One-dimensional random field Kac's model:

localization of the phases *

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Abstract We study the typical profiles of a one dimensional random field Kac model, for values of the temperature and magnitude of the field in the region of two absolute minima for the free energy of the corresponding random field Curie Weiss model. We show that, for a set of realizations of the random field of overwhelming probability, the localization of the two phases corresponding to the previous minima is completely determined. Namely, we are able to construct random intervals tagged with a sign, where typically, with respect to the infinite volume Gibbs measure, the profile is rigid and takes, according to the sign, one of the two values corresponding to the previous minima. Moreover, we characterize the transition from one phase to the other. The analysis extends the one done by Cassandro, Orlandi and Picco in [13].

Key Words and Phrases: phase transition, random walk, random environment, Kac potential.
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1 Introduction

We consider a one-dimensional spin system interacting via a ferromagnetic two-body Kac potential and external random field given by independent Bernoulli variables. Problems where a stochastic contribution is added to the energy of the system arise naturally in condensed matter physics where the presence of the impurities causes the microscopic structure to vary from point to point. Some of the vast literature on these topics may be seen consulting [1-6], [10], [18-21], [23], [32].

Kac's potentials is a short way to denote two-body ferromagnetic interactions with range $\frac{1}{\gamma}$, where γ is a dimensionless parameter such that when $\gamma \to 0$, i.e. very long range, the strength of the interaction becomes very weak, but in such a way that the total interaction between one spin and all the others is finite. They were introduced in [22], and then generalized in [24], to provide a rigorous proof of the validity of the van der Waals theory of a liquid–vapor phase transition. Performing first the thermodynamic limit of the spin system interacting via Kac's potential, and then the limit of infinite range, $\gamma \to 0$, they rigorously derived the Maxwell rule. This implies that the free energy of the system is the convex envelope of the corresponding free energy for the Curie-Weiss model. This leads to two spatially homogeneous phases, corresponding to the two points of minima of the free energy of the Curie-Weiss model. Often we will call + phase the one associated to the positive minimizer, and − phase the one associated to the negative minimizer. For γ fixed and different from zero, there are several papers trying to understand qualitatively and quantitatively the features of systems with long, but finite range interaction. (See for instance [16], [25], [9], [19].) In the one dimensional case, the analysis [15] for Ising spin and [7] for more general spin, gives a satisfactory description of the typical profiles.

Similar type of analysis holds for Ising spin systems interacting via a Kac potential and external random field. The Gibbs measure of this system can be written in terms of a functional over the magnetization profiles obtained through a block spin transformation that maps the microscopic system into a system on $L^\infty(I\!R) \times L^\infty(I\!R)$, for which the length of interaction becomes of order one (the macroscopic system). This functional is a sum of two terms of which one is deterministic and has two minimizers (the above mentioned homogeneous + and − phases). The other term is related to partial sums of the external random magnetic field.

If we consider a finite volume I, on the macroscopic scale, the variance of the stochastic part of the functional is of the order $\gamma|I|$, so that for a volume I with $|I| \approx (\gamma \log\log(1/\gamma))^{-1}$ one can expect to get almost sure fluctuations of order 1 as in the Law of the Iterated Logarithm. These fluctuations of order 1 will compensate the cost for the deterministic part to make a transition from one minimizer to the other ones. In fact in [13] it has been proven that if the system is considered on an interval of length $\frac{1}{\gamma}(\log\frac{1}{\gamma})^p$, $p \ge 2$, then for intervals whose length in macroscopic scale is of order $\frac{1}{\gamma \log\log\frac{1}{\gamma}}$, the typical block spin profile is rigid, taking one of the two values corresponding to the minima of the free energy for the random field Curie Weiss model, or makes at most one transition from one of the minima to the other. This holds for almost all realizations of the field.

It was also proven that the typical profiles are not rigid over any interval of length at least $L_1(\gamma) = \frac{1}{\gamma}(\log\frac{1}{\gamma})(\log\log\frac{1}{\gamma})^{2+\rho}$, for any $\rho > 0$. In [13] the results are shown to be valid for values of the temperature and magnitude of the field in a subset of the region of two absolute minima for the free energy of the corresponding random field Curie Weiss model.

In the present work we show that, on a set of realizations of the random field of probability that goes to 1 when $\gamma \downarrow 0$, we can construct random intervals of length of order $\frac{1}{\gamma}$ to which we associate a sign in such a way that the magnetization profile is rigid on these intervals and, according to the sign, they belong to the + or to the − phase. A description of the transition from one phase to the other is also discussed.

The main problem in the proof of the previous results is the "non locality" of the system, due to the presence of the random field. Within a run of positively magnetized blocks of length 1 in macro scale, the ferromagnetic interaction will favor the persistence of blocks positively magnetized. It is relatively easy to see that the fluctuations of the sum of the random field over intervals of order $\frac{1}{\gamma}$ in macro scale, are the relevant ones. But this is not enough. To determine the beginning, the end, and the sign of a random interval, it is essential to verify other local requirements for the random field. We need a detailed analysis of the sum of the random field in all subintervals of the large interval of order $\frac{1}{\gamma}$. In fact it could happen that even though at large the random field undergoes to a positive (for example) fluctuation, locally there are negative fluctuations which make not convenient for the system to have a magnetization profile close to the + phase in that interval.

Another problem in our analysis is due to the fact that the previously mentioned block-spin transformation gives rise to a random multibody potential. Using a deviation inequality [26], it turns out to be enough to compute the Lipschitz norm of this multibody potential. This is done by using cluster expansion tools to represent this multibody potential as an absolute convergent series.

The plan of the paper is the following:

In section 2 we give a description of the model and present the main results.

In section 3 we implement the block spin transformation and express the Gibbs measure in terms of the above mentioned functional. This functional can be written explicitly as a sum of a deterministic and a stochastic part, and it has been studied in [13] for values of the inverse temperature $\beta > 1$ and magnitude of the field θ sufficiently small. In this section we go deeper into the analysis of the stochastic part extracting the leading part and estimating the remaining by deviation inequalities for Lipschitz functions of Bernoulli random variables. The cluster expansion plays a crucial role in order to get bounds on its Lipschitz norms. In this way we extend the previous results of [13] to the maximal connected region in the β, θ plane, containing $(1, 0)$ in its closure, compatible with the existence of two minimizers for the Random Field Curie Weiss model and control the fluctuation of the stochastic part on a larger scale.

In section 4, we show that the typical block spin profiles are rigid, or make one transition from one of the minima to the other, over a macroscopic scale ϵ/γ, for any $\epsilon > 0$, provided γ is small enough. This is an important intermediate result that extends the results of [13] on the scale $1/(\gamma \log\log(1/\gamma))$ to the larger scale ϵ/γ.

In section 5, we analyze the stochastic contribution on the scale $1/\gamma$ and prove probability estimates which allow us to construct the above mentioned random intervals with corresponding sign.

In section 6 we finally prove the theorems stated in section 2.

In section 7 we prove some technical results on the deterministic part of the functional, used in section 4 and 6.

In section 8 we present a rather short, self contained and complete proof of the convergence of the cluster expansion for our model. This is a standard tool in Statistical Mechanics, but application to this model is new.

In section 9 we discuss some properties of the Random Field Curie Weiss model that are relevant for our paper.

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2 Description of the model and main results

Let $(\Omega, \mathcal{A}, I\!\!P)$ be a probability space on which we have defined $h \equiv \{h_i\}_{i\in Z\!\!\!Z}$, a family of independent, identically distributed Bernoulli random variables with $I\!\!P[h_i = +1] = I\!\!P[h_i = -1] = 1/2$. They represent random signs of external magnetic fields acting on a spin system on $Z\!\!\!Z$, and whose magnitude is denoted by $\theta > 0$. The configuration space is $\mathcal{S} \equiv \{-1, +1\}^{Z\!\!\!Z}$. If $\sigma \in \mathcal{S}$ and $i \in Z\!\!\!Z$, σ_i represents the value of the spin at site i. The pair interaction among spins is given by a Kac potential of the form $J_\gamma(i-j) \equiv \gamma J(\gamma(i-j))$, $\gamma > 0$, on which one requires, for $r \in I\!\!R$: (i) $J(r) \geq 0$ (ferromagnetism); (ii) $J(r) = J(-r)$ (symmetry); (iii) $J(r) \leq ce^{-c'|r|}$ for c, c' positive constants (exponential decay); (iv) $\int J(r)dr = 1$ (normalization). For sake of simplicity we fix $J(r) = 1\!\!1_{[|r|\leq 1/2]}$.

For $\Lambda \subseteq Z\!\!\!Z$ we set $\mathcal{S}_\Lambda = \{-1, +1\}^\Lambda$; its elements are usually denoted by σ_Λ; also, if $\sigma \in \mathcal{S}$, σ_Λ denotes its restriction to Λ. Given $\Lambda \subset Z\!\!\!Z$ finite and a realization of the magnetic fields, the free boundary condition Hamiltonian in the volume Λ is given by

$$H_\gamma(\sigma_\Lambda)[\omega] = -\frac{1}{2} \sum_{(i,j)\in\Lambda\times\Lambda} J_\gamma(i-j)\sigma_i\sigma_j - \theta \sum_{i\in\Lambda} h_i[\omega]\sigma_i, \tag{2.1}$$

which is then a random variable on $(\Omega, \mathcal{A}, I\!\!P)$. In the following we drop the ω from the notation.

The corresponding *Gibbs measure* on the finite volume Λ, at inverse temperature $\beta > 0$ and free boundary condition is then a random variable with values on the space of probability measures on $\mathcal{S}_\Lambda$. We denote it by $\mu_{\beta,\theta,\gamma,\Lambda}$ and it is defined by

$$\mu_{\beta,\theta,\gamma,\Lambda}(\sigma_\Lambda) = \frac{1}{Z_{\beta,\theta,\gamma,\Lambda}} \exp\{-\beta H_\gamma(\sigma_\Lambda)\} \qquad \sigma_\Lambda \in \mathcal{S}_\Lambda, \tag{2.2}$$

where $Z_{\beta,\theta,\gamma,\Lambda}$ is the normalization factor usually called partition function.

To take into account the interaction between the spins in Λ and those outside Λ we set

$$W_\gamma(\sigma_\Lambda, \sigma_{\Lambda^c}) = -\sum_{i\in\Lambda}\sum_{j\in\Lambda^c} J_\gamma(i-j)\sigma_i\sigma_j. \tag{2.3}$$

If $\tilde\sigma \in \mathcal{S}$, the Gibbs measure on the finite volume Λ and boundary condition $\tilde\sigma_{\Lambda^c}$ is the random probability measure on $\mathcal{S}_\Lambda$, denoted by $\mu^{\tilde\sigma_{\Lambda^c}}_{\beta,\theta,\gamma,\Lambda}$ and defined by

$$\mu^{\tilde\sigma_{\Lambda^c}}_{\beta,\theta,\gamma,\Lambda}(\sigma_\Lambda) = \frac{1}{Z^{\tilde\sigma_{\Lambda^c}}_{\beta,\theta,\gamma,\Lambda}} \exp\left\{-\beta(H_\gamma(\sigma_\Lambda) + W_\gamma(\sigma_\Lambda, \tilde\sigma_{\Lambda^c}))\right\}, \tag{2.4}$$

where again the partition function $Z^{\tilde\sigma_{\Lambda^c}}_{\beta,\theta,\gamma,\Lambda}$ is the normalization factor.

Given a realization of h and $\gamma > 0$, there is a unique weak-limit of $\mu_{\beta,\theta,\gamma,\Lambda}$ along a family of volumes $\Lambda_L = [-L, L] \cap Z\!\!\!Z$, $L \in I\!\!N$; such limit is called the infinite volume Gibbs measure $\mu_{\beta,\theta,\gamma}$. The limit does not depend on the boundary conditions, which may be taken h-dependent, but it is a random element, i.e., different realizations of h give a priori different infinite volume Gibbs measures.

As in [15] and [13], our analysis of the large scale profiles under $\mu_{\beta,\theta,\gamma}$ in the limit of $\gamma \downarrow 0$ involves a block spin transformation, which transforms our microscopic system on $Z\!\!\!Z$ into a *macroscopic* system on $I\!\!R$. Since the interaction length is γ^{-1}, one starts by a suitable scale transformation such that on the new scale, which we call *the macroscopic scale*, the interaction length becomes one. Therefore, a macroscopic volume, always taken as an interval $I \subseteq I\!R$, corresponds to the microscopic volume $\Lambda(I) = \gamma^{-1}I \cap \mathbb{Z}$. The results will always be expressed in the macroscopic scale. The block spin transformation involves a "coarse graining". Before making this precise let us set some notations and basic definitions, mostly from [13].

Given a rational positive number δ, $\mathcal{D}_\delta$ denotes the partition of $I\!R$ into (macroscopic) intervals $\tilde{A}_\delta(x) = ((x-1)\delta, x\delta]$ where $x \in \mathbb{Z}$. If $I \subseteq I\!R$ denotes a macroscopic interval we let

$$\mathcal{C}_\delta(I) = \{x \in \mathbb{Z}; \tilde{A}_\delta(x) \subseteq I\}. \tag{2.5}$$

In the following we will consider, if not explicitly written, intervals always in macroscopic scale and $\mathcal{D}_\delta$–measurable, i.e., $I = \cup\{\tilde{A}_\delta(x); x \in \mathcal{C}_\delta(I)\}$.

Given a realization of h and for each configuration σ_Λ, we could define for each block $\tilde{A}_\delta(x)$ a pair of numbers where the first is the average magnetization over the sites with positive h and the second to those with negative h. However it appears, [13], to be more convenient to use another random partition of $\tilde{A}_\delta(x) \cap \mathbb{Z}$ into two sets of the same cardinality. This allows to separate on each block the expected contribution of the random field from its local fluctuactions.

The coarse graining will involve a scale $0 < \delta^* = \delta^*(\gamma) < 1$ satisfying certain conditions of smallness and will be the smallest scale. The elements of $\mathcal{D}_{\delta^*}$ will be denoted by $\tilde{A}(x)$, with $x \in \mathbb{Z}$. The blocks $\tilde{A}(x)$ correspond to intervals of length δ^* in the macroscopic scale and induce a partition of $\mathbb{Z}$ into blocks (in microscopic scale) of order $\delta^*\gamma^{-1}$, hereby denoted by $A(x) = \{i \in \mathbb{Z}; i\gamma \in \tilde{A}(x)\} = \{a(x)+1, \dots, a(x+1)\}$; for notational simplicity, if no confusion arises, we omit to write the explicit dependence on γ, δ^*. To avoid rounding problem, we assume , that $\gamma = 2^{-n}$ for some integer n, with δ^* such that $\delta^*\gamma^{-1}$ is an integer, so that $a(x) = x\delta^*\gamma^{-1}$, with $x \in \mathbb{Z}$. We assume that $\delta^*\gamma^{-1} \uparrow \infty$.

Given a realization $h[\omega] \equiv (h_i[\omega])_{i\in\mathbb{Z}}$, we set $A^+(x) = \{i \in A(x); h_i[\omega] = +1\}$ and $A^-(x) = \{i \in A(x); h_i[\omega] = -1\}$. Let $\lambda(x) \equiv \mathrm{sgn}(|A^+(x)| - (2\gamma)^{-1}\delta^*)$, where sgn is the sign function, with the convention that $\mathrm{sgn}(0) = 0$. For convenience we assume $\delta^*\gamma^{-1}$ to be even, in which case:

$$I\!P\,[\lambda(x) = 0] = 2^{-\delta^*\gamma^{-1}} \binom{\delta^*\gamma^{-1}}{\delta^*\gamma^{-1}/2}. \tag{2.6}$$

We note that $\lambda(x)$ is a symmetric random variable. When $\lambda(x) = \pm 1$ we set

$$l(x) \equiv \inf\{l > a(x) \,:\, \sum_{j=a(x)+1}^{l} 1\!\!1_{\{A^{\lambda(x)}(x)\}}(j) \geq \delta^*\gamma^{-1}/2\} \tag{2.7}$$

and consider the following decomposition of $A(x)$: $B^{\lambda(x)}(x) = \{i \in A^{\lambda(x)}(x); i \leq l(x)\}$ and $B^{-\lambda(x)}(x) = A(x) \setminus B^{\lambda(x)}(x)$. When $\lambda(x) = 0$ we set $B^+(x) = A^+(x)$ and $B^-(x) = A^-(x)$. We set $D(x) \equiv A^{\lambda(x)}(x) \setminus B^{\lambda(x)}(x)$. In this way, the sets $B^\pm(x)$ depend on the realizations of ω, but the cardinality $|B^\pm(x)| = \delta^*\gamma^{-1}/2$ is the same for all realizations. We define

$$m^{\delta^*}(\pm, x, \sigma) = \frac{2\gamma}{\delta^*} \sum_{i \in B^\pm(x)} \sigma_i. \tag{2.8}$$

We have

$$\frac{\gamma}{\delta^*} \sum_{i \in A(x)} \sigma_i = \frac{1}{2}(m^{\delta^*}(+, x, \sigma) + m^{\delta^*}(-, x, \sigma)) \tag{2.9}$$

and

$$\frac{\gamma}{\delta^*}\sum_{i\in A(x)} h_i\sigma_i = \frac{1}{2}(m^{\delta^*}(+,x,\sigma) - m^{\delta^*}(-,x,\sigma)) + \lambda(x)\frac{2\gamma}{\delta^*}\sum_{i\in D(x)}\sigma_i. \tag{2.10}$$

Given a volume $\Lambda \subseteq \mathbb{Z}$ in the original microscopic spin system, it corresponds to the macroscopic volume $I = \gamma\Lambda = \{\gamma i; i \in \Lambda\}$, assumed to be $\mathcal{D}_{\delta^*}$–measurable to avoid rounding problems. The block spin transformation, as considered in [13], is the random map which associates to the spin configuration σ_Λ the vector $(m^{\delta^*}(x,\sigma))_{x\in\mathcal{C}_{\delta^*}(I)}$, where $m^{\delta^*}(x,\sigma) = (m^{\delta^*}(+,x,\sigma), m^{\delta^*}(-,x,\sigma))$, with values in the set

$$\mathcal{M}_{\delta^*}(I) \equiv \prod_{x\in\mathcal{C}_{\delta^*}(I)} \left\{-1, -1+\frac{4\gamma}{\delta^*}, -1+\frac{8\gamma}{\delta^*}, \dots, 1-\frac{4\gamma}{\delta^*}, 1\right\}^2. \tag{2.11}$$

As in [13], we use the same notation $\mu_{\beta,\theta,\gamma,\Lambda}$ to denote both, the Gibbs measure on $\mathcal{S}_\Lambda$, and the probability measure it induces on $\mathcal{M}_{\delta^*}(I)$, through the block spin transformation, i.e., a coarse grained version of the original measure. Analogously, the infinite volume limit (as $\Lambda \uparrow \mathbb{Z}$) of the laws of the block spin $(m^{\delta^*}(x))_{x\in\mathcal{C}_{\delta^*}(I)}$ under the Gibbs measure will also be denoted by $\mu_{\beta,\theta,\gamma}$.

If $\lim_{\gamma\downarrow 0}\delta^*(\gamma) = 0$, this limiting measure will be supported by

$$\mathcal{T} = \{m \equiv (m_1, m_2) \in L^\infty(\mathbb{R}) \times L^\infty(\mathbb{R}); \|m_1\|_\infty \vee \|m_2\|_\infty \le 1\}. \tag{2.12}$$

To denote a generic element in $\mathcal{M}_{\delta^*}(I)$ we write

$$m_I^{\delta^*} \equiv (m^{\delta^*}(x))_{x\in\mathcal{C}_{\delta^*}(I)} \equiv (m_1^{\delta^*}(x), m_2^{\delta^*}(x))_{x\in\mathcal{C}_{\delta^*}(I)}. \tag{2.13}$$

Since I is $\mathcal{D}_{\delta^*}$– measurable, we can identify $m_I^{\delta^*}$ with the element of $\mathcal{T}$ which equals $m^{\delta^*}(x)$ on each $\tilde{A}(x) = ((x-1)\delta^*, x\delta^*]$ for $x \in \mathcal{C}_{\delta^*}(I)$, and vanishes outside I. We denote by T the linear bijection on $\mathcal{T}$ defined by

$$(Tm)(x) = (-m_2(x), -m_1(x)) \quad \forall x \in \mathbb{R}. \tag{2.14}$$

As in [13], the description of the profiles is based on the behavior of local averages of $m^{\delta^*}(x)$ over k successive blocks in the block spin representation, where $k \ge 2$ is a positive integer. Let $\delta = k\delta^*$ such that $1/\delta \in \mathbb{N}$. Given $\ell \in \mathbb{Z}$, recalling (2.5), we define the random variable

$$\eta^{\delta,\zeta}(\ell) = \begin{cases} 1 \text{ if } \forall_{u\in\mathcal{C}_\delta((\ell-1,\ell])} \frac{\delta^*}{\delta}\sum_{x\in\mathcal{C}_{\delta^*}(((u-1)\delta,u\delta])} \|m^{\delta^*}(x) - m_\beta\|_1 \le \zeta; \\ \text{-1 if } \forall_{u\in\mathcal{C}_\delta((\ell-1,\ell])} \frac{\delta^*}{\delta}\sum_{x\in\mathcal{C}_{\delta^*}(((u-1)\delta,u\delta])} \|m^{\delta^*}(x) - Tm_\beta\|_1 \le \zeta; \\ 0 \text{ otherwise,} \end{cases} \tag{2.15}$$

where $m_\beta \equiv (m_{\beta,1}, m_{\beta,2})$ and $Tm_\beta \equiv (-m_{\beta,2}, -m_{\beta,1})$, see (9.26), correspond to the equilibrium states for the random field Curie Weiss model and $\zeta \in (0, m_{2,\beta}]$.

We say that a magnetization profile $m^{\delta^*}(\cdot)$, in an interval $I \subseteq \mathbb{R}$, is close to the equilibrium phase τ, $\tau = 1$ or $\tau = -1$, with tolerance ζ, when

$$\{\eta^{\delta,\zeta}(\ell) = \tau, \forall \ell \in I \cap \mathbb{Z}\} \tag{2.16}$$

In the following we will use always the letter ℓ to indicate an element of $\mathbb{Z}$. This will allow to write (2.16) as $\{\eta^{\delta,\zeta}(\ell) = \tau, \forall \ell \in I\}$.

Given a realization of h, we would like to know if "typically" with respect to the Gibbs measure we have $\eta^{\delta,\zeta}(0) = +1$ or $\eta^{\delta,\zeta}(0) = -1$.

In [13], it was shown that for almost all realizations of h, typically with respect to the Gibbs measure, the magnetization profiles exhibit runs of $\eta^{\delta,\zeta} = +1$ (and runs of $\eta^{\delta,\zeta} = -1$) and that a profile with an $\eta^{\delta,\zeta} = 0$ between two runs with the same sign is strongly depressed.

Here we have to understand the localization of the end points of such runs, which is random (h–dependent) even in the limit $\gamma \downarrow 0$. Our main theorem (Theorem 2.1) describes the right scale of the runs and how they are located, for a set of overwhelming probability of realizations of the random magnetic field. It will be clear that one can get almost sure results by just taking suitable subsequences $\gamma \downarrow 0$. However, the estimates reflect our limitation: for such subsequences our description looses in precision in terms of the random Gibbs measures.

The study of the Random Field Curie–Weiss model, see section 9, allows us to exhibit the maximal connected region of the β, θ plane containing $(1,0)$ in its closure, where the system has two and only two equilibrium states. We call this region $\mathcal{E}$ and all our results for the Random Field Kac model will be valid in this region.

In Theorem 2.1, we prove that for all ω in a subset Ω_γ of overwhelming $I\!P$–probability, $\eta^{\delta,\zeta}(0)$ is "deeply" inside a run of order $1/\gamma$ whose sign is well defined. We are not able to pin down exactly the endpoints of a run, but we can identify for each of them a small region that countains it, whose size can be bounded uniformly with respect to the realizations of the magnetic fields by ρ/γ for a suitable $\rho = \rho(\gamma)$ that tends to zero as $\gamma \downarrow 0$.

In Theorem 2.2, we extend our analysis to any finite number of consecutive runs on the left and on the right of the one that contains the origin. We prove that all the runs have size of order $1/\gamma$, two consecutive runs have alternating signs and between them there is a small region bounded by ρ/γ.

In Theorem 2.4, we prove that between two consecutive runs there is just a single run of $\eta^{\delta,\zeta} = 0$ whose length is much smaller that ρ/γ.

Theorem 2.1 . *Given $(\beta, \theta) \in \mathcal{E}$, there exists a $\gamma_0(\beta, \theta)$ such that for all $0 < \gamma \le \gamma_0(\beta, \theta)$ and for suitable values of $\delta^* > 0, \delta > 0, \zeta_4 > 0, a' > 0$, there exists $\Omega_\gamma \subseteq \Omega$ with*

$$I\!P[\Omega_\gamma] \ge 1 - \left(\frac{1}{\tilde{g}(1/\gamma)}\right)^{a'} \tag{2.17}$$

where $\tilde{g}(.)$ is a suitable positive increasing function such that $\lim_{x\uparrow\infty} \tilde{g}(x) = +\infty$.
For all realizations of the fields $\omega \in \Omega_\gamma$, we can construct explicitly a random measurable pair $\big(I_0(\omega), \tau_0(\omega)\big)$ where $\tau_0(\omega) \in \{-1, +1\}$ and $I_0(\omega)$ is a random interval that contains the origin such that for all $x > 0$

$$I\!P(\omega \in \Omega_\gamma : \gamma |I_0(\omega)| > x) \le 4 e^{-xc} \tag{2.18}$$

$$I\!P(\omega \in \Omega_\gamma : \gamma |I_0(\omega)| < x) \le 2 e^{-\frac{c'}{x}}, \tag{2.19}$$

where $c > 0, c' > 0$ are functions of β, θ.

For any $\omega \in \Omega_\gamma$, we have

$$\mu_{\beta,\theta,\gamma}\Big[\forall \ell \in I_0(\omega) \cap Z\!\!Z, \eta^{\delta,\zeta_4}(\ell) = \tau_0(\omega)\Big] \ge 1 - e^{-\frac{\beta}{\gamma}\frac{1}{\tilde{g}(1/\gamma)}}. \tag{2.20}$$

Moreover the interval $I_0(\omega)$ is maximal, in the following sense: If J is an interval, $I_0(\omega) \subseteq J$, $|J \backslash I_0(\omega)| \ge 4\frac{\rho}{\gamma}$, with $\rho = \left(\frac{5}{\tilde{g}(1/\gamma)}\right)^{a''}$ for suitable a''

$$\mu_{\beta,\theta,\gamma}\Big[\forall \ell \in J \cap Z\!\!Z, \eta^{\delta,\zeta_4}(\ell) = \tau_0\Big] \le e^{-\frac{\beta}{\gamma}\frac{1}{\tilde{g}(1/\gamma)}}. \tag{2.21}$$

Remark:

It follows from our analysis that Ω_γ is measurable with respect to the σ–algebra $\sigma(h_i, i \in [-Q/\gamma^2, Q/\gamma^2])$ where $Q = \exp \frac{\log \tilde{g}(1/\gamma)}{\log\log \tilde{g}(1/\gamma)}$. Furthermore $I_0(\omega)$ is measurable with respect to the trace σ–algebra $\sigma(h_i, i \in [-Q/\gamma^2, Q/\gamma^2]) \cap \Omega_\gamma$. Therefore to decide if typically $\eta^{\delta,\zeta}(0) = +1$ or $\eta^{\delta,\zeta}(0) = -1$, it is sufficient to know the realization of the random magnetic field in the region $[-Q/\gamma^2, Q/\gamma^2])$. A possible choice is $\tilde{g}(x) \equiv 1 \vee \log x$, in which case we have $Q = (\log(1/\gamma))^{\frac{1}{\log\log\log(1/\gamma)}}$.

