UNIQUENESS OF THE MINIMIZER FOR A RANDOM NONLOCAL FUNCTIONAL WITH DOUBLE-WELL POTENTIAL IN $d \leq 2$.

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Abstract. We consider a small random perturbation of the energy functional

$$[u]^2_{H^s(\Lambda,\mathbb{R}^d)} + \int_\Lambda W(u(x))dx$$

for $s \in (0,1)$, where the non-local part $[u]^2_{H^s(\Lambda,\mathbb{R}^d)}$ denotes the total contribution from $\Lambda \subset \mathbb{R}^d$ in the $H^s(\mathbb{R}^d)$ Gagliardo semi-norm of u and W is a double well potential. We show that there exists, as Λ invades $\mathbb{R}^d$, for almost all realizations of the random term a minimizer under compact perturbations, which is unique when $d = 2$, $s \in (\frac{1}{2}, 1)$ and when $d = 1$, $s \in [\frac{1}{4}, 1)$. This uniqueness is a consequence of the randomness. When the random term is absent, there are two minimizers which are invariant under translations in space, $u = \pm 1$.

1. Introduction

Non local functionals, related to fractional Levy partial differential equations, appear frequently in many different areas of mathematics and find many applications in engineering, finance [15], physics [13], chemistry [3] and biology [18]. We consider non local functionals representing the free energy of a material with two (or several) phases, see [5], on a a scale, the so-called mesoscopic scale, which is much larger than the atomistic scale so that the adequate description of the state of the material is by a *continuous* scalar order parameter $m : \Lambda \subseteq \mathbb{R}^d \to \mathbb{R}$. The minimizers of these functionals are functions m^* representing the states or phases of the materials.

The natural question that we pose is the following: What happens to these minimizers when an external, even very weak, random potential is added to the deterministic functional? Does the number of minimizers remain the same, i.e will the material always have the same number of states (or phases)? Is there some significant difference in the qualitative properties of the material when the randomness is added? These are standard questions in a calculus of variations framework.

Partial answers to these type of questions were recently given in two papers by the authors in the context of the Ginzburg Landau functional, i.e in the case where the interaction energy is local and it is modelled by $\langle m, (-\Delta)m\rangle$ there $\langle \cdot, \cdot \rangle$ stands for the L^2 scalar product and m is taken in a function space which makes the scalar product finite, see [6] and [7]. Here we consider a functional in which the interaction energy is non local, i.e. the state of the material at site $x \in \Lambda$ depends on the state of the material in all $\mathbb{R}^d$. We model this non local interaction energy using the fractional Laplacian.

This nonlocality of the interaction needs a very different approach compared to [6] and [7] because of the suitable interpretation of "boundary condition" in the case of a long-range interaction. In particular, an extensive part of the analytical work in the present paper is devoted to so-called minimizers under compact perturbations, see Definition 2.4.

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The *interaction energy* is given by $\langle m, (-\Delta)^s m\rangle$ for $0 < s < 1$, the scalar product and the function space for m need to be suitable defined. In the extreme case $s = 1$ one gets the Ginzburg Landau interaction energy and when $s = 0$ one gets $(-\Delta)^s = I$ where I is the identity operator, so m at site x interacts only with itself.

We add to this non local interaction energy which penalizes spatial changes in m a *double-well potential* $W(m)$, i.e. a nonconvex function which has exactly two minimizers, for simplicity $+1$ and -1, modelling a two-phase material.

Finally, we add a term which couples m to a *random field* $\theta g(\cdot, \omega)$ with mean zero, variance θ^2 and unit correlation length; i.e a term which prefers at each point in space one of the two minimizers of $W(\cdot)$ and thus breaks the translational invariance, but is "neutral" in the mean.

A functional with the aforementioned properties is the following functional

$$G_1^{m_0}(m, \omega, \Lambda) = [m]^2_{H^s(\Lambda,\mathbb{R}^d)} + \int_\Lambda W(m(x))\mathrm{dx} - \theta \int_\Lambda \mathrm{g}_1(\mathrm{x}, \omega)\mathrm{m}(\mathrm{x})\mathrm{dx}, \tag{1.1}$$

where

$$[m]^2_{H^s(\Lambda,\mathbb{R}^d)} = \int_\Lambda \mathrm{dx} \int_\Lambda \mathrm{dy} \frac{|\mathrm{m}(\mathrm{x}) - \mathrm{m}(\mathrm{y})|^2}{|\mathrm{x} - \mathrm{y}|^{\mathrm{d}+2\mathrm{s}}} + 2 \int_\Lambda \mathrm{dx} \int_{\mathbb{R}^\mathrm{d}\setminus\Lambda} \mathrm{dy} \frac{|\mathrm{m}(\mathrm{x}) - \mathrm{m}_0(\mathrm{y})|^2}{|\mathrm{x} - \mathrm{y}|^{\mathrm{d}+2\mathrm{s}}} \tag{1.2}$$

denotes the total contribution from Λ to the $H^s(\mathbb{R}^d)$ Gagliardo semi-norm of m, if we set $m = m_0$ in $\mathbb{R}^d \setminus \Lambda$ in (1.2). The Gagliardo semi-norm is given by

$$\int_{\mathbb{R}^d} \mathrm{dx} \int_{\mathbb{R}^\mathrm{d}} \mathrm{dy} \frac{|\mathrm{m}(\mathrm{x}) - \mathrm{m}(\mathrm{y})|^2}{|\mathrm{x} - \mathrm{y}|^{\mathrm{d}+2\mathrm{s}}} = [\mathrm{m}]^2_{\mathrm{H}^\mathrm{s}(\Lambda,\mathbb{R}^\mathrm{d})} + \int_{\mathbb{R}^\mathrm{d}\setminus\Lambda} \mathrm{dx} \int_{\mathbb{R}^\mathrm{d}\setminus\Lambda} \mathrm{dy} \frac{|\mathrm{m}_0(\mathrm{x}) - \mathrm{m}_0(\mathrm{y})|^2}{|\mathrm{x} - \mathrm{y}|^{\mathrm{d}+2\mathrm{s}}}. \tag{1.3}$$

For the minimization problem the term depending only on the value of m_0 in the Gagliardo semi-norm is irrelevant, since this term is kept fixed trough the minimization procedure. For dimensional reason the right hand side of (1.2) should be multiplied by $c_{d,s}$, a normalizing constant which degenerates when $s \to 1$ or $s \to 0$. In the following the constant $c_{d,s}$ does not play any role, so we replace it by 1.

We are interested in determining the *macroscopic minimizers* of (1.1), i.e minimizers of (1.1) over sequences of regions Λ_n so that $\Lambda_n \nearrow \mathbb{R}^d$ as $n \to \infty$. Namely for any given Λ and fixed boundary value m_0 the minimizers of (1.1) over any reasonable set of functions will depend on the boundary value m_0. Physically one is interested in taking Λ large enough and to characterize the minimizers in a region deep inside Λ and detect if, even so deeply inside, the boundary condition is felt. In other word Λ needs to be large to invade $\mathbb{R}^d$ and we are interested in characterizing the macroscopic minimizer which we construct by a limit procedure using minimization on a sequence of finite subsets of $\mathbb{R}^d$.

When $\theta = 0$, i.e without random term, the constant functions equal to ± 1 are the two macroscopic minimizers: One can obtain the $+1$ (-1) minimizer as the limit of the minimizers of (1.1) when $\theta = 0$ with strictly positive (strictly negative) boundary values by making use of the fact that the cost of a "boundary layer" near the boundary of large balls is of smaller order than the volume as the balls invade $\mathbb{R}^d$, a point to which we will come back below, see (1.4).

When the random field is added, the constant functions equal to ± 1 are not minimizers anymore, due to the presence of the random fields. The question is to show whether there are still two macroscopic minimizers, each one close in some topology to the constant minimizers 1 and -1.

We are able to show in $d = 2$ for $s \in (\frac{1}{2}, 1)$ and in $d = 1$ for $s \in [\frac{1}{4}, 1)$ that for almost all the realizations of the randomness, there exists one macroscopic minimizer which is unique under compact perturbations. In this regime the boundary conditions is not felt by the minimizer. This is an example of uniqueness induced by random terms. The uniqueness holds only in the limit $\Lambda \nearrow \mathbb{R}^d$ and is sensitive to the type of randomness added. We will come back to this point in subsection 2.1. For values of d and s different from the ones for which we state the uniqueness result we expect, for almost all the realizations of the randomness, the existence of at least two macroscopic minimizers, one "close" to the constant minimizer 1, the other "close" to the constant minimizer -1. But this issue is still open. The strategy of our proof is based on the following steps. We prove first that for almost all the realizations of the random field there exist two *macroscopic extremal* minimizers $v^{\pm}(\cdot,\omega)$ so that any other macroscopic minimizer under compact perturbations u^* satisfies $v^-(\cdot,\omega) \le u^*(\cdot,\omega) \le v^+(\cdot,\omega)$. This construction requires two limit procedures. First, for any bounded, sufficiently regular subset of $\mathbb{R}^d$, Λ, and for any $K > 0$ we determine the minimizers of G_1 in Λ with boundary condition $v_0 = K$. Since the functional is not convex there might be many minimizers. Because the set of minimizers in a bounded domain Λ is ordered and compact, we can single out one specific minimizer which we call the maximal $K-$ minimizer. Similarly we single out one specific minimizer G_1 in Λ with $v_0 = -K$ boundary condition, which we call the minimal $K-$ minimizer. The maximal $K-$ minimizer and the minimal $K-$ minimizer of G_1 in Λ have the property that any other minimizer of G_1 in Λ with boundary condition v_0, $\|v_0\|_\infty \le K$ is point wise smaller than the maximal $K-$ minimizer and larger than the minimal $K-$ minimizer of G_1 in Λ. Then we let Λ to invade $\mathbb{R}^d$ obtaining two infinite volume functions $u^{\pm,K}$, and we show that they are infinite volume minimizers under compact perturbations of G_1. At last, we define $v^{\pm}(\cdot,\omega)$ as the point wise limit as $K \to \infty$ of $u^{\pm,K}$, proving again that $v^{\pm}(\cdot,\omega)$ are extrema infinite volume minimizers under compact perturbations. Then we show that for any $s \in (0,1)$ there exists a positive constant C, so that for any bounded, sufficiently regular $\Lambda \subset \mathbb{R}^d$, for almost all the realizations of the random field,

$$\left| G_1^{v^+}(v^+,\omega,\Lambda) - G_1^{v^-}(v^-,\omega,\Lambda) \right| \le C|\Lambda|^{\frac{d-1}{d}} 1\!\!\mathrm{I}_{\{s\in(\frac{1}{2},1)\}} + C|\Lambda|^{\frac{d-2s}{d}} 1\!\!\mathrm{I}_{\{s\in(0,\frac{1}{2})\}} + 1\!\!\mathrm{I}_{\{s=\frac{1}{2}\}} |\Lambda|^{\frac{d-1}{d}} \log|\Lambda|. \tag{1.4}$$

The minimizers $v^{\pm}(\cdot,\omega)$ depend in a highly non trivial way on the random fields $\{g(x,\omega)\}_{\{x\in\mathbb{Z}^d\}}$. Therefore also the difference $G_1^{v^+}(v^+,\omega,\Lambda) - G_1^{v^-}(v^-,\omega,\Lambda)$ depends on the random fields in all of $\mathbb{Z}^d$. We take a sequence $\Lambda_n \subset \Lambda_{n+1}$ and we show that, conditioning on the random fields in Λ_n (i.e taking the expectation over only the random fields outside Λ_n)

$$F_n(\omega) := \mathbb{E}\left[\left\{ G_1(v^+(\omega),\omega,\Lambda_n) - G_1(v^-(\omega),\omega,\Lambda_n) \right\} | \mathcal{B}_{\Lambda_n} \right]$$

has significant fluctuations, with variance of the order of the volume. Here $\mathcal{B}_{\Lambda_n}$ is the σ algebra generated by the random field in Λ_n. Namely we show that

$$\mathbb{E}\left[F_n(\cdot)\right] = 0,$$

and for $t \in \mathbb{R}$

$$\liminf_{n\to\infty} \mathbb{E}\left[e^{t\frac{F_n}{\sqrt{\Lambda_n}}} \right] \ge e^{\frac{t^2 D^2}{2}}, \tag{1.5}$$

where D^2 is given in (4.60). This holds in all dimensions and for all $s \in (0,1)$. In $d = 1$ and for $s \in [\frac{1}{4},1)$, in $d = 2$ and for $s \in (\frac{1}{2},1)$ the bound (1.5) generates a contradiction with the bound (1.4), unless $D^2 = 0$. When $D^2 = 0$ we show that $M = \mathbb{E}[\int_{Q(0)} v^+] - \mathbb{E}[\int_{Q(0)} v^-] = 0$. Further, we show that point-wise $v^+ \ge v^-$, therefore $\mathbb{E}[\int_{Q(0)} v^+] = \mathbb{E}[\int_{Q(0)} v^-] = 0$ and $v^+(\cdot,\omega) = v^-(\cdot,\omega)$, for almost all realizations of the random field. The probabilistic argument has been already applied by Aizenman and Wehr, [1], in the context of Ising spin systems with random external field, see also the monograph by Bovier, [2], for a survey on this subject.

It is instructive to understand what one can say about the functional (1.1) when $\theta = 0$. Denote $J^{m_0}(m,\Lambda)$ the functional (1.1) when $\theta = 0$. In this case the constants $m(x) = \tau$ for $x \in \mathbb{R}^d$ and $\tau = \pm 1$ are the only *bounded* global macroscopic minimizers under compact perturbations. To pass to a so-called *macroscopic* scale, which is coarser than the mesoscopic scale, we rescale space with a small parameter ϵ. If $\mathcal{D} = \epsilon\Lambda$ and $u(z) = m(\epsilon^{-1}z)$ and $u_0(z) = m_0(\epsilon^{-1}z)$ we obtain

$$\tilde{J}_\epsilon^{u_0}(u,\mathcal{D}) = \epsilon^{2s-d}[u]^2_{H^s(\mathcal{D},\mathbb{R}^d)} + \epsilon^{-d}\int_{\mathcal{D}} W(u(z))\mathrm{dz}. \tag{1.6}$$

Functionals with a finite energy on this scale must be Lebesgue almost everywhere close to one of the two minimizers. The second step is to determine the cost of forming an interface between the spatial regions occupied by these two different minimizers.

As in the case of the corresponding local functional one needs to normalize $\tilde{J}^{u_0}_\epsilon(u,\mathcal{D})$ by a power of ϵ related to the dimension of the interface, which is not necessarily an integer in this case, see also Lemma 3.2. Computations similar to the ones done to obtain (1.4) give for $\theta=0$ a factor of ϵ^{-d+1} when $s\in(\frac{1}{2},1)$, ϵ^{-d+2s} when $s\in(0,\frac{1}{2})$, and by $\epsilon^{-d+1}\log\frac{1}{\epsilon}$ when $s=\frac{1}{2}$. Therefore we obtain

$$J^{u_0}_\epsilon(u,\mathcal{D})=\begin{cases}\epsilon^{2s-1}[u]^2_{H^s(\mathcal{D},\mathbb{R}^d)}+\epsilon^{-1}\displaystyle\int_{\mathcal{D}}W(u(z))\mathrm{dz}, & \mathrm{s}\in(\frac{1}{2},1)\\ [u]^2_{H^s(\mathcal{D},\mathbb{R}^d)}+\epsilon^{-2s}\displaystyle\int_{\mathcal{D}}W(u(z))\mathrm{dz}, & \mathrm{s}\in(0,\frac{1}{2})\\ \dfrac{\epsilon^{2s}}{\epsilon\log\epsilon}[u]^2_{H^s(\mathcal{D},\mathbb{R}^d)}+\dfrac{1}{\epsilon\log\epsilon}\displaystyle\int_{\mathcal{D}}W(u(z))\mathrm{dz}, & \mathrm{s}=\dfrac{1}{2}.\end{cases}\tag{1.7}$$

The Γ-convergence for the functional (1.7) has been studied by Savin and Valdinoci, [16]. They show that the functional $J^{u_0}_\epsilon(u,\mathcal{D})$ $\Gamma-$ converges to the classical minimal surface functional when $s\in[\frac{1}{2},1)$ while, when $s\in(0,\frac{1}{2})$ the functional $\Gamma-$ converges to the nonlocal minimal surface functional. There are in the literature other results dealing with $\Gamma-$ convergence of non local functionals, see e.g. [8], [9], [10] and references therein, but they are different from the deterministic part of the functional that we are considering, either for the explicit form or because they do not consider the full interaction of Λ with all of $\mathbb{R}^d$. Physically this implies that the particles in the domain Λ interact with all the particles in $\mathbb{R}^d$ and not only with those ones in Λ, i.e. a sort of nonlocal Dirichlet boundary condition.

2. Notations and Results

We denote by $\Lambda\subset\mathbb{R}^d$ a generic open subset of $\mathbb{R}^d$, by $\partial\Lambda$ the boundary of Λ and by $\Lambda^c=\mathbb{R}^d\setminus\Lambda$. When Λ is a bounded subset of $\mathbb{R}^d$ we write $\Lambda\Subset\mathbb{R}^d$. We denote by $|x|$ the euclidean norm of $x\in\mathbb{R}^d$, by $|\Lambda|$ the volume of Λ, by $\mathrm{diam}(\Lambda)=\sup\{|x-y|,\quad x \text{ and } y\in\Lambda\}$ and by $d_{\partial\Lambda}(x)$ the euclidean distance from x to $\partial\Lambda$. We will consider domain Λ with Lipschitz boundary regularity, i.e the boundary can be thought of as locally being the graph of a Lipschitz continuous function, see for example [4]. It is useful to introduce the following definition. We say that a set with Lipschitz boundary $\Lambda\Subset\mathbb{R}^d$ is *cube-like* if $\mathcal{H}^{d-1}(\partial\Lambda)\le C|\Lambda|^{\frac{d-1}{d}}$ and $\mathrm{diam}(\Lambda)\le C|\Lambda|^{\frac{1}{d}}$, where $\mathcal{H}^{d-1}$ is the $d-1$-dimensional Hausdorff measure and $C>0$ is a constant depending only on the dimension d.

For t and s in $\mathbb{R}$ we denote $s\wedge t=\min\{s,t\}$ and $s\vee t=\max\{s,t\}$. For $\Lambda\subset\mathbb{R}^d$, we denote by $C^{k,\alpha}(\Lambda)$, $k\ge0$ an integer, $\alpha\in(0,1]$ the set of functions continuous and having continuous derivatives up to order k, such that the k-th partial derivatives are Hölder continuous with exponent α.

