_0|$.

Remark: The contours introduced in [5] have the property that, given a contour, its interaction with all the others contours can be made arbitrary small with a suitable choice of C, see Theorem 3.2 of [5], condition (3.15). The reduced contours that we introduce in this paper do not share this property, but allow to single out a set of triangles containing T_0 and to estimate a lower bound for their contribution to the energy that is uniform for all compatible configurations. This is a consequence of properties (1)–(4) that allow to apply Lemma 5.1.

Let Γ_0 be the contour that contains T_0. We identify in Γ_0 families of triangles having the same mass, rearranged in increasing order, see (4.10) and (4.11). By construction we have that $k_{\Gamma_0} = |T_0|$ and $n_{k_{\Gamma_0}}(\Gamma_0) = 1$. We write $\Gamma_0 = (\underline{T}^{(1)}, \dots, \underline{T}^{(k_{\Gamma_0}-1)}, T_0)$ *i.e* if $T_\ell \in \underline{T}^{(\ell)}$, for $\ell \in \{1, \dots k_{\Gamma_0} - 2\}$ we have

$$|T_\ell| < |T_{\ell+1}| < |T_0|. \tag{4.20}$$

Notice that T_0 is the only triangle in Γ_0 having mass strictly bigger than the mass of any other triangle. This holds for any Γ_0 constructed in such a way. For $\ell \in \{1, \dots, k_{\Gamma_0} - 1\}$, $n_\ell(\Gamma_0)$, i.e. the number of triangles having mass ℓ, depends on the Γ_0 we are considering. Properties (1)–(4) above entail to apply Lemma 5.1 when $\alpha \in (0, \frac{1}{2})$. We obtain

$$H_0^+(\underline{T}' \cup T_0) - H_0^+((\underline{T}' \setminus \underline{T}^{(1)}) \cup T_0) \ge \zeta |\underline{T}^{(1)}|^\alpha \tag{4.21}$$

and iterating ℓ times

$$H_0^+(\underline{T}'\cup T_0)-H_0^+((\underline{T}'\setminus\cup_{k=1}^{\ell}\underline{T}^{(k)})\cup T_0)\geq\zeta\sum_{k=1}^{\ell}|\underline{T}^{(k)}|^{\alpha}. \tag{4.22}$$

The last iteration gives

$$H_0^+(\underline{T}'\cup T_0)-H_0^+(\underline{T}'\setminus\Gamma_0)\geq\zeta\sum_{k=1}^{k_{\Gamma_0}-1}|\underline{T}^{(k)}|^{\alpha}+\zeta|T_0|^{\alpha}. \tag{4.23}$$

Given T_0, let $\mathcal{C}_{T_0}$ be the set of contours so that if $\Gamma_0\in\mathcal{C}_{T_0}$, then $T_0\in\Gamma_0$, the (1)– (4) and (4.20) are satisfied. The $\mathcal{C}_{T_0}$ is the the set of *reduced contours* containing T_0. We can then write

$$\mu_\Lambda^+(\mathcal{A}_1)=\frac{1}{Z_\Lambda^+}\sum_{\underline{T}'\in\mathcal{A}_1\setminus T_0}e^{-\beta H^+(\underline{T}'\cup T_0)[\omega]}=\sum_{\Gamma_0\in\mathcal{C}_{T_0}}\frac{1}{Z_\Lambda^+}\sum_{\underline{T}'\sim\Gamma_0}e^{-\beta H^+(\underline{T}'\cup\Gamma_0)[\omega]}, \tag{4.24}$$

where $\underline{T}'\sim\Gamma_0$ means that the configuration of triangles $\underline{S}=\underline{T}'\cup(\cup_{T\in\Gamma_0}T)$, is such that $\underline{S}\in\mathcal{A}_1$ and the family of reduced contours with basis $J(T_0,\underline{S})$ contains Γ_0. We set

$$\mu_\Lambda^+(\Gamma_0)=\frac{1}{Z_\Lambda^+}\sum_{\underline{T}'\sim\Gamma_0}e^{-\beta H^+(\underline{T}'\cup\Gamma_0)[\omega]}. \tag{4.25}$$

We apply, although in a different context, the method used in [8] which consists of 4 steps. We consider first the case $0<\alpha<1/2$, the case $\alpha=0$ and $\alpha=\frac{1}{2}$ will be discussed later.

Step I

For a fixed $\Gamma_0=(\underline{T}^{(1)},\dots,\underline{T}^{(k_{\Gamma_0}-1)},T_0)$, see (4.10), for each $j=\{1,\dots,k_{\Gamma_0}-1\}$ we extract a term $\sum_{k=1}^{j}n_k(\Gamma_0)k^{\alpha}$ from the deterministic part of the Hamiltonian, *i.e.* using (4.22) we write

$$\mu_\Lambda^+(\Gamma_0)\leq e^{-\beta\zeta(\sum_{k=1}^{j}n_k(\Gamma_0)k^{\alpha})}\frac{1}{Z_\Lambda^+[\omega]}\sum_{\underline{T}'\sim\Gamma_0}e^{-\beta H_0^+(\underline{T}'\cup(\Gamma_0\setminus\cup_{k=1}^{j}\underline{T}^{(k)}))+\beta\theta G(\sigma(\underline{T}'\cup\Gamma_0))[\omega]}. \tag{4.26}$$

We add to this list of $k_{\Gamma_0}-1$ inequalities the one we get after extracting the whole Γ_0 *i.e.* using (4.23)

$$\mu_\Lambda^+(\Gamma_0)\leq e^{-\beta\zeta(\sum_{k=1}^{k_{\Gamma_0}-1}n_k(\Gamma_0)k^{\alpha}+|T_0|^{\alpha})}\frac{1}{Z_\Lambda^+[\omega]}\sum_{\underline{T}'\sim\Gamma_0}e^{-\beta H_0^+(\underline{T}')+\beta\theta G(\sigma(\underline{T}'\cup\Gamma_0))[\omega]}. \tag{4.27}$$

Observing the right hand side of (4.26) and (4.27), one notes that the H_0^+ and G are not evaluated at the same configuration of triangles. In the next step we compensate this discrepancy by a corrective term.