Our next result is a simple extension of the previous theorem.

Theorem 2.2 . *Under the same hypothesis of Theorem 2.1 and with the same notations, for all* $k \in I\!N$, *there exists* $\Omega_{\gamma,k} \subseteq \Omega_\gamma$, *with*

$$I\!P[\Omega_{\gamma,k}] \geq 1 - k\left(\frac{1}{\tilde{g}(1/\gamma)}\right)^{a'} \tag{2.22}$$

such that for $\omega \in \Omega_{\gamma,k}$, *we can construct explicitly a random* $(2k+2)$*–tuples*

$$\Big(I_{-k}(\omega), \ldots, I_k(\omega), \tau_0(\omega)\Big) \tag{2.23}$$

where $I_j(\omega), -k \leq j \leq k$ *are disjoint random intervals,* $I_0(\omega)$ *contains the origin and they satisfy for all* $x > 0$

$$I\!P\left[\omega \in \Omega_{\gamma,k} : \sup_{-k \leq j \leq k} \gamma |I_j(\omega)| > x\right] \leq 4(2k+1)e^{-xc} \tag{2.24}$$

$$I\!P\left[\omega \in \Omega_{\gamma,k} : \inf_{-k \leq j \leq k} \gamma |I_j(\omega)| < x\right] \leq (2k+1)2e^{-\frac{c'}{x}}.$$

Moreover

$$\Big|[\inf(I_{-k}(\omega)), \sup(I_k(\omega))] \setminus \bigcup_{j=-k}^{k} I_j(\omega)\Big| \leq (2k+1)\frac{\rho}{\gamma}, \tag{2.25}$$

where ρ *is given just above (2.21). For any* $\omega \in \Omega_{\gamma,k}$, *we have*

$$\mu_{\beta,\theta,\gamma}\Big[\forall j \in \{-k,+k\},\, \forall \ell \in I_j(\omega),\, \eta^{\delta,\zeta_4}(\ell) = (-1)^j \tau_0(\omega)\ \Big] \geq 1 - 2ke^{-\frac{\beta}{\gamma}\frac{1}{\tilde{g}(1/\gamma)}}. \tag{2.26}$$

In the previous theorem nothing is said about what happens in the region between two consecutive intervals with different signs, a region that has a macroscopic length smaller than ρ/γ by (2.25). To describe it we need to introduce the notion of a single change of phases in a given interval.

Definition 2.3 . *Given an interval* $[\ell_1, \ell_2]$ *and a positive integer* $R_2 < |\ell_2 - \ell_1|$, *we say that a single change of phases occurs within* $[\ell_1, \ell_2]$ *on a length* R_2 *if there exists* $\ell_0 \in [\ell_1, \ell_2]$ *so that* $\eta^{\delta,\zeta}(\ell) = \eta^{\delta,\zeta}(\ell_1) \in \{-1, +1\}, \forall \ell \in [\ell_1, \ell_0 - R_2]$*;* $\eta^{\delta,\zeta}(\ell) = \eta^{\delta,\zeta}(\ell_2) = -\eta^{\delta,\zeta}(\ell_1), \forall \ell \in [\ell_0 + R_2, \ell_2]$, *and* $\{\ell \in [\ell_0 - R_2, \ell_0 + R_2] : \eta^{\delta,\zeta}(\ell) = 0\}$ *is a set of consecutive integers. We denote by* $\mathcal{W}_1([\ell_1, \ell_2], R_2, \zeta)$ *the set of all configurations* $\eta^{\delta,\zeta}$ *with these properties.*

In other words, there is an unique run of $\eta^{\delta,\zeta} = 0$, with no more than R_2 elements, inside the interval $[\ell_1, \ell_2]$. Our next result is

Theorem 2.4 . *Under the same hypothesis of Theorem 2.2 and with the same notations, for*

$$R_2 = c'' \left(\tilde{g}(1/\gamma)\right)^{7/2} \tag{2.27}$$

for a suitable c'' that depends on β, θ, for any $\omega \in \Omega_{\gamma,k}$, we have

$$\mu_{\beta,\theta,\gamma}\Big[\bigcap_{-k\le j\le k-1} \mathcal{W}_1([\sup(I_j(\omega)), \inf(I_{j+1})], R_2, \zeta_4)\Big] \ge 1 - 2ke^{-\frac{\beta}{\gamma}\frac{1}{\tilde{g}(1/\gamma)}}. \tag{2.28}$$

Note that the regions where the changes of phases occur have at most length R_2 (in macroscopic units) and we are able to localize it only within an interval of length $\rho/\gamma >> R_2$. This means that up to a small probability subset, we are able to construct an interval of length ρ/γ where does it occur within a scale R_2, but we are not able to determine where it occurs within this interval.

Remark. (Choice of the parameters) In this remark we will clarify the meaning of "suitable" that appears in the statements of the previous theorems.

The main parameters appearing in the problem, besides β, θ and γ, are δ^*, δ and ζ_4. The proof of each theorem, proposition or lemma in the sequel will require some specific bounds on these and other auxiliary parameters. The bounds involved are listed in section 6, see (6.2) until (6.7) and (6.11), which give a set of inequalities that intertwine all of them. Therefore it is necessary to check their consistency.

A way to implement these bounds is to express all auxiliary parameters, see (6.66) until (6.69), in terms of a "suitable" positive increasing function $g(\delta^*/\gamma)$, with $\lim_{x\uparrow\infty} g(x) = \infty$. The main reason to do this is to have the simplest expression for the Gibbs measure estimate (2.20) and the probability estimate (2.17).

After this choice we are left with the following conditions: there exists a $\zeta_0 = \zeta_0(\beta, \theta)$ such that

$$\frac{1}{[\kappa(\beta,\theta)]^{1/3} g^{1/6}(\frac{\delta^*}{\gamma})} < \zeta_4 \le \zeta_0, \tag{2.29}$$

where $\kappa(\beta, \theta)$ satisfies (9.25),

$$\frac{(\delta^*)^2}{\gamma} g^{3/2}(\frac{\delta^*}{\gamma}) \le \frac{1}{\beta\kappa(\beta,\theta)e^3 2^{13}}, \tag{2.30}$$

and

$$(\frac{2\gamma}{\delta^*})^{1/2} \left(\log\frac{1}{\gamma\delta^*} + \frac{\log g(\delta^*/\gamma)}{\log\log g(\delta^*/\gamma)}\right) \le \frac{1}{32}. \tag{2.31}$$

where g is such that $g(x) > 1, g(x)/x \le 1, \forall x > 1$ and $\lim_{x\uparrow\infty} x^{-1} g^{38}(x) = 0$.

The condition $\lim_{x\uparrow\infty} x^{-1} g^{38}(x) = 0$ comes from an explicit choice of δ and of the auxiliary parameter ζ_5 as functions of g, and the constraint (6.5) that has to be satisfied.

A possibility is to start choosing $\delta^* = \gamma^{\frac{1}{2}+d^*}$ for some $0 < d^* < 1/2$, having in mind (2.30), and then choose g. Any positive increasing function g, slowly varying at infinity, can be modified in a finite region of the positive axis to satisfy the above conditions. When g is slowly varying at infinity, the function $\tilde{g}(1/\gamma)$ that appears in Theorem 2.1 is just $g(\gamma^{-\frac{1}{2}-d^*})$. When g is not slowly varying but is a suitable polynomial, $\tilde{g}(1/\gamma)$ can still be simply expressed as a power of $g(\delta^*/\gamma)$. A possible choice is $g(x) = 1 \vee \log x$ or any iterated of it.

Note that the convergence of the cluster expansion requires only $(\delta^*)^2/\gamma < 1/(6e^3\beta)$, cf. Theorem 8.1 but the actual constraint (2.30) is stronger to suitably bound the Lipschitz norm of the multibody potential.

The value of δ^* in this paper is different from the one chosen in [13]. There, to control the many body potential induced by the block spin transformation, a rough estimate proportional to the volume was used.

$\delta^* \approx \gamma \log\log(1/\gamma)$ was taken to describe the behavior of the system in volumes of the order $1/(\gamma \log\log(1/\gamma))$. Note that such a choice of δ^* is incompatible with (2.31).

In this paper, via the cluster expansion, we get an explicit expression of the many body potential and therefore more freedom in the choice of δ^*. The results of [13] were valid only for θ suitably small and this freedom in the choice of δ^* is crucial to get the extension to value of β, θ in the whole region $\mathcal{E}$, see Proposition 3.5 and comments before it.

The constraint (2.29) allows to take ζ_4 as a function of δ^*/γ that goes to zero when $\gamma \downarrow 0$.

Finally the choice of the numerical constants (such as 2^{13}) is never critical and largely irrelevant. We have made no efforts to make the choices close to optimal.

3 The block spin representation

With $C_{\delta^*}(V)$ as in (2.5), let $\Sigma_V^{\delta^*}$ denote the sigma–algebra of $\mathcal{S}$ generated by $m_V^{\delta^*}(\sigma) \equiv (m^{\delta^*}(x,\sigma), x \in \mathcal{C}_{\delta^*}(V))$, where $m^{\delta^*}(x,\sigma) = (m^{\delta^*}(+,x,\sigma), m^{\delta^*}(-,x,\sigma))$, cf. (2.8).

We take $I = (i^-, i^+] \subseteq I\!R$ with $i^\pm \in Z\!\!Z$. The interval I is assumed to be $\mathcal{D}_{\delta^*}$–measurable and we set $\partial^+ I \equiv \{x \in I\!R: i^+ < x \le i^+ + 1\}$, $\partial^- I \equiv \{x \in I\!R: i^- - 1 < x \le i^-\}$, and $\partial I = \partial^+ I \cup \partial^- I$.

For $(m_I^{\delta^*}, m_{\partial I}^{\delta^*})$ in $\mathcal{M}_{\delta^*}(I \cup \partial I)$, cf. (2.11), we set $\widetilde{m}^{\delta^*}(x) = (m_1^{\delta^*}(x) + m_2^{\delta^*}(x))/2$,

$$E(m_I^{\delta^*}) \equiv -\frac{\delta^*}{2} \sum_{(x,y)\in\mathcal{C}_{\delta^*}(I)\times\mathcal{C}_{\delta^*}(I)} J_{\delta^*}(x-y)\tilde{m}^{\delta^*}(x)\tilde{m}^{\delta^*}(y), \tag{3.1}$$

and

$$E(m_I^{\delta^*}, m_{\partial^\pm I}^{\delta^*}) \equiv -\delta^* \sum_{x\in\mathcal{C}_{\delta^*}(I)} \sum_{y\in\mathcal{C}_{\delta^*}(\partial^\pm I)} J_{\delta^*}(x-y)\tilde{m}^{\delta^*}(x)\tilde{m}^{\delta^*}(y), \tag{3.2}$$

where $J_{\delta^*}(x) = \delta^* J(\delta^* x)$. It is easy to see that

$$H_\gamma(\sigma_{\gamma^{-1}I}) + \theta \sum_{i\in\gamma^{-1}I} h_i\sigma_i = \frac{1}{\gamma} E(m_I^{\delta^*}) + \frac{1}{\beta}\log\Big[\prod_{x\in\mathcal{C}_{\delta^*}(I)}\prod_{y\in\mathcal{C}_{\delta^*}(I)} e^{\beta U(\sigma_{A(x)},\sigma_{A(y)})}\Big], \tag{3.3}$$

where

$$U(\sigma_{A(x)},\sigma_{A(y)}) = -\sum_{i\in A(x), j\in A(y)} \gamma\big[J(\gamma|i-j|) - J(\delta^*|x-y|)\big]\sigma_i\sigma_j. \tag{3.4}$$

Since the interaction is only between adjacent blocks of macroscopic length 1, see (2.3), we see that for all intervals I, for $s=+$ or $s=-$

$$\sup_{\sigma_{\gamma^{-1}I}\in M^{\delta^*}(m_I^{\delta^*})} \sup_{\sigma_{\gamma^{-1}\partial^s I}\in M^{\delta^*}(m_{\partial^s I}^{\delta^*})} \Big| W_\gamma(\sigma_{\gamma^{-1}I}|\sigma_{\gamma^{-1}\partial^s I}) - \frac{1}{\gamma}E(m_I^{\delta^*}, m_{\partial^s I}^{\delta^*})\Big| \le \delta^*\gamma^{-1}, \tag{3.5}$$

where $M^{\delta^*}(m_I^{\delta^*}) \equiv \{\sigma \in \gamma^{-1}I : m^{\delta^*}(x,\sigma) = m^{\delta^*}(x), \forall x \in \mathcal{C}_{\delta^*}(I)\}$.

Recalling (2.10), and using (3.3) and (3.5), if F^{δ^*} is a $\Sigma_I^{\delta^*}$-measurable bounded function and $m_{\partial I}^{\delta^*} \in$

$\mathcal{M}_{\delta^*}(\partial I)$, and $\mu_{\beta,\theta,\gamma}\left(F|\Sigma_{\partial I}^{\delta^*}\right)$ denotes the conditional expectation of F^{δ^*} given the σ–algebra $\Sigma_{\partial I}^{\delta^*}$, we have

$$\begin{aligned}\mu_{\beta,\theta,\gamma}\left(F^{\delta^*}\mid\Sigma_{\partial I}^{\delta^*}\right)(m_{\partial I}^{\delta^*}) &= \frac{e^{\pm\frac{\beta}{\gamma}2\delta^*}}{Z_{\beta,\theta,\gamma,I}(m_{\partial I}^{\delta^*})}\times\\ &\times\sum_{m_I^{\delta^*}\in\mathcal{M}_{\delta^*}(I)}F^{\delta^*}(m_I^{\delta^*})e^{-\frac{\beta}{\gamma}\left(E(m_I^{\delta^*})+E(m_I^{\delta^*},m_{\partial I}^{\delta^*})-\frac{\theta\delta^*}{2}\sum_{x\in\mathcal{C}_{\delta^*}(I)}(m_1^{\delta^*}(x)-m_2^{\delta^*}(x))\right)}\\ &\times\sum_{\sigma_{\gamma^{-1}I}}\prod_{x_1\in\mathcal{C}_{\delta^*}(I)}1\!\!1_{\{m^{\delta^*}(x_1,\sigma)=m^{\delta^*}(x_1)\}}e^{2\beta\theta\lambda(x_1)\sum_{i\in D(x_1)}\sigma_i}\\ &\times\prod_{x_2\in\mathcal{C}_{\delta^*}(I)}\prod_{y_2\in\mathcal{C}_{\delta^*}(I)}e^{-\beta U(\sigma_{A(x_2)},\sigma_{A(y_2)})},\end{aligned}\tag{3.6}$$

where

$$\begin{aligned}Z_{\beta,\gamma,\theta,I}(m_{\partial I}^{\delta^*}) = &\sum_{m_I^{\delta^*}\in\mathcal{M}_{\delta^*}(I)}e^{-\frac{\beta}{\gamma}\left(E(m_I^{\delta^*})+E(m_I^{\delta^*},m_{\partial I}^{\delta^*})-\frac{\theta\delta^*}{2}\sum_{x\in\mathcal{C}_{\delta^*}(I)}(m_1^{\delta^*}(x)-m_2^{\delta^*}(x))\right)}\\ &\times\sum_{\sigma_{\gamma^{-1}I}}\prod_{x_1\in\mathcal{C}_{\delta^*}(I)}1\!\!1_{\{m^{\delta^*}(x_1,\sigma)=m^{\delta^*}(x_1)\}}e^{2\beta\theta\lambda(x_1)\sum_{i\in D(x_1)}\sigma_i}\\ &\times\prod_{x_2\in\mathcal{C}_{\delta^*}(I)}\prod_{y_2\in\mathcal{C}_{\delta^*}(I)}e^{-\beta U(\sigma_{A(x_2)},\sigma_{A(y_2)})}.\end{aligned}\tag{3.7}$$

Equality (3.6) has to be interpreted as an upper bound for $\pm=+1$ and a lower bound for $\pm=-1$. Given $m_I^{\delta^*}$, we define the probability measure on $\{-1,+1\}^{\gamma^{-1}I}$ by

$$I\!\!E_{m_I^{\delta^*}}[f]\equiv\frac{\sum_{\sigma_{\gamma^{-1}I}}\prod_{x_1\in\mathcal{C}_{\delta^*}(I)}1\!\!1_{\{m^{\delta^*}(x_1,\sigma)=m^{\delta^*}(x_1)\}}e^{2\beta\theta\lambda(x_1)\sum_{i\in D(x_1)}\sigma_i}f(\sigma)}{\sum_{\sigma_{\gamma^{-1}I}}\prod_{x_1\in\mathcal{C}_{\delta^*}(I)}1\!\!1_{\{m^{\delta^*}(x_1,\sigma)=m^{\delta^*}(x_1)\}}e^{2\beta\theta\lambda(x_1)\sum_{i\in D(x_1)}\sigma_i}}.\tag{3.8}$$

Inside the sum $\sum_{m_I^{\delta^*}}$ in (3.6), we divide and multiply by

$$\sum_{\sigma_{\gamma^{-1}I}}\prod_{x_3\in\mathcal{C}_{\delta^*}(I)}1\!\!1_{\{m^{\delta^*}(x_3,\sigma)=m^{\delta^*}(x_3)\}}e^{2\beta\theta\lambda(x_3)\sum_{i\in D(x_3)}\sigma_i}$$

to get

$$\begin{aligned}\mu_{\beta,\theta,\gamma}\left(F^{\delta^*}\mid\Sigma_{\partial I}\right)(m_{\partial I}^{\delta^*}) &= \frac{e^{\pm\frac{\beta}{\gamma}2\delta^*}}{Z_{\beta,\theta,\gamma,I}(m_{\partial I}^{\delta^*})}\\ &\times\sum_{m_I^{\delta^*}\in\mathcal{M}_{\delta^*}(I)}F^{\delta^*}(m^{\delta^*})e^{-\frac{\beta}{\gamma}\left(E(m_I^{\delta^*})+E(m_I^{\delta^*},m_{\partial I}^{\delta^*})-\frac{\theta\delta^*}{2}\sum_{x\in\mathcal{C}_{\delta^*}(I)}(m_1^{\delta^*}(x)-m_2^{\delta^*}(x))\right)}\\ &\times e^{\log I\!\!E_{m_I^{\delta^*}}[\prod_{x_2\neq y_2}e^{-\beta U(\sigma_{A(x_2)},\sigma_{A(y_2)})}]}\\ &\times\sum_{\sigma_{\gamma^{-1}I}}\prod_{x_3\in\mathcal{C}_{\delta^*}(I)}1\!\!1_{\{m^{\delta^*}(x_3,\sigma)=m^{\delta^*}(x_3)\}}e^{2\beta\theta\lambda(x_3)\sum_{i\in D(x_3)}\sigma_i}.\end{aligned}\tag{3.9}$$

Notice that for each $x \in \mathcal{C}_{\delta^*}(I)$

$$\sum_{\sigma \in \mathcal{S}_{\delta^*\gamma^{-1}}} 1\!\!\mathrm{I}_{\{m^{\delta^*}(x,\sigma)=m^{\delta^*}(x)\}} = \binom{\delta^*\gamma^{-1}/2}{\frac{1+m_1^{\delta^*}(x)}{2}\delta^*\gamma^{-1}/2}\binom{\delta^*\gamma^{-1}/2}{\frac{1+m_2^{\delta^*}(x)}{2}\delta^*\gamma^{-1}/2}, \tag{3.10}$$

Note that the last sum $\sum_{\sigma_{\gamma^{-1}I}}$ in (3.9) factors out into a product over the intervals of length $\delta^*\gamma^{-1}$, indexed by $\mathcal{C}_{\delta^*}(I)$. Let us call

$$\mathcal{G}(m_I^{\delta^*}) \equiv \sum_{x \in \mathcal{C}_{\delta^*}(I)} \mathcal{G}_{x,m^{\delta^*}(x)}(\lambda(x)), \tag{3.11}$$

where for each $x \in \mathcal{C}_{\delta^*}(I)$, $\mathcal{G}_{x,m^{\delta^*}(x)}(\lambda(x))$ is the value of the cumulant generating function:

$$\mathcal{G}_{x,m^{\delta^*}(x)}(\lambda(x)) \equiv -\frac{1}{\beta}\log I\!\!E^{\delta^*}_{x,m^{\delta^*}(x)}(e^{2\beta\theta\lambda(x)\sum_{i\in D(x)}\sigma_i}), \tag{3.12}$$

of the "canonical" measure on $\{-1,+1\}^{A(x)}$, defined through

$$I\!\!E^{\delta^*}_{x,m^{\delta^*}(x)}(\varphi) = \frac{\sum_\sigma \varphi(\sigma) 1\!\!\mathrm{I}_{\{m^{\delta^*}(x,\sigma)=m^{\delta^*}(x)\}}}{\sum_\sigma 1\!\!\mathrm{I}_{\{m^{\delta^*}(x,\sigma)=m^{\delta^*}(x)\}}}, \tag{3.13}$$

the sum being over $\sigma \in \{-1,+1\}^{A(x)}$.

With these notations, (3.9) becomes

$$\mu_{\beta,\theta,\gamma}\left(F^{\delta^*} \mid \Sigma_{\partial I}\right)(m^{\delta^*}_{\partial I}) = \frac{e^{\pm\frac{\beta}{\gamma}2\delta^*}}{Z_{\beta,\theta,\gamma,I}(m^{\delta^*}_{\partial I})} \sum_{m_I^{\delta^*}\in\mathcal{M}_{\delta^*}(I)} F^{\delta^*}(m^{\delta^*}) e^{-\frac{\beta}{\gamma}\left\{\widehat{\mathcal{F}}(m_I^{\delta^*}|m^{\delta^*}_{\partial I})+\gamma\mathcal{G}(m_I^{\delta^*})+\gamma V(m_I^{\delta^*})\right\}}, \tag{3.14}$$

where

$$\begin{aligned}\widehat{\mathcal{F}}(m_I^{\delta^*}|m^{\delta^*}_{\partial I}) =& E(m_I^{\delta^*}) + E(m_I^{\delta^*}, m^{\delta^*}_{\partial I}) - \frac{\theta\delta^*}{2}\sum_{x\in\mathcal{C}_{\delta^*}(I)}(m_1^{\delta^*}(x) - m_2^{\delta^*}(x)) \\ &- \delta^* \sum_{x\in\mathcal{C}_{\delta^*}(I)} \frac{\gamma}{\beta\delta^*}\log\binom{\delta^*\gamma^{-1}/2}{\frac{1+m_1^{\delta^*}(x)}{2}\delta^*\gamma^{-1}/2}\binom{\delta^*\gamma^{-1}/2}{\frac{1+m_2^{\delta^*}(x)}{2}\delta^*\gamma^{-1}/2},\end{aligned} \tag{3.15}$$

and

$$V(m_I^{\delta^*}) \equiv V_I(m_I^{\delta^*}, h) = -\frac{1}{\beta}\log I\!\!E_{m^{\delta^*}(I)}[\prod_{\substack{x\neq y\\ x,y\in\mathcal{C}_{\delta^*}(I)\times\mathcal{C}_{\delta^*}(I)}} e^{-\beta U(\sigma_{A(x)},\sigma_{A(y)})}]. \tag{3.16}$$

That is, up to the error terms $e^{\pm\frac{\beta}{\gamma}2\delta^*}$, we have been able to describe our system in terms of the block spin variables giving a rather explicit form to the deterministic and the stochastic part.

The following lemma gives an explicit integral form of the deterministic part of the block spins system. For $m \in \mathcal{T}$, let us call

$$\begin{aligned}\tilde{\mathcal{F}}(m_I|m_{\partial I}) = &\int_I f_{\beta,\theta}(m(x))\,dx + \frac{1}{4}\int_I\int_I J(x-y)[\tilde{m}(x)-\tilde{m}(y)]^2\,dxdy \\ &+ \frac{1}{2}\int_I dx \int_{I^c} J(x-y)[\tilde{m}(x)-\tilde{m}(y)]^2\,dy.\end{aligned} \tag{3.17}$$

Lemma 3.1 . *If $m^{\delta^*}_{I\cup\partial I}\in\mathcal{M}_{\delta^*}(I\cup\partial I)$ and $m(r)=m^{\delta^*}(x)$ for $r\in((x-1)\delta^*,x\delta^*]$ and $x\in\mathcal{C}_{\delta^*}(I\cup\partial I)$, one has*

$$|\widehat{\mathcal{F}}(m^{\delta^*}_I|m^{\delta^*}_{\partial I})-\tilde{\mathcal{F}}(m_I|m_{\partial I})+\frac{\delta^*}{2}\sum_{y\in\mathcal{C}_{\delta^*}(\partial I)}\left[\tilde m^{\delta^*}(y)\right]^2\sum_{x\in\mathcal{C}_{\delta^*}(I)}J_{\delta^*}(x-y)|\le|I|\frac{\gamma}{\delta^*}\log\frac{\delta^*}{\gamma}. \tag{3.18}$$

Proof: Since

$$|\mathbb{1}_{\{\gamma|i-j|\le 1/2\}}-\mathbb{1}_{\{\delta^*|x-y|\le 1/2\}}|\le\mathbb{1}_{\{-\delta^*+1/2\le\delta^*|x-y|\le\delta^*+1/2\}} \tag{3.19}$$

we have that

$$|U(\sigma_{A(x)},\sigma_{A(y)})|\le\gamma(\frac{\delta^*}{\gamma})^2\mathbb{1}_{\{1/2-\delta^*\le\delta^*|x-y|\le 1/2+\delta^*\}}. \tag{3.20}$$

Given $m^{\delta^*}_I\in\mathcal{M}_{\delta^*}(I)$, we easily obtain from (3.20) that, on $M^{\delta^*}(m^{\delta^*}_I)$:

$$\left|H(\sigma_{\gamma^{-1}I})+\theta\sum_{i\in\gamma^{-1}I}h_i\sigma_i-\frac{1}{\gamma}E(m^{\delta^*}_I)\right|=\frac{1}{\beta}\left|\log\Big[\prod_{x\in\mathcal{C}_{\delta^*}(I)}\prod_{y\in\mathcal{C}_{\delta^*}(I)}e^{\beta U(\sigma_{A(x)},\sigma_{A(y)})}\Big]\right|\le|I|\delta^*\gamma^{-1}. \tag{3.21}$$

Using Stirling formula, see [30], we get

$$\begin{aligned}&\Big|\delta^*\sum_{x\in\mathcal{C}_{\delta^*}(I)}\frac{1}{2\beta}\Big(\mathcal{I}(m^{\delta^*}_1)+\mathcal{I}(m^{\delta^*}_2)\Big)-\delta^*\sum_{x\in\mathcal{C}_{\delta^*}(I)}\frac{\gamma}{\beta\delta^*}\log\binom{\delta^*\gamma^{-1}/2}{\frac{1+m^{\delta^*}_1(x)}{2}\delta^*\gamma^{-1}/2}\binom{\delta^*\gamma^{-1}/2}{\frac{1+m^{\delta^*}_2(x)}{2}\delta^*\gamma^{-1}/2}\Big|\\&\qquad\le\frac{1}{\beta}|I|\frac{\gamma}{\delta^*}\log\frac{\delta^*}{\gamma},\end{aligned} \tag{3.22}$$

where $\mathcal{I}(\cdot)$ is defined after (9.6). Recalling the definition of $f_{\beta,\theta}(m)$, cf. (9.6) and elementary arguments to approximate the sum in (3.1), (3.2), and (3.15) by the corresponding terms in the integral (3.17), one get (3.18). The lemma is proven. ■

Concerning the stochastic part in (3.14), note that there are two random terms in (3.14): $\mathcal{G}(m^{\delta^*}_I)$ and $V(m^{\delta^*}_I)$. To treat them we will use the following classical deviation inequality for Lipschitz function of Bernoulli random variables. See [26] or [13] for a short proof.