2.1. **The disorder.** The disorder or random field is constructed with the help of a family of independent, identically distributed random variables with mean zero and variance equal to 1. We assume that each random variable has distribution absolutely continuous with respect to the Lebesgue measure and that the Lebesgue density is a symmetric, compactly supported function on $\mathbb{R}$. The corresponding infinite product measure on $\mathbb{R}^{\mathbb{Z}^d}$ will be denoted by $\mathbb{P}$ and by $\mathbb{E}[\cdot]$ the mean with respect to $\mathbb{P}$. We denote this family of random variables by $\{g(z,\omega)\}_{z\in\mathbb{Z}^d}$, $\omega\in\Omega$ where we identify Ω with $\mathbb{R}^{\mathbb{Z}^d}$. These assumptions imply that there exists a finite $A>0$ so that

$$\mathbb{E}[g(z)]=0,\quad \mathbb{E}[g^2(z)]=1,\quad \forall z\in\mathbb{Z}^d \quad\text{and}\quad \|g\|_\infty=\sup_z|g(z,\omega)|=A,\quad \mathbb{P}\text{ - a.s.}\tag{2.1}$$

The boundedness assumption is not essential. Different choices of g could be handled by minor modifications provided g is still a random field with finite correlation length, invariant under (integer) translations and such that $g(z,\cdot)$ has a symmetric distribution, absolutely continuous w.r.t the Lebesgue measure and $\mathbb{E}[g(z)^{2+\eta}]<\infty$, $z\in\mathbb{Z}^d$ for $\eta>0$. The method does not apply when g has atoms, i.e. its distribution is not absolutely continuous with respect to the Lebesgue measure, see Remark 4.15. It is not clear to us if this requirement is purely technical or if the discrete distribution of the random field may cause a degeneracy of the ground state like in the Ising spin systems [1].

The symmetry of the measure $\mathbb{P}$ is essential for obtaining the result. Namely if $\mathbb{P}$ does not have a symmetric distribution, it would be no longer natural to compare the qualitative properties of the functional (1.1) for $\theta \neq 0$ with the functional (1.1) with $\theta = 0$. Therefore in the following we always assume that $\mathbb{P}$ is symmetric.

We denote by $\mathcal{B}$ the product $\sigma-$algebra and by $\mathcal{B}_\Lambda$, $\Lambda \subset \mathbb{Z}^d$, the $\sigma-$ algebra generated by $\{g(z,\omega) : z \in \Lambda\}$. In the following we often identify the random field $\{g(z,\cdot) : z \in \mathbb{Z}^d\}$ with the coordinate maps $\{g(z,\omega) = \omega(z) : z \in \mathbb{Z}^d\}$. To use ergodicity properties of the random field it is convenient to equip the probability space $(\Omega, \mathcal{B}, \mathbb{P})$ with some extra structure. First, we define the action T of the translation group $\mathbb{Z}^d$ on Ω. We will assume that $\mathbb{P}$ is invariant under this action and that the dynamical system $(\Omega, \mathcal{B}, \mathbb{P}, T)$ is stationary and ergodic. In our model the action of T is for $y \in \mathbb{Z}^d$

$$(g(z_1, [T_y\omega]), ..., g(z_n, [T_y\omega])) = (g(z_1 + y, \omega), ..., g(z_n + y, \omega)). \tag{2.2}$$

The disorder or random field in the functional will be obtained setting for $x \in \Lambda$

$$g_1(x,\omega) := \sum_{z\in\mathbb{Z}^d} g(z,\omega) 1\!\!1_{(z+[-\frac{1}{2},\frac{1}{2}]^d)\cap\Lambda}(x), \tag{2.3}$$

where for any Borel-measurable set A

$$1\!\!1_A(x) := \begin{cases} 1, \text{if } x \in A \\ 0 \text{ if } x \notin A. \end{cases}$$

2.2. **The double well potential.** Next we define the "double-well potential" W:

Assumption (H1) $W \in C^2(\mathbb{R})$, $W \geq 0$, $W(t) = 0$ iff $t \in \{-1, 1\}$, $W(t) = W(-t)$ and $W(t)$ is strictly decreasing in $[0, 1]$. Moreover there exists δ_0 and $C_0 > 0$ so that

$$W(t) = \frac{1}{2C_0}(t-1)^2 \qquad \forall t \in (1-\delta_0, \infty). \tag{2.4}$$

Note that W is slightly different from the standard choice $W(u) = (1-u^2)^2$. Our choice simplifies some proofs because it makes the Euler-Lagrange equation linear provided solutions stay in one "well." Note that in order to obtain our uniqueness result we could replace the equality in (2.4) by a lower bound on $W(t)$ of the same form.

2.3. **The functional.** We start introducing the functional spaces in which we define the nonlocal interaction term.

Definition 2.1. Fractional Sobolev spaces *Let $D \subset \mathbb{R}^d$ be an open domain and $s \in (0,1)$. We define the fractional Sobolev space $H^s(D)$ as the set of functions $f \in L^2(D)$ so that*

$$\int_{D\times D} \frac{(f(x)-f(y))^2}{|x-y|^{d+2s}} dxdy < \infty.$$

This space, endowed with the norm

$$\|f\|_{H^s(D)} = \|f\|_{L^2(D)} + \left(\int_{D\times D} \frac{(f(x)-f(y))^2}{|x-y|^{d+2s}} dxdy\right)^{\frac{1}{2}}$$

is an Hilbert space. We will say that $f \in H^s_{loc}(\mathbb{R}^d)$, $s \in (0,1)$, if $f \in H^s(B_R)$ for any ball of radius R in $\mathbb{R}^d$.

For $v \in H^s_{loc}(\mathbb{R}^d)$, $\Lambda \Subset \mathbb{R}^d$ denote

$$\mathcal{K}_1(v,\omega,\Lambda) = \int_\Lambda \mathrm{dx} \int_\Lambda \mathrm{dy} \frac{|\mathrm{v(x)} - \mathrm{v(y)}|^2}{|\mathrm{x}-\mathrm{y}|^{\mathrm{d+2s}}} + \int_\Lambda \mathrm{W(v(x))dx} - \theta \int_\Lambda \mathrm{g_1(x,\omega)v(x)dx}. \tag{2.5}$$

Now we introduce some definitions needed to specify "boundary conditions" in a sense appropriate for nonlocal functionals.
For any $\Lambda \Subset \mathbb{R}^d$ and $\Lambda_1 \subset \mathbb{R}^d$, $\Lambda_1 \cap \Lambda = \emptyset$, for v and u in $H^s_{loc}(\mathbb{R}^d)$ denote

$$\mathcal{W}((v,\Lambda),(u,\Lambda_1)) = 2\int_\Lambda \mathrm{dx} \int_{\Lambda_1} \mathrm{dy} \frac{|\mathrm{v(x)} - \mathrm{u(y)}|^2}{|\mathrm{x}-\mathrm{y}|^{\mathrm{d+2s}}} \tag{2.6}$$

the interaction between the function v in Λ and the function u in Λ_1. Note that if Λ_1 is not a bounded set, the term in (2.6) might not be finite. We will show in Lemma 3.2 that when $v \in H^s_{loc}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ then $\mathcal{W}((v,\Lambda),(v,\Lambda_1))$ is bounded, the bound depends on $|\Lambda|$. When $\Lambda_1 = \Lambda^c$ and $u = v$ we simply write

$$\mathcal{W}(v,\Lambda) = 2\int_\Lambda \mathrm{dx} \int_{\Lambda^c} \mathrm{dy} \frac{|\mathrm{v(x)} - \mathrm{v(y)}|^2}{|\mathrm{x}-\mathrm{y}|^{\mathrm{d+2s}}}. \tag{2.7}$$

Definition 2.2. The Functional *For any $\Lambda \subset \mathbb{R}^d$, $v \in H^s_{loc}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ we define*

$$G_1(v,\omega,\Lambda) = \mathcal{K}_1(v,\omega,\Lambda) + \mathcal{W}(v,\Lambda). \tag{2.8}$$

Whenever we need to stress the dependence of G_1 on the value of v outside Λ, i.e. $v(y) = v_0(y), y \in \Lambda^c$, we will write

$$G_1^{v_0}(v,\omega,\Lambda) = \mathcal{K}_1(v,\omega,\Lambda) + \mathcal{W}((v,\Lambda)(v_0,\Lambda^c)). \tag{2.9}$$

We list some useful properties of the functionals G_1 and $\mathcal{K}_1$ that follow immediately from the definitions.

Lemma 2.3.

- *$\mathcal{K}_1$ is superadditive, i.e. if A and B are disjoint sets then*

$$\mathcal{K}_1(v,\omega,A\cup B) \geq \mathcal{K}_1(v,\omega,A) + \mathcal{K}_1(v,\omega,B),$$

- *G_1 is subadditive, i.e. if A and B are disjoint sets then*

$$G_1(v,\omega,A\cup B) \leq G_1(v,\omega,A) + G_1(v,\omega,B). \tag{2.10}$$

Definition 2.4. The minimizers

(1) *We say that $u \in H^s_{loc}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ is a minimizer under compact perturbations for G_1 in $\Lambda \subset \mathbb{R}^d$ if for any compact subdomain $U \subset \Lambda$ we have*

$$G_1(u,\omega,U) < \infty, \quad \mathbb{P} \text{ a.s.}$$

and

$$G_1(u,\omega,U) \leq G_1(v,\omega,U) \quad \mathbb{P} \text{ a.s.}$$

for any v which coincides with u in $\mathbb{R}^d \setminus U$.

(2) *Let $v_0 \in L^\infty(\mathbb{R}^d)$ be independent of $\omega \in \Omega$. We say that $u \in H^s_{loc}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ is a v_0-minimizer for G_1 in $\Lambda \subset \mathbb{R}^d$ if for any compact subdomain $U \subset \Lambda$ we have*

$$G_1^{v_0}(u,\omega,U) < \infty, \quad \mathbb{P}\, a.s.$$

and

$$G_1^{v_0}(u,\omega,U) \leq G_1(v,\omega,U) \quad \mathbb{P}\, a.s.$$

for any v which coincides with v^0 in $\mathbb{R}^d \setminus U$.

(3) *We say u is a free minimizer on Λ if it minimizes $\mathcal{K}_1(\cdot,\omega,\Lambda)$ in $H^s(\Lambda)$.*

Note that v_0 will usually be a constant function.

Remark 2.5 (Existence). *Existence of v_0-minimizers (for sufficiently regular v^0) and free minimizers in a bounded Lipschitz set $\Lambda \subset \mathbb{R}^d$ follows from the compact embedding of $H^s(\Lambda)$ in $L^2(\Lambda)$ and the lower semicontinuity of the H^s-norm. We prove the existence of a v_0-minimizer in Lemma 6.1 and Lemma 6.2 in the Appendix. The existence of exactly one minimizer under compact perturbations is a consequence of the main theorem.*

Definition 2.6. Translational covariant states *We say that the function $v : \mathbb{R}^d \times \Omega \to \mathbb{R}$ is translational covariant if*

$$v(x+y,\omega) = v(x,[T_{-y}\omega]) \quad \forall y \in \mathbb{Z}^d, \quad x \in \mathbb{R}^d. \tag{2.11}$$

Our main result is the following.

Theorem 2.7. *Take $d = 2$ and $s \in (\frac{1}{2}, 1)$ or $d = 1$ and $s \in [\frac{1}{4}, 1)$, θ strictly positive. Let $n \in \mathbb{N}$, $\Lambda_n = (-\frac{n}{2}, \frac{n}{2})^d$ [1] , $v_0 \in L^\infty(\mathbb{R}^d)$ and u_n^* be a v_0-minimizer of G_1 in Λ_n according to Definition 2.4. Then $\mathbb{P}$ a.s. there exists a* unique $u^*(\cdot,\omega)$, independent of the choice of v_0, *defined as*

$$\lim_{n\to\infty} u_n^*(x,\omega) = u^*(x,\omega) \qquad \text{(uniformly on compacts in } x) \tag{2.12}$$

so that

- *$u^*(\cdot,\omega)$ is translation covariant, see (2.11).*
- *$\|u^*(\cdot,\omega)\|_\infty \le 1 + C_0\theta\|g\|_\infty$ where C_0 is the constant in (2.4).*
- *$u^*(\cdot,\omega) \in C^{0,\alpha}_{\mathrm{loc}(\mathbb{R}^d)}$ for any $\alpha < 2s$ when $2s \le 1$, $u^* \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^d)$ for any $\alpha < 2s-1$, when $2s > 1$.*
- $$\mathbb{E}[u^*(x,\cdot)] = 0, \qquad \forall x \in \mathbb{R}^d.$$

Remark 2.8. *Since for any set $\Lambda \Subset \mathbb{R}^d$, $C^{0,\alpha}(\Lambda) \subset C^{0,\beta}(\Lambda)$ for $\beta < \alpha$ and the inclusion is compact, the convergence in (2.12) holds in $C^{0,\beta}$, $\beta < 2s$ when $s \in (0, \frac{1}{2}]$, because we can find α with $\beta < \alpha < 2s$. Similarly one obtains convergence of (2.12) in $C^{1,\beta}$, $\beta < 2s-1$ when $s \in (\frac{1}{2}, 1)$.*

Remark 2.9. *When $\theta = 0$ in (2.5), i.e the random field is absent, the minimum value of $\mathcal{K}_1(\cdot,\cdot,\Lambda)$ is zero for any bounded Λ and there are exactly two translation covariant minimizers under compact perturbations, the constant functions identically equal to 1 or to -1.*

3. Finite volume Minimizers

In this section we state properties for minimizers of the functional G_1 in any bounded set $\Lambda \subset \mathbb{R}^d$. These properties hold in all dimensions d, for all bounded Λ with Lipschitz boundary and for every $\omega \in \Omega$. The ω plays the role of a parameter. We start showing that to determine the minimizers of $\mathcal{K}_1$ in Λ it is sufficient to consider functions v satisfying a uniform L^∞-bound.

For any $t > 0$ denote by $v^t = t \wedge v \vee (-t)$.

Lemma 3.1. *Let the double well potential W satisfy Assumption (H1).*

1. *For all $\omega \in \Omega$, for all $v \in H^s(\Lambda)$ and all $t \ge 1 + C_0\theta\|g\|_\infty$*
$$\mathcal{K}_1(v,\omega,\Lambda) - \mathcal{K}_1(v^t,\omega,\Lambda) \ge \int_{\Lambda_t} \left(C_0^{-1}(t-1) - \theta\|g\|_\infty\right)(|v(y)| - t), \tag{3.1}$$
where C_0 is the constant in (2.4) and $\Lambda_t = \{y \in \Lambda : |v(y)| > t\}$.
2. *Take $v_0 \in H^s_{loc}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and $t \ge \max\{\|v_0\|_\infty, 1 + C_0\theta\|g\|_\infty\}$. The result stated in (3.1) holds for $G_1^{v_0}(v,\omega,\Lambda)$. This implies in particular that minimizers of $G_1^{v_0}$ are bounded uniformly by $\max\{\|v^0\|_\infty, 1 + C_0\theta\|g\|_\infty\}$.*

[1]One could take any increasing, *cube-like*, sequences of sets $\{\Lambda_n\}_n$, $\Lambda_n \subset \mathbb{R}^d$ invading $\mathbb{R}^d$. The proof goes in the same way.

Proof. We have that for x and y and any function v and w

$$[v(x) - w(y)]^2 \geq [v^t(x) - w^t(y)]^2.$$

We immediately obtain

$$\mathcal{K}_1(v,\omega,\Lambda) - \mathcal{K}_1(v^t,\omega,\Lambda) \geq \int_{\Lambda_t} (W(v(y)) - W(t))\, dy - \theta \int_{\Lambda_t} dy g_1(y,\omega)[v(y) - \mathrm{sign}(v(y))t],$$

and from Assumption (H1) and the L^∞-bound on g we derive (3.1). The proof of (2) is a consequence of (1) by choosing $t \geq \max\{\|v_0\|_\infty, 1 + C_0\theta\|g\|_\infty\}$. □

Next we show that the functional (2.8) is finite when $v \in H^s_{loc}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. To this aim it is sufficient to show that $\mathcal{W}(v,\Lambda)$, defined in (2.7), is finite.

Lemma 3.2. *Let $v \in H^s_{loc}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, $\Lambda \Subset \mathbb{R}^d$ and $C = C(\|v\|_\infty, d, s)$ be a generic constant which might change from one occurrence to the other. Suppose that Λ is* cube-like.[2] *Then we have*

$$\mathcal{W}(v,\Lambda) \leq C|\Lambda|^{\frac{d-2s}{d}}, \qquad s \in (0, \frac{1}{2}). \tag{3.2}$$

When $s \in [\frac{1}{2}, 1)$ denote by $B_1(\partial\Lambda) = \{x \in \mathbb{R}^d : d_{\partial\Lambda}(x) \leq 1\}$ we have

$$\mathcal{W}(v,\Lambda) \leq \|v\|_{H^s(B_1(\partial\Lambda))} + \begin{cases} C|\Lambda|^{\frac{d-1}{d}}, & s \in (\frac{1}{2}, 1), \\ C|\Lambda|^{\frac{d-1}{d}} \log(|\Lambda|), & s = \frac{1}{2}. \end{cases} \tag{3.3}$$

When $s \in [\frac{1}{2}, 1)$ and $v \in C^{0,\alpha}(B_1(\partial\Lambda))$ for $\alpha > s - \frac{1}{2}$

$$\mathcal{W}(v,\Lambda) \leq \begin{cases} C|\Lambda|^{\frac{d-1}{d}}, & s \in (\frac{1}{2}, 1), \\ C|\Lambda|^{\frac{d-1}{d}} \log(|\Lambda|), & s = \frac{1}{2}. \end{cases} \tag{3.4}$$

Proof. For any $s \in (0, \frac{1}{2})$ we have

$$\begin{aligned} &\int_\Lambda \mathrm{dx} \int_{\Lambda^c} \mathrm{dy} \frac{|\mathrm{v(x)} - \mathrm{v(y)}|^2}{|\mathrm{x} - \mathrm{y}|^{\mathrm{d+2s}}} \\ &\leq C \int_\Lambda \mathrm{dx} \int_{\Lambda^c} \mathrm{dy} \frac{1}{|\mathrm{x} - \mathrm{y}|^{\mathrm{d+2s}}} \leq \mathrm{C} \int_\Lambda \mathrm{dx} \int_{\{\mathrm{y} \in \mathbb{R}^\mathrm{d}:\ |\mathrm{x}-\mathrm{y}| \geq \mathrm{d}_{\partial\Lambda}(\mathrm{x})\}} \frac{1}{|\mathrm{x} - \mathrm{y}|^{\mathrm{d+2s}}} \mathrm{dy} \leq \mathrm{C} \int_\Lambda |\mathrm{d}_{\partial\Lambda}(\mathrm{x})|^{-2\mathrm{s}} \mathrm{dx} \\ &\leq C(\mathrm{diam}(\Lambda))^{1-2s} \mathcal{H}^{d-1}(\partial\Lambda) \leq C|\Lambda|^{\frac{d-2s}{d}}. \end{aligned} \tag{3.5}$$

Note that for cubes $\mathrm{diam}(\Lambda) \leq C|\Lambda|^{1/d}$, where the constant C depends only on the dimension.