Step II

For each $j\in\{1,\dots,k_{\Gamma_0}-1\}$ we multiply and divide (4.26) by

$$\sum_{\underline{T}'\sim\Gamma_0}e^{-\beta H_0^+(\underline{T}'\cup(\Gamma_0\setminus\cup_{k=1}^{j}\underline{T}^{(k)}))+\beta\theta G(\sigma(\underline{T}'\cup(\Gamma_0\setminus\cup_{k=1}^{j}\underline{T}^{(k)})))[\omega]} \tag{4.28}$$

and when $j=k_{\Gamma_0}$, see (4.27) by

$$\sum_{\underline{T}'\sim\Gamma_0}e^{-\beta H_0^+(\underline{T}')+\beta\theta G(\sigma(\underline{T}'))[\omega]}. \tag{4.29}$$

Setting for $j \in \{1, \ldots, k_{\Gamma_0} - 1\}$

$$F_j[\omega] = \frac{1}{\beta} \log \left\{ \frac{\sum_{\underline{T}' \sim \Gamma_0} e^{-\beta H_0^+(\underline{T}' \cup (\Gamma_0 \setminus \cup_{k=1}^{j} \underline{T}^{(k)})) + \beta\theta G(\sigma(\underline{T}' \cup \Gamma_0))[\omega]}}{\sum_{\underline{T}' \sim \Gamma_0} e^{-\beta H_0^+(\underline{T}' \cup (\Gamma_0 \setminus \cup_{k=1}^{j} \underline{T}^{(k)})) + \beta\theta G(\sigma(\underline{T}' \cup (\Gamma_0 \setminus \cup_{k=1}^{j} \underline{T}^{(k)})))[\omega]}} \right\} \tag{4.30}$$

and for $j = k_{\Gamma_0}$

$$F_{k_{\Gamma_0}}[\omega] = \frac{1}{\beta} \log \left\{ \frac{\sum_{\underline{T}' \sim \Gamma_0} e^{-\beta H_0^+(\underline{T}') + \beta\theta G(\sigma(\underline{T}' \cup \Gamma_0))[\omega]}}{\sum_{\underline{T}' \sim \Gamma_0} e^{-\beta H_0^+(\underline{T}') + \beta\theta G(\sigma(\underline{T}'))[\omega]}} \right\}, \tag{4.31}$$

we have the following set of inequalities: for $j \in \{1, \ldots, k_{\Gamma_0}\}$

$$\mu_\Lambda^+(\Gamma_0) \le e^{-\beta\zeta \sum_{\ell=1}^{j} n_\ell(\Gamma_0)\ell^\alpha + \beta F_j[\omega]} \mu_\Lambda^+(\Gamma_0 \setminus \cup_{k=1}^{j} \underline{T}^{(k)}) \le e^{-\beta\zeta \sum_{k=1}^{j} n_k(\Gamma_0) k^\alpha + \beta F_j[\omega]}. \tag{4.32}$$

Step III

We make a partition of the probability space to take into account the fluctuations of the F_i in (4.32). For each Γ_0 we write

$$\Omega = \cup_{j=0}^{k_{\Gamma_0}} B_j, \tag{4.33}$$

where, recalling (4.11), for $j \in \{1, \ldots, k_{\Gamma_0} - 1\}$

$$B_j = B_j(\Gamma_0) = \{\omega\, : F_j[\omega] \le \frac{\zeta}{2} \sum_{k=1}^{j} n_k(\Gamma_0)\, k^\alpha, \text{ and for } \forall i \in \{j+1, \ldots k_{\Gamma_0}\}, F_i[\omega] > \frac{\zeta}{2} \sum_{k=1}^{i} n_k(\Gamma_0)\, k^\alpha\}; \tag{4.34}$$

$$B_{k_{\Gamma_0}} = B_{k_{\Gamma_0}}(\Gamma_0) = \left\{ \omega\, : F_{k_{\Gamma_0}}[\omega] \le \frac{\zeta}{2} \left(\sum_{k=1}^{k_{\Gamma_0}-1} n_k(\Gamma_0)\, k^\alpha + |T_0|^\alpha \right) \right\}; \tag{4.35}$$

$$B_0 = B_0(\Gamma_0) = \{\omega\, : \forall i \in \{1, \ldots, k_{\Gamma_0}\}, F_i[\omega] > \frac{\zeta}{2} \sum_{k=1}^{i} n_k(\Gamma_0)\, k^\alpha\}. \tag{4.36}$$

The point is that using exponential inequalities for Lipschitz function of subgaussian random variables, see [8], Section 4 for details, one has : for all $\alpha \in (0, 1)$, for $0 \le j \le k_{\Gamma_0} - 1$,

$$I\!E\left[1\!\!1_{B_j}\right] \le e^{-\frac{\zeta^2}{2^{10}\theta^2}\left(\sum_{k=j+1}^{k_{\Gamma_0}-1} n_k(\Gamma_0)\, k^{2\alpha-1} + |T_0|^{2\alpha-1}\right)}, \tag{4.37}$$

with the convention that an empty sum is zero. For $j = k_{\Gamma_0}$ we use $I\!E\left[1\!\!1_{B_{k_{\Gamma_0}}}\right] \le 1$.