Lemma 3.2 . *Let N be a positive integer and F be a real function on $\mathcal{S}_N=\{-1,+1\}^N$ and for all $i\in\{1,\dots,N\}$ let*

$$\|\partial_iF\|_\infty\equiv\sup_{(h,\tilde h):h_j=\tilde h_j,\forall j\ne i}\frac{\left|F(h)-F(\tilde h)\right|}{|h_i-\tilde h_i|}. \tag{3.23}$$

If $I\!P$ is the symmetric Bernoulli measure and $\|\partial(F)\|^2_\infty=\sum_{i=1}^N\|\partial_i(F)\|^2_\infty$ then, for all $t>0$

$$I\!P\left[F-I\!E(F)\ge t\right]\le e^{-\frac{t^2}{4\|\partial(F)\|^2_\infty}} \tag{3.24}$$

and also

$$I\!P\left[F-I\!E(F)\le -t\right]\le e^{-\frac{t^2}{4\|\partial(F)\|^2_\infty}}. \tag{3.25}$$

When considering volumes I that are not too large, we use the following simple fact that follows from (3.11) and (3.12)

$$|\mathcal{G}(m^{\delta^*}_I)|\le 2\theta\sup_{\sigma_I\in\{-1,+1\}^{I/\gamma}}\sum_{x\in\mathcal{C}_{\delta^*}(I)}\Big|\sum_{i\in D(x)}\sigma_i\Big|\le 2\theta\sum_{x\in\mathcal{C}_{\delta^*}(I)}|D(x)|. \tag{3.26}$$

Lemma 3.2 implies the following rough estimate:

Lemma 3.3 . (**The rough estimate**) *For all $\delta^* > \gamma > 0$ and for all positive integer p, that satisfy*

$$12(1+p)\delta^* \log\frac{1}{\gamma} \le 1 \tag{3.27}$$

there exists $\Omega_{RE} = \Omega_{RE}(\gamma,\delta^*,p) \subseteq \Omega$ with $I\!P[\Omega_{RE}] \ge 1-\gamma^2$ such that on Ω_{RE} we have:

$$\sup_{I\subseteq[-\gamma^{-p},\gamma^{-p}]} \frac{\sum_{x\in\mathcal{C}_{\delta^*}(I)}(|D(x)| - I\!E[|D(x)|])}{\sqrt{|I|}} \le \frac{\sqrt{3(1+p)}}{\gamma}\sqrt{\gamma\log\frac{1}{\gamma}} \tag{3.28}$$

and, uniformly with respect to all intervals $I \subseteq [-\gamma^{-p},\gamma^{-p}]$,

$$\sup_{m_I^{\delta^*}\in\mathcal{M}_{\delta^*}(I)} \gamma|\mathcal{G}(m_I^{\delta^*})|| \le 2\theta\left(\frac{|I|}{2}\sqrt{\frac{\gamma}{\delta^*}} + \sqrt{3(1+p)}\sqrt{|I|\gamma\log\frac{1}{\gamma}}\right) \le 2\theta|I|\sqrt{\frac{\gamma}{\delta^*}}. \tag{3.29}$$

This Lemma is a direct consequence of Lemma 3.2, since $|D(x)| = (|D(x)| - I\!E[|D(x)|]) + I\!E[|D(x)|]$, $|D(x)| = |\sum_{i\in A(x)} h_i|/2$, and $I\!E[|D(x)|] \le \frac{1}{2}\sqrt{\delta^*/\gamma}$ by Schwarz inequality.

When we use the estimate (3.29), $V(m_I^{\delta^*})$ is estimated using (3.21) and one has

$$\sup_{m_I^{\delta^*}\in\mathcal{M}_{\delta^*}(I)} \gamma|V(m_I^{\delta^*})| \le \delta^*|I|. \tag{3.30}$$

However when (3.29) and (3.30) give useless results, one can use Lemma 3.2 to estimate $V(m_I^{\delta^*})$ and at some point $\|\partial_i V(m^{\delta^*}\|_\infty$ will be needed.

In Theorem 8.1, with the help of the cluster expansion, we prove the following

Lemma 3.4 . *For any finite interval I, let*

$$\|\partial_i V_I\|_\infty \equiv \sup_{(h,\tilde h):h_j=\tilde h_j,\forall j\neq i} \frac{\left|V_I(m_I^{\delta^*},h) - V_I(m_I^{\delta^*},\tilde h)\right|}{|h_i - \tilde h_i|}. \tag{3.31}$$

Then, for all $\beta > 0$, for all $\delta^* > \gamma > 0$, such that

$$\frac{(\delta^*)^2}{\gamma} \le \frac{1}{6e^3\beta} \tag{3.32}$$

we have

$$\sup_{I\subseteq Z\!\!\!Z}\sup_{i\in I}\|\partial_i V_I\|_\infty \le \frac{1}{\beta}\frac{S}{1-S}, \tag{3.33}$$

where S is given in (8.4), $0 < S \le 6e^3\beta\frac{(\delta^*)^2}{\gamma}$.

Together with the above estimates for V_I, we also need an explicit expression for $\mathcal{G}(m_I^{\delta^*})$. Since $D(x) \subseteq B^{-\lambda(x)}(x)$, $\mathcal{G}_{x,m^{\delta^*}(x)}(\lambda(x))$, see (3.12), depends only on one component of $m^{\delta^*}(x)$, precisely on $m^{\delta^*}_{\frac{3+\lambda(x)}{2}}$. In fact, we have

$$\mathcal{G}_{x,m^{\delta^*}(x)}(\lambda(x)) = -\frac{1}{\beta}\log\frac{\sum_{\sigma\in\{-1,+1\}^{B^{-\lambda(x)}(x)}} 1\!\!1_{\{m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x,\sigma)=m^{\delta^*}_{\frac{3+\lambda(x)}{2}}\}} e^{2\beta\theta\lambda(x)\sum_{i\in D(x)}\sigma_i}}{\sum_{\sigma\in\{-1,+1\}^{B^{-\lambda(x)}(x)}} 1\!\!1_{\{m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x,\sigma)=m^{\delta^*}_{\frac{3+\lambda(x)}{2}}\}}}, \tag{3.34}$$

since the sums over the spin configurations in $\{-1,+1\}^{B^{\lambda(x)}(x)}$ – the ones that depend on $m^{\delta^*}_{\frac{3-\lambda(x)}{2}}$ – cancel out between the numerator and denominator in (3.13).

The explicit expression of $\mathcal{G}_{x,m^{\delta^*}(x)}(\lambda(x))$ that one gets using (3.12) and (3.13) is almost useless. One can think about making an expansion in $\beta\theta$ as we basically did in [13], Proposition 3.1 where $\beta\theta$ was assumed to be as small as needed. Since here we assume $(\beta,\theta)\in\mathcal{E}$, one has to find another small quantity. Looking at the term $\sum_{i\in D(x)}\sigma_i$ in (3.12) and setting

$$p(x)\equiv p(x,\omega)=|D(x)|/|B^{\lambda(x)}(x)|=2\gamma|D(x)|/\delta^*, \tag{3.35}$$

it is easy to see that for $I\subseteq I\!R$, if

$$\left(\frac{2\gamma}{\delta^*}\right)^{1/2}\log\frac{|I|}{\delta^*}\le\frac{1}{32}, \tag{3.36}$$

we have

$$I\!P\left[\sup_{x\in\mathcal{C}_{\delta^*}(I)}p(x)>(2\gamma/\delta^*)^{\frac14}\right]\le e^{-\frac{1}{32}\left(\frac{\delta^*}{2\gamma}\right)^{\frac12}}. \tag{3.37}$$

Depending on the values of $m^{\delta^*}_{\frac{3+\lambda(x)}{2}}$, $\mathcal{G}_{x,m^{\delta^*}(x)}(\lambda(x))$ has a behavior that corresponds to the classical Gaussian, Poissonian, or Binomial regimes, as explained in [13]. However, as we shall see in Remark 4.11, we need accurate estimates only in the Gaussian regime.

Let $g_0(n)$ be a positive increasing real function with $\lim_{n\uparrow\infty}g_0(n)=\infty$ such that $g_0(n)/n$ is decreasing to 0 when $n\uparrow\infty$.

Proposition 3.5 . *For all $(\beta,\theta)\in\mathcal{E}$, there exist $\gamma_0=\gamma_0(\beta,\theta)$ and $d_0(\beta)>0$ such that for $0<\gamma\le\gamma_0$, $\gamma/\delta^*\le d_0(\beta)$, on the set $\{\sup_{x\in\mathcal{C}_{\delta^*}(I)}p(x)\le(2\gamma/\delta^*)^{1/4}\}$, if*

$$|m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x)|\le 1-\left(\frac{g_0(\delta^*\gamma^{-1}/2)}{\delta^*\gamma^{-1}/2}\vee\frac{16p(x)\beta\theta}{1-\tanh(2\beta\theta)}\right), \tag{3.38}$$

then

$$\begin{aligned}\mathcal{G}_{x,m^{\delta^*}(x)}(\lambda(x))=&-\frac{1}{\beta}\log\frac{\Psi_{\lambda(x)2\beta\theta,p(x),m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x),\nu_2}}{\Psi_{0,0,m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x),\nu_1}}\\&-\frac{1}{\beta}|D(x)|\left[\log\cosh(2\beta\theta)+\log\left(1+\lambda(x)m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x)\tanh(2\beta\theta)\right)+\hat\varphi(m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x),2\lambda(x)\beta\theta,p(x))\right],\end{aligned} \tag{3.39}$$

where

$$\left|\hat\varphi(m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x),2\lambda(x)\beta\theta,p(x))\right|\le\left(\frac{2\gamma}{\delta^*}\right)^{1/4}\frac{32\beta\theta(1+\beta\theta)}{(1-|m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x)|)^2(1-\tanh(2\beta\theta))} \tag{3.40}$$

and

$$\left|\log\frac{\Psi_{\lambda(x)2\beta\theta,p(x),m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x),\nu_2}}{\Psi_{0,0,m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x),\nu_1}}\right|\le\frac{18}{g_0(\delta^*\gamma^{-1}/2)}+\left(\frac{2\gamma}{\delta^*}\right)^{1/4}c(\beta\theta), \tag{3.41}$$

with $c(\beta\theta)$ given in (3.55).

Proof: The general strategy of the proof is similar to that of Proposition 3.1 in [13]. However, since there are important differences we give some details. We introduce the "grand canonical" measure on $\{-1,+1\}^{B^{-\lambda(x)}(x)}$, with chemical potential $\nu \in I\!R$, given by

$$I\!E_{x,\nu}(f) = \frac{I\!E_{\sigma_{B^{-\lambda(x)}(x)}}\left[f(\sigma)e^{\nu\sum_{i\in B^{-\lambda(x)}(x)}\sigma_i}\right]}{I\!E_{\sigma_{B^{-\lambda(x)}(x)}}\left[e^{\nu\sum_{i\in B^{-\lambda(x)}(x)}\sigma_i}\right]}, \tag{3.42}$$

where $I\!E_{\sigma_{B^{-\lambda(x)}(x)}}$ is the Bernoulli uniform on $\{-1,+1\}^{B^{-\lambda(x)}(x)}$.

Note that taking $\nu = \nu_1$ with $m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x) = \tanh \nu_1$ in (3.42) and calling

$$m_{B^{-\lambda(x)}(x)}(\sigma) \equiv \frac{1}{|B^{-\lambda(x)}(x)|}\sum_{i\in B^{-\lambda(x)}(x)}\sigma_i \tag{3.43}$$

one gets

$$I\!E_{x,\nu_1}\left[m_{B^{-\lambda(x)}(x)}(\sigma)\right] = m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x). \tag{3.44}$$

On the other hand, for ν_2 such that

$$m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x) = p(x)\tanh(\nu_2 + \lambda(x)2\beta\theta) + (1-p(x))\tanh\nu_2, \tag{3.45}$$

one gets

$$\frac{I\!E_{x,\nu_2}\left[m_{B^{-\lambda(x)}(x)}(\sigma)e^{\lambda(x)2\beta\theta(\sum_{i\in D(x)}\sigma_i)}\right]}{I\!E_{x,\nu_2}\left[e^{\lambda(x)2\beta\theta(\sum_{i\in D(x)}\sigma_i)}\right]} = m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x) \tag{3.46}$$

and

$$\begin{aligned}&\frac{I\!E_{x,\nu_2}\left[\left(m_{B^{-\lambda(x)}(x)}(\sigma) - m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x)\right)^2 e^{\lambda(x)2\beta\theta(\sum_{i\in D(x)}\sigma_i)}\right]}{I\!E_{x,\nu_2}\left[e^{\lambda(x)2\beta\theta(\sum_{i\in D(x)}\sigma_i)}\right]} = \\ &= p(x)\frac{1}{\cosh^2(\nu_2+\lambda(x)2\beta\theta)} + (1-p(x))\frac{1}{\cosh^2(\nu_2)} \equiv \sigma^2_{\lambda(x)2\beta\theta}.\end{aligned} \tag{3.47}$$

Thus if we define

$$\Psi_{\lambda(x)2\beta\theta,p(x),m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x),\nu_2} \equiv \frac{I\!E_{x,\nu_2}\left[e^{\lambda(x)2\beta\theta\sum_{i\in D(x)}\sigma_i}1\!\!1_{\{\sqrt{\frac{2\gamma}{\delta^*}}\sum_{i\in B^{-\lambda(x)}(x)}(\sigma_i - m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x))=0\}}\right]}{I\!E_{x,\nu_2}\left[e^{\lambda(x)2\beta\theta\sum_{i\in D(x)}\sigma_i}\right]} \tag{3.48}$$

and

$$\begin{aligned}&\phi(m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x),\lambda(x)2\beta\theta,p(x))\\ &\equiv \frac{\delta^*}{2\gamma}\left((\nu_1-\nu_2)m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x) + p(x)\log\frac{\cosh(\nu_2+\lambda(x)2\beta\theta)}{\cosh(\nu_1)} + (1-p(x))\log\frac{\cosh(\nu_2)}{\cosh(\nu_1)}\right),\end{aligned} \tag{3.49}$$

a simple computation gives

$$\mathcal{G}_{x,m^{\delta^*}(x)}(\lambda(x)) = -\frac{1}{\beta}\log\frac{\Psi_{\lambda(x)2\beta\theta,p(x),m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x),\nu_2}}{\Psi_{0,0,m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x),\nu_1}} - \frac{1}{\beta}\phi(m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x),\lambda(x)2\beta\theta,p(x)). \tag{3.50}$$

By using elementary formulae on hyperbolic tangents and cosines, one can check the following identity

$$\begin{aligned}&\phi(m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x),\lambda(x)2\beta\theta,p(x))\\ &=|D(x)|\left[\log\cosh 2\beta\theta+\log\left(1+\lambda(x)m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x)\tanh(2\beta\theta)\right)+\hat\varphi(m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x),2\lambda(x)\beta\theta,p(x))\right],\end{aligned} \tag{3.51}$$

where

$$\begin{aligned}&|D(x)|\hat\varphi(m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x),2\lambda(x)\beta\theta,p(x))\\ &\quad=\frac{\delta^*}{2\gamma}(\nu_1-\nu_2)m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x)+\frac{\delta^*}{2\gamma}\log\left(1+m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x)\tanh(\nu_2-\nu_1)\right)\\ &\quad+\frac{\delta^*}{2\gamma}\log\cosh(\nu_2-\nu_1)\\ &\quad+\frac{\delta^*}{2\gamma}p(x)\log\left[1+\frac{\lambda(x)\tanh(2\beta\theta)(1-(m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x))^2)\tanh(\nu_2-\nu_1)}{\left(1+\lambda(x)m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x)\tanh(2\beta\theta)\right)\left(1+m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x)\tanh(\nu_2-\nu_1)\right)}\right].\end{aligned} \tag{3.52}$$

To study (3.51), we need extensions of results proved in [13]. Recalling (3.47) and using again elementary formulae on hyperbolic tangents and cosines one can check that

$$\sigma^2_{\lambda(x)2\beta\theta}=\left(1-(m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x))^2\right)\left[1-p(x)(1-p(x))S(p(x),m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x))\right], \tag{3.53}$$

where

$$0\le S(p(x),m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x))\le\left(1-(m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x))^2\right)c(\beta\theta), \tag{3.54}$$

with

$$c(\beta\theta)=\frac{\tanh^2(2\beta\theta)(1+\tanh^2(2\beta\theta))^2}{[1-\tanh^2(2\beta\theta)]^2[1-\tanh(2\beta\theta)]^6}. \tag{3.55}$$

Following the arguments of the proof of Lemma 3.3 in [13] and assuming $\gamma/\delta^*<d_0(\beta)$ for some well chosen $d_0(\beta)$, it is long but not too difficult to check that

$$|\nu_2-\nu_1|\le\frac{4p(x)\beta\theta}{1-(m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x))^2}. \tag{3.56}$$

Using the fact that (3.38) implies that $\frac{4p(x)\beta\theta}{(1-(m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x))^2)(1-\tanh(2\beta\theta))}\le\frac{1}{4}$, recalling (3.51), and using Taylor expansion we get

$$\begin{aligned}&\left|\frac{\phi(m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x),\lambda(x)2\beta\theta,p(x))}{\frac{\delta^*}{2\gamma}}-p(x)\left[\log\cosh 2\beta\theta+\log\left(1+\lambda(x)m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x)\tanh(2\beta\theta)\right)\right]\right|\\ &\qquad\le\frac{32p^2(x)\beta\theta(1+\beta\theta)}{(1-|m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x)|)^2(1-\tanh(2\beta\theta))}.\end{aligned} \tag{3.57}$$

For the proof of (3.41), use the following estimates:

$$\Psi_{\lambda(x)2\beta\theta,p(x),m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x),\nu_2} = \frac{2}{\sqrt{2\pi|B^{-\lambda(x)}(x)|}\sigma_{l(x)2\beta\theta}}\left(1 \pm \frac{66}{g_0(\delta^*\gamma^{-1}/2)} \pm \left(\frac{2\gamma}{\delta^*}\right)^{1/4} c(\beta\theta)\right) \tag{3.58}$$

with $c(\beta\theta)$ given in (3.55). This estimate replaces the one in [13], see Proposition 3.4 there, where a factor 2 is missing.

To get (3.58), we take

$$\rho_1 = \frac{1}{\sigma_{\lambda(x)2\beta\theta}\sqrt{\delta^*(2\gamma)^{-1}}}\sqrt{2(1-\frac{\pi^2}{12})\log g_0(\delta^*(2\gamma)^{-1})}$$

and call

$$\Psi_{\lambda(x)2\beta\theta,p(x),m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x),\nu_2}(\rho_1) = \frac{1}{2\pi}\int_{-\rho_1}^{+\rho_1} e^{ik|B^{-\lambda(x)}(x)|m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x)}\Phi(\lambda(x)2\beta\theta,p(x),k)\,dk, \tag{3.59}$$

where

$$\Phi(\lambda(x)2\beta\theta,p(x),k) \equiv \left[\frac{\cosh(\lambda(x)2\beta\theta+\nu_2-ik)}{\cosh(\lambda(x)2\beta\theta+\nu_2)}\right]^{|D(x)|}\left[\frac{\cosh(\nu_2-ik)}{\cosh(\nu_2)}\right]^{|B^{-\lambda(x)}(x)\setminus D(x)|}. \tag{3.60}$$

We introduce the two quantities

$$\begin{aligned} I_2 &= \frac{1}{2\pi}\int_{-\pi}^{-\pi+\rho_1} e^{ik|B^{-\lambda(x)}(x)|m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x)}\Phi(\lambda(x)2\beta\theta,p(x),k)\,dk,\\ I_3 &= \frac{1}{2\pi}\int_{\pi-\rho_1}^{\pi} e^{ik|B^{-\lambda(x)}(x)|m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x)}\Phi(\lambda(x)2\beta\theta,p(x),k)\,dk. \end{aligned} \tag{3.61}$$

After simple algebra, using that $m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x) = -1+\frac{2l}{|B|}$ for some $l \in \mathbb{Z}$ and elementary change of variables, one gets the crucial relation

$$I_2 + I_3 = \Psi_{\lambda(x)2\beta\theta,p(x),m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x)}(\rho_1). \tag{3.62}$$

Now $\Psi_{\lambda(x)2\beta\theta,p(x),m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x),\nu_2}$ defined in (3.48) satisfies

$$\Psi_{\lambda(x)2\beta\theta,p(x),m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x),\nu_2} = 2\Psi_{\lambda(x)2\beta\theta,p(x),m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x),\nu_2}(\rho_1) + \widetilde{\mathcal{E}}_{\rho_1}, \tag{3.63}$$

where

$$\widetilde{\mathcal{E}}_{\rho_1} \equiv \frac{1}{2\pi}\int_{-\pi}^{\pi} 1\!\!1_{\{\rho_1<|k|\le\pi-\rho_1\}}\Phi(\lambda(x)2\beta\theta,p(x),k)e^{ik|B^{-\lambda(x)}(x)|m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x)}\,dk. \tag{3.64}$$

To estimate (3.64), one use the fact that

$$\left|\frac{\cosh(x-ik)}{\cosh(x)}\right| \le \exp\left[-\frac{1-\cos(2k)}{4\cosh^2 x}\right] \tag{3.65}$$

and that for $k \in [0,\pi]$, $1-\cos(2k) \geq 2(1-\frac{\pi^2}{12})(k^2 \wedge (k-\pi)^2)$, instead of the wrong (3.39) in [13].

Using standard Gaussian estimates, one gets

$$\widetilde{\mathcal{E}}_{\rho_1} \leq \frac{1}{\sqrt{2\pi|B^{-\lambda(x)}(x)|}\sigma_z} \left(\frac{2\sqrt{2}}{\sqrt{\pi}(1-\frac{\pi^2}{12})\rho_1\sigma_z\sqrt{|B^{-\lambda(x)}(x)|}} \right) \exp\left[-\frac{\rho_1^2\sigma_z^2|B^{-\lambda(x)}(x)|(1-\frac{\pi^2}{12})}{2} \right] \tag{3.66}$$

To estimate $\Psi_{\lambda(x)2\beta\theta,p(x),m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x),\nu_2}(\rho_1)$ one makes exactly the same tedious computations as in [13] pg 1434 and one gets at the end (3.58). ■

4 The ϵ–rigidity

We start by defining the set of profiles having runs of + or of −, with length at least $\frac{\epsilon}{\gamma}$.

Definition 4.1 . *Given $\frac{\epsilon}{\gamma} > \delta^*$, an interval $\Delta_Q \equiv [Q_1, Q_2]\gamma^{-1}$ of length in macroscopic units $\frac{Q}{\gamma} = \frac{(Q_2-Q_1)}{\gamma}$, $Q > 0$ such that $\frac{Q_1}{\epsilon}$ and $\frac{Q_2}{\epsilon}$ are integers, $\zeta_4 > \zeta_1 > 8\gamma/\delta^*$, $1 > \delta > \delta^* > 0$, $R_1 > 0$, $\eta = \pm 1$, we define $\mathcal{A}_1(\Delta_Q,\eta) = \mathcal{A}_1(\Delta_Q,\delta,\zeta_1,\zeta_4,\delta^*,\gamma,\epsilon,R_1,\eta)$ as*

$$\begin{aligned} \mathcal{A}_1(\Delta_Q,\eta) = \Big\{ m^{\delta^*}_{\Delta_Q} \colon & \exists k \in I\!N, \exists r_1,\dots,r_k \in \Big\{ \frac{Q_1}{\epsilon}+1, \frac{Q_1}{\epsilon}+2,\dots,\frac{Q_2}{\epsilon}-2,\frac{Q_2}{\epsilon}-1 \Big\}; \\ & r_0 = \frac{Q_1}{\epsilon}, r_{k+1} = \frac{Q_2}{\epsilon}, r_1 < \ldots < r_k, \exists\, q_i \in [r_i\frac{\epsilon}{\gamma},(r_i+1)\frac{\epsilon}{\gamma}] \quad \text{s.t.} \\ & \eta^{\delta,\zeta_4}(\ell) = \eta(-1)^{i-1}\, \forall \ell \in \mathcal{C}_1([(r_{i-1}+1)\frac{\epsilon}{\gamma}, q_i - R_1]), \\ & \eta^{\delta,\zeta_1}(q_i - R_1) = (-1)^{i-1}\eta, \eta^{\delta,\zeta_1}(q_i+R_1) = (-1)^i\eta, \\ & \eta^{\delta,\zeta_4}(\ell) = \eta(-1)^i\ \forall \ell \in \mathcal{C}_1[(q_i+R_1)\wedge\frac{Q_2}{\gamma},\frac{\epsilon}{\gamma}(r_{i+1})], \quad \text{for } i=1,\dots,k \Big\} \end{aligned} \tag{4.1}$$

and

$$\mathcal{A}_1(\Delta_Q) \equiv \cup_{\eta\in\{-1,+1\}} \mathcal{A}_1(\Delta_Q,\eta). \tag{4.2}$$

Remark.

- The integer $k \geq 0$ represents the number of blocks of length R_1 within Δ_Q where there is at least one change of phases which means that $\eta^{\delta,\zeta_1}(q_i - R_1) = (-1)^{i-1}\eta$, $\eta^{\delta,\zeta_1}(q_i+R_1) = (-1)^i\eta$. There are no restrictions on the profiles within the interval $[q_i - R_1 + 1, q_i + R_1 - 1]$.
- r_i is the index of the i–th block of length ϵ/γ in macroscopic units such that in $[q_i - R_1, q_i + R_1] \subseteq [r_i\frac{\epsilon}{\gamma} - R_1, (r_i+1)\frac{\epsilon}{\gamma} + R_1]$ we see at least one change of phases.
- R_1 will be chosen as an upper bound for the length of the longest interval where the system can stay out of "equilibrium", that is to have a run of $\eta^{\delta,\zeta_1} = 0$. This length is related to the parameters ζ_1, δ, by $R_1 \approx (\delta\zeta_1^3)^{-1}$, see (4.22).

Another definition is needed to describe what happens in the intervals $[q_i - R_1, q_i + R_1]$.

Definition 4.2 . *Let $\Delta_L = [\ell_1,\ell_2]$ be an interval of length L in macroscopic units and $\delta > 0$, $\zeta_4 > \zeta_1 > 8\gamma/\delta^*$ as above. For $\eta = +1$ or $\eta = -1$ we set*

$$\mathcal{W}^{\zeta_1,\zeta_4}(\Delta_L,\eta) \equiv \Big\{ m^{\delta^*}_{\Delta_L} \colon \eta^{\delta,\zeta_1}(\ell_1) = \eta^{\delta,\zeta_1}(\ell_2) = \eta,\ \exists\tilde\ell,\ \ell_1 < \tilde\ell < \ell_2 \quad \eta^{\delta,\zeta_4}(\tilde\ell) = -\eta \Big\} \tag{4.3}$$

and $\mathcal{W}^{\zeta_1,\zeta_4}(\Delta_L) \equiv \mathcal{W}^{\zeta_1,\zeta_4}(\Delta_L,+1) \cup \mathcal{W}^{\zeta_1,\zeta_4}(\Delta_L,-1)$.