When $d \geq 1$ and $s \in [\frac{1}{2}, 1)$, $d_{\partial\Lambda}(x)^{-2s}$ is not integrable anymore over Λ. So we split the integral as follows:

$$\begin{aligned} &\int_\Lambda \mathrm{dx} \int_{\Lambda^c} \mathrm{dy} \frac{|\mathrm{v(x)} - \mathrm{v(y)}|^2}{|\mathrm{x} - \mathrm{y}|^{\mathrm{d+2s}}} \\ &= \int_{\{x \in \Lambda: d_{\partial\Lambda}(x) \leq 1\}} \mathrm{dx} \int_{\mathrm{y} \in \Lambda^c} \mathrm{dy} \frac{|\mathrm{v(x)} - \mathrm{v(y)}|^2}{|\mathrm{x} - \mathrm{y}|^{\mathrm{d+2s}}} + \int_{\{\mathrm{x} \in \Lambda: \mathrm{d}_{\partial\Lambda}(\mathrm{x}) > 1\}} \mathrm{dx} \int_{\mathrm{y} \in \Lambda^c} \mathrm{dy} \frac{|\mathrm{v(x)} - \mathrm{v(y)}|^2}{|\mathrm{x} - \mathrm{y}|^{\mathrm{d+2s}}}. \end{aligned} \tag{3.6}$$

[2] The lemma holds for Lipschitz domains Λ, but then the generic constant C depends on the shape of the domain.

For the last integral, since $|x-y|\geq 1$, we obtain proceeding as in (3.5)

$$\begin{aligned}&\int_{\{x\in\Lambda:d_{\partial\Lambda}(x)>1\}}\mathrm{dx}\int_{\mathrm{y}\in\Lambda^c}\mathrm{dy}\frac{|\mathrm{v(x)-v(y)}|^2}{|\mathrm{x-y}|^{\mathrm{d+2s}}}\\&\leq C\int_{\{x\in\Lambda:d_{\partial\Lambda}(x)>1\}}d_{\partial\Lambda}(x)^{-2s}\mathrm{dx}\leq\begin{cases}C|\Lambda|^{\frac{d-1}{d}} & s\in(\frac{1}{2},1)\\ C|\Lambda|^{\frac{d-1}{d}}\log|\Lambda|, & s=\frac{1}{2}.\end{cases}\end{aligned}\tag{3.7}$$

We split the first integral of (3.6) as

$$\begin{aligned}&\int_{\{x\in\Lambda:d_{\partial\Lambda}(x)\leq 1\}}\mathrm{dx}\int_{\mathrm{y}\in\Lambda^c}\mathrm{dy}\frac{|\mathrm{v(x)-v(y)}|^2}{|\mathrm{x-y}|^{\mathrm{d+2s}}}\\&=\int_{\{x\in\Lambda:d_{\partial\Lambda}(x)\leq 1\}}\mathrm{dx}\int_{\{\mathrm{y}\in\Lambda^c:\mathrm{d}_{\partial\Lambda}(\mathrm{y})\leq 1\}}\mathrm{dy}\frac{|\mathrm{v(x)-v(y)}|^2}{|\mathrm{x-y}|^{\mathrm{d+2s}}}\\&+\int_{\{x\in\Lambda:d_{\partial\Lambda}(x)\leq 1\}}\mathrm{dx}\int_{\{\mathrm{y}\in\Lambda^c:\mathrm{d}_{\partial\Lambda}(\mathrm{y})>1\}}\mathrm{dy}\frac{|\mathrm{v(x)-v(y)}|^2}{|\mathrm{x-y}|^{\mathrm{d+2s}}}.\end{aligned}\tag{3.8}$$

For the last term of (3.8), since $|x-y|\geq 1$, we get

$$\int_{\{x\in\Lambda:d_{\partial\Lambda}(x)\leq 1\}}\mathrm{dx}\int_{\{\mathrm{y}\in\Lambda^c:\mathrm{d}_{\partial\Lambda}(\mathrm{y})>1\}}\mathrm{dy}\frac{|\mathrm{v(x)-v(y)}|^2}{|\mathrm{x-y}|^{\mathrm{d+2s}}}\leq\mathrm{C}\int_{\{\mathrm{x}\in\Lambda:\mathrm{d}_{\partial\Lambda}(\mathrm{x})\leq 1\}}\mathrm{dx}\int_1^\infty\mathrm{r}^{-1-2\mathrm{s}}\mathrm{dr}\leq\mathrm{C}|\Lambda|^{\frac{\mathrm{d}-1}{\mathrm{d}}}\qquad\mathrm{s}\in[\frac{1}{2},1).$$

The first term of the right hand side of (3.8) is obviously bounded when $v\in H^s_{loc}(\mathbb{R}^d)$

$$\int_{\{x\in\Lambda:d_{\partial\Lambda}(x)\leq 1\}}\mathrm{dx}\int_{\{\mathrm{y}\in\Lambda^c:\mathrm{d}_{\partial\Lambda}(\mathrm{y})\leq 1\}}\frac{|\mathrm{v(x)-v(y)}|^2}{|\mathrm{x-y}|^{\mathrm{d+2s}}}\mathrm{dy}\leq\|\mathrm{v}\|_{\mathrm{H^s(B_1(\partial\Lambda))}}.$$

When $v\in C^{0,\alpha}(B_1(\partial\Lambda))$ for $\alpha>s-\frac{1}{2}$ then again arguing as in (3.5)

$$\int_{\{x\in\Lambda:d_{\partial\Lambda}(x)\leq 1\}}\mathrm{dx}\int_{\{\mathrm{y}\in\Lambda^c:\mathrm{d}_{\partial\Lambda}(\mathrm{y})\leq 1\}}\mathrm{dy}\frac{|\mathrm{v(x)-v(y)}|^2}{|\mathrm{x-y}|^{\mathrm{d+2s}}}\leq\mathrm{C}|\Lambda|^{\frac{\mathrm{d}-1}{\mathrm{d}}},\qquad\mathrm{s}\in[\frac{1}{2},1).\tag{3.9}$$

□

Next we prove an energy decreasing rearrangement which allows to show a strong maximum principle, see Lemma 3.4: Minimizers of $G_1(\cdot,\omega,\Lambda)$ corresponding to ordered boundary conditions on Λ^c are ordered as well, i.e they do not intersect. In particular if there exists more than one minimizer corresponding to the same boundary condition they do not intersect.

Lemma 3.3. *Let u and v be in $H^s_{loc}(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$. Then for all $\omega\in\Omega$ and $\Lambda\subset\mathbb{R}^d$*

$$G_1(u\vee v,\omega,\Lambda)+G_1(u\wedge v,\omega,\Lambda)\leq G_1(u,\omega,\Lambda)+G_1(v,\omega,\Lambda).\tag{3.10}$$

When $u=v$ on Λ^c, the equality holds in (3.10) *if and only if*

$$u(x)\leq v(x)\quad or\quad v(x)\leq u(x),\quad a.s.\quad x\in\Lambda.\tag{3.11}$$

When $u\leq v$ on Λ^c and $u<v$ for some open set in Λ^c the equality holds in (3.10) *if and only if*

$$u(x)\leq v(x)\quad a.s.\quad x\in\Lambda.\tag{3.12}$$

Proof. Since u and v are in $H^s_{loc}(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$, G_1 is finite. Let $M(x)=\max\{u(x),v(x)\}$ and $m(x)=\min\{u(x),v(x)\}$. It is immediate to verify that the local part of the functional G_1 satisfies (3.10) with the equality. For the interaction term, for x and y in $\mathbb{R}^d$, we have that

$$[m(x)-m(y)]^2+[M(x)-M(y)]^2\leq[u(x)-u(y)]^2+[v(x)-v(y)]^2.\tag{3.13}$$

Namely if both the minimum values in x and y are reached by the same function either u or v then the equality holds in (3.13). If $m(x) = u(x) < v(x)$ and $m(y) = v(y) < u(y)$ then the left hand side of (3.13) is equal to

$$[u(x) - u(y)]^2 + [v(x) - v(y)]^2 + [u(x) - v(x)][u(y) - v(y)]$$

with $[u(x) - v(x)][u(y) - v(y)] < 0$. The same holds when $m(x) = v(x)$ and $m(y) = u(y)$. In these last case we will have a strict inequality in (3.13), and therefore in (3.10).

Next we prove (3.11). If $u(x) \le v(x)$ or $u(x) \ge v(x)$ for all $x \in \mathbb{R}^d$ then the equality holds in (3.10). When $u = v$ on Λ^c we have also the reverse implication for $x \in \Lambda$. Namely it is immediate to verify that in such case (no matter which value of u or v correspond to M or m)

$$\mathcal{W}(M, \Lambda) + \mathcal{W}(m, \Lambda) = \mathcal{W}(u, \Lambda) + \mathcal{W}(v, \Lambda). \tag{3.14}$$

The equality in (3.10) implies the equality in (3.13), then (3.11) holds. Next we prove (3.12). It is immediate to verify that if (3.14) holds we must have $M(x) = v(x)$ and $m(x) = u(x)$ for $x \in \Lambda$. □

Lemma 3.4. *Let u and v in $H^s_{loc}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ be minimizers of G_1 in Λ, so that $u \le v$ on Λ^c. Then, for all $\omega \in \Omega$, $u = v$ or $|u(x) - v(x)| > 0$ for all $x \in \mathrm{int}(\Lambda)$. If $u < v$ in an open set in Λ^c, then $u < v$ everywhere in $\mathrm{int}(\Lambda)$.*

Proof. Since the result holds for any realization of the random field and Λ is fixed we avoid to explicitly write in G_1 the dependence on ω and Λ. By Lemma 3.3

$$G_1(u \vee v) + G_1(u \wedge v) \le G_1(u) + G_1(v). \tag{3.15}$$

The conditions on u and v in Λ^c yield $u \vee v = v, u \wedge v = u$ on Λ^c, and by the minimization properties of u and v we get $G_1(u \vee v) \ge G_1(v)$, $G_1(u \wedge v) \ge G_1(u)$. This and (3.15) imply that $G_1(u \vee v) + G_1(u \wedge v) = G_1(u) + G_1(v)$, actually that $G_1(u \vee v) = G_1(v)$ and $G_1(u \wedge v) = G_1(u)$. Therefore $u \vee v$ is a minimizer with condition v on Λ^c, and $u \wedge v$ is a minimizer with condition u on Λ^c. Obviously the function $w := u - u \wedge v \ge 0$ in Λ and in particular $w = 0$ on Λ^c. Further since u by assumption is a minimizer and $u \wedge v$ is also a minimizer, they are both solutions of problem (6.6) and the regularity results of Proposition 6.3 hold.

Therefore by construction $w \in C^{0,\alpha}(\Lambda)$, $\alpha < 2s$, when $2s \le 1$ and $w \in C^{1,\alpha}(\Lambda)$, $\alpha < 2s - 1$, when $2s > 1$. On the other hand, w solves

$$\begin{aligned} &(-\Delta)^s w = V(x) \quad \text{in } \Lambda, \\ &w = 0 \qquad \text{on} \qquad \Lambda^c \end{aligned} \tag{3.16}$$

where

$$V(x) = \frac{1}{2}[W'(u(x)) - W'(u(x) \wedge v(x))].$$

Since $W \in C^2(\mathbb{R})$, see Assumption (H1), by the regularity of u and $u \wedge v$ we have that $V \in C^{0,\alpha}(\Lambda)$, $0 < \alpha < 2s$, when $2s \le 1$ and $V \in C^{1,\alpha}(\Lambda)$, $0 < \alpha < 2s - 1$, when $2s > 1$. By [17, Proposition 2. 8] w being solution of (3.16) is in $C^{0,\alpha+2s}$ when $\alpha + 2s \le 1$ and in $C^{1,\alpha+2s-1}$ when $\alpha + 2s > 1$. In both cases the following argument holds. Suppose there exists $x_0 \in \Lambda$ with $u(x_0) = u(x_0) \wedge v(x_0)$, i.e $w(x_0) = 0$. By the regularity of w we have that

$$(-\Delta)^s w(x_0) = \int_\Lambda \mathrm{dy} \frac{[\mathrm{w(x_0) - w(y)}]}{|\mathrm{x_0 - y}|^{\mathrm{d+2s}}} = -\int_\Lambda \mathrm{dy} \frac{\mathrm{w(y)}}{|\mathrm{x_0 - y}|^{\mathrm{d+2s}}} < 0$$

being equal to zero only when $w(x) = 0$ for almost all $x \in \mathbb{R}^d$. Notice that the integral is well defined for any $s \in (0, 1)$ since w is is in $C^{0,\alpha+2s}$ when $\alpha + 2s \le 1$ and in $C^{1,\alpha+2s-1}$ when $\alpha + 2s > 1$. Since $V(x_0) = 0$ by construction, if $(-\Delta)^s w(x_0) < 0$ we have a contradiction with (3.16).

Therefore in the interior of Λ either $u = u \wedge v$ (in which case $u \le v$) or $u > u \wedge v$, i.e. $v < u$. By assumption $u \le v$ in Λ^c and by Lemma 3.3 $v < u$ in the interior of Λ is only possible if $u = v$ on Λ^c. Next we show that when $u = u \wedge v$, then either $u = v$ in Λ (and this is possible only when $u = v$ on

Λ^c) or $u(x) < v(x)$ for x in the interior of Λ. Denote by $w = u - v \geq 0$. As before, we have that w is a solution of

$$\begin{aligned} &(-\Delta)^s w = V(x) \quad \text{in } \Lambda, \\ &w = w_0 \geq 0 \qquad \text{on} \qquad \Lambda^c, \end{aligned} \tag{3.17}$$

where we set $w_0 = v - u$, the difference of the boundary data, which by assumption is positive. Arguing as before, assume that there exists x_0 in the interior of Λ so that $w(x_0) = 0$. By the regularity of w we have that

$$\begin{aligned} (-\Delta)^s w(x_0) &= \int_\Lambda \mathrm{dy} \frac{[\mathrm{w(x_0) - w(y)}]}{|\mathrm{x_0 - y}|^{\mathrm{d+2s}}} + \int_{\Lambda^c} \mathrm{dy} \frac{[\mathrm{w(x_0) - w_0(y)}]}{|\mathrm{x_0 - y}|^{\mathrm{d+2s}}} \\ &= -\int_\Lambda \mathrm{dy} \frac{\mathrm{w(y)}}{|\mathrm{x_0 - y}|^{\mathrm{d+2s}}} - \int_{\Lambda^c} \mathrm{dy} \frac{\mathrm{w_0(y)}}{|\mathrm{x_0 - y}|^{\mathrm{d+2s}}} < 0. \end{aligned} \tag{3.18}$$

Since $V(x_0) = 0$ by construction, if $(-\Delta)^s w(x_0) < 0$ we have a contradiction with (3.17). Therefore if $w_0 = 0$ in Λ^c, in the interior of Λ either $w = 0$ (in which case $u = v$) or $v > u$. If $w_0 > 0$ in some subset of Λ^c the only possibility is $v > u$ in the interior of Λ. □

Note that there may be a priori several minimizers with the same boundary conditions, as our functional is not convex.

Next, given $v_0 \in H^s_{loc}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, we single out two special minimizers of G_1 in Λ, one is the largest minimizer of G_1 in Λ with v_0 boundary conditions (defined a pointwise supremum), the other is the smallest minimizer of G_1 in Λ with $-v_0$ boundary conditions. We call them the v_0- maximal and the v_0- minimal minimizer of G_1 in Λ.

Lemma 3.5. Existence of maximal/minimal minimizers.
Let $\Lambda \Subset \mathbb{R}^d$ be a Lipschitz bounded open set and $v_0 \in H^s_{loc}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$.

1. *The set of minimizer of $G_1^{v_0}$ on Λ is compact, i.e. any sequence of minimizers has a limit in $C^{0,\alpha}(\Lambda)$, $\alpha < 2s$ for $s \in (0, 1/2]$ or $C^{1,\alpha}(\Lambda)$, $\alpha < 2s - 1$ for $s \in (1/2, 1)$, which is still a minimizer.*
2. *The set of minimizers has a maximal and minimal element with respect to point-wise ordering of functions*

Proof: A sequence of minimizers of $G_1^{v_0}$ on Λ is a sequence of functions with energies converging to the infimum, so the same techniques as in the proof of the existence of minimizers apply.

For the second part, let us define a function $\bar{u} : \Lambda \to \mathbb{R}$ by $\bar{u}(x) := \sup\{v(x) : \ v \text{ minimizer}\}$. We have to show that $\bar{u}$ is a minimizer, in particular that it has sufficient regularity. Fix a point x_0 in the interior of Λ. We can find a sequence of minimizers $\{v_n\}_{n\in\mathbb{N}}$ (which for the moment may still depend on x_0) such that $v_n(x_0) \to \bar{u}(x_0)$ and such that the sequence $v_n(x_0)$ is increasing. By Lemma 3.4, $v_n(x) \leq v_m(x)$ for all $m \geq n$ and *all* $x \in \Lambda$. Define now $\bar{v}(x) := \lim_{n\to\infty} v_n(x)$. We know from the first part of the Lemma that the sequence of minimizers $\{v_n\}_{n\in\mathbb{N}}$ has a convergent subsequence which converges to a minimizer. So the point-wise limit $\bar{v}$ must be minimizer, moreover $\bar{v} \leq \bar{u}$.
If there exists $x_1 \in \Lambda$ such that $\bar{v}(x_1) < \bar{u}(x_1)$, then there must be a minimizer w such that $w(x_1) > \bar{v}(x_1)$. But $\bar{v}(x_0) = \bar{u}(x_0) \geq w(x_0)$, contradicting Lemma 3.4. So $\bar{v} = \bar{u}$, which is therefore the maximal minimizer and point-wise maximum over the set of minimizers. The proof for the minimal element is done in the same way.

This allows us to define the following object:

Definition 3.6. *Given $v_0 \in H^s_{loc}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, we say that u^+ (u^-) is the v_0- maximal (v_0- minimal) minimizer of G_1 in Λ if*

- *$u^+(x) = v_0(x)$, ($u^-(x) = -v_0(x)$) for $x \in \Lambda^c$,*
- *u^+, (u^-) is a miminizer of G_1 in Λ according to (2) of Definition 2.4,*

- *if* $\tilde{u}$ *is any other minimizer (if more than one) of* G_1 *in* Λ *so that* $\tilde{u}(x) = v_0(x)$, $(\tilde{u}(x) = -v_0(x))$ *for* $x \in \Lambda^c$, *then* $\tilde{u}(x) < u^+(x)$ $(\tilde{u}(x) > u^-(x))$ *for* $x \in \Lambda$.

4. Infinite volume covariant states

In this section we construct two functions $v^{\pm}(\cdot,\omega)$ on $\mathbb{R}^d$ which we denote *macroscopic extrema minimizers or infinite-volume states.* They are obtained, as explained in the introduction, through a two limits procedure. We first show that for any $K \geq 1 + C_0\theta\|g\|_\infty$, where C_0 is the constant in (2.4), the $K-$ maximal and minimal minimizers of G_1^K in Λ_n as $n \to \infty$ converge in a suitable way to $u^{\pm,K}$. Then we define the $v^{\pm}(\cdot,\omega)$ as the point-wise limit, when $K \to \infty$ of $u^{\pm,K}$. We show that the $v^{\pm}(\cdot,\omega)$, constructed in such a way, are minimizers under compact perturbations and they do not depend on the boundary values.