Step IV

By (4.33) we have

$$I\!E\left[\mu_\Lambda^+(\Gamma_0)\right] = \sum_{j=0}^{k_{\Gamma_0}} I\!E\left[\mu_\Lambda^+(\Gamma_0) 1\!\!1_{\{B_j\}}\right]. \tag{4.38}$$

For $j \in \{1, \ldots, k_{\Gamma_0}\}$, (4.32) entails

$$I\!E\left[\mu_\Lambda^+(\Gamma_0) 1\!\!1_{\{B_j\}}\right] \le e^{-\beta\zeta\left(\sum_{k=1}^{j} n_k(\underline{\Gamma}_0)\, k^\alpha\right)} I\!E\left[e^{\beta F_j} 1\!\!1_{\{B_j\}}\right]. \tag{4.39}$$

Recalling (4.34) and (4.35) on B_j we have

$$F_j \le \frac{\zeta}{2}\sum_{k=1}^{j} n_k(\Gamma_0)\, k^{\alpha}. \tag{4.40}$$

This with (4.39) and (4.37) gives

$$I\!E\left[\mu^+_\Lambda(\Gamma_0)1\!\!1_{\{B_j\}}\right] \le e^{-\beta\frac{\zeta}{2}\sum_{k=1}^{j} n_k(\Gamma_0)\,k^\alpha} e^{-\frac{\zeta^2}{2^{10}\theta^2}\left(\sum_{k=j+1}^{k_{\Gamma_0}-1} n_k(\Gamma_0)\,k^{2\alpha-1}+|T_0|^{2\alpha-1}\right)}. \tag{4.41}$$

Taking into account that for the set B_0, defined in (4.36), the estimate (4.37) holds, from (4.38) we get

$$\begin{aligned} I\!E\left[\mu^+_\Lambda(\Gamma_0)\right] &\le \sum_{j=0}^{k_{\Gamma_0}} e^{-\frac{\beta\zeta}{4}\sum_{k=1}^{j} n_k(\Gamma_0)\,k^\alpha} e^{-\frac{\zeta^2}{2^{10}\theta^2}\left(\sum_{k=j+1}^{k_{\Gamma_0}-1} n_k(\Gamma_0)\,k^{2\alpha-1}+|T_0|^{2\alpha-1}\right)} \\ &\le (k_{\Gamma_0}+1)e^{-\bar b\left(\sum_{k=1}^{k_{\Gamma_0}-1} n_k(\Gamma_0)\,k^{2\alpha-1}+|T_0|^{2\alpha-1}\right)}. \end{aligned} \tag{4.42}$$

We adopted, as before, the convention that an empty sum is zero, and set

$$\bar b = \min\left(\frac{\beta\zeta}{2}, \frac{\zeta^2}{2^{10}\theta^2}\right). \tag{4.43}$$

Final conclusions To estimate (4.13) we take into account the partition in (4.19) and for each $i \in V$ we write

$$I\!E\left[\mu^+_\Lambda(\nu_i(L,-))\right] \le I_1(i) + I_2(i) + I_3(i), \tag{4.44}$$

where $I_1(i)$ is defined in (4.45) and it is the contribution of the first term in (4.19), $I_2(i)$ is defined in (4.51) and it is the contribution of the second term in (4.19) and $I_3(i)$ is defined in a similar way as $I_2(i)$ and it is the contribution of the third term in (4.19). By (4.24), (4.25) and (4.42) we have

$$\begin{aligned} I_1(i) &\equiv \sum_{\ell_0=1}^{L}\ \sum_{T_0: T_0\ni i, |T_0|=\ell_0}\ \sum_{\Gamma_0\in\mathcal{C}_{T_0}, \Gamma_0\ni T_0} I\!E\left[\mu^+_\Lambda(\Gamma_0)\right] \\ &\le \sum_{\ell_0=1}^{L}\ \sum_{T_0: T_0\ni i, |T_0|=\ell_0}\ \sum_{\Gamma_0\in\mathcal{C}_{T_0}, \Gamma_0\ni T_0} (\ell_0+2)e^{-\bar b\left(\sum_{k=1}^{k_{\Gamma_0}-1} n_k(\Gamma_0)\,k^{2\alpha-1}+\ell_0^{2\alpha-1}\right)}. \end{aligned} \tag{4.45}$$

Since all the triangles in Γ_0 are smaller than ℓ_0, we have

$$\begin{aligned} \sum_{k=1}^{k_{\Gamma_0}-1} n_k(\Gamma_0)\,k^{2\alpha-1} + \ell_0^{2\alpha-1} &\ge \frac{1}{\ell_0^{1-2\alpha}(4+\log\ell_0)}\left(\sum_{k=1}^{k_{\Gamma_0}-1} n_k(\Gamma_0)(4+\log k) + (4+\log\ell_0)\right) \\ &= \frac{1}{\ell_0^{1-2\alpha}(4+\log\ell_0)}\left(\sum_{k=1}^{k_{\Gamma_0}} n_k(\Gamma_0)(4+\log k)\right) \end{aligned} \tag{4.46}$$

where we used that $k_{\Gamma_0} = |T_0| = \ell_0$ and $n_{k_{\Gamma_0}}(\Gamma_0) = 1$ by construction. Therefore, using (4.46), we have