Given an interval I and a positive integer L_2 we denote by

$$\mathcal{W}^{\zeta_1,\zeta_4}(I,L_2) \equiv \bigcup_{L:\, 2\le L\le L_2} \ \bigcup_{\Delta_L\subseteq I} \mathcal{W}^{\zeta_1,\zeta_4}(\Delta_L). \tag{4.4}$$

The profiles in the complementary of this set do not have two changes of phases within an interval of length smaller than L_2, uniformly along intervals that are within I. We set

$$\mathcal{A}(\Delta_Q) = \mathcal{A}_1(\Delta_Q) \setminus \mathcal{W}^{\zeta_1,\zeta_4}(\Delta_Q, L_2). \tag{4.5}$$

If $L_2 > 2R_1$ the profiles in $\mathcal{A}(\Delta_Q)$ have exactly one change of phase within each interval $[q_i - R_1, q_i + R_1]$. The main result of this Section is the following:

Theorem 4.3 . *Let $(\beta,\theta) \in \mathcal{E}$. We take $\kappa(\beta,\theta) > 0$ verifying* (9.25)*, $\mathcal{F}^*$ is defined in (7.5), and $V(\beta,\theta)$ given by* (4.56)*. There exist $0 < \gamma_0 = \gamma_0(\beta,\theta) < 1$, $0 < d_0 = d_0(\beta,\theta) < 1$, and $0 < \zeta_0 = \zeta_0(\beta,\theta) < 1$, such that for all $0 < \gamma \le \gamma_0$, for all $\delta^*, \delta, \zeta_4, \zeta_1$ with $\delta^* \ge \gamma$, $\gamma/\delta^* \le d_0$, $1 > \delta > \delta^* > 0$, $\zeta_0 \ge \zeta_4 > \zeta_1 > 8\gamma/\delta^*$, and $Q > 3$ that satisfy the following conditions*

$$\frac{32}{\kappa(\beta,\theta)}\zeta_1 \le \delta\zeta_4^3, \tag{4.6}$$

$$\frac{128(1+\theta)}{\kappa(\beta,\theta)}\frac{2(5+\mathcal{F}^*)}{\mathcal{F}^*}\sqrt{\frac{\gamma}{\delta^*}} < \delta\zeta_1^3, \tag{4.7}$$

$$\zeta_1 \ge \left(5184(1+c(\beta\theta))^2\sqrt{\frac{\gamma}{\delta^*}}\right) \vee \left(12\frac{e^3\beta}{c(\beta,\theta)}\frac{(\delta^*)^2}{\gamma}\right)^2 \tag{4.8}$$

for constants $c(\beta,\theta)$ given in (4.65)*, and $c(\beta\theta)$ given in* (3.55)*,*

$$\sqrt{\gamma}\log Q \le \frac{1}{48}\sqrt{\frac{12e^3\beta}{c(\beta,\theta)}}, \tag{4.9}$$

if we call

$$R_1 = \frac{4(5+\mathcal{F}^*)}{\kappa(\beta,\theta)\delta\zeta_1^3} \tag{4.10}$$

and

$$L_2 = \frac{\mathcal{F}^*}{32(1+\theta)}\sqrt{\frac{\delta^*}{\gamma}}, \tag{4.11}$$

then for any interval Δ_Q of length $\frac{Q}{\gamma}$ and any $\epsilon > \gamma\delta^*$, there exists $\Omega_4 = \Omega_4(\gamma,\delta^*,\Delta_Q,\epsilon,\delta,\zeta_1,\zeta_4)$ with

$$I\!P[\Omega_4] > 1 - 6\gamma^2 - \frac{6Q}{\epsilon}\exp\left\{-\frac{(\mathcal{F}^*)^2}{\epsilon 2^{10}V^2(\beta,\theta)}\right\} \tag{4.12}$$

and for all $\omega \in \Omega_4$, we have

$$\mu_{\beta,\theta,\gamma}\left(\mathcal{A}(\Delta_Q)\right) \ge 1 - (\frac{3Q}{\gamma^2})^5 e^{-\frac{\beta}{\gamma}\left[(\frac{\kappa(\beta,\theta)}{4}\delta\zeta_4^3)\wedge\mathcal{F}^*\right]}. \tag{4.13}$$

To prove Theorem 4.3, we represent the system in terms of block spins. This representation was used also in [13]. However, the way to treat some error terms that appear at the very beginning of the computations is different, see (3.3) and (3.4). We first define the subsets of the complementary of $\mathcal{A}(\Delta_Q)$ where the above mentioned changes do not affect the results already obtained in [13].

Let $\Delta_L = [\ell_1, \ell_2]$ be an interval of length $L = \ell_2 - \ell_1 \in \mathbb{N}$. Let $\delta > \delta^*$, $\zeta_4 > \zeta_1 > 8\gamma/\delta^*$ be positive real numbers.

Definition 4.4 . *We set*

$$\mathcal{O}_0^{\delta,\zeta_1}(\Delta_L) \equiv \left\{\eta^{\delta,\zeta_1}(\ell) = 0, \quad \forall \ell \in \Delta_L \cap \mathbb{Z}\right\}. \tag{4.14}$$

Taking $\tilde{L} \le L$ a positive integer, let $\Delta_{\tilde{L}} = [\tilde{\ell}_1, \tilde{\ell}_2]$, $\Delta_{\tilde{L}} \subseteq \Delta_L$. Define for $\eta = +1$ or $\eta = -1$:

$$\mathcal{R}_{0,\eta}^{\delta,\zeta_1,\zeta_4}(\Delta_L, \tilde{L}) \equiv \left\{\eta^{\delta,\zeta_1}(\ell_1) = \eta^{\delta,\zeta_1}(\ell_2) = \eta; \right\} \cap \mathcal{O}_0^{\delta,\zeta_1}([\ell_1+1, \ell_2-1]) \cap \left(\cup_{\Delta_{\tilde{L}} \subset \Delta_L} \mathcal{O}_0^{\delta,\zeta_4}(\Delta_{\tilde{L}})\right) \tag{4.15}$$

and $\mathcal{R}_0^{\delta,\zeta_1,\zeta_4}(\Delta_L, \tilde{L}) \equiv \mathcal{R}_{0,+}^{\delta,\zeta_1,\zeta_4}(\Delta_L, \tilde{L}) \cup \mathcal{R}_{0,-}^{\delta,\zeta_1,\zeta_4}(\Delta_L, \tilde{L})$.

Note that $\mathcal{R}_{0,\eta}^{\delta,\zeta_1,\zeta_4}(\Delta_L, \tilde{L})$ decreases in $\tilde{L}$, therefore $\cup_{\tilde{L}: 1 \le \tilde{L} \le L} \mathcal{R}_{0,\eta}^{\delta,\zeta_1,\zeta_4}(\Delta_L, \tilde{L}) = \mathcal{R}_{0,\eta}^{\delta,\zeta_1,\zeta_4}(\Delta_L, 1)$.

We set

$$\mathcal{R}_0^{\delta,\zeta_1,\zeta_4}(I) \equiv \bigcup_{L:\, 2 \le L \le |I|} \ \bigcup_{\Delta_L \subseteq I} \mathcal{R}_0^{\delta,\zeta_1,\zeta_4}(\Delta_L, 1), \tag{4.16}$$

$$\mathcal{O}_0^{\delta,\zeta_1}(I, R_1) \equiv \bigcup_{R:\, R_1 \le R \le |I|} \ \bigcup_{\Delta_R \subseteq I} \mathcal{O}_0^{\delta,\zeta_1}(\Delta_R). \tag{4.17}$$

Theorem 4.5 . *Let $(\beta, \theta) \in \mathcal{E}$. There exist $\gamma_0 = \gamma_0(\beta,\theta) > 0$, $d_0 = d_0(\beta,\theta) > 0$, and $0 < \zeta_0(\beta,\theta) < 1$ such that if $0 < \gamma \le \gamma_0$, $\delta^* > \gamma$, $\gamma/\delta^* \le d_0$, and p is a positive integer such that*

$$(1+p)\delta^* \log\frac{1}{\gamma} \le \frac{1}{12} \tag{4.18}$$

there exists $\Omega_{RE} = \Omega_{RE}(\gamma, \delta^*, p)$ with $\mathbb{P}[\Omega_{RE}] \ge 1 - \gamma^2$, such that for all δ, ζ_1, ζ_4 with $1 > \delta > \delta^* > 0$, $\zeta_0(\beta,\theta) > \zeta_4 > \zeta_1 > 8\gamma/\delta^*$, and

$$\delta\zeta_1^3 > \frac{128(1+\theta)}{\kappa(\beta,\theta)}(\delta^* \vee \sqrt{\frac{\gamma}{\delta^*}}), \tag{4.19}$$

$$\delta\zeta_4^3 > \frac{32}{\kappa(\beta,\theta)}\zeta_1, \tag{4.20}$$

where $\kappa(\beta,\theta) > 0$ satisfies (9.25), on Ω_{RE} we have

$$\mu_{\beta,\theta,\gamma}\left(\cup_{I \subset [-\gamma^{-p}, \gamma^{-p}]}\left(\mathcal{O}_0^{\delta,\zeta_1}(I, R_1) \cup \mathcal{W}^{\zeta_1,\zeta_4}(I, L_2) \cup \mathcal{R}_0^{\delta,\zeta_1,\zeta_4}(I)\right)\right) \le \frac{3^4}{\gamma^{5p}} e^{-\frac{\beta}{\gamma}\left[\left(\frac{\kappa(\beta,\theta)}{4}\delta\zeta_4^3\right) \wedge \left(\mathcal{F}^*\right)\right]}, \tag{4.21}$$

with $\mathcal{F}^*$ given in (7.5),

$$R_1 = \frac{4(5+\mathcal{F}^*)}{\kappa(\beta,\theta)\delta\zeta_1^3}, \tag{4.22}$$

and

$$L_2 = \frac{\mathcal{F}^*}{64(1+\theta)} \frac{1}{\delta^* \vee \sqrt{\frac{\gamma}{\delta^*}}}. \tag{4.23}$$

The proof of Theorem 4.5 is the same as the proof of Corollary 5.2, Corollary 5.4, and Corollary 5.6 in [13], with $\Delta\mathcal{F}$ in [13] is equal to $2\mathcal{F}^*$ here. Moreover with a little work, one can make explicit the constants depending on β, θ that appear in [13]. Note that the condition $(\beta,\theta) \in \mathcal{E}$ is weaker than the condition used in [13], however this will make no difference at all since we just use the rough estimate, see Lemma 3.3 to treat the random field.

Let $\mathcal{B}_0([-\gamma^{-p},\gamma^{-p}],R_1,L_2)\equiv\cap_{I\subset[-\gamma^{-p},\gamma^{-p}]}\big(\mathcal{O}_0^{\delta,\zeta_1}(I,R_1)\cup\mathcal{W}^{\zeta_1,\zeta_4}(I,L_2)\cup\mathcal{R}_0^{\delta,\zeta_1,\zeta_4}(I)\big)^c$.On this set we can only have runs of $\eta^{\delta,\zeta_1}=0$, with length at most R_1 and runs of $\eta^{\delta,\zeta_4}(\ell)=\eta\in\{-1,+1\}$, with length at least L_2. The next step is to prove that the length of the previous runs of $\eta^{\delta,\zeta_4}=\eta\in\{-1,+1\}$ is indeed bounded from below by ϵ/γ.

Definition 4.6 . *For $\eta\in\{+1,-1\}$, $\ell_1<\tilde\ell_1<\tilde\ell_2<\ell_2$ with $3\le\tilde\ell_1-\ell_1\le R_1$ $3\le\ell_2-\tilde\ell_2\le R_1$, let*

$$\widetilde{\mathcal{W}}_\eta^{\zeta_1,\zeta_4}(\ell_1,\tilde\ell_1,\tilde\ell_2,\ell_2)\equiv\big\{m^{\delta^*}_{[\ell_1,\ell_2]}\colon\eta^{\delta,\zeta_1}(\ell_1)=\eta^{\delta,\zeta_1}(\ell_1+1)=\eta^{\delta,\zeta_1}(\ell_2-1)=\eta^{\delta,\zeta_1}(\ell_2)=\eta,\ \eta^{\delta,\zeta_4}(\ell)=-\eta,\ \forall\ell\in[\tilde\ell_1-1,\tilde\ell_2+1]\big\}. \tag{4.24}$$

Proposition 4.7 . *Let $(\beta,\theta)\in\mathcal{E}$. We take $\kappa(\beta,\theta)>0$ as in (9.25), $\mathcal{F}^*>0$ as in (7.5), $V(\beta,\theta)$ as in (4.56), and $c(\beta)$ as in (3.55). There exist $\gamma_0=\gamma_0(\beta,\theta)>0$, $d_0=d_0(\beta,\theta)>0$, and $0<\zeta_0=\zeta_0(\beta,\theta)<1$ such that if $0<\gamma\le\gamma_0$, $\delta^*>\gamma$, $\gamma/\delta^*\le d_0$, and $0<\zeta_1<\zeta_4<\zeta_0$, $1>\delta>\delta^*>0$ verify the following conditions*

$$\delta\zeta_1^3\ge\frac{128(1+\theta)(5+\mathcal{F}^*)}{\kappa(\beta,\theta)\mathcal{F}^*}\sqrt{\frac{\gamma}{\delta^*}} \tag{4.25}$$

$$\zeta_1\ge\left(5184(1+c(\beta\theta))^2\sqrt{\frac{\gamma}{\delta^*}}\right)\vee\left(12\frac{e^3\beta}{c(\beta,\theta)}\frac{(\delta^*)^2}{\gamma}\right)^2 \tag{4.26}$$

for a constant $c(\beta,\theta)$ given in (4.65), if Δ_Q is an interval containing the origin, of length Q/γ in macroscopic units, with

$$\sqrt{\gamma}\log Q\le\frac{1}{12}\sqrt{\frac{12e^3\beta}{c(\beta,\theta)}}, \tag{4.27}$$

and $\epsilon>\gamma\delta^*$, then there exists $\Omega_4=\Omega_4(\beta,\theta,\gamma,\zeta,\delta,\Delta_Q,\epsilon)$ with

$$I\!\!P[\Omega_4]\ge1-3\gamma^2-\frac{2Q}{\epsilon}e^{-\frac{(\mathcal{F}^*)^2}{\epsilon2^{11}\zeta_4c^2(\beta,\theta)}}-\frac{4Q}{\epsilon}e^{-\frac{(\mathcal{F}^*)^2}{\epsilon2^{10}V^2(\beta,\theta)}} \tag{4.28}$$

such that on Ω_4, we have, for $\eta=\pm1$

$$\mu_{\beta,\theta,\gamma}\left(\cup_{\substack{[\ell_1,\ell_2]\subseteq\Delta_Q\\|\ell_2-\ell_1|\le\epsilon\gamma^{-1}}}\cup^{**}_{[\tilde\ell_1,\tilde\ell_2]\subseteq[\ell_1,\ell_2]}\widetilde{\mathcal{W}}_\eta^{\zeta_1,\zeta_4}(\ell_1,\tilde\ell_1,\tilde\ell_2,\ell_2)\right)\le\frac{R_1^2Q}{\gamma^3}e^{-\frac{\beta}{\gamma}\mathcal{F}^*}. \tag{4.29}$$

In (4.29), the union $\cup^{**}$ refers to the extra constraints $2\le\tilde\ell_1-\ell_1\le R_1$, $\ell_2-\tilde\ell_2\le R_1$, with R_1 given by (4.22).

Remark.

- The constraint (4.27) is present since we use the rough estimate, Lemma 3.3, to control some terms. Note that taking $p=1+[\log Q/\log(1/\gamma)]$, (4.26) and (4.27) imply $12(1+p)\delta^*\log(1/\gamma)\le1$, when γ_0 is small enough which is the condition (3.27) for the rough estimate. We will see that $\Omega_4\subseteq\Omega_{\rm RE}$.
- The constraint $\ell_2-\ell_1\le\epsilon\gamma^{-1}$ enters into play in (4.28), giving the terms proportional to ϵ^{-1} into the exponential.
- The uniformity with respect to the intervals inside Δ_Q gives the prefactors $\frac{Q}{\epsilon}$ in (4.28) and not $\frac{Q}{\gamma}$, since a maximal inequality is used. The union in (4.29) contains at most $R_1^2\epsilon^2Q\gamma^{-3}$ terms.

Proof: We split it in 4 steps.

Step 1: reduction to finite volume

Recalling (4.24), we define

$$\mathcal{R}(\eta) \equiv \mathcal{R}^{\delta,\zeta_4}(\eta) \equiv \mathcal{R}^{\delta,\zeta_4}(\tilde\ell_1,\tilde\ell_2,\eta) = \left\{ m^{\delta^*}_{[\tilde\ell_1,\tilde\ell_2]} \colon \eta^{\delta,\zeta_4}(\ell) = \eta, \forall \ell \in [\tilde\ell_1,\tilde\ell_2] \right\}, \tag{4.30}$$

and

$$\mathcal{W}^{\zeta_1,\zeta_4}_\eta(\ell_1+1,\tilde\ell_1,\tilde\ell_2,\ell_2-1) \equiv \left\{ \eta^{\delta,\zeta_1}(\ell_1+1) = \eta^{\delta,\zeta_1}(\ell_2-1) = \eta \right\} \cap \mathcal{R}(-\eta). \tag{4.31}$$

We can write

$$\widetilde{\mathcal{W}}^{\zeta_1,\zeta_4}_\eta(\ell_1,\tilde\ell_1,\tilde\ell_2,\ell_2) = \left\{ \eta^{\delta,\zeta_1}(\ell_1) = \eta^{\delta,\zeta_1}(\ell_2) = \eta \right\} \cap \mathcal{W}^{\zeta_1,\zeta_4}_\eta(\ell_1+1,\tilde\ell_1,\tilde\ell_2,\ell_2-1). \tag{4.32}$$

Let us first consider a volume Λ such that $\gamma\Lambda \supset \Delta_Q$. Recalling (2.3) and (2.4), multiplying and dividing by $Z^{\sigma_{\gamma^{-1}\partial[\ell_1+1,\ell_2-1]}}_{\beta,\theta,\gamma,\gamma^{-1}[\ell_1+1,\ell_2-1]}$ we have

$$\begin{aligned} &\mu_{\beta,\theta,\gamma,\Lambda}\left(\widetilde{\mathcal{W}}^{\zeta_1,\zeta_4}_\eta(\ell_1,\tilde\ell_1,\tilde\ell_2,\ell_2) \right) = \\ &\frac{1}{Z_{\beta,\theta,\gamma,\Lambda}} \sum_{\sigma_{\Lambda\setminus\gamma^{-1}[\ell_1+1,\ell_2-1]}} e^{-\beta H(\sigma_{\Lambda\setminus\gamma^{-1}[\ell_1+1,\ell_2-1]})} \mathbb{1}_{\{\eta^{\delta,\zeta_1}(\ell_1)=\eta^{\delta,\zeta_1}(\ell_2)=\eta\}} Z^{\sigma_{\gamma^{-1}\partial[\ell_1+1,\ell_2-1]}}_{\beta,\theta,\gamma,\gamma^{-1}[\ell_1+1,\ell_2-1]} \\ &\sum_{\sigma_{[\ell_1+1,\ell_2-1]}} \mathbb{1}_{\mathcal{W}^{\zeta_1,\zeta_4}_\eta(\ell_1+1,\tilde\ell_1,\tilde\ell_2,\ell_2-1)} \frac{e^{-\beta H_\gamma(\sigma_{\gamma^{-1}[\ell_1+1,\ell_2-1]}) - \beta W_\gamma(\sigma_{\gamma^{-1}[\ell_1+1,\ell_2-1]},\sigma_{\gamma^{-1}\partial[\ell_1+1,\ell_2-1]})}}{Z^{\sigma_{\gamma^{-1}\partial[\ell_1+1,\ell_2-1]}}_{\beta,\theta,\gamma,\gamma^{-1}[\ell_1+1,\ell_2-1]}}. \end{aligned} \tag{4.33}$$

Since $\eta^{\delta,\zeta_1}(\ell_1) = \eta^{\delta,\zeta_1}(\ell_1+1) = \eta^{\delta,\zeta_1}(\ell_2-1) = \eta^{\delta,\zeta_1}(\ell_2) = \eta$, using (3.5) and recalling (3.6) , we get

$$\begin{aligned} &\sum_{\sigma_{[\ell_1+1,\ell_2-1]}} \mathbb{1}_{\mathcal{W}^{\zeta_1,\zeta_4}_\eta(\ell_1+1,\tilde\ell_1,\tilde\ell_2,\ell_2-1)} \frac{e^{-\beta H_\gamma(\sigma_{\gamma^{-1}[\ell_1+1,\ell_2-1]}) - \beta W_\gamma(\sigma_{\gamma^{-1}[\ell_1+1,\ell_2-1]},\sigma_{\gamma^{-1}\partial[\ell_1+1,\ell_2-1]})}}{Z^{\sigma_{\gamma^{-1}\partial[\ell_1+1,\ell_2-1]}}_{\beta,\theta,\gamma,\gamma^{-1}[\ell_1+1,\ell_2-1]}} \\ &\quad \le e^{+\frac{\beta}{\gamma}4(\delta^*+\zeta_1)} \mu_{\beta,\theta,\gamma}\left(\mathbb{1}_{\mathcal{W}^{\zeta_1,\zeta_4}_\eta(\ell_1+1,\tilde\ell_1,\tilde\ell_2,\ell_2-1)} \middle| \Sigma^{\delta^*}_{\partial[\ell_1+1,\ell_2-1]} \right) (m^{\delta^*}_{\partial[\ell_1+1,\ell_2-1]} = m_\eta), \end{aligned} \tag{4.34}$$

where m_+ (m_-) is the constant function on $\partial^+ I$ or $\partial^- I$ with value $m^{\delta^*}_\beta$ (resp. $Tm^{\delta^*}_\beta$).

Notice that for any Λ such that $\gamma\Lambda \supset \Delta_Q$

$$\begin{aligned} &\frac{1}{Z_{\beta,\theta,\gamma,\Lambda}} \sum_{\sigma_{\Lambda\setminus\gamma^{-1}[\ell_1+1,\ell_2-1]}} e^{-\beta H(\sigma_{\Lambda\setminus\gamma^{-1}[\ell_1+1,\ell_2-1]})} \mathbb{1}_{\{\eta^{\delta,\zeta_1}(\ell_1)=\eta^{\delta,\zeta_1}(\ell_2)=\eta\}} Z^{\sigma_{\gamma^{-1}\partial[\ell_1+1,\ell_2-1]}}_{\beta,\theta,\gamma,\gamma^{-1}[\ell_1+1,\ell_2-1]} \\ &\quad \le \mu_{\beta,\gamma,\theta,\Lambda}(\mathbb{1}_{\{\eta^{\delta,\zeta_1}(\ell_1)=\eta^{\delta,\zeta_1}(\ell_2)=\eta\}}) \le 1. \end{aligned} \tag{4.35}$$

Therefore, inserting (4.34) in (4.33) and taking the limit $\Lambda \uparrow \mathbb{Z}$ we get

$$\begin{aligned} &\mu_{\beta,\theta,\gamma}\left(\tilde{\mathcal{W}}^{\zeta_1,\zeta_4}_\eta(\ell_1,\tilde\ell_1,\tilde\ell_2,\ell_2) \right) \\ &\quad \le e^{\frac{\beta}{\gamma}4(\zeta_1+\delta^*)} \mu_{\beta,\theta,\gamma}\left(\mathcal{W}^{\zeta_1,\zeta_4}_\eta(\ell_1+1,\tilde\ell_1,\tilde\ell_2,\ell_2-1) \mid \Sigma^{\delta^*}_{\partial[\ell_1+1,\ell_2-1]} \right) (m^{\delta^*}_{\partial[\ell_1+1,\ell_2-1]} = m_\eta). \end{aligned} \tag{4.36}$$

To continue, recalling (3.7) and writing $m^{\delta^*}_{\partial I} = (m^{\delta^*}_{\partial^- I}, m^{\delta^*}_{\partial^+ I})$, we set simply

$$Z_{\beta,\theta,\gamma,I}\left(m^{\delta^*}_{\partial^- I} = m_{s_1}, m^{\delta^*}_{\partial^+ I} = m_{s_2} \right) \equiv Z^{m_{s_1},m_{s_2}}_I \tag{4.37}$$

when $(m_{s_1}, m_{s_2}) \in \{m_-, 0, m_+\}^2$, where m_+ and m_- are as above, and for $m_{s_1} = 0$, we set in (3.7) $E(m_I^{\delta^*}, m_{\partial-I}^{\delta^*}) = 0$ while for $m_{s_2} = 0$ we set $E(m_I^{\delta^*}, m_{\partial+I}^{\delta^*}) = 0$. In a similar way, recalling (3.14), if F is $\Sigma_I^{\delta^*}$–measurable we set

$$\frac{Z_I^{m_{s_1},m_{s_2}}(F)}{Z_I^{m_{s_1},m_{s_2}}} \equiv \frac{\sum_{m_I^{\delta^*}\in\mathcal{M}_{\delta^*}(I)} F(m_I^{\delta^*}) e^{-\frac{\beta}{\gamma}\left\{\widehat{\mathcal{F}}(m_I^{\delta^*}|m_{\partial-I}^{\delta^*}=m_{s_1} m_{\partial+I}^{\delta^*}=m_{s_2})+\gamma\mathcal{G}(m_I^{\delta^*})+\gamma V(m_I^{\delta^*})\right\}}}{Z_I^{m_{s_1},m_{s_2}}}. \tag{4.38}$$

Using the fact that $\eta^{\delta,\zeta}(\tilde\ell_1) = \eta^{\delta,\zeta}(\tilde\ell_1 - 1)$ and $\eta^{\delta,\zeta}(\tilde\ell_2 + 1) = \eta^{\delta,\zeta}(\tilde\ell_2)$ we can decouple the contribution coming from the interval $[\tilde\ell_1 - 1, \tilde\ell_2 + 1]$ and restrict the configuration in the denominator in a suitable way to get

$$\begin{aligned} &\mu_{\beta,\theta,\gamma}\left(\mathcal{W}_\eta^{\zeta_1,\zeta_4}(\ell_1+1,\tilde\ell_1,\tilde\ell_2,\ell_2-1) \;\middle|\; \Sigma^{\delta^*}_{\partial[\ell_1+1,\ell_2-1]}\right)(m^{\delta^*}_{\partial[\ell_1+1,\ell_2-1]} = m_\eta) \\ &\quad\le e^{\frac{\beta}{\gamma}8\zeta_1} \frac{Z^{m_\eta,m_{-\eta}}_{[\ell_1+1,\tilde\ell_1-1]}(\eta^{\delta,\zeta_1}(\ell_1+1)=\eta)}{Z^{m_\eta,m_\eta}_{[\ell_1+1,\tilde\ell_1-1]}(\eta^{\delta,\zeta_1}(\ell_1+1)=\eta)} \frac{Z^{0,0}_{[\tilde\ell_1,\tilde\ell_2]}(\mathcal{R}(-\eta))}{Z^{0,0}_{[\tilde\ell_1,\tilde\ell_2]}(\mathcal{R}(\eta))} \frac{Z^{m_{-\eta},m_\eta}_{[\tilde\ell_2+1,\ell_2-1]}(\eta^{\delta,\zeta_1}(\ell_2-1)=\eta)}{Z^{m_\eta,m_\eta}_{[\tilde\ell_2+1,\ell_2-1]}(\eta^{\delta,\zeta_1}(\ell_2-1)=\eta)}. \end{aligned} \tag{4.39}$$

The first and the third ratio on the right hand side of (4.39) are easily estimated. Since $0 < \ell_1 - \tilde\ell_1 \le R_1$, $0 < \ell_2 - \tilde\ell_2 \le R_1$ with R_1 given by (4.22), using the rough estimate Lemma 3.3 and (3.30), it can be checked that on Ω_{RE}, uniformly over all intervals $[\ell_1, \tilde\ell_1] \subseteq [-\gamma^{-p}, \gamma^{-p}]$, we have

$$\frac{Z^{m_\eta,m_{-\eta}}_{[\ell_1+1,\tilde\ell_1-1]}(\eta^{\delta,\zeta_1}(\ell_1+1)=\eta)}{Z^{m_\eta,m_\eta}_{[\ell_1+1,\tilde\ell_1-1]}(\eta^{\delta,\zeta_1}(\ell_1+1)=\eta)} \le e^{\frac{\beta}{\gamma}(8(1+\theta)R_1\sqrt{\frac{\gamma}{\delta^*}})} \frac{e^{-\frac{\beta}{\gamma}\inf_{m^{\delta^*}_{[\ell_1+1,\tilde\ell_1-1]}\in\{\eta^{\delta,\zeta_1}(\ell_1+1)=\eta\}}\tilde{\mathcal{F}}(m^{\delta^*}_{[\ell_1+1,\tilde\ell_1-1]}|m_\eta,m_{-\eta})}}{e^{-\frac{\beta}{\gamma}\tilde{\mathcal{F}}(T^{\frac{1-\eta}{2}}m^{\delta^*}_{\beta,[\ell_1+1,\tilde\ell_1-1]}|m_\eta,m_\eta)}}, \tag{4.40}$$

where $\tilde{\mathcal{F}}(\cdot)$ is given in (3.17) and we have used the fact that since $\tilde m_\beta^{\delta^*} = -\widetilde{Tm_\beta^{\delta^*}}$ the boundary terms, see (3.18),

$$\frac{\delta^*}{2} \sum_{y\in\mathcal{C}_{\delta^*}(\partial[\ell_1+1,\tilde\ell_1-1])} \left[\tilde m^{\delta^*}(y)\right]^2 \sum_{x\in\mathcal{C}_{\delta^*}([\ell_1+1,\tilde\ell_1-1])} J_{\delta^*}(x-y) \tag{4.41}$$

cancel between the numerator and the denominator in (4.40).