Theorem 4.1. *[infinite-volume states] For almost all* $\omega \in \Omega$, *there exist two functions* $v^+(x,\omega)$, $v^-(x,\omega)$, $x \in \mathbb{R}^d$, *having the following properties.*

- *If* $2s \leq 1$, *then* $v^{\pm}(\cdot,\omega) \in C^{\alpha}_{loc}(\mathbb{R}^d)$ *for all* $\alpha < 2s$. *If* $s \in (\frac{1}{2}, 1)$, *then* $v^{\pm}(\cdot,\omega) \in C^{1,\alpha}_{loc}(\mathbb{R}^d)$ *for all* $\alpha < 2s - 1$.
- $v^{\pm}(\cdot,\omega)$ *are translation covariant.*
- $$v^+(x,\omega) = -v^-(x,-\omega) \quad x \in \mathbb{R}^d. \tag{4.1}$$
- $v^{\pm}$ *are minimizers under compact perturbations in the sense of Def. 2.4, (1).*
- $$\|v^{\pm}(\omega)\|_\infty \leq 1 + C_0\theta\|g\|_\infty, \tag{4.2}$$
  where C_0 *is the constant in* (2.4).
- *Let* $\Lambda_n = (-\frac{n}{2}, \frac{n}{2})^d$, $n \in \mathbb{N}$, *we have*
  $$\lim n^{-d} \int_{\Lambda_n} v^{\pm}(x,\omega)\mathrm{d}x = m^{\pm}, \tag{4.3}$$
  where $m^{\pm} = \mathbb{E}\left[\int_{[-\frac{1}{2},\frac{1}{2}]^d} v^{\pm}(x,\cdot)\mathrm{d}x\right]$, *and* $m^+ = -m^- \geq 0$.
- *Given* $v_0 \in L^\infty(\mathbb{R}^d)$, *let* $\bar{w}_n(\cdot,\omega)$ *be a minimizer of* $G_1^{v_0}(v,\omega,\Lambda_n)$ *according to Definition 2.4, then uniformly on* v_0

$$v^-(x,\omega) \leq \liminf_{n\to\infty} \bar{w}_n(x,\omega) \leq \limsup_{n\to\infty} \bar{w}_n(x,\omega) \leq v^+(x,\omega), \quad \text{(uniformly on compacts in } x\text{).} \tag{4.4}$$

These $v^{\pm}(\cdot,\omega)$ infinite volume minimizers will be obtained as limits of the so-called K-maximal/minimal minimizers.

Proposition 4.2. *Let* $K \in \mathbb{R}$, $K \geq 1 + C_0\theta\|g\|_\infty$ *and* $u_n^{\pm,K} \in H^s_{loc}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ *be respectively the* $K-$*maximal and the* $K-$*minimal minimizers of* G_1 *in* $\Lambda_n = (-\frac{n}{2}, \frac{n}{2})^d$. *We have that* $\mathbb{P}-a.s.$

$$\lim_{n\to\infty} u_n^{\pm,K}(x,\omega) = u^{\pm,K}(x,\omega) \qquad \text{point-wise and uniformly on compacts in} \quad x. \tag{4.5}$$

Further

- *If* $2s \leq 1$, *then* $u^{\pm,K}(\cdot,\omega) \in C^{\alpha}_{loc}(\mathbb{R}^d)$ *for all* $\alpha < 2s$. *If* $s \in (\frac{1}{2}, 1)$, *then* $u^{\pm,K}(\cdot,\omega) \in C^{1,\alpha}_{loc}(\mathbb{R}^d)$ *for all* $\alpha < 2s - 1$.
- $u^{\pm,K}(\cdot,\omega)$ *are translation covariant.*
- $$u^{+,K}(\cdot,\omega) = -u^{-,K}(\cdot,-\omega), \quad \mathbb{P} - a.s. \tag{4.6}$$
  .

Remark 4.3. *As in Remark 2.8 the convergence in* (4.5) *holds in* $C^{0,\beta}$, $\beta < \alpha < 2s$ *when* $s \in (0, \frac{1}{2}]$ *and in* $C^{1,\beta}$, $\beta < \alpha$, $\alpha < 2s-1$ *when* $s \in (\frac{1}{2}, 1)$.

Proof. We start proving the existence of $u^{\pm,K}$. For $z \in \mathbb{Z}^d$, denote by $u_n^{z,+,K} := u_n^{z,+,K}(\cdot,\omega)$ the maximal minimizer of G_1 in the domain $z+\Lambda_n$, so that $u_n^{z,+,K}(\cdot,\omega) = K$ in $\mathbb{R}^d \setminus (z+\Lambda_n)$ and respectively $u_n^{z,-,K} := u_n^{z,-,K}(\cdot,\omega)$ the minimal minimizer of G_1 in the domain $z+\Lambda_n$, so that $u_n^{z,-,K}(\cdot,\omega) = -K$ in $\mathbb{R}^d \setminus (z+\Lambda_n)$. If $z = 0$ we write $u_n^{\pm,K}$. Without loss of generality we assume for the next paragraph $z = 0$.

By Lemma 3.1, (2), $\|u_n^{\pm,K}\|_\infty \leq K$ for any n. Therefore $u_m^{+,K} \leq K$ on $\Lambda_m \setminus \Lambda_n$ for $m > n$. Lemma 3.4 implies that for any x and ω (and $n > n_0(x)$) the sequence $\{u_n^{+,K}(x,\omega)\}_n$ is decreasing. Moreover it is bounded from below by $-K$. Hence, reasoning in a similar manner for $u_n^{-,K}$,

$$u^{\pm,K}(x,\omega) := \lim_n u_n^{\pm,K}(x,\omega)$$

exist and are measurable as function of ω. We start analyzing the case $2s \leq 1$. As the $u_n^{\pm,K}$ are bounded and minimizers, they are on each fixed cube Q Hölder continuous of order $\alpha < 2s$ for any $2s \leq 1$, uniformly in n, provided $Q \subseteq \Lambda_n$, see Proposition 6.3. This implies that subsequences converge locally uniformly to a Hölder function of order $\alpha < 2s$. As the entire sequence converges point-wise, the limit of any subsequence must coincide with $u^{\pm,K}$, which is therefore a locally Hölder continuous function of order $\alpha < 2s$. The same argument for general z yields monotone limits $u^{z,\pm,K}$. When $s \in (\frac{1}{2}, 1)$ the argument goes in the same way, the only difference is that by Proposition 6.3 the minimizers $u_n^{\pm,K}$ are uniformly bounded and uniformly with respect to n in $C^{1,\alpha}$ with $\alpha < 2s-1$ on each fixed cube Q which does not depend on n.

To show that $u^{\pm,K}$ are translational covariant, notice that, by (2.2)

$$u_n^{0,+,K}(0,\omega) = u_n^{z,+,K}(z, T_{-z}\omega).$$

Take m large enough so that $\Lambda_n + z \subseteq \Lambda_m$. We have that $u_n^{z,+,K}(z, T_{-z}\omega) = u_n^{0,+,K}(0,\omega) \geq u_m^{0,+,K}(0,\omega)$, since $m > n$. Then letting first $n \to \infty$ and then $m \to \infty$ we get $u^{z,+,K}(z, T_{-z}\omega) \geq u^{0,+,K}(0,\omega)$. The opposite equality follows in the same way by taking $\Lambda_m \subseteq \Lambda_n + z$. Note that we used in the proof that the boundary condition is translation invariant. Next we prove (4.6). It is immediate to verify that

$$G_1^K(v,\omega,\Lambda_n) = G_1^{-K}(-v,-\omega,\Lambda_n) = G_1^{-K}(w,-\omega,\Lambda_n), \tag{4.7}$$

(see notation (2.9)), if we set $-v = w$. Therefore if $u_n^{+,K}(\cdot,\omega)$ is the maximal minimizer of $G_1^K(v,\omega,\Lambda_n)$ we have that $w_n(\cdot,-\omega) = -u_n^{+,K}(\cdot,\omega)$ is the minimal minimizer of $G_1^{-K}(w,-\omega,\Lambda_n)$ in Λ_n, i.e $w_n(\cdot,-\omega) = u_n^{-,K}(\cdot,-\omega)$. Then letting $n \to \infty$ we get (4.6). □

Next we show that the states $u^{\pm,K}$ are indeed minimizers under compact perturbations. In the proof we will only use that the boundary condition is bounded by K and has the regularity of a minimizer, but not that it is actually a constant.

Proposition 4.4. *Let* $K \in \mathbb{R}$, $K \geq 1 + C_0\theta\|g\|_\infty$ *and* $u^{\pm,K}(\cdot,\omega)$ *be the functions constructed in Proposition 4.2. Then, for any* $\Lambda \Subset \mathbb{R}^d$, *we have that*

$$G_1^{u^{+,K}}(u^{+,K},\omega,\Lambda) \leq G_1^{u^{+,K}}(u,\omega,\Lambda),$$

for any measurable function u *which coincides with* $u^{+,K}(\cdot,\omega)$ *in* Λ^c. *The same holds for* $u^{-,K}$.

Proof. Denote shortly $u^{+,K} = u^*$. We argue by contradiction. Assume that there exists a bounded set Λ and a measurable function u so that $G_1^{u^*}(u,\omega,\Lambda) < G_1^{u^*}(u^*,\omega,\Lambda)$. Let Λ_n be so large that $\Lambda \subset \Lambda_n$ and let $u_n^{+,K}$ be the $K-$maximal minimizer of G_1 in Λ_n, see Definition 3.6.

For simplicity we drop the dependence on ω and denote

$$E_1 := G_1^{u^*}(u^*,\Lambda), \qquad E_2 := G_1^{u^*}(u,\Lambda), \qquad E_n := G_1^K(u_n^{+,K},\Lambda_n).$$

By assumption there exists a $\delta > 0$ such that $E_2 + \delta < E_1$. The aim is to construct a function $\widetilde{u_n}$ such that if $E_2 + \delta < E_1$ then $G_1^K(\widetilde{u_n}, \Lambda_n) < E_n$ for some n large enough, which gives a contradiction.

Step 1: By (2.8)

$$\begin{aligned} E_1 &= \mathcal{K}_1(u^*, \Lambda) + \mathcal{W}((u^*, \Lambda), (u^*, \Lambda^c)) && (4.8)\\ E_2 &= \mathcal{K}_1(u, \Lambda) + \mathcal{W}((u, \Lambda), (u^*, \Lambda^c)) && (4.9)\\ E_n &= \mathcal{K}_1(u_n^{+,K}, \Lambda) + \mathcal{W}((u_n^{+,K}, \Lambda), (u_n^{+,K}, \Lambda_n \setminus \Lambda)) + \mathcal{K}_1(u_n^{+,K}, \Lambda_n \setminus \Lambda)\\ &+ \mathcal{W}((u_n^{+,K}, \Lambda_n \setminus \Lambda), (K, \Lambda_n^c)). && (4.10) \end{aligned}$$

Step 2: Next we show that for any $\epsilon > 0$ there exists n_ϵ s.t. for $n \geq n_\epsilon$

$$|\mathcal{K}_1(u_n^{+,K}, \Lambda) - \mathcal{K}_1(u^*, \Lambda)| < \epsilon, \tag{4.11}$$

$$A \equiv |\mathcal{W}((u^*, \Lambda), (u^*, \Lambda_n \setminus \Lambda)) - \mathcal{W}((u_n^{+,K}, \Lambda), (u_n^{+,K}, \Lambda_n \setminus \Lambda))| < \epsilon. \tag{4.12}$$

The bound (4.11) follows immediately from Proposition 5.2 with $D = \Lambda$, the regularity properties of the minimizers and Remark 4.3. To show (4.12), fix $R > 0$ so that $\Lambda \subset B_{R/2}(0)$ and require n so large that $B_R(0) \subset \Lambda_n$. Note that we can choose such R to be bounded uniformly in n. We upper bound A in (4.12) as following:

$$\begin{aligned} &A \leq |I_1| + |I_2|,\\ &I_1 = \int_\Lambda \int_{B_R(0)\setminus\Lambda} \frac{|u^*(z) - u^*(z')|^2 - |u_n^{+,K}(z) - u_n^{+,K}(z')|^2}{|z - z'|^{d+2s}} dz dz',\\ &I_2 = \int_\Lambda \int_{\Lambda_n \setminus B_R(0)} \frac{|u^*(z) - u^*(z')|^2 - |u_n^{+,K}(z) - u_n^{+,K}(z')|^2}{|z - z'|^{d+2s}} dz dz'. \end{aligned}$$

I_1 is estimated (in a very rough way) by Proposition 5.2 with $D = B_R$. For I_2, since $|u^*| \leq K,\ |u_n^{+,K}| \leq K$ we have

$$|I_2| \leq \int_\Lambda \int_{\mathbb{R}^d \setminus B_R(0)} \frac{8K}{|z - z'|^{d+2s}} dz dz' \leq 8KC(d)|\Lambda| \int_{R/2}^\infty r^{-2s-1} \leq K|\Lambda|C'(d)R^{-2s}.$$

Here we used the integrability of the kernel at infinity. In conclusion, by choosing first R sufficiently large, depending on ϵ, and then n_ϵ large depending on R we obtain (4.11) and (4.12) for all $n \geq n_\epsilon$.

Step 3: In the same way as I_2 above we use the integrability of the kernel at infinity to get

$$|\mathcal{W}((u^*, \Lambda), (u^*, \Lambda_n \setminus \Lambda)) - \mathcal{W}_1((u^*, \Lambda), (u^*, \mathbb{R}^d \setminus \Lambda))| \leq 4KC(d)|\Lambda| \int_{R/2}^\infty r^{-2s-1} \leq K|\Lambda|C'(d)R^{-2s} < \epsilon$$

for R and n sufficiently large. So

$$\begin{aligned} E_n >& E_1 - 3\epsilon + \mathcal{K}_1(u_n^{+,K}, \Lambda_n \setminus \Lambda) + \mathcal{W}((u_n^{+,K}, \Lambda_n \setminus \Lambda), (K, \Lambda_n^c))\\ >& E_2 + \mathcal{K}_1(u_n^{+,K}, \Lambda_n \setminus \Lambda) + \mathcal{W}_1((u_n^{+,K}, \Lambda_n \setminus \Lambda), (K, \Lambda_n^c)) + \delta - 3\epsilon. \end{aligned} \tag{4.13}$$

Step 4 Now we construct a function on Λ_n such that its energy in this cube with K b.c. approximates the first three terms in the last line of (4.13), which will lead to a contradiction. Define a function $\widetilde{u_n}$ which is equal to u in Λ and equal to $u_n^{+,K}$ outside a boundary layer of width 1 of Λ:

$$\widetilde{u_n}(x) := \begin{cases} u(x), & \text{if } x \in \Lambda,\\ u_n^{+,K}(x) & \text{if } x \in \mathbb{R}^d:\ \operatorname{dist}(x, \Lambda) > 1,\\ u^*(x) + \Psi(x)(u_n^{+,K}(x) - u^*(x)) & \text{else} \end{cases} \tag{4.14}$$

where $\Psi : \mathbb{R}^d \to [0, 1]$ is a smooth cut-off function nondecreasing in $\operatorname{dist}(x, \Lambda)$ with $\Psi(x) = 0$ if $\operatorname{dist}(x, \Lambda) < 1/2$ and $\Psi(x) = 1$ if $\operatorname{dist}(x, \Lambda) > 1$. Notice that $\widetilde{u_n} - u^* \to 0$ in $C^{0,\alpha}(\Lambda_n \setminus \Lambda)$ for $\alpha < 2s$.

By the equality $u^*(x)+\Psi(x)(u_n^{+,K}(x)-u^*(x)) = u_n^{+,K}(x)+[1-\Psi(x)](u^*(x)-u_n^{+,K}(x))$ which we will use in the following we get also that $\widetilde{u_n}-u_n^{+,K}\to 0$ in $C^{0,\alpha}(\Lambda_n\setminus\Lambda)$ for $\alpha<2s$. Set

$$\begin{aligned}I_3 &= |\mathcal{W}((u,\Lambda),(u^*,\Lambda^c))-\mathcal{W}((u,\Lambda),(\widetilde{u_n},\Lambda^c))|\\ &= \left|\int_\Lambda dz\int_{\Lambda^c}dz'\,\frac{|u(z)-u^*(z')|^2-|u(z)-\tilde{u}_n(z')|^2}{|z-z'|^{d+2s}}\right|\\ &= \left|\int_\Lambda dz\int_{\Lambda^c}dz'\,\frac{2u(z)[\widetilde{u_n}(z')-u^*(z')]+[(\widetilde{u_n}(z'))^2-(u^*(z'))^2]}{|z-z'|^{d+2s}}\right|.\end{aligned}$$

As $\widetilde{u_n}(x)=u^*(x)$ for $x\in\Lambda^c$ and $\mathrm{dist}(x,\Lambda)<1/2$, the integrand vanishes unless $|z-z'|>1/2$. For R as in Step 2 we estimate I_3 by splitting $\Lambda^c=(\Lambda^c\cap B_R(0))\cup(\Lambda^c\setminus B_R(0))$

$$I_3\le C(d)|\Lambda|R^d\|u^*-u_n^{+,K}\|_{L^\infty(B_R)}+|\Lambda|C(d)R^{-2s}K.$$

Choosing first R large and then n_0 depending on R and ϵ, we obtain that for $n\ge n_0$, $|I_3|<\epsilon$ and hence, see (4.9),

$$E_2\ge\mathcal{K}_1(u,\Lambda)+\mathcal{W}((u,\Lambda),(\widetilde{u_n},\Lambda^c))-\epsilon. \tag{4.15}$$

By definition of $\widetilde{u_n}$

$$\mathcal{W}((u,\Lambda),(\widetilde{u_n},\Lambda^c))=\mathcal{W}((u,\Lambda),(\widetilde{u_n},\Lambda_n\setminus\Lambda))+\mathcal{W}_1(u,\Lambda),(K,\Lambda_n^c),$$

we therefore obtain

$$E_2\ge\mathcal{K}_1(u,\Lambda)+\mathcal{W}_1((u,\Lambda),(\widetilde{u_n},\Lambda_n\setminus\Lambda))+\mathcal{W}((u,\Lambda),(K,\Lambda_n^c))-\epsilon. \tag{4.16}$$

Step 5 By (4.14) and (4.16)

$$\begin{aligned}G_1^K(\widetilde{u_n},\Lambda_n) &= \mathcal{K}_1(u,\Lambda)+\mathcal{W}((u,\Lambda),(\widetilde{u_n},\Lambda_n\setminus\Lambda))+\mathcal{W}((u,\Lambda),(K,\Lambda_n^c))\\ &\quad+\mathcal{K}_1(\widetilde{u_n},\Lambda_n\setminus\Lambda)+\mathcal{W}((\widetilde{u_n},\Lambda_n\setminus\Lambda),(K,\Lambda_n^c))\\ &\le E_2+\epsilon+\mathcal{K}_1(\widetilde{u_n},\Lambda_n\setminus\Lambda)+\mathcal{W}((\widetilde{u_n},\Lambda_n\setminus\Lambda),(K,\Lambda_n^c)).\end{aligned}$$

Therefore

$$E_2\ge G_1^K(\widetilde{u_n},\Lambda_n)-\epsilon-\mathcal{K}_1(\widetilde{u_n},\Lambda_n\setminus\Lambda)-\mathcal{W}((\widetilde{u_n},\Lambda_n\setminus\Lambda),(K,\Lambda_n^c)). \tag{4.17}$$

By (4.13) if we show that

$$\left|\mathcal{K}_1(\widetilde{u_n},\Lambda_n\setminus\Lambda)-\mathcal{K}_1(u_n^{+,K},\Lambda_n\setminus\Lambda)\right|<\epsilon \tag{4.18}$$

$$\left|\mathcal{W}((\widetilde{u_n},\Lambda_n\setminus\Lambda),(K,\Lambda_n^c))-\mathcal{W}((u_n^{+,K},\Lambda_n\setminus\Lambda),(K,\Lambda_n^c))\right|<\epsilon, \tag{4.19}$$

then

$$E_n>-6\epsilon+\delta+G_1^K(\widetilde{u_n},\Lambda_n)$$

for n sufficiently large. As ϵ was arbitrary and E_n is minimal value with K-boundary conditions, this means $\delta=0$ and hence u^* is a minimizer under compact perturbations. Next we prove (4.18) and (4.19). The (4.18) follows by applying Proposition 5.2 since $\widetilde{u_n}-u_n^{+,K}\to 0$ in $C^{0,\alpha}(\Lambda_n\setminus\Lambda)$ for $\alpha<2s$. Note that the difference is equal to zero for $\mathrm{dist}(x,\Lambda)>1$. The (4.19) follows by

$$\begin{aligned}&\int_{(\Lambda_n\setminus\Lambda)\times\Lambda_n^c}dzdz'\,\frac{\left||\widetilde{u_n}(z)-K|^2-|u_n^{+,K}(z)-K|^2\right|}{|z-z'|^{d+2s}}\le 4K\int_{\{\mathrm{dist}(x,\Lambda)\le 1\}\cap(\Lambda_n\setminus\Lambda)\times B_R(0)^c}dzdz'\,\frac{|\widetilde{u_n}(z)-u_n^{+,K}(z)|}{|z-z'|^{d+2s}}\\ &\le|2\Lambda|4K\|u_n^{+,K}-u^*\|_{L^\infty(2\Lambda)}C(d)\int_R^\infty r^{-2s-1}dr\le|2\Lambda|4K\|u_n^{+,K}-u^*\|_{L^\infty(2\Lambda)}C'(d)R^{-2s}\end{aligned}$$

where we used that $\widetilde{u_n}=u_n^{+,K}$ for $\mathrm{dist}(x,\Lambda)>1$ and R is chosen as large as possible with $B_R(0)\subseteq\Lambda_n$.