$$\begin{aligned} I_1(i) &\le \sum_{\ell_0=1}^{L}(\ell_0+2)\sum_{T_0: T_0\ni 0, |T_0|=\ell_0}\ \sum_{\Gamma_0\in\mathcal{C}_{T_0}, \Gamma_0\ni T_0} e^{-\frac{\bar b}{\ell_0^{1-2\alpha}(4+\log\ell_0)}\left(\sum_{k=1}^{k_{\Gamma_0}} n_k(\Gamma_0)(4+\log k)\right)} \\ &\le \sum_{\ell_0=1}^{L}(\ell_0+2)\sum_{\Gamma:\Gamma\ni 0, |\Gamma|\ge\ell_0} e^{-\frac{\bar b}{\ell_0^{1-2\alpha}(4+\log\ell_0)}\left(\sum_{k=1}^{k_{\underline{\Gamma}}} n_k(\underline{\Gamma})(4+\log k)\right)} \\ &\le \sum_{\ell_0=1}^{L}(\ell_0+2)\sum_{m=\ell_0}^{\infty}\ \sum_{\Gamma:\Gamma\ni 0, |\Gamma|=m} e^{-\frac{\bar b}{\ell_0^{1-2\alpha}(4+\log\ell_0)}\left(\sum_{k=1}^{k_{\underline{\Gamma}}} n_k(\underline{\Gamma})(4+\log k)\right)}, \end{aligned} \tag{4.47}$$

where for each $\ell_0 \in \{1, \ldots, L\}$, the sum over $\Gamma : \Gamma \ni 0, |\Gamma| \geq \ell_0$ is in fact over the contours defined with a $C = \ell_0$ and mass at least ℓ_0.

To apply Theorem 5.2 to the last sum in (4.47), we need to impose that condition (5.9) holds when $C = |T_0|$ and $b \equiv b(T_0) = \frac{\bar{b}}{|T_0|^{1-2\alpha}(4+\log|T_0|)}$ for $|T_0| = \ell_0 \in \{1, \ldots, L\}$. By Remark 5.3 it is enough to take $b \geq D + (\log C)/4$ where $D \geq D_0 = \max(8, \frac{1}{4}\log\frac{2}{d_0})$, where d_0 is the quantity introduced in Theorem 5.2. Therefore, taking into account that $|T_0| = \ell_0$ we should require

$$\frac{\bar{b}}{\ell_0^{1-2\alpha}(4+\log \ell_0)} \geq D + (\log \ell_0)/4, \qquad \forall \ell_0 \in \{1, \ldots, L\}. \tag{4.48}$$

We impose a condition stronger than (4.48) which holds uniformly with respect to $1 \leq \ell_0 \leq L$. We require

$$\frac{\bar{b}}{|L|^{1-2\alpha}(4+\log|L|)} \geq Dg_2(\bar{b}, \alpha) \geq D_0 + \frac{\log L}{4}, \tag{4.49}$$

where the function $g_2(\bar{b}, \alpha)$, $\lim_{x\to\infty} g_2(x, \alpha) = \infty$, is introduced to get probabilities estimates comparable with those obtained in the upper bound. The actual choice of $g_2(\bar{b}, \alpha)$ is done later. The maximum value of L satisfying condition (4.49) is the L_{min} given in (4.7).

By Theorem 5.2 we can then estimate the last sum in (4.47) obtaining

$$I_1(i) \leq \sum_{\ell_0=1}^{L} (\ell_0 + 2) \sum_{m=\ell_0}^{\infty} 2me^{-(Dg_2(\bar{b},\alpha))(\log m+4)} \leq 10e^{-4Dg_2(\bar{b},\alpha)}. \tag{4.50}$$

Next we estimate the contribution of the second term in (4.44), the third term can be estimated in the same way. For each triangle T_{left} and for each contour Γ so that $T_{left} \in \Gamma$ we apply the estimates (4.42) and we obtain:

$$\begin{aligned} I_2(i) &\equiv \sum_{\ell_0=1}^{L} \sum_{\Delta:\Delta\ni i,\,|\Delta|=\ell_0} \sum_{\ell_1=1}^{\ell_0} \sum_{T_{left}:|T_{left}|=\ell_1} 1\!\!1_{\{T_{left} \lhd \Delta\}} \sum_{\Gamma_{left}\in \mathcal{C}_{T_{left}} \Gamma_{left}\ni T_{left}} I\!\!E\left[\mu_\Lambda^+(\Gamma_{left})\right] \\ &\leq \sum_{\ell_0=1}^{L} \ell_0 \sum_{\ell_1=1}^{\ell_0} (\ell_1+2) \sum_{\Gamma:\Gamma\ni 0;|\Gamma|\geq \ell_1} e^{-\bar{b}\left(\sum_{k=1}^{k_\Gamma-1} n_k(\Gamma)\,k^{2\alpha-1} + \ell_1^{2\alpha-1}\right)}. \end{aligned} \tag{4.51}$$

As before the k_Γ appearing in the previous formula is by construction $k_\Gamma = |T_{left}| = \ell_1$ and $n_{k_\Gamma}(\Gamma) = 1$. We can repeat the argument as in (4.46) and (4.47) obtaining

$$I_2(i) \leq \sum_{\ell_0=1}^{L} \ell_0 \sum_{\ell_1=1}^{\ell_0} (\ell_1+2) \sum_{m=\ell_1}^{\infty} \sum_{\Gamma:\Gamma\ni 0,|\Gamma|=m} e^{-\frac{\bar{b}}{\ell_1^{1-2\alpha}(4+\log \ell_1)}\left(\sum_{k=1}^{k_\Gamma} n_k(\underline{\Gamma})(4+\log k)\right)}. \tag{4.52}$$