Using the same arguments as in the proof of Lemma 7.3, see after (7.31), taking $d = 2$, it can be proved that

$$\begin{aligned} &\inf_{1\le\tilde\ell_1-\ell_1\le R_1} \inf_{m^{\delta^*}_{[\ell_1+1,\tilde\ell_1-1]}\in\{\eta^{\delta,\zeta_1}(\ell_1+1)=\eta\}} \tilde{\mathcal{F}}(m^{\delta^*}_{[\ell_1+1,\tilde\ell_1-1]}|m_\eta,m_{-\eta}) - \tilde{\mathcal{F}}(T^{\frac{1-\eta}{2}}m^{\delta^*}_{\beta,[\ell_1+1,\tilde\ell_1-1]}|m_\eta,m_\eta) \\ &\quad\ge \mathcal{F}^* - (4L_0 + 2R_1)(1+\theta)\left(\delta^* \vee \sqrt{\frac{\gamma}{\delta^*}}\right), \end{aligned} \tag{4.42}$$

where $\mathcal{F}^*$ is defined in (7.5) and $L_0 = \frac{2}{\alpha(\beta,\theta)}\log\frac{\delta^*}{\gamma}$ with $\alpha(\beta,\theta)$ as in (7.4). A similar argument can be used for the third ratio in (4.39), and we get

$$\frac{Z^{m_\eta,m_{-\eta}}_{[\ell_1+1,\tilde\ell_1-1]}(\eta^{\delta,\zeta_1}(\ell_1+1)=\eta)}{Z^{m_\eta,m_\eta}_{[\ell_1+1,\tilde\ell_1-1]}(\eta^{\delta,\zeta_1}(\ell_1+1)=\eta)} \frac{Z^{m_{-\eta},m^\eta}_{[\tilde\ell_2+1,\ell_2-1]}(\eta^{\delta,\zeta_1}(\ell_2-1)=\eta)}{Z^{m_\eta,m_\eta}_{[\tilde\ell_2+1,\ell_2-1]}(\eta^{\delta,\zeta_1}(\ell_2-1)=\eta)} \le e^{-\frac{\beta}{\gamma}(2\mathcal{F}^*-32(1+\theta)(R_1+L_0)\sqrt{\frac{\gamma}{\delta^*}})}. \tag{4.43}$$

It remains to treat the second ratio in (4.39), that is

$$\frac{Z^{0,0}_{[\tilde\ell_1,\tilde\ell_2]}(\mathcal{R}(-\eta))}{Z^{0,0}_{[\tilde\ell_1,\tilde\ell_2]}(\mathcal{R}(\eta))} \equiv \frac{\sum_{m^{\delta^*}_{\tilde I_{12}}\in\mathcal{M}_{\delta^*}(\tilde I_{12})} \mathbb{1}_{\{\mathcal{R}(-\eta)\}} e^{-\frac{\beta}{\gamma}\left\{\widehat{\mathcal{F}}(m^{\delta^*}_{\tilde I_{12}}|0)+\gamma\mathcal{G}(m^{\delta^*}_{\tilde I_{12}})+\gamma V(m^{\delta^*}_{\tilde I_{12}})\right\}}}{\sum_{m^{\delta^*}_{\tilde I_{12}}\in\mathcal{M}_{\delta^*}(\tilde I_{12})} \mathbb{1}_{\{\mathcal{R}(\eta)\}} e^{-\frac{\beta}{\gamma}\left\{\widehat{\mathcal{F}}(m^{\delta^*}_{\tilde I_{12}}|0)+\gamma\mathcal{G}(m^{\delta^*}_{\tilde I_{12}})+\gamma V(m^{\delta^*}_{\tilde I_{12}})\right\}}}, \tag{4.44}$$

where $\widehat{\mathcal{F}}_\gamma(m^{\delta^*}_{\tilde I_{12}}|0)$ is as (3.15) for $I=\tilde I_{12}=[\tilde\ell_1,\tilde\ell_2]$ but with the term $E(m^{\delta^*}_I, m^{\delta^*}_{\partial I})\equiv 0$ and, recalling (2.14), we have $T\mathcal{R}(\eta)=\mathcal{R}(-\eta)$ and $\mathbb{1}_{\{\mathcal{R}(-\eta)\}}\equiv\mathbb{1}_{\{\mathcal{R}(-\eta)\}}(m^{\delta^*}_{\tilde I_{12}})$.

Notice that if we flip h_i to $-h_i$, for all i, then $\lambda(x)\to-\lambda(x)$, $B^+(x)\to B^-(x)$ while $D(x)$ does not change. Therefore,

$$\frac{Z^{0,0}_{[\tilde\ell_1,\tilde\ell_2]}(\mathcal{R}(-\eta))}{Z^{0,0}_{[\tilde\ell_1,\tilde\ell_2]}(\mathcal{R}(\eta))}(h) = \frac{Z^{0,0}_{[\tilde\ell_1,\tilde\ell_2]}(\mathcal{R}(\eta))}{Z^{0,0}_{[\tilde\ell_1,\tilde\ell_2]}(\mathcal{R}(-\eta))}(-h), \tag{4.45}$$

which implies that $\log\frac{Z^{0,0}_{[\tilde\ell_1,\tilde\ell_2]}(\mathcal{R}(-\eta))}{Z^{0,0}_{[\tilde\ell_1,\tilde\ell_2]}(\mathcal{R}(\eta))}(h)$ has a symmetric distribution around the origin in particular has mean zero.

Step 2: Extraction of the leading stochastic part.

Let $m^{\delta^*}_\beta$ be one of the points in $\left\{-1,-1+\frac{4\gamma}{\delta^*},\ldots,1-\frac{4\gamma}{\delta^*},1\right\}^2$ which is closest to m_β. Given an interval I we let $m^{\delta^*}_{\beta,I}$ be the function which coincides with $m^{\delta^*}_\beta$ on I and vanishes outside I.

Recalling (3.11), we introduce

$$\Delta^\eta\mathcal{G}(m^{\delta^*}_{\beta,\tilde I_{12}}) \equiv \eta\left[\mathcal{G}(m^{\delta^*}_{\beta,\tilde I_{12}})-\mathcal{G}(Tm^{\delta^*}_{\beta,\tilde I_{12}})\right]. \tag{4.46}$$

Using the fact that the functional $\widehat{\mathcal{F}}$ is left invariant by T, we write

$$\frac{Z^{0,0}_{[\tilde\ell_1,\tilde\ell_2]}(\mathcal{R}(-\eta))}{Z^{0,0}_{[\tilde\ell_1,\tilde\ell_2]}(\mathcal{R}(\eta))}(h) \equiv e^{\beta\Delta^\eta\mathcal{G}(m^{\delta^*}_{\beta,\tilde I_{12}})}\frac{Z_{-\eta,0,\delta,\zeta_4}(\tilde I_{12})}{Z_{\eta,0,\delta,\zeta_4}(\tilde I_{12})} \tag{4.47}$$

where

$$\frac{Z_{-\eta,0,\delta,\zeta_4}(\tilde I_{12})}{Z_{\eta,0,\delta,\zeta_4}(\tilde I_{12})} \equiv \frac{\sum_{m^{\delta^*}_{\tilde I_{12}}\in\mathcal{M}^{\delta^*}(\tilde I_{12})} \mathbb{1}_{\{\mathcal{R}^{\delta,\zeta_4}(\eta)\}} e^{-\frac{\beta}{\gamma}\left\{\widehat{\mathcal{F}}(m^{\delta^*}_{\tilde I_{12}},0)+\gamma\Delta_0^{-\eta}\mathcal{G}(m^{\delta^*}_{\tilde I_{12}})+\gamma V(Tm^{\delta^*}_{\tilde I_{12}})\right\}}}{\sum_{m^{\delta^*}_{\tilde I_{12}}\in\mathcal{M}_{\delta^*}(\tilde I_{12})} \mathbb{1}_{\{\mathcal{R}^{\delta,\zeta_4}(\eta)\}} e^{-\frac{\beta}{\gamma}\left\{\widehat{\mathcal{F}}(m^{\delta^*}_{\tilde I_{12}},0)+\gamma\Delta_0^{\eta}\mathcal{G}(m^{\delta^*}_{\tilde I_{12}})+\gamma V(m^{\delta^*}_{\tilde I_{12}})\right\}}} \tag{4.48}$$

with

$$\Delta_0^\eta\mathcal{G}(m^{\delta^*}_{\tilde I_{12}}) \equiv \sum_{x\in\mathcal{C}_{\delta^*}(\tilde I_{12})}\Delta_0^\eta\mathcal{G}^h_{x,m^{\delta^*}(x)} \tag{4.49}$$

and, recalling (3.12),

$$\Delta_0^\eta\mathcal{G}^h_{x,m^{\delta^*}(x)} = \mathcal{G}_{x,T^{\frac{1-\eta}{2}}m^{\delta^*}(x)}(\lambda(x)) - \mathcal{G}_{x,T^{\frac{1-\eta}{2}}m^{\delta^*}_\beta(x)}(\lambda(x)) \tag{4.50}$$

with T^0 equal to the identity.

Note that by definition, $|m^{\delta^*}_\beta-m_\beta|\le 8\gamma/\delta^*$ and taking d_0 small enough (4.26) implies $|m^{\delta^*}_\beta-m_\beta|\le 8\gamma/\delta^*\le\zeta_1$. Thus, the block spin configuration constantly equal to $m^{\delta^*}_\beta$ (resp. $Tm^{\delta^*}_\beta$) is in $\mathcal{R}^{\delta,\zeta_4}(+1)$, (resp $\mathcal{R}^{\delta,\zeta_4}(-1)$).

Step 3: Control of the remaining stochastic part.

To estimate the last term in (4.47), we use Lemma 3.2. A control of the Lipschitz norm is needed. Since it is rather involved to do it, we postpone the proof of the next Lemma to the end of the section.

Lemma 4.8 . *Given $(\beta,\theta)\in\mathcal{E}$, there exist $\gamma_0=\gamma_0(\beta,\theta)>0$, $d_0=d_0(\beta,\theta)>0$, and $\zeta_0=\zeta_0(\beta,\theta)$ such that for all $0<\gamma\le\gamma_0$, for all $\delta^*>\gamma$ with $\gamma/\delta^*\le d_0$, for all $0<\zeta_4<\zeta_0$ that satisfy the following condition*

$$\zeta_4 \ge \left(5184(1+c(\beta\theta))^2(\frac{\gamma}{\delta^*})^{1/2}\right)\vee\left(12\frac{e^3\beta}{c(\beta,\theta)}\frac{(\delta^*)^2}{\gamma}\right)^2 \tag{4.51}$$

where $c(\beta\theta)$ is given in (3.55) and $c(\beta,\theta)$ is given in (4.65), then for all $a>0$,

$$I\!P\left[\max_{I\subseteq\Delta_Q}{}^*\max_{\tilde I_{12}\subseteq I}\left|\log\frac{Z_{-\eta,0,\delta,\zeta_4}(\tilde I_{12})}{Z_{\eta,0,\delta,\zeta_4}(\tilde I_{12})}\right|\ge\beta\frac{4a+12\zeta_4}{\gamma}\right]\le\frac{2Q}{\epsilon}\frac{e^{-\frac{u}{\epsilon}}}{1-e^{-\frac{u}{\epsilon}}} \tag{4.52}$$

where $\max_{I\subseteq\Delta_Q}{}^*$ denote the maximum over the intervals $I\subseteq\Delta_Q$ such that $|I|=\epsilon\gamma^{-1}$ and $u\equiv\frac{a^2\beta^2}{8\zeta_4c^2(\beta,\theta)}$.

Step 4 Control of the leading stochastic part.

To estimate the first term in the right hand side of (4.47), we denote $\Delta^\eta\mathcal{G}(m^{\delta^*}_{\beta,\tilde I_{12}})=-\eta\sum_{x\in\mathcal{C}_{\delta^*}(\tilde I_{12})}X(x)$. where using Proposition 3.5, on the set $\{p(x)\le(2\gamma/\delta^*)^{1/4}\}$, $X(x)$ is defined by

$$X(x)=-\lambda(x)|D(x)|\left[\log\frac{1+m^{\delta^*}_{\beta,2}\tanh(2\beta\theta)}{1-m^{\delta^*}_{\beta,1}\tanh(2\beta\theta)}+\Xi_1(x,\beta\theta,p(x))\right]-\lambda(x)\Xi_2(x,\beta\theta,p(x)) \tag{4.53}$$

where Ξ_1 and Ξ_2 are easily obtained from (3.39). Furthermore, choosing $g_0(n)=n^{1/4}$ in Proposition 3.5, it follows that

$$|\Xi_1(x,\beta\theta,p(x))|\le 64\frac{\beta\theta(1+\beta\theta)}{(1-m_{\beta,1})^2(1-\tanh(2\beta\theta))}(2\frac{\gamma}{\delta^*})^{1/4}. \tag{4.54}$$

and

$$|\Xi_2(x,\beta\theta,p(x)|\le(2\frac{\gamma}{\delta^*})^{1/4}\left[36+2c(\beta\theta)\right] \tag{4.55}$$

where $c(\beta\theta)$ is given in (3.55).

Thus, calling

$$V(\beta,\theta)=\log\frac{1+m_{\beta,2}\tanh(2\beta\theta)}{1-m_{\beta,1}\tanh(2\beta\theta)}, \tag{4.56}$$

on the event $\{p(x)\le(2\gamma/\delta^*)^{\frac14}\}$, when $\frac{\gamma}{\delta^*}\downarrow 0$ the leading term in (4.53) is simply

$$-\lambda(x)|D(x)|V(\beta,\theta)=-\frac{V(\beta,\theta)}{2}\sum_{i\in A(x)}h_i, \tag{4.57}$$

and, from (4.53), we have

$$\begin{aligned} I\!E[X(x)1\!\!1_{\{p(x)\le(2\gamma/\delta^*)^{1/4}\}}]&=0,\\ I\!E[X^2(x)1\!\!1_{\{p(x)\le(2\gamma/\delta^*)^{1/4}\}}]&=\frac{\delta^*}{\gamma}c(\beta,\theta,\gamma/\delta^*)\end{aligned} \tag{4.58}$$

where, if $\gamma/\delta^*<d_0(\beta,\theta)$ for suitable $0<d_0(\beta,\theta)$, $c(\beta,\theta,\gamma/\delta^*)$ satisfies:

$$\frac{V_-^2}{4}\equiv\frac{V^2(\beta,\theta)}{4}\left[1-(\gamma/\delta^*)^{1/5}\right]^2\le c(\beta,\theta,\gamma/\delta^*)\le\frac{V^2(\beta,\theta)}{4}\left[1+(\gamma/\delta^*)^{1/5}\right]^2\equiv\frac{V_+^2}{4}. \tag{4.59}$$

Using Lemma 5.4, exponential Markov inequality, and the Levy inequality we get

$$I\!P\Big[\max^*_{I\subseteq \mathcal{J}} \max_{\tilde I_{12}\subseteq I} \big|\gamma \sum_{x\in \mathcal{C}_{\delta^*}(I)} X(x)\big| \geq 2s\Big] \leq \frac{4Q}{\epsilon} e^{-\frac{2s^2}{\epsilon V_+^2}}. \tag{4.60}$$

Then we collect (4.52), (4.60) and make the choice $a = \mathcal{F}^*/16$, $s = \mathcal{F}^*/32$. Using the hypothesis (4.25) and the definition (4.22), choosing d_0 small enough, we get $32(1+\theta)(R_1+L_0)\sqrt{\gamma/\delta^*} + 4\delta^* \leq \mathcal{F}^*/2$. Taking ζ_0 small enough to have $28\zeta_4 \leq \mathcal{F}^*/8$, we get

$$\mu_{\beta,\theta,\gamma}\big(\widetilde{\mathcal{W}}_\eta^{\zeta_1,\zeta_4}(\ell_1,\tilde\ell_1,\tilde\ell_2,\ell_2)\big) \leq e^{-\frac{\beta}{\gamma}(2\mathcal{F}^* - 32(1+\theta)(R_1+L_0)\sqrt{\frac{\gamma}{\delta^*}} - 4(\zeta_1+\delta^*) - 24\zeta - 4a - 4s)} \leq e^{-\frac{\beta}{\gamma}\mathcal{F}^*} \tag{4.61}$$

with $I\!P$–probability at least

$$1 - 3\gamma^2 - \frac{2Q}{\epsilon}\frac{e^{-\frac{u}{\epsilon}}}{1-e^{-\frac{u}{\epsilon}}} - \frac{4Q}{\epsilon} e^{-\frac{(\mathcal{F}^*)^2}{2^9 \epsilon V_+^2}}, \tag{4.62}$$

where

$$u \equiv \frac{(\mathcal{F}^*)^2}{2^{11}\zeta_4 c^2(\beta,\theta)}. \tag{4.63}$$

The unions in (4.29) involves at most $R_1^2\epsilon^2 Q\gamma^{-3}$ terms. This ends the proof of Proposition 4.7. ■

Proof of Theorem 4.3: It is an immediate consequence of Theorem 4.5 that will allow to restrict ourself to $\mathcal{B}_0([-\gamma^{-p},\gamma^{-p}],R_1,L_2)$ (see above definition 4.6) where the length of the runs of ± 1 are at least $L_2 > 2$, and then Proposition 4.7, assuming ζ_0 small enough to have $u \geq (\mathcal{F}^*)^2/(2^6 V_+^2)$. ■

Lemma 3.2 is the basic ingredient to prove Lemma 4.8. An estimate of Lipschitz norms is given in the next lemma. Then an Ottaviani type inequality will be used to take care of the max in (4.52). We state Lemma 4.9 for a general ζ since it will be used in Section 5 with a ζ different from ζ_4.

Lemma 4.9 . *Let $(\beta,\theta)\in\mathcal{E}$. We take $c(\beta)$ as in (3.55). There exist $\gamma_0=\gamma_0(\beta,\theta)>0$, $d_0(\beta,\theta)>0$, and $\zeta_0(\beta,\theta)$ such that for all $0<\gamma\leq\gamma_0$, for all $\delta^*>\gamma$ with $\gamma/\delta^*<d_0$, and for all $0<\zeta\leq\zeta_0$, that satisfy*

$$\zeta > \Big(5184(1+c(\beta\theta))^2(\frac{\gamma}{\delta^*})^{1/2}\Big) \vee \left(\frac{12e^3\beta}{c(\beta,\theta)}\frac{(\delta^*)^2}{\gamma}\right)^2 \tag{4.64}$$

where $c(\beta\theta)$ is defined in (3.55) and

$$c(\beta,\theta) = 257\left(\frac{1}{(1-\tanh(2\beta\theta))^2} + \frac{1}{1-m_{\beta,1}}\right) + e^{4\beta\theta}\frac{1+\tanh(2\beta\theta)}{1-\tanh(2\beta\theta)} e^{257\left(\frac{1}{(1-\tanh(2\beta\theta))^2}+\frac{1}{1-m_{\beta,1}}\right)} \tag{4.65}$$

then

$$\left\|\partial_i \log\frac{Z_{+,0,\delta,\zeta}(\tilde I_{12})}{Z_{-,0,\delta,\zeta}(\tilde I_{12})}\right\|_\infty \leq \sqrt{\zeta}c(\beta,\theta) + 12e^3\beta\frac{(\delta^*)^2}{\gamma} \leq 2\sqrt{\zeta}c(\beta,\theta), \tag{4.66}$$

where $\frac{Z_{+,0,\delta,\zeta}(\tilde I_{12})}{Z_{-,0,\delta,\zeta}(\tilde I_{12})}$ is defined as in (4.48) with ζ_4 replaced by ζ.

The proof of Lemma 4.9 is done similarly to the corresponding estimates in Section 4 of [13]. The main differences is that the explicit form of $\Delta_0^\eta\mathcal{G}$ in (4.48) is not the same, and we use the cluster expansion method to estimate the Lipschitz factors coming from $V(m^{\delta^*}_{\tilde I_{12}})$. Since we did not see a simple way to modify the proof given in [13] we prefer to start from the very beginning of the computations .

Given $i \in \gamma^{-1}\tilde I_{12}$, let $x(i) = [\gamma i/\delta^*]$ be the index of the block of length δ^* that contains γi, and let $u(i) = [x(i)\delta^*/\delta]$ be the index of the block of length δ that contains $x(i)$.

Let us denote

$$\mathcal{C}_{\delta/\delta^*}(u(i)) \equiv \mathcal{C}_{\delta/\delta^*}(i) \equiv \left\{ x \in \mathbb{Z}, \left[\frac{x(i)\delta^*}{\delta}\right]\frac{\delta}{\delta^*} < x \le \left[\frac{x(i)\delta^*}{\delta}\right]\frac{\delta}{\delta^*} + \frac{\delta}{\delta^*} \right\} \tag{4.67}$$

i.e., the set of indices of those blocks of length δ^* that are inside the block of length δ indexed by $u(i)$.

Given a sample of h, let us denote $h^{(i)}$ the configuration $h^{(i)}_j = h_j$ for $j \neq i$, $h^{(i)}_i = -h_i$. To simplify the notations, we do not write explicitly the δ, ζ dependence of $Z_{\pm,0,\delta,\zeta}$ and we write the Lipschitz factors as

$$\partial_i \log \frac{Z_{+,0,\delta,\zeta}}{Z_{-,0,\delta,\zeta}} = \log \frac{Z_{+,0}(\tilde I_{12})(h)}{Z_{+,0}(\tilde I_{12})(h^{(i)})} - \log \frac{Z_{-,0}(\tilde I_{12})(h)}{Z_{-,0}(\tilde I_{12})(h^{(i)})}. \tag{4.68}$$

To continue we need a simple observation: if $\sum_{x\in\mathcal{C}_{\delta/\delta^*}(i)} \|m^{\delta^*}(x) - m_\beta\|_1 \le \frac{\delta}{\delta^*}\zeta$, then, given $g_1(\zeta)$ decreasing such that $\lim_{\zeta\downarrow 0} g_1(\zeta) = 0$ but $\frac{\zeta}{g_1(\zeta)} < 1$, and if $\zeta \le 1$, we have

$$\sum_{x\in\mathcal{C}_{\delta/\delta^*}(i)} 1\!\!\mathrm{I}_{\{\|m^{\delta^*}(x)-m_\beta\|_1 \le g_1(\zeta)\}} \ge \frac{\delta}{\delta^*}(1 - \frac{\zeta}{g_1(\zeta)}). \tag{4.69}$$

This suggests to make a partition of $\mathcal{C}_{\delta/\delta^*}(i)$ into two sets,

$$\mathcal{K}(m^{\delta^*}) \equiv \left\{ x \in \mathcal{C}_{\delta/\delta^*}(i) : \|m^{\delta^*}(x) - m_\beta\|_1 \le g_1(\zeta) \right\}. \tag{4.70}$$

and $\mathcal{B}(m^{\delta^*}) = \mathcal{C}_{\delta/\delta^*}(i) \setminus \mathcal{K}(m^{\delta^*})$. Let $\ell(i) = [i\gamma]$, for all $m^{\delta^*} \equiv m^{\delta^*}_{\ell(i)}$ we write

$$1\!\!\mathrm{I}_{\{\eta^{\delta,\zeta}(\ell(i))=1\}}(m^{\delta^*}) = \sum_{X\subset\mathcal{C}_{\delta/\delta^*}(i)} 1\!\!\mathrm{I}_{\{\mathcal{K}=X\}}(m^{\delta^*}) 1\!\!\mathrm{I}_{\{\mathcal{B}=X^c\}}(m^{\delta^*}) 1\!\!\mathrm{I}_{\{\eta^{\delta,\zeta}(\ell(i))=1\}}(m^{\delta^*}) \tag{4.71}$$

where the sum is over all the subsets of $\mathcal{C}_{\delta/\delta^*}(i)$ and $X^c \equiv \mathcal{C}_{\delta/\delta^*}(i)\setminus X$. It follows from (4.69) that $\eta_{\delta,\zeta}(\ell(i)) = 1$ and $|X| \le \frac{\delta}{\delta^*}(1 - \frac{\zeta}{g_1(\zeta)})$ are incompatible. Therefore we can impose that $|X| \ge \frac{\delta}{\delta^*}(1 - \frac{\zeta}{g_1(\zeta)})$ in (4.71). Let

$$\mathcal{N}(\zeta) = \sum_{X\subset\mathcal{C}_{\delta/\delta^*}(i)} 1\!\!\mathrm{I}_{\{|X|\ge\frac{\delta}{\delta^*}(1-\frac{\zeta}{g_1(\zeta)})\}} = \sum_{k=\frac{\delta}{\delta^*}(1-\frac{\zeta}{g_1(\zeta)})}^{\frac{\delta}{\delta^*}} \binom{\frac{\delta}{\delta^*}}{k}, \tag{4.72}$$

and notice that (4.68) is the same as

$$\log \frac{Z_{+,0}(\tilde I_{12})(h)}{\mathcal{N}(\zeta)^{\frac12} Z_{+,0}(\tilde I_{12})(h^{(i)})} - \log \frac{Z_{-,0}(\tilde I_{12})(h)}{\mathcal{N}(\zeta)^{\frac12} Z_{-,0}(\tilde I_{12})(h^{(i)})}. \tag{4.73}$$