□

Next we show that $\|u^{+,K}\|_\infty$ is bounded uniformly on K.

Lemma 4.5. *Let $u^{\pm,K}(x,\omega)$ the functions constructed in Proposition 4.2, see* (4.5). *Then* uniformly in K

$$\|u^{\pm,K}(\omega)\|_\infty \le 1 + C_0\theta\|g\|_\infty \qquad \mathbb{P}-a.s., \tag{4.20}$$

where C_0 is the constant in (2.4).

Proof. Take $\Lambda_0 = [-\frac{1}{2}, \frac{1}{2}]^d$ and $\Lambda_n = (-\frac{n}{2}, \frac{n}{2})^d$, i.e. $|\Lambda_n| = n^d|\Lambda_0|$. Define for $z \in \mathbb{Z}^d$ and for any $C^+ \ge 1 + C_0\theta\|g\|_\infty$

$$\Lambda_z := \Lambda_0 + z, \quad B^n(\omega) = |\{x \in \Lambda_n : |u^{+,K}(x,\omega)| > C^+\}| = \sum_{z\in\Lambda_n\cap\mathbb{Z}^d} |\{x \in \Lambda_z : |u^{+,K}(x,\omega)| > C^+\}|.$$

Since $u^{+,K}$ is translation covariant

$$|\{x \in \Lambda_z : |u^{+,K}(x,\omega)| > C^+\}| = |\{x \in \Lambda_0 : |u^{+,K}(x, T_z\omega)| > C^+\}|.$$

Hence we obtain

$$B^n(\omega) = \sum_{z\in\Lambda_n\cap\mathbb{Z}^d} |\{x \in \Lambda_0 : |u^{+,K}(x, T_z\omega)| > C^+\}|.$$

If $|\{x \in \Lambda_0 : |u^{+,K}(x,\omega)| > C^+\}| = 0$, $\mathbb{P}-$ almost surely then $B^n(\omega) = 0$, $\mathbb{P}-$ a. s. for all n, and we obtain the claim. Suppose that the claim is false. Assume that for some $\eta > 0$

$$\mathbb{E}\left\{|\{x \in \Lambda_0 : |u^{+,K}(x,\omega)|\right\} = |\Lambda_0|\eta.$$

Therefore by the ergodic theorem $\frac{B^n(\omega)}{n^d} \to \eta$ almost surely. Fix an ω in the set of full measure where this holds, and treat it from now on as parameter. There exists n_0 (depending on ω), such that for $n \ge n_0$, $B^n(\omega) > n^d\eta/2 > 0$.

Now define a function

$$v(x) := \begin{cases} C^+ \wedge u^{+,K} \vee (-C^+), & \text{if } \{x : \text{ dist}(x, \Lambda_n^c) > 2\}, \\ u^{+,K} & \text{if } x \in \mathbb{R}^d : \{x : \text{ dist}(x, \Lambda_n^c) \le 1\} \cup \Lambda_n^c \\ \Phi(x) & \text{else,} \end{cases} \tag{4.21}$$

where $\Phi(x)$ is a smooth interpolation between $u^{+,K}$ and $C^+ \wedge u^{+,K} \vee (-C^+)$. By Lemma 3.1 there exists a constant $c > 0$ which depends on K, θ, C_0 and $\|g\|_\infty$ such that

$$\mathcal{K}_1(u^{+,K}, \Lambda_n) \ge \mathcal{K}_1(v, \Lambda_n) + c|B^n| > \mathcal{K}_1(v, \Lambda_n) + c\frac{\eta}{2}n^d. \tag{4.22}$$

Note that $u^{+,K}$ is minimizer and has therefore higher regularity. The cutting and interpolation procedure retains Holder regularity. (For the cutting, note that it is the application of a Lipschitz function. For the interpolation, note that the cut-off can be chosen smooth, with a uniform bound on the first derivative.) So we have sufficient regularity to apply Lemma 5.1 for Λ_n, and we know that for any given ϵ there exists n_ϵ sufficiently large so that for $n \ge n_\epsilon$

$$\mathcal{W}((u^{+,K}, \Lambda_n), (u^{+,K}, \Lambda_n^c)) - \mathcal{W}((v, \Lambda_n), (u^{+,K}, \Lambda_n^c)) \ge -2\epsilon|\Lambda_n|. \tag{4.23}$$

From (4.22) adding and subtracting $\mathcal{W}((u^{+,K}, \Lambda_n), (u^{+,K}, \Lambda_n^c))$ we get

$$G_1^{u^{+,K}}(u^{+,K}, \Lambda_n) \ge \mathcal{K}_1(v, \Lambda_n) + \mathcal{W}((u^{+,K}, \Lambda_n), (u^{+,K}, \Lambda_n^c)) + c\eta/2(2n)^d.$$

Taking into account (4.23) we obtain

$$G_1^{u^{+,K}}(u^{+,K}, \Lambda_n) \ge G_1^{u^{+,K}}(v, \Lambda_n) - 2\epsilon|\Lambda_n| + c\eta/2(2n)^d.$$

Choosing n sufficiently large we get $G_1^{u^{+,K}}(v, \Lambda_n) < G_1^{u^{+,K}}(u^{+,K}, \Lambda_n)$, which contradicts the fact that $u^{+,K}$ is a minimizer under compact perturbations. Note that the proof works for all $C^+ \ge 1 + C_0\theta\|g\|_\infty$, which proves (4.20). $\square$

Definition 4.6. Infinite volume states *Let $K \in \mathbb{R}$, $K \geq 1 + C_0\theta\|g\|_\infty$ and $u^{\pm,K}(\cdot,\omega)$ the functions constructed in Proposition 4.2. We define the infinite volume states $v^{\pm}(\cdot,\omega)$ be the following point-wise limit:*

$$\lim_{K\to\infty} u^{\pm,K} = v^{\pm}, \qquad \mathbb{P}-a.s. \tag{4.24}$$

The limit is well defined since $\|u^{\pm,K}(\cdot,\omega)\|_\infty \leq 1 + C_0\theta\|g\|_\infty$ and the sequence $\{u^{+,K}(\cdot,\omega)\}_K$ is increasing ($\{u^{-,K}(\cdot,\omega)\}_K$ decreasing) in K.

In the next lemma we show that the $v^{\pm}$ inherit the regularity of $u^{\pm,K}$ and that convergence in (4.24) holds in a stronger norm.

Lemma 4.7. *Let $K \in \mathbb{R}$, $K \geq 1 + C_0\theta\|g\|_\infty$ and $u^{\pm,K}(\cdot,\omega)$ the functions constructed in Proposition 4.2. Then $v^{\pm}(\cdot,\omega)$ defined in (4.24) are in $C^{0,\alpha}_{loc}(\mathbb{R}^d)$ for any $\alpha < 2s$ for $2s \leq 1$, and $C^{1,\alpha}_{loc}(\mathbb{R}^d)$ for any $\alpha < 2s-1$ for $2s > 1$. Further for any $\Lambda \Subset \mathbb{R}^d$ the convergence in (4.24) holds in $C^{0,\beta}(\Lambda)$, $\beta < \alpha < 2s$ when $s \in (0, \frac{1}{2}]$, and in $C^{1,\beta}(\Lambda)$, $\beta < \alpha < 2s-1$ when $s \in (\frac{1}{2}, 1)$.*

Proof. By Proposition 4.2 $\{u^{\pm,K}(\cdot,\omega)\}_K$ is bounded and in $C^{0,\alpha}_{loc}$ for any $\alpha < 2s$ for $2s \leq 1$, and $C^{1,\alpha}_{loc}$ for any $\alpha < 2s-1$ for $2s > 1$. This implies that subsequences converge locally uniformly to an Holder function of order $\alpha < 2s$ when $2s \leq 1$ and when $2s > 1$ to a function in $C^{1,\alpha}_{loc}$ for $\alpha < 2s-1$. As the entire sequence converges point-wise, the limit of any subsequence must coincide with $v^{\pm}$, which is therefore a locally Holder continuous function of order $\alpha < 2s$ when $2s \leq 1$ or when $2s > 1$ a function in $C^{1,\alpha}_{loc}$ with $\alpha < 2s-1$. From this and the compact embedding of Holder spaces, see Remark 2.8, we deduce that $u^{\pm,K}$ converge to $v^{\pm}$ on any compact set Λ in $C^{0,\beta}(\Lambda)$, $\beta < \alpha < 2s$ when $2s \leq 1$ and on $C^{1,\beta}(\Lambda)$, $\beta < \alpha$ when $\alpha < 2s-1$. □

The following Lemma states that point-wise limits of minimizers under compact perturbations are minimizers under compact perturbations. As we could not find an appropriate result in the literature, we prove it here in the form needed for this paper.

Lemma 4.8. *Let $\Psi^k : \mathbb{R}^d \to \mathbb{R}$ be a family of uniformly bounded (in L^∞) minimizers under compact perturbations of G_1, see Definition (2.4). Assume that $\{\Psi^k\}$ converges point-wise to a function $\Psi : \mathbb{R}^d \to \mathbb{R}$. Then Ψ is a minimizer of G_1 under compact perturbations.*

Proof. In the following ω is a parameter, so we avoid to write it explicitly. We show the lemma by contradiction. Assume that Ψ is not a minimizer under compact perturbation. Then there exists a compact set (which we may assume to be a cube) Λ and a measurable function u so that $G_1^{\Psi}(u,\omega,\Lambda) < G_1^{\Psi}(\Psi,\omega,\Lambda)$. Denote $\Lambda_1 = \Lambda \cup \{x \in \mathbb{R}^d : dist(x,\Lambda) \leq 2\}$

$$E_1 := G_1^{\Psi}(\Psi,\omega,\Lambda), \qquad E_2 := G_1^{\Psi}(u,\omega,\Lambda), \qquad E_k := G_1^{\Psi^k}(\Psi^k,\omega,\Lambda_1).$$

By assumption there exists a $\delta > 0$ such that $E_2 + \delta < E_1$. The aim is to construct a function $\widetilde{\Psi^k}$, for some k large enough, such that if $E_2 + \delta < E_1$ then $G_1^{\Psi^k}(\widetilde{\Psi^k},\Lambda_1) < E_k$, which gives a contradiction, since Ψ^k is by assumption a minimizer under compact perturbations. The proof is similar to the one in Proposition 4.4.

Step 1: By (2.8)

$$E_1 = \mathcal{K}_1(\Psi,\Lambda) + \mathcal{W}((\Psi,\Lambda),(\Psi,\Lambda^c)) \tag{4.25}$$
$$E_2 = \mathcal{K}_1(u,\Lambda) + \mathcal{W}((u,\Lambda),(\Psi,\Lambda^c)) \tag{4.26}$$
$$E_k = \mathcal{K}_1(\Psi^k,\Lambda_1) + \mathcal{W}((\Psi^k,\Lambda_1),(\Psi^k,\Lambda_1^c)) \tag{4.27}$$

We write E_k as

$$E_k = \mathcal{K}_1(\Psi^k,\Lambda) + \mathcal{W}((\Psi^k,\Lambda),(\Psi^k,\Lambda^c)) + B_k,$$

where

$$B_k = \mathcal{K}_1(\Psi^k,\Lambda_1 \setminus \Lambda) + \mathcal{W}((\Psi^k,\Lambda_1 \setminus \Lambda),(\Psi^k,\Lambda_1^c)).$$

Step 2: Next we show that for any $\epsilon > 0$ there exists k_ϵ s.t. for $k \geq k_\epsilon$

$$|\mathcal{K}_1(\Psi^k, \Lambda) - \mathcal{K}_1(\Psi, \Lambda)| < \epsilon, \tag{4.28}$$

$$A \equiv |\mathcal{W}((\Psi, \Lambda), (\Psi, \Lambda^c)) - \mathcal{W}((\Psi^k, \Lambda), (\Psi^k, \Lambda^c))| < \epsilon. \tag{4.29}$$

The (4.28) follows immediately from Proposition 5.2 with $D = \Lambda$, the regularity property of the minimizers, see Lemma 4.7. For (4.29), fix $R > 0$ so that $\Lambda \subset B_{R/2}(0)$. We upper bound A in (4.29) as following:

$$\begin{aligned} &A \leq |I_1| + |I_2|, \\ &I_1 = \int_\Lambda \int_{B_R(0)\setminus\Lambda} \frac{|\Psi(z) - \Psi(z')|^2 - |\Psi^k(z) - \Psi^k(z')|^2}{|z - z'|^{d+2s}} dz dz', \\ &I_2 = \int_\Lambda \int_{\Lambda^c\setminus B_R(0)} \frac{|\Psi(z) - \Psi(z')|^2 - |\Psi^k(z) - \Psi^k(z')|^2}{|z - z'|^{d+2s}} dz dz'. \end{aligned}$$

I_1 is estimated (in a very rough way) by Proposition 5.2 with $D = B_R$. For I_2, since $|\Psi| \leq C^+$, $|\Psi^k| \leq C^+$ we have

$$|I_2| \leq \int_\Lambda \int_{\mathbb{R}^d\setminus B_R(0)} \frac{8C^+}{|z - z'|^{d+2s}} dz dz' \leq 8C^+ C(d)|\Lambda| \int_{R/2}^\infty r^{-2s-1} \leq C^+|\Lambda|C'(d)R^{-2s}.$$

Here we used the integrability of the kernel at infinity. In conclusion, by choosing first R sufficiently large, depending on ϵ, and then k_ϵ large depending on R we obtain (4.28) and (4.29) for all $k \geq k_\epsilon$.

Step 3: By (4.28) and (4.29) for k sufficiently large

$$E_k > E_1 - 2\epsilon + B_k > E_2 + \delta - 2\epsilon + B_k. \tag{4.30}$$

Step 4 Define a function $\widetilde{\Psi^k}$ which is equal to u in Λ and equal to Ψ^k outside a boundary layer of width 1 of Λ.

$$\widetilde{\Psi^k}(x) := \begin{cases} u(x), & \text{if } x \in \Lambda, \\ \Psi^k(x) & \text{if } x \in \mathbb{R}^d: \ \mathrm{dist}(x, \Lambda) > 1 \\ \Psi(x) + \Phi(x)(\Psi^k(x) - \Psi(x)) & \text{else} \end{cases} \tag{4.31}$$

where $\Phi : \mathbb{R}^d \to [0, 1]$ is a smooth cut-off function nondecreasing in $\mathrm{dist}(x, \Lambda)$ with $\Phi(x) = 0$ if $\mathrm{dist}(x, \Lambda) < 1/2$ and $\Phi(x) = 1$ if $\mathrm{dist}(x, \Lambda) > 1$. Then

$$\begin{aligned} &I_3 := |\mathcal{W}((u, \Lambda), (\Psi, \Lambda^c)) - \mathcal{W}((u, \Lambda), (\widetilde{\Psi^k}, \Lambda^c))| \\ &= \left| \int_\Lambda \mathrm{d}z \int_{\Lambda^c} \mathrm{d}z' \frac{|u(z) - \Psi(z')|^2 - |u(z) - \widetilde{\Psi^k}(z')|^2}{|z - z'|^{d+2s}} \right| \\ &= \left| \int_\Lambda \mathrm{d}z \int_{\Lambda^c} \mathrm{d}z' \frac{2u(z)[\widetilde{\Psi^k}(z') - \Psi(z')] + [\widetilde{\Psi^k}(z') - \Psi(z')][\widetilde{\Psi^k}(z') + \Psi(z')]}{|z - z'|^{d+2s}} \right| \\ &= \left| \int_\Lambda \mathrm{d}z \int_{\Lambda^c} \mathrm{d}z' 1\!\!1_{|z-z'|>1/2|} \frac{2u(z)[\widetilde{\Psi^k}(z') - \Psi(z')] + [\widetilde{\Psi^k}(z') - \Psi(z')][\widetilde{\Psi^k}(z') + \Psi(z')]}{|z - z'|^{d+2s}} \right|. \end{aligned}$$

The last equality holds since $\widetilde{\Psi^k}(x) = \Psi(x)$ for $x \in \Lambda^c$ and $\mathrm{dist}(x, \Lambda) < 1/2$, therefore the integrand vanishes unless $|z - z'| > 1/2$. Take R so large that $\Lambda \subset B_{\frac{R}{2}}(0)$ and split $\Lambda^c = (\Lambda^c \cap B_R(0)) \cup (\Lambda^c \setminus B_R(0))$. We obtain

$$I_3 \leq C(d)|\Lambda|R^d \|\Psi - \Psi^k\|_{L^\infty(B_R)} + |\Lambda|C(d, \theta, C_0, \|g\|_\infty)R^{-2s}.$$

For any ϵ take $R_0(\epsilon)$ so that for $R \geq R_0(\epsilon)$ $|\Lambda| C(d, \theta, C_0, \|g\|_\infty) R^{-2s}. \leq \frac{\epsilon}{2}$, then take K_0 depending on ϵ, so that that for $K \geq K_0$, $|I_3| < \epsilon$ and hence

$$E_2 = \mathcal{K}_1(u, \Lambda) + \mathcal{W}((u, \Lambda), (\Psi, \Lambda^c)) \geq \mathcal{K}_1(u, \Lambda) + \mathcal{W}((u, \Lambda), (\widetilde{\Psi^k}, \Lambda^c)) - \epsilon. \tag{4.32}$$