To apply Theorem 5.2 to the last sum of (4.52) we need a condition similar to (4.48) which holds now uniformly with respect to $\ell_1 \in \{1, \ldots, \ell_0\}$ and $\ell_0 \in \{1, \ldots, L\}$. We obtain

$$I_2(i) = \sum_{\ell_0=1}^{L} \ell_0 \sum_{\ell_1=1}^{\ell_0} (\ell_1+2) \sum_{m=\ell_1}^{\infty} 2me^{-Dg_2(\bar{b},\alpha)(\log m+4)} \leq 10L^2 e^{-4Dg_2(\bar{b},\alpha)}. \tag{4.53}$$

Collecting (4.47), (4.53) and adding the contribution from $I_3(i)$ we get

$$\mathbb{E}\left[\mu_\Lambda^+(\nu_i(L,-))\right] \le 30L^2 e^{-4Dg_2(\bar b,\alpha)}. \tag{4.54}$$

By Markov inequality, on a probability subset $\Omega_4 = \Omega_4(L,i)$ with

$$\mathbb{P}[\Omega(L,i)] \ge 1 - 6Le^{-2Dg_2(\bar b,\alpha)}, \tag{4.55}$$

one gets

$$\mu_\Lambda^+(\nu_i(L,-)) \le 6Le^{-2D_0 g_2(\bar b,\alpha)}. \tag{4.56}$$

Recalling the definition of $\mathcal{V}(V,L)$, see (4.3), one gets that on a probability subset $\Omega_5 = \Omega_5(V)$ with

$$\mathbb{P}[\Omega_5] \ge 1 - 6|V|Le^{-2Dg_2(\bar b,\alpha)} \tag{4.57}$$

$$\mu_\Lambda^+(\mathcal{V}(V,L)) \le 6|V|Le^{-2Dg_2(\bar b,\alpha)}. \tag{4.58}$$

The choice of parameters

• $0 < \alpha < \frac{1}{2}$.

Choosing $L = L_{\min}(\alpha)$ as in (4.7) and $g_2(\bar b,\alpha) \ge 1 + \frac{1}{4}\frac{1}{1-2\alpha}\log \bar b$, see (4.4), we have that the inequalities of (4.49) are satisfied. The estimates in (4.6) and (4.9) will follow from (4.57) and (4.58) taking the interval V as in (4.8).

• $\alpha = 0$.

Going back to (4.26), the modifications are the following : each time k^α, respectively $|T|^\alpha$, appears replace it by $(4+\log k)$, respectively by $(4+\log|T|)$. The events defined in step III are modified in the same way. The only mathematical difference comes with (4.37) replaced by

$$\mathbb{E}\left[1\!\!1_{B_j}\right] \le e^{-\frac{\zeta^2}{2^{10}\theta^2}\left(\sum_{k=j+1}^{k_{\Gamma_0}-1} n_k(\Gamma_0)\frac{(4+\log k)^2}{k} + \frac{(4+\log|T_0|)^2}{|T_0|}\right)}. \tag{4.59}$$

Taking into account that

$$\frac{(4+\log k)^2}{k} \ge \frac{(4+\log \ell_0)}{\ell_0}(4+\log k) \tag{4.60}$$

the formula (4.46) is replaced by

$$\sum_{k=1}^{k_{\Gamma_0}-1}\frac{(4+\log k)^2}{k} + \frac{(4+\log|T_0|)^2}{|T_0|} \ge \frac{(4+\log|T_0|)}{|T_0|}\left(\sum_{k=1}^{k_{\Gamma_0}} n_k(\Gamma_0)(4+\log k)\right). \tag{4.61}$$

The requirements in (4.49) become

$$\bar b\,\frac{4+\log L}{L} \ge Dg_2(\bar b,0) \tag{4.62}$$

and

$$g_2(\bar b,0) \ge 1 + \frac{\log L}{4} \tag{4.63}$$

where as before $D \ge D_0 = \max(8, \frac{1}{4}\log\frac{2}{d_0})$ and $\lim_{x\to\infty} g_2(x,0) = \infty$. The conditions (4.62) and (4.63) are satisfied choosing

$$\frac{3}{2}\log\frac{2}{3}\frac{\bar b}{\log \bar b} \le g_2(\bar b,0), \tag{4.64}$$

$$L \equiv L_{\min} = \frac{\bar{b}}{Dg_2(\bar{b},0)}\left(4 + \log\frac{\bar{b}}{Dg_2(\bar{b},0)}\right) \tag{4.65}$$

and assuming that $\bar{b}/Dg_2(\bar{b},0) \geq 1$. Taking

$$\mathrm{diam}(V_{min}(0)) = \frac{\bar{b}}{Dg_2(\bar{b},0)}\left(4 + \log\left[\frac{\bar{b}}{Dg_2(\bar{b},0)}\right]\right) \tag{4.66}$$

the estimates (4.6) and (4.9) follow.