The two terms are estimated in the same way. We consider the first one. It is easy to see that, with self–explanatory notation,

$$\frac{Z_{+,0}(\tilde I_{12})(h)}{\mathcal{N}(\zeta)^{\frac12} Z_{+,0}(\tilde I_{12})(h^{(i)})} = \frac{1}{\mathcal{N}(\zeta)^{\frac12}} \mathcal{Q}_+ \left[e^{\frac{\beta}{\gamma}\left(\gamma\Delta^+_0 \mathcal{G}^h_{x(i)} - \gamma\Delta^+_0 \mathcal{G}^{h^{(i)}}_{x(i)}\right)} e^{\frac{\beta}{\gamma}\left(\gamma V(\tilde I_{12},h) - \gamma V(\tilde I_{12},h^{(i)})\right)} \right], \tag{4.74}$$

where $\mathcal{Q}$ is the probability measure

$$\mathcal{Q}_+[\Psi] = \frac{\sum_{m^{\delta^*}(\tilde I_{12})\in\mathcal{M}_{\delta^*}(\tilde I_{12})} \Psi(m^{\delta^*}) 1\!\!1_{\{\mathcal{R}(+)\}} e^{-\frac{\beta}{\gamma}\left\{\widehat{\mathcal{F}}(m^{\delta^*}_{\tilde I_{12}},0)+\gamma\Delta_0^+\mathcal{G}^{h^{(i)}}(m^{\delta^*}_{\tilde I_{12}})+\gamma V(m^{\delta^*}_{\tilde I_{12}},h^{(i)})\right\}}}{\sum_{m^{\delta^*}(\tilde I_{12})\in\mathcal{M}_{\delta^*}(I_{12})} 1\!\!1_{\{\mathcal{R}(+)\}} e^{-\frac{\beta}{\gamma}\left\{\widehat{\mathcal{F}}(m^{\delta^*}_{\tilde I_{12}},0)+\gamma\Delta_0^+\mathcal{G}^{h^{(i)}}(m^{\delta^*}_{\tilde I_{12}})+\gamma V(m^{\delta^*}_{\tilde I_{12}},h^{(i)})\right\}}}. \tag{4.75}$$

Applying Schwartz inequality to (4.74) we obtain

$$\frac{Z_{+,0}(\tilde I_{12})(h)}{\mathcal{N}(\zeta)^{\frac12} Z_{+,0}(\tilde I_{12})(h^{(i)})} \le \left(\frac{1}{\mathcal{N}(\zeta)}\mathcal{Q}_+\left[e^{\frac{\beta}{\gamma}2\left(\gamma\Delta_0^+\mathcal{G}^h_{x(i)}-\gamma\Delta_0^+\mathcal{G}^{h^{(i)}}_{x(i)}\right)}\right]\right)^{\frac12}\left(\mathcal{Q}_+\left[e^{\frac{\beta}{\gamma}2\left(\gamma V(\tilde I_{12},h)-\gamma V(\tilde I_{12},h^{(i)})\right)}\right]\right)^{\frac12}. \tag{4.76}$$

The last term on the right hand side of (4.76), can be immediately estimated through Lemma 3.4, and we obtain

$$\left|\frac12\log\mathcal{Q}_+\left[e^{\frac{\beta}{\gamma}2\left(\gamma V(\tilde I_{12},h)-\gamma V(\tilde I_{12},h^{(i)})\right)}\right]\right| \le 6e^3\beta\frac{(\delta^*)^2}{\gamma}. \tag{4.77}$$

The needed estimates for the first term in the right hand side of (4.76) are summarized in the next Lemma

Lemma 4.10 . *Let ζ and $g_1(\zeta)$ be the quantities defined before (4.69). For all $(\beta,\theta)\in\mathcal{E}$, there exist $\zeta_0(\beta\theta)$ and $d_0(\beta\theta)$ such that for all $0<\zeta\le\zeta_0(\beta\theta)$, for all $\gamma/\delta^*\le d_0(\beta,\theta)$, for all increasing $g_0(n)$ such that $\lim_{n\uparrow\infty} g_0(n)=\infty$ but $g_0(n)/n$ is decreasing with $\lim_{n\uparrow\infty} g_0(n)/n=0$ we have that*

$$\left|\frac12\log\frac{1}{\mathcal{N}(\zeta)}\mathcal{Q}_+\left[e^{\frac{\beta}{\gamma}2\left(\gamma\Delta_0\mathcal{G}^h_{x(i)}-\gamma\Delta_0\mathcal{G}^{h^{(i)}}_{x(i)}\right)}\right]\right| \le f_1(\zeta)+\frac{\zeta}{g_1(\zeta)}e^{|f_2-f_1(\zeta)|}, \tag{4.78}$$

where

$$f_1(\zeta)\le\|h-h^{(i)}\|256 g_1(\zeta)\left(\frac{1}{(1-\tanh(2\beta\theta))^2}+\frac{1}{1-m_{\beta,1}}\right)+\frac{72}{g_0(\delta^*\gamma^{-1}/2)}+\left(\frac{2\gamma}{\delta^*}\right)^{1/4}4c(\beta\theta) \tag{4.79}$$

with $c(\beta\theta)$ given in (3.55) and

$$f_2\equiv f_2(\beta,\theta)\le\|h-h^{(i)}\|\Big(\log\frac{1+\tanh(2\beta\theta)}{1-\tanh(2\beta\theta)}+4\beta\theta\Big). \tag{4.80}$$

Proof: We insert (4.71) within the [.] in the left hand side of (4.78). Then, see (4.56) in [13], it can be checked that if we have an estimate of the form

$$\left|\Delta_0^+\mathcal{G}^h_{x(i)}-\Delta_0^+\mathcal{G}^{h^{(i)}}_{x(i)}\right|\le f_1(\zeta)1\!\!1_{\{x(i)\in\mathcal{K}\}}+f_2 1\!\!1_{\{x(i)\in\mathcal{B}\}}. \tag{4.81}$$

From (4.72) we then get

$$\left|\log\frac{1}{\mathcal{N}(\zeta)}\mathcal{Q}_+\left[e^{\frac{\beta}{\gamma}2\left(\gamma\Delta_0^+\mathcal{G}^h_{x(i)}-\gamma\Delta_0^+\mathcal{G}^{h^{(i)}}_{x(i)}\right)}\right]\right|\le f_1(\zeta)+\frac{\zeta}{g_1(\zeta)}e^{|f_2-f_1(\zeta)|}. \tag{4.82}$$

To get (4.81) with $f_1(\zeta)$ that satisfies (4.79) and f_2 that satisfies (4.80), we recall (3.34) and denote

$$\mathcal{G}_{x,m^{\delta^*}(x)}(\lambda(x))\equiv-\frac{1}{\beta}\log L^{\delta^*}_{x,m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x)}(\lambda(x)2\beta\theta,D(x)), \tag{4.83}$$

so that

$$\beta\left(\Delta_0^+\mathcal{G}^h_{x(i)}-\Delta_0^+\mathcal{G}^{h^{(i)}}_{x(i)}\right)=-\log\frac{L^{\delta^*}_{x(i),m^{\delta^*}_{\frac{3+\lambda(x(i))}{2}}(x(i))}(\lambda(x(i))2\beta\theta,D(x(i)))}{L^{\delta^*}_{x(i),m^{\delta^*}_{\frac{3+\lambda^{(i)}(x(i))}{2}}(x(i))}(\lambda^{(i)}(x(i))2\beta\theta,D^{(i)}(x(i)))}$$
$$+\log\frac{L^{\delta^*}_{x(i),m^{\delta^*}_{\beta,\frac{3+\lambda(x(i))}{2}}(x(i))}(\lambda(x(i))2\beta\theta,D(x(i)))}{L^{\delta^*}_{x(i),m^{\delta^*}_{\beta,\frac{3+\lambda^{(i)}(x(i))}{2}}(x(i))}(\lambda^{(i)}(x(i))2\beta\theta,D^{(i)}(x(i)))},\tag{4.84}$$

where $\lambda^{(i)}(x(i))$ and $D^{(i)}(x(i))$ are the respective images of $\lambda(x(i))$ and $D(x(i))$ by the map $h\to h^{(i)}$.

The first case to consider is when $\lambda^{(i)}(x(i))=-\lambda(x(i))$, in which case $|D(x(i))|=|D^{(i)}(x(i))|=1$ and, using (3.34), it can be checked that

$$\beta\left(\Delta_0^+\mathcal{G}^h_{x(i)}-\Delta_0^+\mathcal{G}^{h^{(i)}}_{x(i)}\right)$$
$$=\log\frac{1+\lambda(x)m^{\delta^*}_{\frac{3+\lambda(x(i))}{2}}(x(i))\tanh(\lambda(x(i))2\beta\theta)}{1+\lambda(x)m^{\delta^*}_{\beta,\frac{3+\lambda(x(i))}{2}}(x(i))\tanh(\lambda(x(i))2\beta\theta)}\frac{1-\lambda(x)m^{\delta^*}_{\beta,\frac{3-\lambda(x(i))}{2}}(x(i))\tanh(-\lambda(x(i))2\beta\theta)}{1-\lambda(x)m^{\delta^*}_{\frac{3-\lambda(x(i))}{2}}(x(i))\tanh(-\lambda(x(i))2\beta\theta)}\tag{4.85}$$

Now if ζ_0 is chosen in such a way that $g_1(\zeta)\le(1-\tanh(2\beta\theta))/2$, noticing that $(\beta,\theta)\in\mathcal{E}$ implies $0<\tanh(2\beta\theta)<1$ when $1<\beta<\infty$, a simple computation gives that $||m^{\delta^*}(x(i))-m^{\delta^*}_\beta||_1\le g_1(\zeta)$ implies

$$\left|\beta(\Delta_0^+\mathcal{G}^h_{x(i)}-\Delta_0^+\mathcal{G}^{h^{(i)}}_{x(i)})\right|\le\frac{4||m^{\delta^*}(x(i))-m^{\delta^*}_\beta||_1}{1-\tanh(2\beta\theta)}\le\frac{4g_1(\zeta)}{1-\tanh(2\beta\theta)}\tag{4.86}$$

while without condition on $||m^{\delta^*}(x(i))-m^{\delta^*}_\beta||_1$ we have

$$\left|\beta(\Delta_0^+\mathcal{G}^h_{x(i)}-\Delta_0^+\mathcal{G}^{h^{(i)}}_{x(i)})\right|\le\log\frac{1+\tanh(2\beta\theta)}{1-\tanh(2\beta\theta)}.\tag{4.87}$$

Therefore (4.79) and (4.80) are satisfied in this particular case.

The other case to study is when $\lambda^{(i)}(x(i))=\lambda(x(i))$ and therefore $\big||D(x(i))|-|D^{(i)}(x(i))|\big|=1$.

If $x(i)\in\mathcal{B}$, recalling (4.82), we do not need a very accurate estimate for the terms in (4.84). Recalling (3.34), it is not difficult to see that each term in term in the right hand side of (4.84) is bounded by $2\beta\theta$, so we get

$$\beta\left|\Delta_0^+\mathcal{G}^h_{x(i)}-\Delta_0^+\mathcal{G}^{h^{(i)}}_{x(i)}\right|\le4\beta\theta.\tag{4.88}$$

Collecting (4.87) and (4.88) we have proven (4.80).

It remains to consider the case where $x(i)\in\mathcal{K}$. Recalling (4.81) and (4.82) this will give us the term $f_1(\zeta)$. Here we want use the explicit form of $\mathcal{G}_{x,m^{\delta^*}}$ given in Proposition 3.5. To check that (3.38) is satisfied, let us first note that since $g_1(x)$ and $g_0(x)/x$ are decreasing, $\lim_{x\downarrow0}g_1(x)=0$ and $\lim_{n\uparrow\infty}g_0(n)/n=0$, if we choose $\zeta_0=\zeta_0(\beta,\theta)$ such that

$$g_1(\zeta_0)+\frac{\zeta_0g_0(4/\zeta_0)}{4}\vee\frac{16(\zeta_0/4)^{1/4}\beta\theta}{1-\tanh(2\beta\theta)}\le1-m_{\beta,1}\tag{4.89}$$

and then we choose d_0 such that $\gamma(\delta^*)^{-1} < d_0$ and (4.64) implies $\zeta > 8\gamma(\delta^*)^{-1}$, we get

$$g_1(\zeta) + \frac{g_0(\delta^*\gamma^{-1}/2)}{\delta^*\gamma^{-1}/2} \vee \frac{16(2\gamma/\delta^*)^{1/4}\beta\theta}{1-\tanh(2\beta\theta)} \leq 1 - m_{\beta,1} \tag{4.90}$$

which implies that on $\mathcal{K}(m^{\delta^*})$ and on the set $\{sup_{x\in\mathcal{C}_{\delta^*}(I)}p(x) \leq (2\gamma/\delta^*)^{1/4}\}$, we have (3.38).

Remark 4.11 . The fact that it is enough to have accurate estimates only in the Gaussian case comes from the previous sentence together with (4.81), (4.82) and (4.88).

To estimate (4.84), we first notice that the contribution to $\beta\left|\Delta_0^+\mathcal{G}^h_{x(i)} - \Delta_0^+\mathcal{G}^{h^{(i)}}_{x(i)}\right|$ coming from the terms that correspond to (3.41) is bounded by

$$\frac{72}{g_0(\delta^*\gamma^{-1}/2)} + \left(\frac{2\gamma}{\delta^*}\right)^{1/4} 4c(\beta\theta) \tag{4.91)}$$

with $c(\beta\theta)$ the positive constant given in (3.55). The terms in (4.84) that come from

$$-|D(x)|\left[\log\cosh(2\beta\theta) + \log\left(1 + \lambda(x)m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x)\tanh(2\beta\theta)\right)\right] \tag{4.92}$$

in (3.39) give a contribution that is bounded by

$$\frac{8g_1(\zeta)}{1-\tanh(2\beta\theta)} \tag{4.93}$$

when $||m^{\delta^*}(x(i)) - m^{\delta^*}_\beta||_1 \leq g_1(\zeta)$. It remains to estimate the contribution to (4.84) of the terms that come from

$$|D|\hat\varphi(m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x), 2\lambda(x)\beta\theta, p(x)) \tag{4.94}$$

in (3.39). Unfortunately the estimate (3.40) is useless and we have to consider the explicit form of $\hat\varphi$, see (3.52). The contribution of $\hat\varphi$ in (4.84) can be bounded by

$$\int_{p(x(i))\wedge p^{(i)}}^{p(x(i))\vee p^{(i)}} \int_{m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x)\wedge m^{\delta^*}_{\frac{3+\lambda(x)}{2},\beta}(x)}^{m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x)\vee m^{\delta^*}_{\frac{3+\lambda(x)}{2},\beta}(x)} \left|\frac{\partial^2\left[p|B|\hat\varphi(m,2\lambda(x)\beta\theta,p)\right]}{\partial m\partial p}\right| dp\,dm. \tag{4.95}$$

It is just a long task to compute the previous partial derivative, using (3.45), (3.47) and (3.55) and to check that the following estimates are valid if ζ is such that $g_1(\zeta) \leq (1-\tanh(2\beta\theta))/2$

$$\begin{aligned} &\frac{\partial\nu_2}{\partial p} \leq \frac{2}{\sigma_m^2}, \quad \frac{\partial\nu_2}{\partial m} = \frac{1}{\sigma_m^2}, \\ &\left|\frac{\partial^2\nu_2}{\partial p\partial m}\right| \leq \frac{4}{\sigma_m^2}, \quad 0 < \frac{1}{\sigma_m^2} - \frac{1}{1-m^2} \leq \frac{pc(\beta\theta)}{\sigma_m^2}. \end{aligned} \tag{4.96}$$

It is clear that unpleasant looking terms like $(1 + m\tanh(\nu_2 - \nu_1))^{-1}$ appear in the computations. Using (3.56), the fact that we can assume that $\zeta_0 = \zeta_0(\beta,\theta)$ is small enough to get that if $\zeta \leq \zeta_0$ then $||m - m_\beta||_1 \leq g_1(\zeta)$ implies $1 - |m| \geq (1 - m_{\beta,1})/2$. Then, assuming $d_0(\beta,\theta)$ to be small enough in order to have that $\gamma/\delta^* \leq d_0(\beta,\theta)$ implies $4\beta\theta(\gamma/\delta^*)^{1/4}/(1 - m_{\beta,1}) \leq 1/2$, we get

$$1 + m\tanh(\nu_2 - \nu_1)) > 1 - \frac{4m_{\beta,1}\beta\theta p(x)}{1 - m_{\beta,1}} > \frac{1}{4} \tag{4.97}$$

for all m and p that occur in the integral in (4.95). So, these terms do not present any problem. We get

$$\left|\frac{\partial^2\left[p|B|\hat{\varphi}(m,2\lambda(x)\beta\theta,p)\right]}{\partial m\partial p}\right| \le |B|256\left(\frac{1}{(1-\tanh(2\beta\theta))^2}+\frac{1}{1-m_{\beta,1}}\right). \tag{4.98}$$

Notice that

$$\int_{p(x(i))\wedge p^{(i)}}^{p(x(i))\vee p^{(i)}}\int_{m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x)\wedge m^{\delta^*}_{\frac{3+\lambda(x)}{2},\beta}(x)}^{m^{\delta^*}_{\frac{3+\lambda(x)}{2}}(x)\vee m^{\delta^*}_{\frac{3+\lambda(x)}{2},\beta}(x)} dp\,dm \le ||m^{\delta^*}-m^{\delta^*}_{\beta}||_1\frac{2}{B}. \tag{4.99}$$

Thus, inserting (4.98) in (4.95), using (4.99) and then collecting (4.91) and (4.93) we get (4.79). ∎

Proof of Lemma 4.9 We recall (4.68), (4.73), and (4.76) and apply Lemma 4.10 and (4.77). The presence of ζ in (4.79) and $\zeta/g_1(\zeta)$ in (4.78) suggests to take $g_1(\zeta)=\sqrt{\zeta}$. The presence of $(g_0(\delta^*\gamma^{-1}/2))^{-1}$ and $(2\gamma/\delta^*)^{1/4}$ in (4.79) suggests to choose $g_0(n)=n^{1/4}$. Thus, calling

$$c_1 \equiv c_1(\beta,\theta) \equiv 256\left(\frac{1}{(1-\tanh(2\beta\theta))^2}+\frac{1}{1-m_{\beta,1}}\right) \tag{4.100}$$

and

$$c_2 \equiv c_2(\beta,\theta) = e^{4\beta\theta}\frac{1+\tanh(2\beta\theta)}{1-\tanh(2\beta\theta)} \tag{4.101}$$

we get that the left hand side of (4.78) is bounded by

$$\sqrt{\zeta}\left(c_1+c_2e^{\sqrt{\zeta}c_1+72(1+c(\beta\theta))(\frac{2\gamma}{\delta^*})^{1/4}}\right)+72(1+c(\beta\theta))(\frac{2\gamma}{\delta^*})^{1/4} \tag{4.102}$$

from which we get the first term on the right hand side of(4.66) with the $c(\beta,\theta)$ given in (4.65). ∎

Proof of Lemma 4.8

Using Lemma 3.2 and Lemma 4.9, we get after a simple computation, for all $a>0$, for all intervals $\tilde{I}_{12}=[\tilde{\ell}_1,\tilde{\ell}_2]$

$$I\!P\left[\left|\log\frac{Z_{-\eta,0,\delta,\zeta_4}(\tilde{I}_{12})}{Z_{\eta,0,\delta,\zeta_4}(\tilde{I}_{12})}\right|\ge\frac{a}{\gamma}\right]\le\exp\left(-\frac{a^2}{8\gamma|\tilde{\ell}_1-\tilde{\ell}_2|\zeta c^2(\beta,\theta)}\right). \tag{4.103}$$

To get (4.52), we need the following modification of the Ottaviani inequality done in [13], see Lemma (5.8) there. Given an interval $\tilde{I}\subseteq I$, calling $Y(\tilde{I})\equiv\log\frac{Z_{-\eta,0,\delta,\zeta}(\tilde{I})}{Z_{\eta,0,\delta,\zeta}(\tilde{I})}$, then for all $a>0$, for all $\zeta>8\gamma(\delta^*)^{-1}$, we have

$$I\!P\left[\max_{\tilde{I}_{1,2}\subseteq I}\left|Y(\tilde{I}_{12})\right|\ge\beta\frac{4a+12\zeta}{\gamma}\right]\le\frac{I\!P\left[|Y(I)|\ge\beta\frac{a}{\gamma}\right]}{\inf_{\tilde{I}_{12}\subseteq I}I\!P\left[\left|Y(\tilde{I}_{12})\right|\le\beta\frac{a}{\gamma}\right]}. \tag{4.104}$$

Then for all $a>0$, setting $\tilde{x}=4a+12\zeta$, we obtain

$$I\!P\left[{\max_{I\subseteq\Delta_Q}}^*\max_{\tilde{I}_{12}\subseteq I}|Y(\tilde{I}_{12})|\ge\beta\frac{\tilde{x}}{\gamma}\right]\le\frac{2Q}{\epsilon}I\!P\left[\max_{\tilde{I}_{12}\subseteq\widehat{I}_{[0,2]}}|Y(\tilde{I}_{12})|\ge\beta\frac{\tilde{x}}{\gamma}\right], \tag{4.105}$$

where $\widehat{I}_{[0,2]}=[0,2\epsilon\gamma^{-1}]$. This implies (4.52) after a short computation. ∎

5 Probabilistic estimates

In this section we construct a random interval $J(\omega)$, to which the interval $I(\omega)$ appearing in Theorem 2.1 is simply related.

Our final aim is to control the behavior of the random field over intervals of (macroscopic) length of order larger or equal to $\frac{1}{\gamma}$. To achieve this, it is convenient to consider blocks of (macroscopic) length ϵ/γ, with the basic assumption that $\epsilon/\gamma > \delta^*$. To avoid rounding problems we assume $\epsilon/\gamma\delta^* \in I\!\!N$ and we define, for $\alpha \in Z\!\!Z$

$$\chi^{(\epsilon)}(\alpha) \equiv \gamma \sum_{x;\delta^* x \in \tilde{A}_{\epsilon/\gamma}(\alpha)} X(x) 1\!\!\mathrm{I}_{\{p(x) \le (2\gamma/\delta^*)^{1/4}\}}, \tag{5.1}$$

where, according to the previous notation $\tilde{A}_{\epsilon/\gamma}(\alpha) = ((\alpha-1)\frac{\epsilon}{\gamma}, \alpha\frac{\epsilon}{\gamma}] \subseteq I\!\!R$ and for sake of simplicity the γ, δ^* dependence is not explicit. To simplify further, and if no confusion arises, we shall write simply $\chi(\alpha)$. Note that $\chi(\alpha)$ is a symmetric random variable and assuming that $I \supset \tilde{A}_{\epsilon/\gamma}(\alpha)$ for all α under consideration

$$\begin{aligned} I\!\!E(\chi(\alpha)) &= 0, \\ I\!\!E(\chi^2(\alpha)) &= \epsilon c(\beta, \theta, \gamma/\delta^*), \end{aligned} \tag{5.2}$$

as it follows from (4.58) since there are $\epsilon(\gamma\delta^*)^{-1}$ terms in the sum in (5.1).

The construction of $J(\omega)$ involves a discrete random walk obtained from the variables $\chi(\alpha), \alpha \in Z\!\!Z$, defined by (5.1) and satisfying (5.2). If Δ is a finite interval in $Z\!\!Z$ we set $\mathcal{Y}(\Delta) = \sum_{\tilde{\alpha}\in\Delta} \chi(\tilde{\alpha})$. For convenience we write

$$\mathcal{Y}_\alpha \equiv \begin{cases} \mathcal{Y}(\{1,\dots,\alpha\}), & \text{if } \alpha \ge 1; \\ 0 & \text{if } \alpha = 0; \\ -\mathcal{Y}(\{\alpha+1,\dots,0\}), & \text{if } \alpha \le -1, \end{cases} \tag{5.3}$$

so that if $\Delta = \{\alpha_1+1, \dots, \alpha_2\} \equiv (\alpha_1, \alpha_2]$, with $\alpha_1 < \alpha_2$ integers, we have $\mathcal{Y}(\Delta) = \mathcal{Y}_{\alpha_2} - \mathcal{Y}_{\alpha_1}$.

As $\gamma \downarrow 0$, we assume $\epsilon \downarrow 0$ but $\epsilon/\gamma\delta^* \uparrow +\infty$. In this regime, $\mathcal{Y}_{[\cdot/\epsilon]}$ converges in law to a bilateral Brownian motion (no drift, diffusion coefficient $V(\beta,\theta)$).

Given a real positive number f, $0 < f < \mathcal{F}^*/4$ where $\mathcal{F}^*$ is defined in (7.5), we denote

$$\mathcal{D}(f,+) \equiv \mathcal{D}(f,+,\omega) \equiv \left\{\Delta\colon \mathcal{Y}(\Delta) \ge 2\mathcal{F}^* + f\,,\ \inf_{\Delta' \subseteq \Delta} \mathcal{Y}(\Delta') \ge -2\mathcal{F}^* + f\right\}, \tag{5.4}$$

the set of random (finite) intervals $\Delta \subseteq Z\!\!Z$ with an (uphill) increment of size at least $2\mathcal{F}^* + f$, and such that no interval within Δ presents a (downhill) increment smaller than $-2\mathcal{F}^* + f$. Such an interval $\Delta \subseteq Z\!\!Z$ is said to give rise to a *positive elongation*, and we set $\mathrm{sgn}\,\Delta = +1$.

Similarly,

$$\mathcal{D}(f,-) \equiv \mathcal{D}(f,-,\omega) \equiv \left\{\Delta\colon \mathcal{Y}(\Delta) \le -2\mathcal{F}^* - f\,,\ \sup_{\Delta' \subseteq \Delta} \mathcal{Y}(\Delta') \le 2\mathcal{F}^* - f\right\}, \tag{5.5}$$

and such an interval is said to give rise to a *negative elongation*. If $\Delta \in \mathcal{D}(f,-)$, we set $\mathrm{sgn}\,\Delta = -1$. We call

$$\mathcal{D}(f,\omega) \equiv \mathcal{D}(f,+,\omega) \cup \mathcal{D}(f,-,\omega). \tag{5.6}$$

Remark: $\mathcal{D}(f,+) \cap \mathcal{D}(f,-) = \emptyset$ since $f > 0$, so that the above definition of $\mathrm{sgn}\Delta$ is well posed. However, we may have intervals $\Delta_1 \in \mathcal{D}(f,+)$ and $\Delta_2 \in \mathcal{D}(f,-)$ such that $\Delta_1 \cap \Delta_2 \neq \emptyset$.

Given $Q > 0$ and writing $A^c = \Omega \setminus A$, we let

$$\mathcal{P}_0(f,Q) = \{\exists \Delta \in \mathcal{D}(f,\omega),\ \Delta \subseteq [-Q/\epsilon, Q/\epsilon]\}^c\,, \tag{5.7}$$

be the set of realizations of the random field that neither give rise to a positive nor to a negative elongation in the interval $[-Q/\epsilon, Q/\epsilon]$. As we will see later, cf. Theorem 5.1, $I\!P[\mathcal{P}_0(f,d)]$ is small provided Q is large, *uniformly* on $0 < f \le \mathcal{F}^*/4$. (The uniformity is trivial since from the definitions $\mathcal{D}(f,\pm) \subseteq \mathcal{D}(\tilde f,\pm)$ if $0 < \tilde f < f$.)