By definition of $\widetilde{\Psi^k}$

$$\mathcal{W}((u, \Lambda), (\widetilde{\Psi^k}, \Lambda^c)) = \left[\mathcal{W}((u, \Lambda), (\widetilde{\Psi^k}, \Lambda_1 \setminus \Lambda)) + \mathcal{W}_1(u, \Lambda), (\Psi^k, \Lambda_1^c)\right],$$

we therefore obtain

$$E_2 \geq \mathcal{K}_1(u, \Lambda) + \left[\mathcal{W}((u, \Lambda), (\widetilde{\Psi^k}, \Lambda_1 \setminus \Lambda)) + \mathcal{W}_1(u, \Lambda), (\Psi^k, (\Lambda_1)^c)\right] - \epsilon. \tag{4.33}$$

Step 5 By (4.31) and (4.33)

$$\begin{aligned} G_1^{\Psi^k}(\widetilde{\Psi^k}, \Lambda_1) &= \mathcal{K}_1(u, \Lambda) + \mathcal{W}((u, \Lambda), (\widetilde{\Psi^k}, \Lambda_1 \setminus \Lambda)) + \mathcal{W}((u, \Lambda), (\Psi^k, \Lambda_1^c)) \\ &+ \mathcal{K}_1(\widetilde{\Psi^k}, \Lambda_1 \setminus \Lambda) + \mathcal{W}((\widetilde{\Psi^k}, \Lambda_1 \setminus \Lambda), (\Psi^k, \Lambda_1^c)) \\ &\leq E_2 + \epsilon + \mathcal{K}_1(\widetilde{\Psi^k}, \Lambda_1 \setminus \Lambda) + \mathcal{W}((\widetilde{\Psi^k}, \Lambda_1 \setminus \Lambda), (\Psi^k, \Lambda_1^c)). \end{aligned} \tag{4.34}$$

Next we show that for any $\epsilon > 0$ there exists k_ϵ so that for $k \geq k_\epsilon$

$$\left|\mathcal{K}_1(\widetilde{\Psi^k}, \Lambda_1 \setminus \Lambda) - \mathcal{K}_1(\Psi^k, \Lambda_1 \setminus \Lambda)\right| < \epsilon \tag{4.35}$$

$$\left|\mathcal{W}((\widetilde{\Psi^k}, \Lambda_1 \setminus \Lambda), (\Psi^k, \Lambda_1^c)) - \mathcal{W}((\Psi^k, \Lambda_1 \setminus \Lambda), (\Psi^k, \Lambda_1^c))\right| < \epsilon. \tag{4.36}$$

Assuming that (4.35) and (4.36) hold, we obtain from (4.30) and (4.34) that

$$E_k > E_2 + \delta - 2\epsilon + B_k \geq -4\epsilon + \delta + G_1^{\Psi^k}(\widetilde{\Psi^k}, \Lambda_1)$$

for k sufficiently large. As ϵ was arbitrary and E_k is minimal value with Ψ^k-boundary conditions, $\delta = 0$ and hence Ψ is a minimizer under compact perturbations.

To prove (4.35), we notice that $\widetilde{\Psi^k}(x) = \Psi(x) + \Phi(x)(\Psi^k(x) - \Psi(x)) = \Psi^k(x) + (1 - \Phi(x))(\Psi(x) - \Psi^k(x))$ and $\Phi(x) = 1$ when $\mathrm{dist}(x, \Lambda) \geq 1$ and $\|\widetilde{\Psi^k} - \Psi^k\|_{C^{0,\beta}(\Lambda_1 \setminus \Lambda)} \to 0$ for $\beta < \alpha < 2s$ when $s \in (0, \frac{1}{2}]$ and $\|\widetilde{\Psi^k} - \Psi^k\|_{C^{1,\beta}(\Lambda_1 \setminus \Lambda)} \to 0$ for $\beta < \alpha$ when $s \in (\frac{1}{2}, 1)$. Therefore by Proposition 5.2 for k large enough

$$\left|\mathcal{K}_1(\widetilde{\Psi^k}, \Lambda_1 \setminus \Lambda) - \mathcal{K}_1(\Psi^k, \Lambda_1 \setminus \Lambda)\right| \leq \epsilon$$

Note that the difference is equal to zero for $\mathrm{dist}(x, \Lambda) > 1$. Next we prove (4.36). We have

$$\begin{aligned} &\int_{(\Lambda_1 \setminus \Lambda) \times \Lambda_1^c} \frac{\left||\widetilde{\Psi^k}(z) - \Psi^k(z')|^2 - |\Psi^k(z) - \Psi^k(z')|^2\right|}{|z - z'|^{d+2s}} \\ &= \int_{(\Lambda_1 \setminus \Lambda) \times \Lambda_1^c} \mathbb{I}_{\{dist(z, \Lambda) \leq 1\}} \frac{\left||\widetilde{\Psi^k}(z) - \Psi^k(z')|^2 - |\Psi^k(z) - \Psi^k(z')|^2\right|}{|z - z'|^{d+2s}} \\ &\leq C \int_{(\Lambda_1 \setminus \Lambda) \times \Lambda_1^c} \mathbb{I}_{\{dist(z, \Lambda) \leq 1\}} \frac{|\Psi(z) - \Psi^k(z)|}{|z - z'|^{d+2s}} \\ &\leq C |\Lambda|^{\frac{d-1}{d}} \|\Psi - \Psi^k\|_{L^\infty(\Lambda_1)} \int_1^\infty r^{-2s-1} dr \leq C |\Lambda|^{\frac{d-1}{d}} \|\Psi - \Psi^k\|_{L^\infty(\Lambda_1)} \leq \epsilon \end{aligned}$$

if $k \geq k_\epsilon$.

□

Now we can prove the main theorem:

Proof of Theorem 4.1 Let $v^\pm$ be the infinite volume states defined in (4.24). The existence and the first three properties of $v^\pm$ are established in Proposition 4.2 for $u^{\pm,K}$ and they are inherited by the limit. Lemma 4.5 establishes the L^∞ bound for $u^{\pm,K}$ which is inherited by the limit as well. The proof that $v^\pm$ are minimizers under compact perturbation is done in Lemma 4.8. Next we prove (4.3). We have

$$\begin{aligned}\int_{\Lambda_n} v^\pm(x,\omega)\mathrm{d}x &= \sum_{z\in\Lambda_n\cap\mathbb{Z}^d}\int_{\{z+[-\frac12,\frac12]^d\}} v^\pm(x,\omega)\mathrm{d}x\\ &= \sum_{z\in\Lambda_n\cap\mathbb{Z}^d}\int_{[-\frac12,\frac12]^d} v^\pm(T_z x,\omega)\mathrm{d}x = \sum_{z\in\Lambda_n\cap\mathbb{Z}^d}\int_{[-\frac12,\frac12]^d} v^\pm(x,T_{-z}\omega)\mathrm{d}x.\end{aligned} \tag{4.37}$$

Since $|v^\pm(x,\omega)|\le(1+C_0\theta\|g\|_\infty)$, by the Birkhoff's ergodic theorem, see for example [12], we have $\mathbb{P}-$ a.s

$$\begin{aligned}&\lim\frac{1}{n^d}\int_{\Lambda_n} v^\pm(x,\omega)\mathrm{d}x = \lim\frac{1}{n^d}\sum_{z\in\Lambda_n\cap\mathbb{Z}^d}\int_{[-\frac12,\frac12]^d} v^\pm(x,T_{-z}\omega)\mathrm{d}x\\ &=\mathbb{E}\left[\int_{[-\frac12,\frac12]^d} v^\pm(x,\cdot)\mathrm{d}x\right] = m^\pm.\end{aligned} \tag{4.38}$$

It remains to show (4.4). Let $\bar w_n$ be as in the statement of the theorem and fix $x\in\Lambda_n$. Denote $K=\max\{\|\bar v_0\|_\infty,(1+C_0\theta\|g\|_\infty)\}$. Let $u_n^{\pm,K}$ the $K-$ maximal and the $K-$ minimal minimizer of G_1 in Λ_n, see Definition 3.6. By Lemma 3.4 we get that $u_n^{-,K}(x,\omega)\le\bar w_n(x,\omega)\le u_n^{+,K}(x,\omega)$ for $x\in\mathbb{R}^d$. Then, by (4.5), uniformly for any compact set of $\mathbb{R}^d$ containing x we have

$$v^-(x,\omega)\le u^{-,K}(x,\omega)\le\liminf_n\bar w_n(x,\omega)\le\limsup_n\bar w_n(x,\omega)\le u^{+,K}(x,\omega)\le v^+(x,\omega).$$

The first and last inequality hold since $\{u^{+,K}\}_K$ is increasing ($\{u^{-,K}\}_K$ decreasing) in K. The (4.4) follows. □

In the next Lemma we bound uniformly in ω the difference between the energy of the two extrema macroscopic minimizers $v^\pm$.

Lemma 4.9. *Let $\Lambda\Subset\mathbb{R}^d$, cube-like, $v^\pm$ be the infinite volume states constructed in Theorem 4.1. There exists a positive constant C depending on θ, d, s, C_0 and $\|g\|_\infty$, so that $\mathbb{P}-$ a.s.*

$$\big|G_1(v^+,\omega,\Lambda)-G_1(v^-,\omega,\Lambda)\big|\le\begin{cases}C|\Lambda|^{\frac{d-2s}{d}}, & s\in(0,\frac12),\\ C|\Lambda|^{\frac{d-1}{d}}, & s\in(\frac12,1),\\ C|\Lambda|^{\frac{d-1}{d}}\log|\Lambda|, & s=\frac12.\end{cases} \tag{4.39}$$

Proof. Let the cut-off function $\Psi:\mathbb{R}^d\to\mathbb{R}$ be a smooth nondecreasing function in $\mathrm{dist}(x,\Lambda^c)$ with $\Psi(x)=1$ if $\mathrm{dist}(x,\Lambda^c)\ge1$ and $\Psi(x)=0$ if $\mathrm{dist}(x,\Lambda^c)=0$. Set

$$\tilde u:=\Psi v^+ + (1-\Psi\ )\,v^-. \tag{4.40}$$

The function $\tilde u$ is equal to v^- when $x\in\Lambda^c$ and equal to v^+ when $x\in\Lambda$, $\mathrm{dist}(x,\Lambda^c)>1$ and interpolates in a smooth way between these values. Since v^- is the minimal $-$ minimizer in Λ we have

$$G_1(v^-,\omega,\Lambda)\le G_1^{v^-}(\tilde u,\omega,\Lambda). \tag{4.41}$$

We will show that

$$G_1^{v^-}(\tilde u,\omega,\Lambda)\le G_1(v^+,\omega,\Lambda)+M(s) \tag{4.42}$$

where we denote shortly by $M(s)$ the right hand side of (4.39). Therefore from (4.41)

$$G_1(v^-,\omega,\Lambda) - G_1(v^+,\omega,\Lambda) \le M(s). \tag{4.43}$$

In a similar way we can show that

$$G_1(v^+,\omega,\Lambda) - G_1(v^-,\omega,\Lambda) \le M(s). \tag{4.44}$$

Then, from (4.43) and (4.44) we get (4.39). Next we show (4.42). By definition

$$G_1^{v^-}(\tilde u,\omega,\Lambda) = \mathcal{K}_1(\tilde u,\omega,\Lambda) + \mathcal{W}((\tilde u,\Lambda)(v^-,\Lambda^c)). \tag{4.45}$$

Denote by

$$\partial_\Lambda = \{x \in \Lambda : \mathrm{dist}(x,\Lambda^c) \le 1\}.$$

By definition of $\tilde u$, see (4.40), we have

$$\mathcal{K}_1(\tilde u,\omega,\Lambda) = \mathcal{K}_1(v^+,\omega,\Lambda\setminus\partial_\Lambda) + \mathcal{K}_1(\tilde u,\omega,\partial_\Lambda) + \mathcal{W}((v^+,\Lambda\setminus\partial_\Lambda),(\tilde u,\partial_\Lambda)). \tag{4.46}$$

By adding and subtracting $\mathcal{K}_1(v^+,\omega,\partial_\Lambda)$ and the interaction term $\mathcal{W}((v^+,\Lambda\setminus\partial_\Lambda),(v^+,\partial_\Lambda))$ we get

$$\begin{aligned}\mathcal{K}_1(\tilde u,\omega,\Lambda) &= \mathcal{K}_1(v^+,\omega,\Lambda) + \left[\mathcal{K}_1(\tilde u,\omega,\partial_\Lambda) - \mathcal{K}_1(v^+,\omega,\partial_\Lambda)\right]\\ &\quad + \left[\mathcal{W}((v^+,\Lambda\setminus\partial_\Lambda),(\tilde u,\partial_\Lambda)) - \mathcal{W}((v^+,\Lambda\setminus\partial_\Lambda),(v^+,\partial_\Lambda))\right].\end{aligned} \tag{4.47}$$

For the second term of (4.45) we add and subtract $\mathcal{W}((v^+,\Lambda),(v^+,\Lambda^c))$ obtaining

$$\mathcal{W}((\tilde u,\Lambda)(v^-,\Lambda^c)) = \mathcal{W}((v^+,\Lambda),(v^+,\Lambda^c)) + \left[\mathcal{W}((\tilde u,\Lambda),(v^-,\Lambda^c)) - \mathcal{W}((v^+,\Lambda),(v^+,\Lambda^c))\right]. \tag{4.48}$$

Taking into account (4.45), (4.47) (4.48) we get that

$$G_1^{v^-}(\tilde u,\omega,\Lambda) = G_1^{v^+}(v^+,\omega,\Lambda) + \mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3 \tag{4.49}$$

where

$$\begin{aligned}\mathcal{R}_1 &= \left[\mathcal{K}_1(\tilde u,\omega,\partial_\Lambda) - \mathcal{K}_1(v^+,\omega,\partial_\Lambda)\right],\\ \mathcal{R}_2 &= \left[\mathcal{W}((v^+,\Lambda\setminus\partial_\Lambda),(\tilde u,\partial_\Lambda)) - \mathcal{W}((v^+,\Lambda\setminus\partial_\Lambda),(v^+,\partial_\Lambda))\right],\\ \mathcal{R}_3 &= \left[\mathcal{W}((\tilde u,\Lambda),(v^-,\Lambda^c)) - \mathcal{W}((v^+,\Lambda),(v^+,\Lambda^c))\right].\end{aligned} \tag{4.50}$$

Since $\mathcal{R}_2$ and $\mathcal{R}_3$ are difference of positive terms and $\tilde u$, v^- and v^+ are smooth enough we can apply (3.4) of Lemma 3.2 to each single term obtaining

$$|\mathcal{R}_2| \le M(s), \qquad |\mathcal{R}_3| \le M(s).$$

Next we estimate $\mathcal{R}_1$. We have

$$\begin{aligned}|\mathcal{R}_1| &\le \int_{\partial_\Lambda} dx \int_{\partial_\Lambda} dy \frac{\left|(\tilde u(x)-\tilde u(y))^2 - (v^+(x)-v^+(y))^2\right|}{|x-y|^{d+2s}}\\ &+ \int_{\partial_\Lambda} \left|W(\tilde u(x)) - W(v^+(x))\right| \mathrm{dx} + \theta \int_{\partial_\Lambda} \left|\mathrm{g_1(x,\omega)}\left[\mathrm{\tilde u(x) - v^+(x)}\right]\right| \mathrm{dx}\\ &\le \int_{\partial_\Lambda} dx \int_{\partial_\Lambda} dy \frac{\left|(\tilde u(x)-\tilde u(y))^2 - (v^+(x)-v^+(y))^2\right|}{|x-y|^{d+2s}}\\ &+ C(C_0,\theta,\|g\|_\infty)|\Lambda|^{\frac{d-1}{d}}\end{aligned} \tag{4.51}$$

where $C(C_0,\theta,\|g\|_\infty)$ is a constant which depends only on θ, the bound on the random field, see (2.1) and the interaction W. We need some care to estimate the integral term in (4.51) since the integral might be singular. We exploit the regularity of the minimizers. Recall that for $s \in (0,\frac{1}{2}]$, $v^+ \in C^{0,\alpha}_{loc}(\mathbb{R}^d)$

for $\alpha < 2s$ and for $s \in (\frac{1}{2}, 1)$, $v^+ \in C^{1,\alpha}_{loc}(\mathbb{R}^d)$ for $\alpha < 2s-1$. The same regularity holds by construction for $\tilde{u}$. Therefore

$$\int_{\partial_\Lambda}\int_{\partial_\Lambda} \frac{\left[(\tilde{u}(x)-\tilde{u}(y))^2 - (v^+(x)-v^+(y))^2\right]}{|x-y|^{d+2s}} \leq \begin{cases} 2C\int_{\partial_\Lambda}\int_{\partial_\Lambda} \frac{1}{|x-y|^{d+2s-2\alpha}} & s \in (0, \frac{1}{2}] \\ 2C\int_{\partial_\Lambda}\int_{\partial_\Lambda} \frac{1}{|x-y|^{d+2s-2}} & s \in (\frac{1}{2}, 1). \end{cases} \tag{4.52}$$

We have that when $s \in (0, \frac{1}{2}]$, $2s - 2\alpha < 0$ and when $s \in (\frac{1}{2}, 1)$, $2s-2 < 0$. Therefore both terms on the right hand side of (4.52) are integrable and bounded by $C|\Lambda|^{\frac{d-1}{d}}$.

□

The quantity defined next plays a fundamental role.

Definition 4.10.

(1) For a cube $\Lambda \subseteq \mathbb{R}^n$ we define $\mathcal{B}_\Lambda$ as the σ-algebra generated by the random field in Λ.
(2) Let $v^\pm(\omega)$ be the infinite volume states constructed before. We define

$$F_n(\omega) := \mathbb{E}\left[\left\{G_1(v^+(\cdot), \cdot, \Lambda_n) - G_1(v^-(\cdot), \cdot, \Lambda_n)\right\} | \mathcal{B}_{\Lambda_n}\right]. \tag{4.53}$$

Remark 4.11. *By definition $F_n(\cdot)$ is $\mathcal{B}_{\Lambda_n}$ measurable and by the symmetry assumption on the random field $\{g(z,\cdot), z \in \mathbb{Z}^d\}$*

$$\mathbb{E}\left[F_n(\cdot)\right] = 0. \tag{4.54}$$

Namely $v^+(x,\omega) = -v^-(x,-\omega)$ for $x \in \mathbb{R}^d$. This implies that

$$G_1(v^+(\omega), \omega, \Lambda_n) = G_1(v^-(-\omega), -\omega, \Lambda_n) \tag{4.55}$$

and by the symmetry of the random field we get (4.54).