• $\alpha = 1/2$.
The estimate (4.37) is replaced by

$$I\!E\left[1\!\!\mathrm{I}_{B_j}\right] \leq e^{-\frac{\zeta^2}{2^{10}\theta^2}\left(1+\sum_{k=j+1}^{k_{\Gamma_0}-1} n_\ell(\underline{\Gamma}_0)\right)}. \tag{4.67}$$

Since for any $\ell > 0$, $1 + \sum_{k=1}^{\ell} n_k(\underline{\Gamma}_0) \geq 1$ the formula (4.42) becomes

$$I\!E[\mu_\Lambda^+(\Gamma_0] \leq (k_{\Gamma_0}+1)e^{-\frac{\bar{b}}{2}}e^{-\frac{\bar{b}}{2}\left(1+\sum_{k=1}^{k_{\Gamma_0}-1} n_k(\Gamma_0)\right)} \leq (k_{\Gamma_0}+1)e^{-\frac{\bar{b}}{2}}e^{-\frac{\bar{b}}{2}\frac{1}{(4+\log\ell_0)}\left(\sum_{k=1}^{k_{\Gamma_0}} n_k(\Gamma_0)(4+\log k)\right)}, \tag{4.68}$$

where in the last inequality we took into account that $k_{\Gamma_0} = \ell_0$ and $n_{k_{\Gamma_0}}(\Gamma_0) = 1$. To apply Theorem 5.2 we assume

$$\frac{\bar{b}}{2}\frac{1}{(4+\log L)} \geq D\,(4+\log L) \tag{4.69}$$

where $D \geq \max(8, \frac{1}{4}\log\frac{2}{d_0})$. For

$$L \equiv L_{\min}(1/2) = e^{-4+\sqrt{\frac{\bar{b}}{2D}}} \tag{4.70}$$

(4.69) is satisfied. Further taking into account that

$$\frac{\bar{b}}{2}\frac{1}{(4+\log L)} = \sqrt{\frac{\bar{b}D}{2}},$$

and estimate (4.68), we get

$$I\!E\left[\mu_\Lambda^+(\nu_i(L,-))\right] \leq 30e^{-\frac{\bar{b}}{2}}L^2e^{-\sqrt{\frac{\bar{b}D}{2}}} = 30e^{-8}e^{-\frac{\bar{b}}{2}}e^{-2\sqrt{\bar{b}}\frac{(2D-1)}{\sqrt{2D}}}. \tag{4.71}$$

By Markov's inequality, see (4.55), on a probability subset $\Omega = \Omega(L,i)$ with

$$I\!P[\Omega(L,i)] \geq 1 - 6e^{-4}e^{-\frac{\bar{b}}{4}}e^{-\sqrt{\bar{b}}\frac{(2D-1)}{\sqrt{2D}}} \tag{4.72}$$

one gets, since $6e^{-4} \leq 1$,

$$\mu_\Lambda^+(\nu_i(L,-)) \leq e^{-\frac{\bar{b}}{4}}e^{-\sqrt{\bar{b}}\frac{(2D-1)}{\sqrt{2D}}}. \tag{4.73}$$

Then, taking $V = e^{\frac{\bar{b}}{4}}$ we get

$$I\!P[\Omega_5] \geq 1 - e^{-\sqrt{\bar{b}}\frac{(2D-1)}{\sqrt{2D}}} \quad \text{and} \quad \mu_\Lambda^+(\mathcal{V}(V,L)) \leq e^{-\sqrt{\bar{b}}\frac{(2D-1)}{\sqrt{2D}}}. \tag{4.74}$$

■

Appendix: Geometrical description of the spin configurations

We will follow the geometrical description of the spin configuration presented in [5] and use the same notations. We will consider homogeneous boundary conditions, i.e the spins in the boundary conditions are either all +1 or all −1. Actually we will restrict ourself to + boundary conditions and consider spin configurations $\sigma = \{\sigma_i, i \in \mathbb{Z}\} \in \mathcal{X}_+$ so that $\sigma_i = +1$ for all $|i|$ large enough.

In one dimension an interface at $(x, x+1)$ means $\sigma_x \sigma_{x+1} = -1$. Due to the above choice of the boundary conditions, any $\sigma \in \mathcal{X}_+$ has a finite, even number of interfaces. The precise location of the interface is immaterial and this fact has been used to choose the interface points as follows: For all $x \in \mathbb{Z}$ so that $(x, x+1)$ is an interface take the location of the interface to be a point inside the interval $[x + \frac{1}{2} - \frac{1}{100}, x + \frac{1}{2} + \frac{1}{100}]$, with the property that for any four distinct points r_i, $i = 1, \dots, 4$ $|r_1 - r_2| \neq |r_3 - r_4|$. This choice is done once for all so that the interface between x and $x+1$ is uniquely fixed. Draw from each one of these interfaces points two lines forming respectively an angle of $\frac{\pi}{4}$ and of $\frac{3}{4}\pi$ with the $\mathbb{Z}$ line. We have thus a bunch of growing $\vee-$ lines each one emanating from an interface point. Once two $\vee-$ lines meet, they are frozen and stop their growth. The other two lines emanating from the the same interface points are erased. The $\vee-$ lines emanating from others points keep growing. The collision of the two lines is represented graphically by a triangle whose basis is the line joining the two interfaces points and whose sides are the two segment of the $\vee-$ lines which meet. The choice done of the location of the interface points ensure that collisions occur one at a time so that the above definition is unambiguous. In general there might be triangles inside triangles. The endpoints of the triangles are suitable coupled pairs of interfaces points. The graphical representation just described maps each spin configuration in $\mathcal{X}_+$ to a set of triangles.

Notation *Triangles will be usually denoted by T, the collection of triangles constructed as above by $\mathcal{T}$ and we will write*

$$|T| = \textit{cardinality of } T \cap \mathbb{Z} = \textit{mass of } T, \tag{5.1}$$

and by supp$(T) \subset \mathbb{R}$ the basis of the triangle.