Deciding if a given interval gives rise to a positive or negative elongation is a local procedure, in the sense that it depends only on the values of $\chi(\alpha)$, with α in the considered interval. But, since our goal is to find the beginning and the end of successive runs of $\eta^{\delta,\zeta} = +1$, and runs of $\eta^{\delta,\zeta} = -1$, we should determine contiguous elongations with alternating signs. For this we first need (not necessarily contiguous) elongations with alternating signs. We set, for $k \in I\!N$:

$$\begin{aligned} B_+(f,k,Q) \equiv \big\{\omega \in \Omega\colon \exists\, 0 \le a_1 < b_1 \le a_2 < b_2 \le \ldots \le a_k < b_k \le Q/\epsilon, (a_i,b_i] \in \mathcal{D}(f),\\ i = 1,..,k;\ \mathrm{sgn}(a_i,b_i] = -\mathrm{sgn}(a_{i+1},b_{i+1}],\ i = 1,..,k-1\big\}, \end{aligned} \tag{5.8}$$

$$\begin{aligned} B_-(f,k,Q) \equiv \big\{\omega \in \Omega\colon \exists\, 0 \ge b_1 > a_1 \ge b_2 > a_2 \ge \ldots \ge b_k > a_k \ge -Q/\epsilon, (a_i,b_i] \in \mathcal{D}(f),\\ i = 1,..,k;\ \mathrm{sgn}(a_i,b_i] = -\mathrm{sgn}(a_{i+1},b_{i+1}],\ i = 1,..,k-1\big\}, \end{aligned} \tag{5.9}$$

and $\mathcal{P}_1(f,k,Q) \equiv (B_+(f,k,Q) \cap B_-(f,k,Q))^c \supseteq \mathcal{P}_0(f,Q)$. In Theorem 5.1 we shall prove that $I\!P[\mathcal{P}_1(f, k,Q)]$ is small, *uniformly* in $0 < f \le \mathcal{F}^*/4$, and $k \ge 1$, provided Q is taken large enough.

For reasons that will be clear later we set:

$$\begin{aligned} \mathcal{P}_2'(f,Q) = \{\exists \alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 \in [-Q/\epsilon, Q/\epsilon]\colon |\mathcal{Y}_{\alpha_1} - \mathcal{Y}_{\alpha_3}| \vee |\mathcal{Y}_{\alpha_2} - \mathcal{Y}_{\alpha_4}| \le 3f,\\ ||\mathcal{Y}_{\alpha_1} - \mathcal{Y}_{\alpha_2}| - 2\mathcal{F}^*| \le 3f,\\ \mathcal{Y}_\alpha \in [\mathcal{Y}_{\alpha_1} \wedge \mathcal{Y}_{\alpha_2} - 3f, \mathcal{Y}_{\alpha_1} \vee \mathcal{Y}_{\alpha_2} + 3f], \forall \alpha \in [\alpha_1,\alpha_4]\} \end{aligned}$$

and

$$\mathcal{P}_2''(f,Q) = \mathcal{P}_2'(f,Q) \cup \{\max_{\alpha \in [-Q/\epsilon, Q/\epsilon]} |\chi(\alpha)| > f\}. \tag{5.10}$$

To construct the previously described $J(\omega)$, with $0 \in J(\omega) \subseteq [-Q/\gamma, Q/\gamma]$, it will suffice to have $\omega \in \big(\mathcal{P}_1(f,3,Q) \cup \mathcal{P}_2''(f,Q)\big)^c$. Having fixed Q sufficiently large so that $I\!P(\mathcal{P}_1(f,3,Q))$ is suitably small for any $0 < f \le \mathcal{F}^*/4$, we shall take f small enough and ϵ suitably small so that $I\!P(\mathcal{P}_2''(f,Q))$ is also suitably small, as stated in Theorem 5.1.

Let $\omega \in (\mathcal{P}_1(f,3,Q) \cup \mathcal{P}_2''(f,Q))^c$. Starting at $\alpha = 0$, and going to the right we tag the "first" interval in $Z\!\!Z$ which provides an elongation. We then use an explicit way to construct *contiguous* intervals that provide elongations with alternating signs. $J(\omega)$ will be defined with the help of such elongations. Having a discrete random walk, different types of ambiguities appear in this construction and we need to estimate the probability of their occurrence. We discuss a possible construction.

Let us define for each $a, b \in [-Q/\epsilon, Q/\epsilon] \cap Z\!\!Z$:

$$\begin{aligned} b_-(a) &\equiv \inf\{b' > a\colon (a,b'] \in \mathcal{D}(f,\omega)\},\\ b_+(a) &\equiv \sup\{b' > a\colon (a,b'] \in \mathcal{D}(f,\omega)\},\\ a_+(b) &\equiv \sup\{a' < b\colon (a',b] \in \mathcal{D}(f,\omega)\},\\ a_-(b) &\equiv \inf\{a' < b\colon (a',b] \in \mathcal{D}(f,\omega)\}, \end{aligned} \tag{5.11}$$

with the infima and suprema taken on $[-Q/\epsilon, Q/\epsilon] \cap \mathbb{Z}$; thus, if the corresponding set is non-empty we have a minimum or maximum; otherwise we make the usual convention: $\inf \emptyset = +\infty$ and $\sup \emptyset = -\infty$.

We see at once:

- if $b_-(a) < +\infty$ then $a_-(b_-(a)) \le a \le a_+(b_-(a))$;
- if $a_+(b) > -\infty$ then $b_-(a_+(b)) \le b \le b_+(a_+(b))$.

Let us set $a_0 \equiv \inf\{a \ge 0\colon b_-(a) < +\infty\}$. Since $\omega \in B_+(f, 3, Q) \subseteq B_+(f, 1, Q)$, we have $0 \le a_0 < b_-(a_0) \equiv b_0 \le Q/\epsilon$, and $(a_0, b_0]$ is an elongation. Also, $(a_-(b_0), b_0] \supseteq (a_0, b_0]$ is an elongation with the same sign. To fix ideas we assume $+1 = \mathrm{sgn}(a_0, b_0]$. This will serve as starting point for the construction. We now set, for $b < b_0$:

$$\begin{aligned}\tilde{a}_+(b) &= \sup\{a < b\colon (a, b] \in \mathcal{D}(f, -)\},\\ b_{-1} &= \sup\{b < b_0\colon \tilde{a}_+(b) > -\infty\}, \qquad \text{and} \qquad a_{-1} = \tilde{a}_+(b_{-1}).\end{aligned} \tag{5.12}$$

Since $\omega \in B_-(f, 3, Q) \subseteq B_-(f, 2, Q)$ we have $-Q/\epsilon \le a_{-1} < b_{-1}$, and from the construction, we easily check $a_{-1} < 0$. Observe that in (5.12) we need to consider $b < b_0$ (instead of $b \le a_0$) due to the possibility of non-empty overlap among elongations with different signs. We make the following:

Claim 1. If $\omega \in (\mathcal{P}_1(f, 3, Q) \cup \mathcal{P}_2''(f, Q))^c$ we have $b_{-1} \ge a_-(b_0)$.

Proof of Claim. We prove it by contradiction. For that, we suppose that $b_{-1} < a_-(b_0)$, and consider two cases:

(I) $\mathcal{Y}_\alpha \le \mathcal{Y}_{a_-(b_0)}$ for some $\alpha \in [-Q/\epsilon, a_-(b_0))$;

(II) $\mathcal{Y}_\alpha > \mathcal{Y}_{a_-(b_0)}$ for all $\alpha \in [-Q/\epsilon, a_-(b_0))$.

In case (I), letting $\alpha_0 = \max\{\alpha < a_-(b_0)\colon \mathcal{Y}_\alpha \le \mathcal{Y}_{a_-(b_0)}\}$, we take: α_3 any point of (global) minimum of $\mathcal{Y}_\cdot$ in $[a_-(b_0), b_0]$; $\alpha_4 = \min\{\alpha \in [\alpha_3, b_0]\colon \mathcal{Y}_\alpha - \mathcal{Y}_{\alpha_3} \ge 2\mathcal{F}^* + f\}$, which exists since $\mathrm{sgn}(a_-(b_0), b_0] = +1$; $\alpha_2 = \max\{\alpha \in [\alpha_0, \alpha_-(b_0)]\colon \mathcal{Y}_{\alpha_3} - \mathcal{Y}_\alpha < -2\mathcal{F}^* + f\}$, which exists in this case, otherwise $(\alpha_0, b_0]$ would be a positive elongation, contradicting the definition of $a_-(b_0)$.

We see that starting from α_2 and moving backwards in time, the process $\mathcal{Y}$ must take a value below $\mathcal{Y}_{\alpha_2} - 2\mathcal{F}^* + 3f$ before it reaches a value above $\mathcal{Y}_{\alpha_2} + 2f$ (otherwise $b_{-1} \ge a_-(b_0)$)); taking α_1 as the "first" (backwards) such time, we are in the situation described in $\mathcal{P}_2'(f, Q)$, contradicting our assumption on ω.

In case (II), let α_4 be any point of minimum of $\mathcal{Y}(\cdot)$ in $[a_-(b_0), b_0]$. Due to the assumption that $\omega \in B_-(f, 3, Q)$, there exists a positive elongation contained in $[-Q/\epsilon, a_-(b_0)]$. Together with the assumption in (II) this allows to define $\alpha_1 = \max\{\alpha < a_-(b_0)\colon \mathcal{Y}_\alpha \ge \mathcal{Y}_{\alpha_4} + 2\mathcal{F}^* + f\}$, and $-Q/\epsilon \le \alpha_1 < a_-(b_0)$. Taking $\alpha_3 = \sup\{\alpha < \alpha_4\colon \mathcal{Y}_\alpha - \mathcal{Y}_{\alpha_4} \ge 2\mathcal{F}^* - f\}$ which exists otherwise $[\alpha_1, \alpha_4]$ would be a negative elongation contradicting $b_{-1} < a_-(b_0)$. Moreover $\alpha_3 \ge \alpha_1$. We see that starting from α_3 and moving "backwards" in time, $\mathcal{Y}_\cdot$ has to make a downwards increment of at least $2\mathcal{F}^* - 3f$ "before" α_1 [otherwise $b_{-1} \ge a_-(b_0)$]. and we get α_2 as the "first" such time, we are in the situation described in $\mathcal{P}_2'(f, Q)$, contradicting our assumption on ω.

Having assumed that $\omega \in (\mathcal{P}_1(f, 3, Q) \cup \mathcal{P}_2''(f, Q))^c$ in this construction, the previous claim tells us that $b_{-1} \ge a_-(b_0)$. For $\mathrm{sgn}(a_0, b_0] = +1$ we define

$$\alpha_0^* = \min\{\alpha \in [a_-(b_0), b_{-1}]\colon \quad \mathcal{Y}_\alpha = \min_{a_-(b_0) \le \tilde{\alpha} \le b_{-1}} \mathcal{Y}(\tilde{\alpha})\}. \tag{5.13}$$

In this situation $(a_{-1}, \alpha_0^*]$ and $(\alpha_0^*, b_0]$ are contiguous elongations, with alternating signs (-1 and $+1$ resp.). The same holds for $(a_-(\alpha_0^*), \alpha_0^*] \supseteq (a_{-1}, \alpha_0^*]$ and $(\alpha_0^*, b_+(\alpha_0^*)] \supseteq (\alpha_0^*, b_0]$.

Remark. Though not needed, one can check that $\mathcal{Y}_{\alpha_0^*} = \min_{a_{-1} \le \alpha \le b_0} \mathcal{Y}_\alpha$.

With $\omega \in (\mathcal{P}_1(f,3,Q) \cup \mathcal{P}_2''(f,Q))^c$ we may proceed one step to the right, where the next "breaking point" will be a maximum in a suitable interval. We first set, for $a > \alpha_0^*$:

$$\begin{aligned} \tilde{b}_-(a) &= \inf\{b > a : (a,b] \in \mathcal{D}(f,-)\} \\ a_1 &= \inf\{a > \alpha_0^* : \tilde{b}_-(a) < +\infty\}, \quad \text{and} \quad b_1 = \tilde{b}_-(a_1) \end{aligned} \tag{5.14}$$

and since $\omega \in B_+(f,3,Q) \subseteq B_+(f,2,Q)$ we have $0 < a_1 < b_1 \le Q/\epsilon$. Moreover, as before we have:
Claim 2. For $\omega \in (\mathcal{P}_1(f,3,Q) \cup \mathcal{P}_2''(f,Q))^c$ we must have $a_1 \le b_+(\alpha_0^*)$.

Claim 2 is proven in the same way as the previous one, and we omit details. It allows to define, for such ω:

$$\alpha_1^* = \min\{\alpha \in [a_1, b_+(\alpha_0^*)] : \mathcal{Y}_\alpha = \max_{a_1 \le \tilde{\alpha} \le b_+(\alpha_0^*)} \mathcal{Y}_{\tilde{\alpha}}\} \tag{5.15}$$

so that $(\alpha_0^*, \alpha_1^*]$, and $(\alpha_1^*, b_1]$ are contiguous elongations with alternating signs ($+1$ and -1 resp.). Also $\text{sgn}(\alpha_1^*, b_+(a_1)] = \text{sgn}(\alpha_1^*, b_1]$, and, similarly to previous observation, we see that $\mathcal{Y}_{\alpha_1^*} = \max_{a_0 \le \alpha \le b_1} \mathcal{Y}_\alpha$.

If $\alpha_0^* < 0$ we set $J(\omega) = (\frac{\epsilon \alpha_0^*}{\gamma}, \frac{\epsilon \alpha_1^*}{\gamma})$. If instead, $\alpha_0^* \ge 0$, in order to determine $J(\omega)$ we need to extend the construction one more step to the left. In this case, we may consider for any $b < \alpha_0^*$:

$$\begin{aligned} \tilde{a}_+(b) &= \sup\{a < b : (a,b] \in \mathcal{D}(f,+)\}, \\ b_{-2} &= \sup\{b < \alpha_0^* : \tilde{a}_+(b) > -\infty\}, \quad \text{and} \quad a_{-2} = \tilde{a}_+(b_{-2}). \end{aligned} \tag{5.16}$$

Since $\alpha_0^* \ge 0$, $\text{sgn}(a_-(\alpha_0^*), \alpha_0^*] = -1$, and $\omega \in B_-(f,3,Q) \subseteq B_-(f,2,Q)$ we have $-Q \le b_{-2} \le \alpha_0^*$ and $-Q \le a_{-2}$. Moreover, from the construction $a_{-2} < a_-(\alpha_0^*) \le a_{-1}$. As before, we can prove the following:
Claim 3. For $\omega \in (\mathcal{P}_1(f,3,Q) \cup \mathcal{P}_2''(f,Q))^c$ we must have $b_{-2} \ge a_-(\alpha_0^*)$.

The proof of Claim 3 is omitted, since it follows the same argument of Claim 1, under the previous assumptions. Having $b_{-2} \ge a_-(\alpha_0^*)$ we may split the intervals through

$$\alpha_{-1}^* = \inf\{\alpha \in [a_-(\alpha_0^*), b_{-2}] : \mathcal{Y}_\alpha = \max_{a_-(\alpha_0^*) \le \tilde{\alpha} \le b_{-2}} \mathcal{Y}_{\tilde{\alpha}}\}, \tag{5.17}$$

so that $(a_{-2}, \alpha_{-1}^*]$ and $(\alpha_{-1}^*, \alpha_0^*]$ are elongations with alternating signs. As in the previous steps, we see that $b_{-2} < a_-(\alpha_0^*)$ is not possible if $\omega \notin \mathcal{P}_2'(f,Q)$. Moreover, from the construction it follows that $\alpha_{-1}^* < 0$, otherwise it would contradict the definition of a_0 and $\text{sgn}(a_0, b_0] = +1$. Thus, for $\alpha_0^* \ge 0$ we set $J(\omega) = (\frac{\epsilon \alpha_{-1}^*}{\gamma}, \frac{\epsilon \alpha_0^*}{\gamma})$. Though not used in the sequel, we may again check that, $\mathcal{Y}_{\alpha_{-1}^*} = \max_{a_{-2} \le \alpha \le b_{-1}} \mathcal{Y}_\alpha$.

Under the assumptions on $\omega \in (\mathcal{P}(f,3,Q) \cup \mathcal{P}_2''(f,Q))^c$ we have constructed contiguous elongations $(a_{-2}, \alpha_{-1}^*]$, $(\alpha_{-1}^*, \alpha_0^*]$, $(\alpha_0^*, \alpha_1^*]$, and $(\alpha_1^*, b_1]$, with alternating signs.

Starting from $(a_-(\alpha_{-1}^*), \alpha_{-1}^*]$ and $(\alpha_1^*, b_+(\alpha_1^*)]$, the construction may be continued to the left and right respectively, if $\omega \notin \mathcal{P}_1(f,k,Q) \cup \mathcal{P}_2''(f,Q)$ for larger k. For Theorem 2.2 it suffices to have $\omega \in (\mathcal{P}_1(f, 3(2k+1), Q) \cup \mathcal{P}_2''(f,Q))^c$.

Remark. We have chosen α_0^*, α_1^*, etc... as the first minimizer or maximizer, respectively, since the random walk may have multiple maximizers on the intervals considered there. In fact the random walk can oscillate, being always below or equal to the maximum. Since in the limit $\epsilon \downarrow 0$, the random walk converges in law to a Brownian motion where the local maxima are always distinct, see [29] p. 108, we can expect that for a random walk such a result holds approximately. A way to do it is to accept an error on the location of the beginning or the end of the runs of $\eta^{\delta,\zeta}(\ell)$. For this we need to prove that if α_1 and α_2 are the locations of two local maxima of $\mathcal{Y}(\cdot)$ and the distance between α_1 and α_2 is larger than ρ/ϵ, then $I\!P[|\mathcal{Y}_{\alpha_1} - \mathcal{Y}_{\alpha_2}| \le \tilde{\delta}]$

goes to zero in the limit $\epsilon \downarrow 0$, for a suitable choice of the parameters $\rho = \rho(\epsilon)$, $\tilde\delta = \tilde\delta(\rho,\epsilon) = \tilde\delta(\epsilon)$ both vanishing as $\epsilon \to 0$.

We define, for ρ and $\tilde\delta$ positive,

$$\begin{aligned}\mathcal{P}_2(f,+,Q,a_{-1},b_0,\rho,\tilde\delta) \equiv \{\omega \in (\mathcal{P}_1(f,3,Q) \cup \mathcal{P}_2''(f,Q))^c;\, \exists\, \tilde\alpha \in [a_{-1},b_0],\\ |\tilde\alpha - \alpha_0^*| > \rho/\epsilon, |\mathcal{Y}_{\tilde\alpha} - \mathcal{Y}_{\alpha_0^*}| \le \tilde\delta\},\end{aligned} \tag{5.18}$$

$$\begin{aligned}\mathcal{P}_2(f,+,Q,a_0,b_1,\rho,\tilde\delta) \equiv \{\omega \in (\mathcal{P}_1(f,3,Q) \cup \mathcal{P}_2''(f,Q))^c;\, \exists\, \tilde\alpha \in [a_0,b_1],\\ |\tilde\alpha - \alpha_1^*| > \rho/\epsilon, |\mathcal{Y}_{\tilde\alpha} - \mathcal{Y}_{\alpha_1^*}| \le \tilde\delta\},\end{aligned} \tag{5.19}$$

and

$$\begin{aligned}\mathcal{P}_2(f,+,Q,a_{-2},b_{-1},\rho,\tilde\delta) \equiv \{\omega \in (\mathcal{P}_1(f,3,Q) \cup \mathcal{P}_2''(f,Q))^c;\, \alpha_0^* > 0, \exists\, \tilde\alpha \in [a_{-2},b_{-1}],\\ |\tilde\alpha - \alpha_{-1}^*| > \rho/\epsilon, |\mathcal{Y}_{\tilde\alpha} - \mathcal{Y}_{\alpha_{-1}^*}| \le \tilde\delta\}.\end{aligned} \tag{5.20}$$

We will show that the previous three sets have $I\!P$-probability as small as we want provided we choose the parameters ϵ, ρ, $\tilde\delta$ in a suitable way.

We recall that we have defined the random interval $J(\omega)$ as follows:

$$J(\omega) = \begin{cases} \left(\frac{\epsilon\alpha_0^*}{\gamma}, \frac{\epsilon\alpha_1^*}{\gamma}\right), & \text{if } \alpha_0^* < 0;\\ \left(\frac{\epsilon\alpha_{-1}^*}{\gamma}, \frac{\epsilon\alpha_0^*}{\gamma}\right), & \text{if } \alpha_0^* \ge 0.\end{cases} \tag{5.21}$$

There is some arbitrariness when $\alpha_0^* = 0$, but accepting to make an error ρ/ϵ on the location of the maximizers or minimizers, we will show that the set

$$\mathcal{P}_3(f,Q,\rho) \equiv \left\{\omega \in (\mathcal{P}(f,3,Q) \cup \mathcal{P}_2''(f,Q))^c; \alpha_0^* \,\text{or}\, \alpha_{-1}^* \in [-2\frac{\rho}{\epsilon}, 2\frac{\rho}{\epsilon}]\right\} \tag{5.22}$$

has a very small probability.

Remark. Always assuming $\omega \in (\mathcal{P}(f,3,Q) \cup \mathcal{P}_2''(f,Q))^c$, for $\text{sgn}(a_0,b_0] = -1$, we perform the obvious modifications of the construction.

Recalling that all over this work, $\beta > 1$ and $\theta > 0 \in \mathcal{E}$, the control on the various exceptional sets is summarized in the following:

Theorem 5.1 . *There exist positive constants $Q_0 = Q_0(\beta,\theta)$, $f_0 = f_0(\beta,\theta)$, $\rho_0 = \rho_0(\beta,\theta)$ and $\gamma_0 = \gamma_0(\beta,\theta)$ such that for all $0 < \gamma \le \gamma_0$, $0 < \rho \le \rho_0$, and $0 < f \le f_0$, for all ϵ such that*

$$\delta^*\gamma < \epsilon \le \frac{2}{V^2(\beta,\theta)\log(1944)}\left(\rho^{4+2a} \wedge f^2\right) \tag{5.23}$$

for an arbitrary given $a > 0$, we have the following: For all integers $k > 1, Q \ge Q_0(\beta,\theta)$,

$$I\!P\left[\mathcal{P}_0(f,Q)\right] \le 3e^{-\frac{Q}{2C_1}} + \frac{1}{\log 2}\frac{2f + 9V(\beta,\theta)\sqrt{\epsilon\log\frac{C_1}{\epsilon}}}{2\mathcal{F}^* - f}\log\frac{2\mathcal{F}^* - f}{2f + 2V(\beta,\theta)\sqrt{\epsilon\log\frac{C_1}{\epsilon}}}, \tag{5.24}$$

where $V(\beta,\theta)$ is given by (4.56) and $C_1 = C_1(\beta,\theta)$ is given in (5.46) with $b = 2\mathcal{F}^*$;

$$I\!P\left[\mathcal{P}_1\left(f,k,Q\right)\right] \le (k+5)e^{-\frac{Q}{2kC_1}} + \frac{k}{\log 2}\frac{2f + 9V(\beta,\theta)\sqrt{\epsilon\log\frac{C_1}{\epsilon}}}{2\mathcal{F}^* - f}\log\frac{2\mathcal{F}^* - f}{2f + 2V(\beta,\theta)\sqrt{\epsilon\log\frac{C_1}{\epsilon}}} \tag{5.25}$$

$$\mathbb{P}\left[\mathcal{P}_2''(f,Q)\right] \le 8(2Q+1)^2 \frac{2\sqrt{2\pi}}{V(\beta,\theta)}(9f)^{a/(2+a)} + (2Q+1)\frac{1296}{V(\beta,\theta)} \frac{9f + (2+V(\beta,\theta))\sqrt{\epsilon \log \frac{C_1}{\epsilon}}}{(9f)^{3/(4+2a)}} + \frac{4Q}{\epsilon} e^{-\frac{f}{4\epsilon V^2(\beta,\theta)}}. \tag{5.26}$$

Moreover, for $\tilde\delta(\rho) = \rho^{2+a}$ *we have*

$$\mathbb{P}\left[\cup_{i=-k}^{k} \cup_{s_1\in\{\pm 1\}} \mathcal{P}_2(f,s_1,Q,a_i,b_{i+1},\rho,\tilde\delta(\rho))\right] \le (4k+2)3G_1(\beta,\theta,\tilde\delta(\rho),\epsilon)\log\frac{4}{G_1(\beta,\theta,\tilde\delta(\rho),\epsilon)}, \tag{5.27}$$

where

$$G_1(\beta,\theta,\tilde\delta(\rho),\epsilon) \equiv \frac{2^{16}C_1}{\sqrt{V(\beta,\theta)}}\left(\rho^{a/2} + \frac{\sqrt{1+V(\beta,\theta)}(\epsilon\log\frac{C_1}{\epsilon})^{1/4}}{\rho^{3/4}}\right) \tag{5.28}$$

with C_1 *as in (5.24), and if* $0<\kappa<1/2$

$$\mathbb{P}\left[\mathcal{P}_3(f,Q,\rho)\right] \le 6\rho^{\frac12-\kappa} + \frac{2}{\Gamma(\frac12-\kappa)}\left(\frac{\epsilon}{\rho}\right)^{\frac12-\kappa} + \frac{\epsilon}{\rho^2}\exp\left(8^{\frac{C(\beta,\theta)}{\kappa^2}} 2\log\frac{C(\beta,\theta)}{\kappa^2}\right), \tag{5.29}$$

where $C(\beta,\theta)$ *is a suitable constant that depends on* $V(\beta,\theta)$ *and* $\Gamma(\cdot)$ *is the Euler Gamma function.*

The proof will be given at the end of this section.

Remark: The quantities a_i and b_i are random variables, but none is a stopping time. As $\epsilon\downarrow 0$, and then $\rho\downarrow 0$ (5.27) reduces to the well known fact that with probability one, the Brownian path does not have two equal local maximum (or minimum) over any finite interval (see [29] pg 108).

To simplify the writing of the above estimates, we made the following choice:

$$\rho = \epsilon^{\frac{1}{4(2+a)}}, \quad f = \epsilon^{\frac14}, \quad \kappa = 1/4. \tag{5.30}$$

Then, calling

$$\begin{aligned}\mathcal{P}(k,\epsilon,Q) =& \mathcal{P}_1(f=\epsilon^{\frac14},k,Q)\cup\mathcal{P}_2''(f=\epsilon^{\frac14},Q)\cup\mathcal{P}_3(f=\epsilon^{\frac14},a_{-2},b_{-1},\rho=\epsilon^{\frac{1}{4(2+a)}})\\ &\cup\left(\cup_{i=-k}^{k}\cup_{s_1\in\{\pm1\}}\mathcal{P}_2(f=\epsilon^{\frac14},s_1,Q,a_i,b_{i+1},\rho=\epsilon^{\frac{1}{4(2+a)}},\tilde\delta(\rho)=\epsilon^{\frac14})\right),\end{aligned} \tag{5.31}$$

after simple estimates one gets

Corollary 5.2 . *There exist positive constants* $Q_0=Q_0(\beta,\theta)$, $\gamma_0=\gamma_0(\beta,\theta)$ *and* $\epsilon_0(\beta,\theta)$ *such that for all* $0<\gamma\le\gamma_0$, *for all* ϵ *that satisfies* $\delta^*\gamma<\epsilon\le\epsilon_0$, *for all* $Q>Q_0$, $k>1$ *we have*

$$\mathbb{P}[\mathcal{P}(k,\epsilon,Q)] \le (k+5)e^{-\frac{Q}{2kC_1}} + k\epsilon^{\frac{a}{16(2+a)}} + Q^2\epsilon^{\frac{a}{8+2a}} + Qe^{-\frac{1}{2\epsilon^{3/4}V^2(\beta,\theta)}}. \tag{5.32}$$

where $a>0$ *is a given arbitrary positive number.*

Recalling (5.21), the following Proposition will be used for proving (2.18) and (2.19). It will be proved at the end of this section.

Proposition 5.3 . *For all* $0<x<(\mathcal{F}^*)^2/(V^2(\beta,\theta)18\log 2)$ *we have*

$$\mathbb{P}[\gamma|J|\le x] \le 2e^{-\frac{(\mathcal{F}^*)^2}{18xV^2(\beta,\theta)}} \tag{5.33}$$

while for all $x>0$ *we have*

$$\mathbb{P}[\gamma|J|\ge x] \le 4e^{-\frac{x}{8C_1(\beta,\theta,\mathcal{F}^*)}(1-\frac{\log 3}{\log 4})}, \tag{5.34}$$

where $C_1(\beta,\theta,\mathcal{F}^*)$ is defined in (5.46).
Remark: Note that for $x \ge (\mathcal{F}^*)^2/(V^2(\beta,\theta)18\log 2)$ the right hand side of (5.33) is larger than 1. Therefore (5.33) is trivially satisfied also in this case.