Next we want to quantify how $v^\pm(\omega)$ changes when the random field is modified only in one site, for example at the site i. We introduce the following notation:

$$\omega^{(i)} : \omega^{(i)}(z) = \omega(z) \quad z \neq i, \qquad \omega = (\omega(i), \omega^{(i)}) \quad i, z \in \mathbb{Z}^d.$$

The $v^+(\cdot, (\omega(0), \omega^{(0)}))$ is then the state v^+ when the random field at the origin is $\omega(0)$, and $v^+(\cdot, (\omega(0) - h, \omega^{(0)}))$ the state v^+ when the random field at the origin is $\omega(0) - h$, and the same definition is used for the infinite volume state $v^-(\cdot, (\cdot, \omega^{(0)}))$ and for the finite volume minimizers $v_n^\pm(\cdot, (\cdot, \omega^{(0)}))$.

Now we are able to state the following lemma:

Lemma 4.12. *For $\Lambda \Subset \mathbb{R}^d$, $0 \in \Lambda$, $h > 0$ we have*

$$\theta h \int_{Q(0)} v^+(\omega(0), \omega^{(0)}) \mathrm{d}x \geq G_1(v^+(\omega(0)-h, \omega^{(0)}), (\omega(0)-h, \omega^{(0)}), \Lambda) - G_1(v^+(\omega(0), \omega^{(0)}), (\omega(0), \omega^{(0)}), \Lambda) \geq \theta h \int_{Q(0)} v^+(\omega(0)-h, \omega^{(0)}) \mathrm{d}x \tag{4.56}$$

where $Q(0) = [-1/2, 1/2]^d$. The same inequalities hold for v^-.

Proof. Let Λ_n be a cube centered at the origin so that $\Lambda \subset \Lambda_n$, $K \geq (1 + C_0\theta\|g\|_\infty)$. Let $v_n^+ = v^{+,K}$ be the $K-$ maximal minimizer of G_1 in Λ_n see Definition 3.6. Remark that v_n^+ is measurable with respect to the random field $g(z,\omega)$, $z \in \Lambda_n \cap \mathbb{Z}^d$. We have

$$\begin{aligned}
&G_1(v_n^+(\omega(0),\omega^{(0)}),(\omega(0),\omega^{(0)}),\Lambda) - G_1(v_n^+(\omega(0)-h,\omega^{(0)}),(\omega(0)-h,\omega^{(0)}),\Lambda)\\
&= G_1(v_n^+(\omega(0),\omega^{(0)}),(\omega(0),\omega^{(0)}),\Lambda) - G_1(v_n^+(\omega(0),\omega^{(0)}),(\omega(0)-h,\omega^{(0)}),\Lambda)\\
&+ G_1(v_n^+(\omega(0),\omega^{(0)}),(\omega(0)-h,\omega^{(0)}),\Lambda) - G_1(v_n^+(\omega(0)-h,\omega^{(0)}),(\omega(0)-h,\omega^{(0)}),\Lambda).
\end{aligned} \tag{4.57}$$

By explicit computation, see (2.5), we have that

$$G_1(v_n^+(\omega(0),\omega^{(0)}),(\omega(0),\omega^{(0)}),\Lambda) - G_1(v_n^+(\omega(0),\omega^{(0)}),(\omega(0)-h,\omega^{(0)}),\Lambda) = -h\theta\int_{Q(0)} v_n^+(\omega(0),\omega^{(0)})dx.$$

The last line in (4.57) is nonnegative, because $v_n^+(\omega(0)-h,\omega^{(0)})$ is a minimizer of G_1 in Λ_n when the random field is $(\omega(0)-h,\omega^{(0)})$. Therefore

$$G_1(v_n^+(\omega(0)-h,\omega^{(0)}),(\omega(0)-h,\omega^{(0)}),\Lambda) - G_1(v_n^+(\omega(0),\omega^{(0)}),(\omega(0),\omega^{(0)}),\Lambda) \leq h\theta\int_{Q(0)} v_n^+(\omega(0),\omega^{(0)})dx.$$

By splitting

$$\begin{aligned}
&G_1(v_n^+(\omega(0),\omega^{(0)}),(\omega(0),\omega^{(0)}),\Lambda) - G_1(v_n^+(\omega(0)-h,\omega^{(0)}),(\omega(0)-h,\omega^{(0)}),\Lambda)\\
&= G_1(v_n^+(\omega(0),\omega^{(0)}),(\omega(0),\omega^{(0)}),\Lambda) - G_1(v_n^+(\omega(0)-h,\omega^{(0)}),(\omega(0),\omega^{(0)}),\Lambda)\\
&+ G_1(v_n^+(\omega(0)-h,\omega^{(0)}),(\omega(0),\omega^{(0)}),\Lambda) - G_1(v_n^+(\omega(0)-h,\omega^{(0)}),(\omega(0)-h,\omega^{(0)}),\Lambda)
\end{aligned}$$

we obtain in a similar way

$$G_1(v_n^+(\omega(0)-h,\omega^{(0)}),(\omega(0)-h,\omega^{(0)}),\Lambda) - G_1(v_n^+(\omega(0),\omega^{(0)}),(\omega(0),\omega^{(0)}),\Lambda) \geq h\theta\int_{Q(0)} v_n^+(\omega(0)-h,\omega^{(0)})dx.$$

To pass to the limit note that the cube $Q(0)$ remains fixed. Denote by M the smallest integer such that $\Lambda \subseteq B_M(0)$, where $B_M(0)$ is a ball centered at the origin of radius M.

By the smoothness of the minimizers, see Proposition 6.3 $v_n^+ \in C^{0,\alpha}(B_M(0))$ with $\alpha < 2s$ when $2s < 1$ and in $C^{1,\alpha}(B_M(0))$, $\alpha < 1-2s$ when $s \in [\frac{1}{2},1)$. Further the sequence $\{v_n^+\}_n$ uniformly converges to $v^{+,K}$ in $B_M(0)$ and $|v^{+,K}| \leq 1 + C_0\theta\|g\|_\infty$ uniformly in n and K. By Lebesgue's Theorem on dominated convergence, we may pass to the limit under the integral as $n \to \infty$. By Definition 4.6 $\{v^{+,K}\}_K$ point-wise converges to v^+ when $K \to \infty$ then applying again the Lebesgue's Theorem on dominated convergence we pass to the limit as $K \to \infty$ and the claim is shown. The corresponding statement for v^- is proved in the same way. ☐

Remark 4.13. *From Lemma 4.12 we have that*

$$\omega(0) \mapsto \int_{Q(0)} v^+(\omega(0),\omega^{(0)})\mathrm{d}x$$

is nondecreasing.

Corollary 4.14. *Let $\omega(i)$ be the random field in the site i which has probability distribution absolutely continuous w.r.t the Lebesgue measure. We have that $G_1(v^+(\omega),\omega,\Lambda)$ is $\mathbb{P}$-a.e. differentiable w.r.t to $\omega(i)$ and*

$$\frac{\partial G_1(v^\pm(\omega),\omega,\Lambda)}{\partial\omega(i)} = -\theta\int_{Q(i)} v^\pm(x,\omega)\mathrm{d}x.$$

Proof. It is sufficient to consider the case $i = 0$. By applying Lemma 4.12 for $\omega(0)$ and $\tilde{\omega}(0) = \omega(0) + h$ we see that left and right derivatives exist and are equal if $s \mapsto \int_{Q(0)} v^+(s, \omega^{(0)}) \mathrm{d}x$ is continuous at $s = \omega(0)$. By Remark 4.13 this happens for Lebesgue almost all s, hence by the assumptions on the random field $\mathbb{P}$-a.e. □

Remark 4.15. *When the distribution of g is not absolutely continuous with respect to Lebesgue measure Corollary 4.14 does not hold. We still can show Lemma 4.12 but we can only estimate from above and below the difference in the energy which appears when the random field is modified in one site.*

Theorem 4.16. *Let $F_n(\cdot)$ be defined in (4.53), we have that* [3]

$$\lim_{n\to\infty} \frac{1}{\sqrt{|\Lambda_n|}} [F_n(\cdot)] \stackrel{D}{=} Z, \tag{4.58}$$

where Z stands for a Gaussian random variable with mean 0 and variance b^2, defined in (4.66) with

$$4\theta^2(1 + C_0\theta\|g\|_\infty)^2 \geq b^2 \geq D^2 \tag{4.59}$$

where

$$D^2 = \mathbb{E}\left[\left(\mathbb{E}\left[F_n|\mathcal{B}(0)\right]\right)^2 \right], \tag{4.60}$$

$\mathcal{B}(0)$ is the sigma -algebra generated by $g(0, \omega)$ and C_0 is given in (2.4).

The proof of this theorem is done invoking the general result presented in the appendix and proceeding in the same way as in [7]. To facilitate the reader we recall below the main steps of the proof.

Proof. We decompose F_n as a martingale difference sequence. We order the points in $\Lambda_n \cap \mathbb{Z}^d$ according to the lexicographic ordering. In the following $i \leq j$ refers to the lexicographic ordering. Any other ordering will be fine but it is convenient to fix one. We introduce the family of increasing $\sigma-$ algebra $\mathcal{B}_{n,i}$, $i \in \Lambda_n \cap \mathbb{Z}^d$ where $\mathcal{B}_{n,i}$ is the $\sigma-$ algebra generated by the random variables $\{g(z), z \in \Lambda_n \cap \mathbb{Z}^d, z \leq i\}$. We denote by

$$\mathcal{B}_{n,0} = (\emptyset, \Omega), \quad \mathcal{B}_{n,i} \subset \mathcal{B}_{n,j} \qquad i \leq j, \quad i \in \Lambda_n \cap \mathbb{Z}^d, \quad j \in \Lambda_n \cap \mathbb{Z}^d.$$

We split

$$F_n = \sum_{i\in\mathbb{Z}^d\cap\Lambda_n} (\mathbb{E}[F_n|\mathcal{B}_{n,i}] - \mathbb{E}[F_n|\mathcal{B}_{n,i-1}]) := \sum_{i\in\mathbb{Z}^d\cap\Lambda_n} Y_{n,i}. \tag{4.61}$$

By construction $\mathbb{E}\left[Y_{n,i}\right] = 0$ for $i \in \mathbb{Z}^d \cap \Lambda_n$, $\mathbb{E}\left[Y_{n,i}|\mathcal{B}_{n,k}\right] = 0$, for all $0 \leq k \leq i - 1$. Denote

$$V_n := \frac{1}{|\Lambda_n \cap \mathbb{Z}^d|} \sum_{i\in\Lambda_n\cap\mathbb{Z}^d} \mathbb{E}\left[Y_{n,i}^2|\mathcal{B}_{n,i-1}\right]. \tag{4.62}$$

By Lemma 4.17 stated below we have that $V_n \to b^2$ in probability and b^2 satisfies (4.59). By Lemma 4.18 stated below we have that for any $a > 0$

$$U_n(a) := \frac{1}{|\Lambda_n \cap \mathbb{Z}^d|} \sum_{i\in\Lambda_n\cap\mathbb{Z}^d} \mathbb{E}[Y_{n,i}^2 1_{\{|Y_{n,i}|\geq a\sqrt{|\Lambda_n\cap\mathbb{Z}^d|}\}}|\mathcal{B}_{n,i-1}] \tag{4.63}$$

converges to 0 in probability. We can then invoke Theorem 5.1, stated in the appendix. The correspondence to the notation used in the appendix is the following. Identify $|\Lambda_n \cap \mathbb{Z}^d|$ with n, $\frac{F_n}{\sqrt{|\Lambda_n\cap\mathbb{Z}^d|}} \leftrightarrow S_n$, $\frac{Y_{n,i}}{\sqrt{|\Lambda_n\cap\mathbb{Z}^d|}} \leftrightarrow X_{n,i}$ and $\mathcal{B}_{n,i} \leftrightarrow \mathcal{F}_{n,i}$. Then (4.58) is obtained. □

[3] $\lim_{n\to\infty} X_n \stackrel{D}{=} Z$ denotes convergence in distribution of the random variables X_n to a random variable Z.

Before stating Lemma 4.17 it is convenient to introduce a new sigma-algebra $\mathcal{B}_i^{\leq}$ generated by the random fields $\{g(z,\omega), z \in \mathbb{Z}^d, z \leq i\}$ where $\leq$ refers to the lexicographic ordering. Define for $i \in \Lambda_n$

$$W_i[\omega] = \mathbb{E}\left[G_1(v^+(\omega),\omega,\Lambda_n) - G_1(v^-(\omega),\omega,\Lambda_n)|\mathcal{B}_i^{\leq}\right] - \mathbb{E}\left[G_1(v^+(\omega),\omega,\Lambda_n) - G_1(v^-(\omega),\omega,\Lambda_n)|\mathcal{B}_{i-1}^{\leq}\right]. \tag{4.64}$$

Note that W_i is a random variable depending on random fields on sites smaller or equal than i under the lexicographic order. In particular it does not depend on the choice of the cube Λ_n provided $i \in \Lambda_n$. The proof of this last statement uses that the random field has a distribution continuous with respect to Lebesgue measure. In particular the proof relies on Corollary 4.14 and it is done in [7, Lemma 4.9].

Lemma 4.17. *Let V_n be the quantity defined in* (4.62)*. For all $\delta > 0$*

$$\lim_{n\to\infty} \mathbb{P}\left[|V_n - b^2| \geq \delta\right] = 0, \tag{4.65}$$

where W_0 is defined in (4.64)

$$b^2 = \mathbb{E}\left[W_0^2\right]. \tag{4.66}$$

Further

$$4\theta^2(1 + C_0\theta\|g\|_\infty)^2 \geq b^2 \geq \mathbb{E}\left[\left(\mathbb{E}\left[F_n|\mathcal{B}(0)\right]\right)^2\right], \tag{4.67}$$

where C_0 is given in (2.4)*.*

Lemma 4.18. *Let $U_n(a)$ defined in* (4.63)*. For any $a > 0$ for any $\delta > 0$*

$$\lim_{n\to\infty} \mathbb{P}\left[U_n(a) \geq \delta\right] = 0.$$

For the proof of Lemma 4.17 and Lemma 4.18 see [7].

Lemma 4.19. *For $\Lambda \subset \mathbb{R}^d$, $0 \in \Lambda$, we have*

$$\frac{\partial}{\partial\omega(0)}\mathbb{E}\left[F_n|\mathcal{B}(0)\right] = -\theta\mathbb{E}\left[\int_{Q(0)} v^+(x,\omega)dx|\mathcal{B}(0)\right] + \theta\mathbb{E}\left[\int_{Q(0)} v^-(x,\omega)dx|\mathcal{B}(0)\right]$$

where $Q(0) := [-1/2, 1/2]^d$. Further

$$\mathbb{E}\left[\frac{\partial}{\partial\omega(0)}\mathbb{E}\left[F_n|\mathcal{B}(0)\right]\right] = -2\theta m^+,$$

where m^+ is defined in (4.3)*.*

Proof. The proof follows from Corollary 4.14 after taking conditional expectations. Further, by Theorem 4.1, we have

$$\begin{aligned}\mathbb{E}\left[\frac{\partial}{\partial\omega(0)}\mathbb{E}\left[F_n|\mathcal{B}(0)\right]\right] &= -\theta\mathbb{E}\left[\mathbb{E}\left[\int_{Q(0)} v^+(x,\omega)dx|\mathcal{B}(0)\right]\right]\\ &+ \theta\mathbb{E}\left[\mathbb{E}\left[\int_{Q(0)} v^-(x,\omega)dx|\mathcal{B}(0)\right]\right] = \theta[-m^+ + m^-] = -2\theta m^+.\end{aligned} \tag{4.68}$$

□

Lemma 4.20. *If*

$$\mathbb{E}\left[\left(\mathbb{E}\left[F_n|\mathcal{B}(0)\right]\right)^2\right] = 0 \tag{4.69}$$

then $m^+ = m^- = 0$, see for the definition (4.3)*.*

Proof. Denote $f(\omega(0)) := \mathbb{E}\left[-F_n|\mathcal{B}(0)\right]$. Set $s = \omega(0)$, (4.69) can be written as $\int f^2(s)\mathbb{P}(ds) = 0$. This implies that $f(s) = 0$ for $\mathbb{P}$ almost all point of continuity of the distribution $g(0)$. By Lemma 4.19 and by bound (4.2) in Theorem 4.1 we have that $(1 + C_0\|g\|_\infty\theta)\theta \geq f'(s) \geq 0$ almost everywhere. If $f(s) = 0$ for $\mathbb{P}$ almost all point of continuity of the distribution g, then $f'(s) = 0$ for $\mathbb{P}$ almost all point of continuity of the distribution of $\omega(0)$. But if $f'(s) = 0$ then from Lemma 4.19 we get $m^+ = m^- = 0$. □

Proof of Theorem 2.7 Applying Theorem 4.16 we get the following lower bound on the Laplace transform of $F_n(\omega)$ defined in Definition 4.10:

$$\liminf_{n\to\infty} \mathbb{E}\left[e^{t\frac{F_n}{\sqrt{\Lambda_n}}}\right] \geq e^{\frac{t^2 D^2}{2}} \tag{4.70}$$

where D^2 is defined in (4.60). It is immediate to realize that (4.70) and the results stated in Lemma 4.9 contradict each other in $d = 2$ for all $s \in (\frac{1}{2}, 1)$ and in $d = 1$ for $s \in [\frac{1}{4}, 1)$ unless $D^2 = 0$. On the other hand when $D^2 = 0$, Lemma 4.20 implies

$$m^+ = -m^- = \mathbb{E}\left[\int_{[-\frac{1}{2},\frac{1}{2}]^d} v^{\pm}(x,\cdot)\mathrm{d}x\right] = 0. \tag{4.71}$$

Now (4.4) implies that $\mathbb{P}$-a.s. $v^+(x,\omega) \geq v^-(x,\omega)$ for all $x \in R^d$. This and (4.71) imply that $v^+(x,\omega) = v^-(x,\omega)$ a.s. By (4.4) $\mathbb{P}-$ a.s. and uniformly for any compact of $\mathbb{R}^d$ containing x we have that

$$v^-(x,\omega) \leq \liminf_{n\to\infty} u_n^*(x,\omega) \leq \limsup_{n\to\infty} u_n^*(x,\omega) \leq v^+(x,\omega).$$

Since $v^- = v^+$, $\mathbb{P}-$ a.s, we obtain that

$$\liminf_{n\to\infty} u_n^*(x,\omega) = \limsup_{n\to\infty} u_n^*(x,\omega) = u^*(x,\omega) = v^{\pm}(x,\omega)$$

uniformly on compact of x. The properties of the minimizer stated in Theorem 2.7 therefore follow from the corresponding properties of $v^{\pm}$, see Theorem 4.1. Further we have

$$\mathbb{E}[v^+(x,\cdot)] =_{\text{symm}} -\mathbb{E}[v^-(x,\cdot)] =_{\text{unique}} -\mathbb{E}[v^+(x,\cdot)], \qquad x \in \mathbb{R}^d.$$

This implies for any $x \in \mathbb{R}^d$, $\mathbb{E}[v^{\pm}(x,\cdot)] = \mathbb{E}[u^*(x,\cdot)] = 0$.

□

5. Technical Lemmas

In this section we collect some lemmas we need to prove the main results.