We have thus represented a configuration $\sigma \in \mathcal{X}_+$ as a collection of $\underline{T} = (T_1, \dots, T_n)$. The above construction defines a one to one map from $\mathcal{X}_+$ onto $\mathcal{T}$. It is easy to see that a triangle configuration $\underline{T}$ belongs to $\mathcal{T}$ iff for any pair T and T' in $\underline{T}$

$$\mathrm{dist}(T, T') \geq \min\{|T|, |T'|\}. \tag{5.2}$$

Here dist(T, T') is the cardinality of $I \cap \mathbb{Z}$ where I is the interval between T and T' if T and T' are disjoint; if T and T' are one contained in the other the I is the smallest interval between the two.

We say that two collections of triangles $\underline{S}'$ and $\underline{S}$ are compatible and we denote it by $\underline{S}' \sim \underline{S}$ iff $\underline{S}' \cup \underline{S} \in \mathcal{T}$ (*i.e.* there exists a configuration in $\mathcal{X}_+$ such that its corresponding collection of triangles is the collection made of all triangles that are obtained by concatenating $\underline{S}'$ and $\underline{S}$.) By an abuse of notation, we write

$$H_0^+(\underline{T}) = H_0^+(\sigma), \quad G(\sigma(\underline{T}))[\omega] = G(\sigma)[\omega], \quad \sigma \in \mathcal{X}_+ \iff \underline{T} \in \mathcal{T}.$$

Contours A contour Γ is a collection $\underline{T}$ of triangles related by a hierarchical network of connections controlled by a positive number C, see (5.4), under which all the triangles of a contour become mutually connected. The constant C must be chosen so that

$$\sum_{m \geq 1} \frac{4m}{[Cm]^3} \leq \frac{1}{2} \tag{5.3}$$

where $[x]$ denotes the integer part of x. Note that $C \geq 4$ implies (5.3). For our construction we need C to satisfy (5.3) and further constraints.

We denote by $T(\Gamma)$ the smallest interval which contains the basis of all triangles of the contour Γ. The right and left endpoints of $T(\Gamma) \cap \mathbb{Z}$ are denoted by $x_{\pm}(\Gamma)$. We denote $|\Gamma|$ the mass of the contour Γ

$$|\Gamma| = \sum_{T \in \Gamma} |T|$$

i.e. $|\Gamma|$ is the sum of the masses of all the triangles belonging to Γ. We denote by $\mathcal{R}(\cdot)$ the algorithm which associates to any configuration $\underline{T}$ a configuration $\{\Gamma_j\}$ of contours with the following properties.

P.0 *Let* $\mathcal{R}(\underline{T}) = (\Gamma_1, \ldots, \Gamma_n)$, $\Gamma_i = \{T_{j,i}, 1 \leq j \leq k_i\}$, *then* $\underline{T} = \{T_{j,i}, 1 \leq i \leq n, 1 \leq j \leq k_i\}$

P.1 *Contours are well separated from each other.* Any pair $\Gamma \neq \Gamma'$ verifies one of the following alternatives.

$$T(\Gamma) \cap T(\Gamma') = \emptyset$$

i.e. $[x_-(\Gamma), x_+(\Gamma)] \cap [x_-(\Gamma'), x_+(\Gamma')] = \emptyset$, in which case

$$dist(\Gamma, \Gamma') := \min_{T \in \Gamma, T' \in \Gamma'} dist(T, T') > C \min\left\{|\Gamma|^3, |\Gamma'|^3\right\} \tag{5.4}$$

where C is a positive number. If

$$T(\Gamma) \cap T(\Gamma') \neq \emptyset,$$

then either $T(\Gamma) \subset T(\Gamma')$ or $T(\Gamma') \subset T(\Gamma)$; moreover, supposing for instance that the former case is verified, (in which case we call Γ an inner contour) then for any triangle $T'_i \in \Gamma'$, either $T(\Gamma) \subset T'_i$ or $T(\Gamma) \cap T'_i = \emptyset$ and

$$dist(\Gamma, \Gamma') > C|\Gamma|^3, \quad \text{if} \quad T(\Gamma) \subset T(\Gamma'). \tag{5.5}$$

P.2 *Independence.* Let $\{\underline{T}^{(1)}, \ldots, \underline{T}^{(k)}\}$, be $k > 1$ configurations of triangles; $\mathcal{R}(\underline{T}^{(i)}) = \{\Gamma_j^{(i)}, j = 1, \ldots, n_i\}$ the contours of the configurations $\underline{T}^{(i)}$. Then if any distinct $\Gamma_j^{(i)}$ and $\Gamma_{j'}^{(i')}$ satisfies **P.1**,

$$\mathcal{R}(\underline{T}^{(1)}, \ldots, \underline{T}^{(k)}) = \{\Gamma_j^{(i)}, j = 1, \ldots, n_i; i = 1, \ldots, k\}.$$

As proven in [5], the algorithm $\mathcal{R}(\cdot)$ having properties **P.0**, **P.1** and **P.2** is unique and therefore there is a bijection between families of triangles and contours.

Next we present in a way more suitable to our needs the results proven in [5]. Lemma 5.1 deals only with triangles, Theorem 5.2 with countours.

Lemma 5.1

Take $\underline{T} \in \mathcal{T}$ *and a triangle* $\widehat{T} \in \underline{T}$ *which does not contain any other triangle and such that*

$$\inf_{T \in \underline{T}} \text{dist}(\widehat{T}, T) \geq |\widehat{T}|. \tag{5.6}$$

For $\alpha \in (0, \frac{\log 3}{\log 2} - 1)$ *we have*

$$H_0^+(\underline{T} \cup \widehat{T}) - H_0^+(\underline{T}) \geq \zeta |\widehat{T}|^{\alpha} \tag{5.7}$$

where $\zeta = (1 - 2(2^\alpha - 1))$. *For* $\alpha = 0$, *we have*

$$H_0^+(\underline{T} \cup \widehat{T}) - H_0^+(\underline{T}) \geq 2\log L + 8. \tag{5.8}$$

Proof: *c.f.* proof of Lemma 2.1 and and Lemma A.1 of [5].

Theorem 5.2

There exists an absolute constant d_0 *such that for all* $C > 1$, *where* C *is the constant in the contour definition, see (5.4), and for all* $b > 0$ *so that*

$$C \sum_{x=1}^{\infty} x^6 e^{-b(\log x + 4)} \leq d_0, \tag{5.9}$$

the following holds: for all integers $m \geq 1$

$$\sum_{\{0 \in \Gamma, |\Gamma| = m\}} w_b^0(\Gamma) \leq 2m e^{-b(\log m + 4)},$$

where

$$w_b^0(\Gamma) = \prod_{T \in \Gamma} e^{-b(\log |T| + 4)}. \tag{5.10}$$

We explicitly quantify the condition (5.9) under which Theorem 4.1 of [5] holds. This can de deduced by looking at the proof of Theorem 4.1, see Section 4.3 of [5].

Remark 5.3 . Notice that for $b \geq D + (\log C)/4$ where $D \geq D_0 = \max(8, \frac{1}{4}\log\frac{2}{d_0})$ the condition (5.9) is satisfied.

References

[1] M. Aizenman, J. Chayes, L. Chayes and C. Newman: Discontinuity of the magnetization in one–dimensional $1/|x-y|^2$ percolation, Ising and Potts models. *J. Stat. Phys.* **50** no. 1-2 1–40 (1988).

[2] M. Aizenman, and J. Wehr: Rounding of first order phase transitions in systems with quenched disorder. *Com. Math. Phys.* **130**, 489–528 (1990).

[3] A. Bovier *Statistical Mechanics of Disordered Systems.* Cambridge Series in Statistical and Probabilistic mathematics., (2006).

[4] J. Bricmont, and A. Kupiainen: Phase transition in the three-dimensional random field Ising model. *Com. Math. Phys.*,**116**, 539–572 (1988).

[5] M. Cassandro, P. A. Ferrari, I. Merola and E. Presutti: Geometry of contours and Peierls estimates in $d = 1$ Ising models with long range interaction. *J. Math. Phys.* **46**, no 5, (2005)

[6] M. Cassandro, E. Orlandi, and P.Picco: Typical configurations for one-dimensional random field Kac model. *Ann. Prob.* **27**, No 3, 1414-1467, (1999).

[7] M. Cassandro, E. Orlandi, P. Picco and M.E. Vares: One-dimensional random field Kac's model: Localization of the Phases *Electron. J. Probab.*, **10**, 786-864, (2005).

[8] M. Cassandro, E. Orlandi, and P.Picco: Phase Transition in the 1d Random Field Ising Model with long range interaction. *Comm. Math. Phy.*, **2**, 731-744 (2009)

[9] Y.S. Chow and H. Teicher *Probability theory. Independence, interchangeability, martingales.* Third edition. Springer Texts in Statistics. Springer-Verlag, New York, 1997.

[10] R. Dobrushin: The description of a random field by means of conditional probabilities and. conditions of its regularity. *Theory Probability Appl.* **13**, 197-224 (1968)

[11] R. Dobrushin: The conditions of absence of phase transitions in one-dimensional classical systems. *Matem. Sbornik*, **93** (1974), N1, 29-49

[12] R. Dobrushin: Analyticity of correlation functions in one-dimensional classical systems with slowly decreasing potentials. *Comm. Math. Phys.* **32** (1973), N4, 269-289

[13] F.J. Dyson: Existence of phase transition in a one-dimensional Ising ferromagnetic. *Comm. Math. Phys.*,**12**,91–107, (1969).

[14] J. Fröhlich and T. Spencer: The phase transition in the one-dimensional Ising model with $\frac{1}{r^2}$ interaction energy. *Comm. Math. Phys.*, **84**, 87–101, (1982).

[15] J.Z. Imbrie: Decay of correlations in the one-dimensional Ising model with $J_{ij} =| i-j |^{-2}$, *Comm. Math. Phys.* **85**, 491–515. (1982).

[16] J.Z. Imbrie and C.M. Newman: An intermediate phase with slow decay of correlations in one-dimensional $1/|x-y|^2$ percolation, Ising and Potts models. *Comm. Math. Phys.* **118** , 303–336 (1988).

[17] J. P. Kahane Propriétés locale des fonctions à séries de Fourier aléatoires *Studia Matematica* **19** 1–25 (1960).

[18] L. Le Cam *Asymptotic methods in statistical decision theory.* Springer Series in Statistics, Springer-Verlag, New York, Berlin, Heidelberg, (1986).

[19] E. Orlandi, and P.Picco: One-dimensional random field Kac's model: weak large deviations principle. *Electronic Journal Probability* **14**, 1372–1416, (2009).

[20] J. B. Rogers and C.J. Thompson: Absence of long range order in one dimensional spin systems. *J. Statist. Phys.* **25**, 669–678 (1981)

[21] D. Ruelle: Statistical mechanics of one-dimensional Lattice gas. *Comm. Math. Phys.* **9**, 267–278 (1968)