Proposition 5.1. *For any $\epsilon > 0$, for all $v \in H^\alpha_{loc} \cap L^\infty(\mathbb{R}^d)$, $s < \alpha$, for all cube Δ large enough*

$$\mathcal{W}(v,\Delta) \leq \epsilon|\Delta|. \tag{5.1}$$

Proof. Let L be the edge of Δ. We have for any $R > 0$, $R \leq \frac{1}{4}diam(\Delta)$

$$\begin{aligned}&\int_\Delta \mathrm{dx} \int_{\Delta^c} \mathrm{dy} \frac{|\mathrm{v(x)} - \mathrm{v(y)}|^2}{|\mathrm{x} - \mathrm{y}|^{\mathrm{d+2s}}} \\ &= \int_{\{x\in\Delta: d_{\partial\Delta}(x)\leq R\}} \mathrm{dx} \int_{\Delta^c} \mathrm{dy} \frac{|\mathrm{v(x)} - \mathrm{v(y)}|^2}{|\mathrm{x} - \mathrm{y}|^{\mathrm{d+2s}}} + \int_{\{\mathrm{x}\in\Delta: \mathrm{d}_{\partial\Delta}(\mathrm{x})>\mathrm{R}\}} \mathrm{dx} \int_{\Delta^c} \mathrm{dy} \frac{|\mathrm{v(x)} - \mathrm{v(y)}|^2}{|\mathrm{x} - \mathrm{y}|^{\mathrm{d+2s}}}.\end{aligned} \tag{5.2}$$

Assume $s < \frac{1}{2}$, then from (5.2) and boundedness of v

$$\begin{aligned}
&\int_\Delta \mathrm{dx} \int_{\Delta^c} \mathrm{dy} \frac{|\mathrm{v(x)-v(y)}|^2}{|\mathrm{x-y}|^{\mathrm{d+2s}}} \\
&\le \int_{\{x\in\Delta: d_{\partial\Delta}(x)\le R\}} \int_{\Delta^c} \mathrm{dy} \frac{\mathrm{C}}{|\mathrm{x-y}|^{\mathrm{d+2s}}} \mathrm{dx} + \int_{\{\mathrm{x}\in\Delta:\mathrm{d}_{\partial\Delta}(\mathrm{x})>\mathrm{R}\}} \mathrm{dx} \int_{\Delta^c} \mathrm{dy} \frac{\mathrm{C}}{|\mathrm{x-y}|^{\mathrm{d+2s}}} \\
&\le \int_{\{x\in\Delta: d_{\partial\Delta}(x)\le R\}} \mathrm{dx}\mathrm{d}_{\partial\Lambda}(\mathrm{x})^{-2s} + |\Delta| \int_{|\mathrm{y}|\ge\mathrm{R}} \mathrm{dy} \frac{\mathrm{C}}{|\mathrm{y}|^{\mathrm{d+2s}}} \\
&\le CR^{1-2s}L^{d-1} + \frac{\epsilon}{2}|\Delta|
\end{aligned} \tag{5.3}$$

if R large enough, since $\int_{|y|\ge1} \mathrm{dy}\frac{\mathrm{C}}{|\mathrm{y}|^{\mathrm{d+2s}}} < \mathrm{C}$. Given such R we then choose L so large that $CL^{d-1}R^{1-2s} \le \frac{\epsilon}{2}L^d$. Hence (5.1) when $s < \frac{1}{2}$. When $s \in [\frac{1}{2}, 1)$ we can still split as in (5.2) and estimate the second integral of (5.2) as done in (5.3). Care needs to be taken to estimate the first integral of (5.2). In this case fix $\rho > 0$

$$\begin{aligned}
&\int_{\{x\in\Delta: d_{\partial\Delta}(x)\le R\}} \mathrm{dx} \int_{\Delta^c} \mathrm{dy} \frac{|\mathrm{v(x)-v(y)}|^2}{|\mathrm{x-y}|^{\mathrm{d+2s}}} \\
&= \int_{\{x\in\Delta: d_{\partial\Delta}(x)\le R\}} \mathrm{dx} \left[\int_{\{\mathrm{y}\in\Delta^c,\mathrm{d}_{\partial\Delta}(\mathrm{y})\le\rho\}} \mathrm{dy} \frac{|\mathrm{v(x)-v(y)}|^2}{|\mathrm{x-y}|^{\mathrm{d+2s}}} + \int_{\{\mathrm{y}\in\Delta^c,\mathrm{d}_{\partial\Delta}(\mathrm{y})\ge\rho\}} \mathrm{dy} \frac{|\mathrm{v(x)-v(y)}|^2}{|\mathrm{x-y}|^{\mathrm{d+2s}}} \right]
\end{aligned} \tag{5.4}$$

The first term in (5.4) is estimated as following. Since $v \in H^s$, $\alpha > s$, set $\alpha = s + \gamma$, $\gamma > 0$, $|v(x) - v(y)| \le |x - y|^{s+\gamma}$

$$\begin{aligned}
&\int_{\{x\in\Delta: d_{\partial\Delta}(x)\le R\}} \mathrm{dx} \int_{\{\mathrm{y}\in\Delta^c,\mathrm{d}_{\partial\Delta}(\mathrm{y})\le\rho\}} \mathrm{dy} \frac{|\mathrm{v(x)-v(y)}|^2}{|\mathrm{x-y}|^{\mathrm{d+2s}}} \\
&\le \int_{\{x\in\Delta: \rho\le d_{\partial\Delta}(x)\le R\}} \mathrm{dx} \int_{\{\mathrm{y}\in\Delta^c,\mathrm{d}_{\partial\Delta}(\mathrm{y})\le\rho\}} \mathrm{dy} \frac{|\mathrm{v(x)-v(y)}|^2}{|\mathrm{x-y}|^{\mathrm{d+2s}}} + \int_{\{\mathrm{x}\in\Delta:\mathrm{d}_{\partial\Delta}(\mathrm{x})\le\rho\}} \mathrm{dx} \int_{\{\mathrm{y}\in\Delta^c,\mathrm{d}_{\partial\Delta}(\mathrm{y})\le\rho\}} \mathrm{dy} \frac{|\mathrm{v(x)-v(y)}|^2}{|\mathrm{x-y}|^{\mathrm{d+2s}}} \\
&\le C(\rho)L^{d-1}R + \int_{\{x\in\Delta: d_{\partial\Delta}(x)\le\rho\}} \mathrm{dx} \int_{\{\mathrm{y}\in\Delta^c,\mathrm{d}_{\partial\Delta}(\mathrm{y})\le\rho\}} \mathrm{dy} \frac{\mathrm{C}}{|\mathrm{x-y}|^{\mathrm{d}-\gamma}} \\
&\le C(\rho)L^{d-1}R + \rho^{1+\gamma}L^{d-1}C \simeq \epsilon|\Lambda|
\end{aligned} \tag{5.5}$$

The second term in (5.4) is bounded as following

$$\begin{aligned}
&\int_{\{x\in\Delta: d_{\partial\Delta}(x)\le R\}} \mathrm{dx} \int_{\{\mathrm{y}\in\Delta^c,\mathrm{d}_{\partial\Delta}(\mathrm{y})\ge\rho\}} \mathrm{dy} \frac{\mathrm{C}}{|\mathrm{x-y}|^{\mathrm{d+2s}}} \\
&\le RL^{d-1} \int_{\{|y|\ge\rho\}} \mathrm{dy} \frac{\mathrm{C}}{|\mathrm{y}|^{\mathrm{d+2s}}} \le \mathrm{RL^{d-1}C}(\rho) \le \frac{\epsilon}{2}|\Delta|
\end{aligned} \tag{5.6}$$

provided L is suitable chosen. □

Proposition 5.2. *Take $D \Subset \mathbb{R}^d$ and assume that $u_n \to u$ in $C^{0,\alpha}(D)$ for $s < \alpha < 2s$, then*

$$I_n := \left| \int_{D\times D} \frac{|u(z) - u(z')|^2 - |u_n(z) - u_n(z')|^2}{|z - z'|^{d+2s}} dzdz' \right| \le C'(d)K|D|(\mathrm{diam}(D))^d \|u - u_n\|_{C^{0,\alpha}} \to 0.$$

Proof. Note that

$$\begin{aligned}
&\left||u(z) - u(z')|^2 - |u_n(z) - u_n(z')|^2\right| = |[u(z) - u(z') + u_n(z) - u_n(z')][u(z) - u(z') - (u_n(z) - u_n(z'))]| \\
\le\ & (|u(z) - u(z')| + |u_n(z) - u_n(z')|)\,(|(u(z) - u_n(z)) - (u(z') - u_n(z'))|) \\
\le\ & (\|u\|_{C^{0,\alpha}} + \|u_n\|_{C^{0,\alpha}})|z - z'|^\alpha \cdot \|u - u_n\|_{C^{0,\alpha}}|z - z'|^\alpha.
\end{aligned}$$

As a convergent sequence is bounded, there is a $K>0$ such that $(\|u\|_{C^{0,\alpha}}+\|u_n\|_{C^{0,\alpha}})<K$. So

$$\begin{aligned} I_n &\le \int_{D\times D}\frac{\big||u(z)-u(z')|^2-|u_n(z)-u_n(z')|^2\big|}{|z-z'|^{d+2s}}dzdz' \le K\|u-u_n\|_{C^{0,\alpha}}\int_{D\times D}|z-z'|^{2(\alpha-s)-d} \\ &\le C(d)K|D|\|u-u_n\|_{C^{0,\alpha}}\int_0^2 r^{\delta-1}dr \le C'(d)K|D|(\mathrm{diam}(D))^d\|u-u_n\|_{C^{0,\alpha}}\to 0 \end{aligned}$$

where $\delta=2(\alpha-s)>0$. Note that we need only $\alpha>s$.

□

6. Appendix

We collect in this section general results about fractional laplacian scattered in the literature and recall the main probabilistic result used to prove Theorem 4.16.

6.1. **Minimizers of the functional** (2.9) **on open bounded Lipschitz sets.** We recall here some basic results assuring that the minimization of the functional (2.9) in an open, bounded Lipschitz set has solution. In the following ω plays the role of a parameter. It is kept fixed and the results hold for all $\omega\in\Omega$.

Proposition 6.1. *Let $\Lambda\Subset\mathbb{R}^d$ be a Lipschitz bounded open set and $u_0:\mathbb{R}^d\to\mathbb{R}$ be a measurable function. Suppose that there exists a measurable function $\tilde u$ which coincides with u_0 in Λ^c and such that $G_1(\tilde u,\omega,\Lambda)<\infty$. Then there exists a measurable function u^* such that*

$$G_1^{u_0}(u^*,\omega,\Lambda)\le G_1^{u_0}(v,\omega,\Lambda)$$

for any measurable function v which coincides with u_0 in Λ^c.

Proof. Take a minimizing sequence, that is, let $u_k=u_0$ in Λ^c so that $G_1(u_k,\omega,\Lambda)\le G_1(\tilde u,\omega,\Lambda)$ and

$$\lim_{k\to\infty}G_1(u_k,\omega,\Lambda)=\inf_v G_1(v,\omega,\Lambda)$$

for any v which coincides with u_0 in in Λ^c. Then by the following compactness result, see Proposition 6.2, up to subsequence, u_k converges almost everywhere to some u^*. By Fatou's Lemma we conclude. □

Proposition 6.2. *Let $\Lambda\Subset\mathbb{R}^d$ be a Lipschitz open set and $\mathcal{F}$ be a bounded subset of $L^2(\Lambda)$. Suppose that*

$$\sup_{f\in\mathcal{F}}\int_\Lambda \mathrm{dx}\int_\Lambda \mathrm{dy}\frac{|\mathrm{f(x)-f(y)}|^2}{|\mathrm{x-y}|^{\mathrm{d+2s}}}<\infty.$$

Then $\mathcal{F}$ is precompact in $L^2(\Lambda)$.

For the proof of Proposition 6.2 see [14, Lemma 6.11]. The proof is based on the classical Riesz-Frechet-Kolmogorov Theorem. Some modifications are needed due to the non-locality of the fractional norm. If Λ is not Lipschitz then Proposition 6.2 does not hold. One can find counterexample, see for example [4, Example 9.2].

Next we show that minimizers of the functional (2.9) solve the Euler -Lagrange equation (6.6) and prove some regularity results.

In the following, Λ, v_0 and ω are kept fixed, therefore we write $G_1^{v_0}(v,\omega,\Lambda)=G_1(v)$. To derive the Euler Lagrange equation for the minimizers of $G_1(v)$ we compute the Frechet derivative of $G_1(v)$. For $w\in C_0^\infty(\Lambda)$ we have that

$$\begin{aligned} G_1(v+tw)&=G_1(v)+2t\int_\Lambda \mathrm{dx}\int_\Lambda \mathrm{dy}\frac{[\mathrm{v(x)-v(y)}]\cdot[\mathrm{w(x)-w(y)}]}{|\mathrm{x-y}|^{\mathrm{d+2s}}} \\ &+t\int_\Lambda \mathrm{dx}[\mathrm{W'(v)}+\theta \mathrm{g_1}]\mathrm{w}+4\mathrm{t}\int_\Lambda \mathrm{dx}\int_{\Lambda^c}\mathrm{dy}\frac{[\mathrm{v(x)-v_0(y)}]\cdot \mathrm{w(x)}}{|\mathrm{x-y}|^{\mathrm{d+2s}}}+\mathrm{O(t^2)}, \end{aligned} \tag{6.1}$$

where $W'(\cdot)$ is the derivative of $W(\cdot)$ with respect to its argument. Then the Frechet derivative computed in v is the following linear operator defined for $w \in C_0^\infty(\Lambda)$ as the following

$$\begin{aligned} D_v G_1(w) &= 2\int_\Lambda \mathrm{dx} \int_\Lambda \mathrm{dy} \frac{[\mathrm{v(x)} - \mathrm{v(y)}]\cdot[\mathrm{w(x)} - \mathrm{w(y)}]}{|\mathrm{x}-\mathrm{y}|^{\mathrm{d+2s}}} \\ &+ \int_\Lambda \mathrm{dx}[\mathrm{W'(v)} + \theta \mathrm{g_1}]\mathrm{w} + 4\int_\Lambda \mathrm{dx}\int_{\Lambda^c} \mathrm{dy} \frac{[\mathrm{v(x)} - \mathrm{v_0(y)}]\mathrm{w(x)}}{|\mathrm{x}-\mathrm{y}|^{\mathrm{d+2s}}}. \end{aligned} \tag{6.2}$$

At this point one is tempted to split the first integral in (6.2) in two terms and exchange x with y in one of the terms to obtain

$$2\int_\Lambda \mathrm{dxw(x)} \int_\Lambda \mathrm{dy} \frac{[\mathrm{v(x)} - \mathrm{v(y)}]}{|\mathrm{x}-\mathrm{y}|^{\mathrm{d+2s}}}. \tag{6.3}$$

However we cannot always do that. The inner integral in the first integral in (6.2) might not be absolutely convergent. So in general it can be defined only as a principal value. In such a case

$$\int_\Lambda \mathrm{dx} \int_\Lambda \mathrm{dy} \frac{[\mathrm{v(x)} - \mathrm{v(y)}]\cdot[\mathrm{w(x)} - \mathrm{w(y)}]}{|\mathrm{x}-\mathrm{y}|^{\mathrm{d+2s}}} = \int_\Lambda \mathrm{dxw(x)} \lim_{\mathrm{r}\to 0} \int_{\Lambda\setminus \mathrm{B_r(x)}} \mathrm{dy} \frac{[\mathrm{v(x)} - \mathrm{v(y)}]}{|\mathrm{x}-\mathrm{y}|^{\mathrm{d+2s}}}, \tag{6.4}$$

where $B_r(x)$ is a ball of radius $r > 0$ centered in x.

From (6.2) and (6.4) we deduce that a minimizer of $G_1^{v_0}(v, \omega, \Lambda)$ is a function $v \in H^s_{loc} \cap L^\infty$ which solves

$$\begin{aligned} &2\int_\Lambda \mathrm{dxw(x)}((-\Delta)^{\mathrm{s}}\mathrm{v})(\mathrm{x}) \\ &+ \int_\Lambda \mathrm{dx}[\mathrm{W'(v)} + \theta \mathrm{g_1}]\mathrm{w} + 4\int_\Lambda \mathrm{dx}\int_{\Lambda^c} \mathrm{dy} \frac{[\mathrm{v(x)} - \mathrm{v_0(y)}]\mathrm{w(x)}}{|\mathrm{x}-\mathrm{y}|^{\mathrm{d+2s}}} = 0. \end{aligned} \tag{6.5}$$

We identify the problem stated in (6.5) to the following Dirichlet boundary value problem for the corresponding Euler-Lagrange equation:

$$\begin{aligned} &(-\Delta)^s v = -\frac{1}{2}[W'(v) + \theta g_1] \quad \text{in } \Lambda, \qquad \omega \in \Omega \\ &v = v_0 \quad \text{in } \Lambda^c. \end{aligned} \tag{6.6}$$

We recall the following regularity result proven in [17, Proposition 2.9].

Proposition 6.3. *Let $w = (-\Delta)^s u$ in $\mathbb{R}^d$ so that $\|u\|_\infty$ and $\|w\|_\infty$ are finite. If $2s \le 1$ then $u \in C^{0,\alpha}$ for any $\alpha < 2s$, and*

$$\|u\|_{C^{0,\alpha}} \le C[\|u\|_\infty + \|w\|_\infty]$$

for a constant $C = C(d, s, \alpha)$.

If $2s > 1$ then $u \in C^{1,\alpha}$ for any $\alpha < 2s - 1$, and

$$\|u\|_{C^{1,\alpha}} \le C[\|u\|_\infty + \|w\|_\infty],$$

for a constant $C = C(d, s, \alpha)$.

We remark that the above results are valid for solution of (6.6) in bounded domains, leading to a local regularity theory.

The main tool to prove Theorem 4.16 is the following general result which we reported from [11], see [11, Theorem 3.2 and Corollary 3.1].

Theorem 6.4. *Let $S_{n,i}$, $i = 1, \dots k_n$ be a double array of zero mean martingales with respect to the filtration $\mathcal{F}_{n,i}$, $\mathcal{F}_{n,i} \subset \mathcal{F}_{n+1,i}$ $i = 1, \dots k_n$ with $S_{n,k_n} = S_n$, so that $S_{n,i} = \mathbb{E}[S_n|\mathcal{F}_{n,i}]$. We assume that $k_n \uparrow \infty$ as $n \uparrow \infty$. Denote*

$$X_{n,i} := S_{n,i} - S_{n,i-1},$$

$$V_n = \sum_{i=1}^{k_n} \mathbb{E}[X_{n,i}^2 | \mathcal{F}_{n,i-1}],$$

$$U_{n,a} = \sum_{i=1}^{k_n} \mathbb{E}[X_{n,i}^2 \mathbb{1}_{\{|[X_{n,i}^2|>a\}} | \mathcal{F}_{n,i-1}].$$

Suppose that

- *for some constant* b^2 *and for all* $\delta > 0$, $\lim_{n\to\infty} \mathbb{P}[|V_n - b^2| \geq \delta] = 0$,
- *for any* $a > 0$ *and for any* $\delta > 0$

$$\lim_{n\to\infty} \mathbb{P}\left[U_n(a) \geq \delta\right] = 0, \qquad \textit{(Lindeberg condition)}$$

then in distribution

$$\lim_{n\to\infty} S_n \stackrel{D}{=} Z,$$

where Z *is a Gaussian random variable with mean equal to zero and variance equal to* b^2.

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