Basic estimates.

Several probabilistic estimates are needed for Theorem 5.1 and are summarized in the following Lemmata and Proposition. The variables $\chi(\alpha)$, $\alpha \in \mathbb{Z}$ defined by (5.1), with $X(x)$ given by (4.53), constitute the basic objects in the following analysis. We recall that we always assume that $\beta > 1, \theta > 0 \in \mathcal{E}$. Recalling (4.59) we set

$$V_-^2 = V^2(\beta,\theta)\Big(1-(\tfrac{\gamma}{\delta^*})^{1/5}\Big)^2 \qquad \text{and} \qquad V_+^2 = V^2(\beta,\theta)\Big(1+(\tfrac{\gamma}{\delta^*})^{1/5}\Big)^2. \tag{5.35}$$

Remark: Throughout this section we shall assume that $0 < \gamma/\delta^* \le d_0(\beta,\theta) \wedge 2^{-5}$ so that $V(\beta,\theta)/2 \le V_- \le \sqrt{c(\beta,\theta,\gamma/\delta^*)} \le V_+ \le 3V(\beta,\theta)/2$ where $V(\beta,\theta)$ is given in (4.56).

We need some further simple estimates concerning the variables $\chi(\alpha)$ that are not difficult to prove just recalling that $\chi(\alpha)$ is a sum over $\epsilon(\gamma\delta^*)^{-1}$ independent symmetric random variables $X(x)$. (5.38) is proved using (5.37).

Lemma 5.4 . *There exists a $d_0(\beta,\theta) > 0$, such that if $\gamma/\delta^* \le d_0(\beta,\theta)$ then*

$$I\!\!E\left[e^{\lambda\chi(\alpha)}\right] \le e^{\frac{\lambda^2}{2}\epsilon V_+^2}, \qquad \forall \lambda \in I\!\!R \tag{5.36}$$

with V_+^2 defined in (5.35). If $0 < \lambda < [\epsilon V_+^2]^{-1}$, we have

$$I\!\!E\left[e^{\frac{\lambda}{2}|\chi(\alpha)|^2}\right] \le \frac{1}{1-\epsilon\lambda V_+^2}. \tag{5.37}$$

For all $k \ge 3$ and $p = 1,2,4$:

$$I\!\!E\left[\max_{\alpha=1,\dots,k} |\chi(\alpha)|^p\right] \le (4\epsilon V_+^2 \log k)^{p/2}\Big(1+\frac{p}{\log k}\Big)^{\frac{p}{2}\vee 1}. \tag{5.38}$$

In order to have an elongation, as previously described, it is necessary to find suitable uphill or downhill increments of height $2\mathcal{F}^* + f$.

A constructive way to locate elongations, though it might miss some of them, is related to the following stopping times:

Given $b > 0$ ($b = \mathcal{F}^* + \frac{f}{2}$ later), we set $\tau_0 = 0$, and define, for $k \ge 1$:

$$\begin{aligned} \tau_k &= \inf\{t > \tau_{k-1} \colon |\sum_{\alpha=\tau_{k-1}+1}^{t} \chi(\alpha)| \ge b\}, \\ \tau_{-k} &= \sup\{t < \tau_{-(k-1)} \colon |\sum_{\alpha=t+1}^{\tau_{-(k-1)}} \chi(\alpha)| \ge b\}. \end{aligned} \tag{5.39}$$

Clearly, the random variables $\Delta\tau_{k+1} := \tau_{k+1} - \tau_k$, $k \in \mathbb{Z}$, are independent and identically distributed. (Recall that $\Delta\tau_1 = \tau_1$ from the definitions.) We define,

$$S_k = \text{sgn}\Big(\sum_{j=\tau_{k-1}+1}^{\tau_k} \chi(j)\Big); \qquad S_{-k} = \text{sgn}\Big(\sum_{j=\tau_{-k}+1}^{\tau_{-k+1}} \chi(j)\Big) \qquad \text{for} \quad k \ge 1. \tag{5.40}$$

We need probabilistic estimates for the variables $\Delta\tau_k$ and τ_k, which are obtained by standard methods. An upper bound on the tail of their distribution can be given as follows:

Lemma 5.5 . *There exists a positive constant $d_0(\beta,\theta)$ such that for all integer v, $\gamma/\delta^* < d_0(\beta,\theta)$ and $0<\epsilon<\epsilon_0(\beta,\theta,b)$ where*

$$\epsilon_0(\beta,\theta,b) := \frac{1}{3^8}\left(I\!P\big[Y \geq \frac{4b}{V(\beta,\theta)}\big]\right)^2, \tag{5.41}$$

we have

$$I\!P\left[\tau_1 \geq \frac{v}{\epsilon}\right] \leq \exp\left(-v I\!P\left[Y \geq \frac{4b}{V(\beta,\theta)}\right]\right), \tag{5.42}$$

where Y is standard Gaussian and $V(\beta,\theta)$ as in (4.56).

Remark: For future use, note that $\epsilon_0(\beta,\theta,b)$ is a decreasing function of b.

Proof: Since the $\chi(\alpha)$ are i.i.d. random variables, for any positive integer v, we have:

$$I\!P\left[\tau_1 \geq \frac{v}{\epsilon}\right] \leq I\!P\left[\max_{k=0,\dots,v-1}\left|\sum_{\alpha=k/\epsilon+1}^{(k+1)/\epsilon}\chi(\alpha)\right| < 2b\right] = \left(I\!P\left[|\mathcal{Y}(1/\epsilon)| \leq 2b\right]\right)^v \tag{5.43}$$

We can use (5.36) to get an estimate of the fourth moment of $\chi(\alpha)$ and apply Berry–Essen Theorem ([17] p. 304) to control the right hand side in (5.43). Consequently, there exists a constant $C_{BE}=C_{BE}(\beta,\theta)$ which, according to Berry-Essen inequality may be taken as

$$C_{BE} = 0.8 \sup_{0<\gamma/\delta^*\leq d_0(\beta,\theta),\epsilon>\delta^*\gamma} I\!E(|\chi(1)|^3)/I\!E(|\chi(1)|^2)^{3/2} \leq 3^4 \tag{5.44}$$

assuming at the last step that $\gamma/\delta^* \leq d_0(\beta,\theta) < (1/2)^5$. Therefore

$$I\!P\left[|\mathcal{Y}(1/\epsilon)| \leq 2b\right]\Big] \leq 1 - 2I\!P\big[Y \geq \frac{2b}{\sqrt{c(\beta,\theta,\gamma/\delta^*)}}\big] + 3^4\sqrt{\epsilon} \leq 1 - I\!P\big[Y \geq \frac{4b}{V(\beta,\theta)}\big], \tag{5.45}$$

where Y is a standard Gaussian, using $0<\epsilon<\epsilon_0(\beta,\theta,b)$ and (5.41) for the last inequality in (5.45). Using $1-x\leq e^{-x}$, we get (5.42) ∎

The following lemma gives bounds for the mean of τ_1 and follows easily from the Wald Identity, see [27], pg 83, and (5.38).

Lemma 5.6 . *If*

$$C_1 = C_1(\beta,\theta,b) = \frac{2}{I\!P[Y > 4b/V(\beta,\theta)]}, \tag{5.46}$$

where Y is standard gaussian and $0<\epsilon<\epsilon_0(\beta,\theta,b)$ cf. (5.41), there exists $d_0(\beta,\theta)$ such that for $\gamma/\delta^* < d_0(\beta,\theta)$ we have

$$\frac{b^2}{\epsilon V^2(\beta,\theta)}(1-(\gamma/\delta^*)^{1/5})^2 \leq I\!E[\tau_1] \leq \frac{b^2}{\epsilon V^2(\beta,\theta)}(1+(\gamma/\delta^*)^{1/5})^2\left(1+9\frac{V(\beta,\theta)}{b}\sqrt{\epsilon\log\frac{C_1}{\epsilon}}\right)^2. \tag{5.47}$$

Remark: For future use, note that $C_1(\beta,\theta,b)$ is increasing with b.

We need exponential estimates for the probability that a Cesàro average over k terms of the previous $\Delta\tau_i$'s is outside an interval that contains the mean $I\!E[\tau_1]$. The result is:

Lemma 5.7 . *For all $0 < s < b^2[4(\log 2)V_+^2]^{-1}$, for all positive integers k we have*

$$I\!P\left[\tau_k \le \frac{ks}{\epsilon}\right] \le e^{-k\frac{b^2}{4sV_+^2}}, \tag{5.48}$$

where V^+ is defined in (5.35). Moreover, for $\epsilon_0 = \epsilon_0(\beta,\theta,b)$ as (5.41), for all $0 < \epsilon < \epsilon_0$, for all positive integers k, and for all $s > 0$ we have

$$I\!P\left[\tau_k \ge \frac{k}{\epsilon}(s + \log 2)C_1\right] \le e^{-sk}, \tag{5.49}$$

where $C_1 = C_1(\beta,\theta,b)$ is given in (5.46).

Proof: (5.48) is an immediate consequence of the Markov exponential inequality together with the exponential Wald identity see [27], pg 81. (5.49) is an immediate consequence of the Markov exponential inequality together with (5.42) to estimate the Laplace transform. ■

As we shall check, the above stopping times with $b = \mathcal{F}^* + \frac{f}{2}$, provide a simple way to catch elongations. It will be enough to find successive indices $k \ge 1$ ($k \le -2$) such that $S_k = S_{k+1}$ and eliminating a set of small probability, see Lemma 5.10, $(\tau_{k-1}, \tau_{k+1}]$ ($(\tau_k, \tau_{k+2}]$ respectively) will provide an elongation which is positive if $S_k = +1$, or negative otherwise. Still, if $S_{-1} = S_1$, then $(\tau_{-1}, \tau_1]$ is an elongation. Not all elongations are of this form, as one simply verifies, but what matters is that this procedure catches enough of them, sufficient to prove Theorem 5.1. The basic ingredient is given in the next two lemmas.

Lemma 5.8 . *Let $\epsilon_0 = \epsilon_0(\beta,\theta,b)$ be given by (5.41). For all $0 < \epsilon < \epsilon_0$, all integer $k \ge 1$, and all $s > 0$ we have*

$$I\!P\left[\tau_k \le \frac{k(s + \log 2)C_1}{\epsilon}; \exists i \in \{1,\ldots,k-1\}, S_i = S_{i+1}\right] \ge \left(1 - e^{-sk}\right)(1 - \frac{1}{2^{k-1}}). \tag{5.50}$$

Proof: It follows at once from the fact, due to the symmetry, that conditionally on $\Delta\tau_i$'s the variables $S_i, i \neq 0$'s form a family of i.i.d. Bernoulli symmetric random variables (see (5.40)), with the trivial observation that for i.i.d. symmetric Bernoulli random variables

$$I\!P\left[\exists i \in \{1,\ldots,k-1\} \colon S_i = S_{i+1}\right] = 1 - \frac{1}{2^{k-1}}. \tag{5.51}$$

Together with (5.49), this entails (5.50). ■

To deal with the case where more than one elongation is involved, we define to the right of the origin

$$\begin{aligned} i_1^* &\equiv \inf\{i \ge 1 : S_i = S_{i+1}\} \\ i_{j+1}^* &\equiv \inf\left\{i \ge (i_j^* + 2) : S_i = S_{i+1} = -S_{i_j^*}\right\} \qquad j \ge 1, \end{aligned} \tag{5.52}$$

and to the left

$$i_{-1}^* \equiv \begin{cases} -1 & \text{if } S_{-1} = S_1 = -S_{i_1^*}, \\ \sup\left\{i \le -2 : S_i = S_{i+1} = -S_{i_1^*}\right\} & \text{if } S_{-1} \neq S_1 \text{ or } S_1 = -S_{i_1^*}, \end{cases}$$

$$i_{-j-1}^* \equiv \sup\left\{i \le i_j^* - 2 : S_i = S_{i+1} = -S_{i_j^*}\right\} \qquad j \ge 1, \tag{5.53}$$

we then have:
Lemma 5.9 . *Let $\epsilon_0 = \epsilon_0(\beta, \theta, b)$ be given by (5.41). For all $0 < \epsilon < \epsilon_0$, all k and L positive integers, L even, (just for simplicity of writing) and all $s > 0$ we have:*

$$I\!P\left[\tau_{kL-1} \le \frac{(kL-1)(s+\log 2)C_1}{\epsilon}, \forall_{1\le j\le k}\ i_j^* < jL\right] \ge \left(1 - e^{-s(kL-1)}\right)\left(1 - \tfrac{1}{2^{L-1}}\right)\left(1 - \left(\tfrac{3}{4}\right)^{L/2}\right)^{k-1} \tag{5.54}$$

and

$$\begin{aligned} &I\!P\left[\tau_{-kL} \ge \frac{-kL(s+\log 2)C_1}{\epsilon},\ \tau_{L-1} \le \frac{(L-1)(s+\log 2)C_1}{\epsilon},\ i_1^* < L,\ \forall_{1\le j\le k}\ i_{-j}^* > -jL\right] \\ &\ge \left(1 - e^{-s(kL-1)}\right)\left(1 - \tfrac{1}{2^{L-1}}\right)\left(1 - \left(\tfrac{3}{4}\right)^{L/2}\right)^{k}. \end{aligned} \tag{5.55}$$

Proof: We prove (5.54); (5.55) is done similarly. We again use that conditionally on $\Delta\tau_i$'s, the variables S_i's are i.i.d. Bernoulli symmetric random variables. Recalling Lemma 5.7, it is then sufficient to prove that

$$I\!P[i_1^* < L, i_2^* < 2L, \dots, i_k^* < kL] \ge \left(1 - \tfrac{1}{2^{L-1}}\right)\left(1 - \left(\tfrac{3}{4}\right)^{L/2}\right)^{k-1}. \tag{5.56}$$

When $k = 1$ this is just (5.51). On the other side, using the above mentioned properties of the random variables S_i we easily see that

$$I\!P[i_{j+1}^* - i_j^* \le L \mid i_1^*, \dots, i_j^*] \ge 1 - \left(\tfrac{3}{4}\right)^{L/2} \qquad \text{a.s.}$$

from where (5.54) follows at once. ■

Next we verify that the above described method provides elongations, with overwhelming probability. Recalling (5.52) let us assume, to fix ideas, that $S_{i_1^*} = S_{i_1^*+1} = 1$. From the definition of τ_i, see (5.39), with $b = \mathcal{F}^* + (f/2)$, we have that

$$\mathcal{Y}((\tau_{\{i_1^*-1\}}, \tau_{\{i_1^*+1\}}]) = \sum_{\alpha=\tau_{\{i_1^*-1\}}+1}^{\tau_{\{i_1^*+1\}}} \chi(\alpha) \ge 2\mathcal{F}^* + f. \tag{5.57}$$

Therefore $(\tau_{\{i_1^*-1\}}, \tau_{\{i_1^*+1\}}]$ automatically satisfies one of the two conditions to give rise to an elongation, cf. (5.4).

Let us see that, except on a set of small probability, the other requirement is fulfilled, i.e.,

$$\inf_{\tau_{\{i_1^*-1\}} < \alpha_1 < \alpha_2 \le \tau_{\{i_1^*+1\}}} \sum_{\alpha=\alpha_1}^{\alpha_2} \chi(\alpha) \ge -2\mathcal{F}^* + f. \tag{5.58}$$

On the event $\{S_i = 1\}$, we readily see that

$$\inf_{\tau_{\{i-1\}}+1 \le \alpha \le \tau_i} \sum_{\bar\alpha=\tau_{\{i-1\}}+1}^{\alpha} \chi(\bar\alpha) \ \ge\ -\mathcal{F}^* - f/2, \text{ and } \inf_{\tau_{\{i-1\}}+1 \le \alpha \le \tau_i} \sum_{\bar\alpha=\alpha}^{\tau_i} \chi(\bar\alpha) \ \ge\ 0. \tag{5.59}$$

Since $\sum_{\alpha=\alpha_1}^{\alpha_2}\chi(\alpha)=\sum_{\alpha=\alpha_1}^{\tau_i}\chi(\alpha)+\sum_{\alpha=\tau_i+1}^{\alpha_2}\chi(\alpha)$, on $\{S_i=S_{i+1}=1\}$ we have

$$\inf_{\tau_{\{i-1\}}+1\le\alpha_1\le\tau_i<\alpha_2\le\tau_{\{i+1\}}}\sum_{\alpha=\alpha_1}^{\alpha_2}\chi(\alpha)\ge\ -\mathcal{F}^*-f/2\ge\ -2\mathcal{F}^*+f. \tag{5.60}$$

In the last inequality we used $f<\mathcal{F}^*/4<2\mathcal{F}^*/3$. Therefore, it remains to evaluate $I\!\!P[\mathcal{J}(i_1^*)\cup\mathcal{J}(i_1^*+1),S_{i_1^*}=1]$, where

$$\mathcal{J}(i):=\left\{\inf_{\tau_{\{i-1\}}+1\le\alpha_1<\alpha_2\le\tau_{\{i\}}}\sum_{\alpha=\alpha_1}^{\alpha_2}\chi(\alpha)<-2\mathcal{F}^*+f\right\}. \tag{5.61}$$

Note that on $\{S_i=1\}$, we have $\inf_{\tau_{\{i-1\}}+1\le\alpha_1<\alpha_2\le\tau_i}\sum_{\tilde\alpha=\alpha_1}^{\alpha_2}\chi(\tilde\alpha)\ge-2\mathcal{F}^*-f$, where we used (5.59) and $\sup_{\tau_{\{i-1\}}+1\le\alpha_1\le\tau_i}\sum_{\tilde\alpha=\tau_{i-1}+1}^{\alpha_1-1}\chi(\tilde\alpha)\le\mathcal{F}^*+\frac{f}{2}$. As a consequence, for any integer i:

$$\{\mathcal{J}(i),S_i=1\}\subseteq\left\{-2\mathcal{F}^*-f\le\inf_{\tau_{\{i-1\}}+1\le\alpha_1<\alpha_2\le\tau_i}\sum_{\tilde\alpha=\alpha_1}^{\alpha_2}\chi(\tilde\alpha)\le-2\mathcal{F}^*+f\right\}.$$

An analogous inequality (with a sup instead of an inf) holds in the case $S_{i_1^*}=-1$. Therefore we need to prove the following:

Lemma 5.10 . *Let $\epsilon_0=\epsilon_0(\beta,\theta,2\mathcal{F}^*)$ be given by (5.41) and $C_1=C_1(\beta,\theta,2\mathcal{F}^*)$ be given by(5.46). For all $0<f<\mathcal{F}^*/4$ and for all $0<\epsilon<\epsilon_0$ we have*

$$\begin{aligned}I\!\!P\left[\cup_{j=i_1^*,i_1^*+1}\left\{2\mathcal{F}^*-f<\sup_{\tau_{j-1}<\alpha_1<\alpha_2\le\tau_j}\left|\sum_{\tilde\alpha=\alpha_1}^{\alpha_2}\chi(\tilde\alpha)\right|<2\mathcal{F}^*+f\right\}\right]\\ \le\frac{2G(\beta,\theta,\epsilon,f)}{\log 2}\log\frac{1}{G(\beta,\theta,\epsilon,f)},\end{aligned} \tag{5.62}$$

where

$$G(\beta,\theta,\epsilon,f)\equiv\frac{2f+9V(\beta,\theta)\sqrt{\epsilon\log\frac{C_1}{\epsilon}}}{2\mathcal{F}^*-f}. \tag{5.63}$$

Remark: Clearly i_1^* is anticipating, and $\tau_{i_1^*-1}$ and $\tau_{i_1^*}$ are not stopping times.
Proof: Since $I\!\!P[i_1^*=i,S_{i_1^*}=1]=2^{-i+1}$, we have

$$I\!\!P\big[\mathcal{J}(i_1^*),S_{i_1^*}=1\big]\le\sum_{i=1}^{i_0}I\!\!P\big[\mathcal{J}(i),S_i=1\big]+2^{-i_0}, \tag{5.64}$$

where i_0 will be suitably chosen. To treat the sum, we define the stopping times

$$T_{\mathcal{F}^*-\frac{3f}{2}}=\inf\big\{\alpha>\tau_{\{i-1\}};\sum_{\tilde\alpha=\tau_{i-1}+1}^{\alpha}\chi(\tilde\alpha)\ge\mathcal{F}^*-\tfrac{3f}{2}\big\}, \tag{5.65}$$

$$T_{\mathcal{F}^*+\frac{f}{2}}=\inf\big\{\alpha>\tau_{\{i-1\}};\sum_{\tilde\alpha=\tau_{\{i-1\}}+1}^{\alpha}\chi(\tilde\alpha)\ge\mathcal{F}^*+\tfrac{f}{2}\big\}, \tag{5.66}$$

$$T^-_{\mathcal{F}^*-\frac{3f}{2}}=\inf\big\{\alpha>T_{\mathcal{F}^*-\frac{3f}{2}};\sum_{\tilde\alpha=\tau_{\{i-1\}}+1}^{\alpha}\chi(\tilde\alpha)\le-\mathcal{F}^*+\tfrac{3f}{2}\big\}. \tag{5.67}$$

By inspection we verify that $\{\mathcal{J}(i), S_i = 1\} \subseteq \mathcal{S}(i) \equiv \{T_{\mathcal{F}^*-\frac{3f}{2}} \le T^-_{\mathcal{F}^*-\frac{3f}{2}} \le T_{\mathcal{F}^*+\frac{f}{2}}\}$, and by the strong Markov property, we have

$$\mathbb{P}\big[\mathcal{S}(i)\big] \le \int_{\mathcal{F}^*-\frac{3f}{2}}^{\mathcal{F}^*+\frac{f}{2}} \mathbb{P}\big[\tilde{T}^-_{\mathcal{F}^*-\frac{3f}{2}+x} < \tilde{T}_{\mathcal{F}^*+\frac{f}{2}-x}\big] \mathbb{P}\Big[\sum_{\alpha=\tau_{i-1}+1}^{T_{\mathcal{F}^*-\frac{3f}{2}}} \chi(\alpha) \in dx\Big] \le \mathbb{P}\big[\tilde{T}^-_{2\mathcal{F}^*-3f} < \tilde{T}_{2f}\big], \tag{5.68}$$

where we have written $\tilde{T}_x \equiv \inf\{\alpha \ge 1 \colon \mathcal{Y}_\alpha \ge x\}$, $\tilde{T}^-_x \equiv \inf\{\alpha \ge 1 \colon \mathcal{Y}_\alpha \le -x\}$.

At this point we need the estimate (5.91), in Lemma 5.13 below, it gives

$$\mathbb{P}\big[\tilde{T}^-_{2\mathcal{F}^*-3f} < \tilde{T}_{2f}\big] \le \frac{2f + 9V(\beta,\theta)\sqrt{\epsilon \log \frac{C_1}{\epsilon}}}{2\mathcal{F}^* - f} \equiv G(\beta,\theta,\epsilon,f). \tag{5.69}$$

with $C_1 = C_1(\beta,\theta,2\mathcal{F}^*) \ge (C_1(\beta,\theta,(2\mathcal{F}^*-3f)\vee(2f))$ if $0 < \epsilon < \epsilon_0(\beta,\theta,2\mathcal{F}^*) \le \epsilon_0(\beta,\theta,(2\mathcal{F}^*-3f)\vee(2f))$. Here we have used that $\epsilon_0(\beta,\theta,b)$ is decreasing with b and that $C_1(\beta,\theta,b)$ is increasing with b.

Consequently, cf. (5.64), (5.68) and (5.69) we have

$$\mathbb{P}\big[\mathcal{J}(i_1^*),\, S_{i_1^*} = 1\big] \le \sum_{i=1}^{i_0} \mathbb{P}\big[\mathcal{S}(i)\big] + 2^{-i_0} \le i_0 G(\beta,\theta,\epsilon,f) + 2^{-i_0}. \tag{5.70}$$

Taking $i_0 = \log \frac{1}{G(\beta,\theta,\epsilon,f)} [\log 2]^{-1}$ we obtain (5.62), since the same works for i_1^*+1. ■

To show that (5.27) holds, we need to bound the probability of finding two extrema in an interval $[\tau_{i_j^*}, \tau_{i_{j+1}^*}]$, at distance larger than ρ/ϵ and whose values are within $\tilde{\delta}$.

We fix the interval $[\tau_{i_{-1}^*}, \tau_{i_1^*}]$ (the peculiarity of having fixed the origin will not bother), and for any given h,k positive integers we denote

$$\mathcal{E}(k,h,+) = \big\{\omega \in \Omega : i_{-1}^* = -h, i_1^* = k, S_k = -1\big\}, \tag{5.71}$$

where for definiteness we are considering only the case of maxima, i.e., we have assumed that $S_k = S_{k+1} = -1, S_{-h} = S_{-h+1} = +1$ on $\mathcal{E}(k,h,+)$. The case of minima is similar. Recall that $\mathbb{P}\left[\mathcal{E}(k,h,+)\right] \le 2^{-(k+h)}$.

The positive integers h,k in (5.71) determine a random interval $\{\tau_{-h},\dots,\tau_{k+1}\} \subseteq \mathbb{Z}$ in which the index α of the variables $\chi(\alpha)$ varies. Using Lemma 5.7, on a set of probability larger than $\left(1-e^{-sk}\right)\left(1-e^{-sh}\right)$, we can replace this random interval by a larger deterministic one. In particular, assuming $s \ge \log 2$, except for a set of probability at most $4e^{-s}$, for all $h,k \ge 1$, $\{\tau_{-h},\dots,\tau_{k+1}\} \subseteq \{\mathcal{L}(-h,\epsilon),\dots,\mathcal{L}(k+1,\epsilon)\}$ where

$$\mathcal{L}(r,\epsilon) \equiv r\frac{(s+\log 2)C_1}{\epsilon} \qquad r \in \mathbb{Z} \tag{5.72}$$

with $C_1 = C_1(\beta,\theta,2\mathcal{F}^*) \ge C_1(\beta,\theta,\mathcal{F}^*+(f/2))$ as in (5.46).

We now partition the interval $[\mathcal{L}(-h,\epsilon),\mathcal{L}(k+1,\epsilon)]$ into blocks of length ρ/ϵ, where ρ was already introduced in (5.30). Assuming, as always, that we do not have rounding off problems, the number of such blocks inside $[\mathcal{L}(-h,\epsilon),\mathcal{L}(k+1,\epsilon)]$ is $\mathcal{L}(k+1,\rho)-\mathcal{L}(-h,\rho)$, i.e., of order $(k+h+1)\rho^{-1}$, with $\mathcal{L}(\cdot,\rho)$ defined as in (5.72) with ϵ replaced by ρ.

Given $\underline{\alpha} \equiv \mathcal{L}(-h,\epsilon) \le \alpha_1 \le \alpha_2 \le \mathcal{L}(k+1,\epsilon)$, let:

$$\mathcal{Y}^*(\underline{\alpha},\alpha_1,\alpha_2) \equiv \max_{\alpha_1\le\tilde{\alpha}\le\alpha_2} \sum_{\alpha=\underline{\alpha}}^{\tilde{\alpha}} \chi(\alpha). \tag{5.73}$$

Given $\tilde{\delta}>0$, $\rho>0$, and ℓ such that $\mathcal{L}(-h,\rho)\le \ell \le \mathcal{L}(k-1,\rho)$, let us define the event

$$\begin{aligned}\mathcal{D}(k,h,\rho,\tilde{\delta},+,\epsilon) \equiv \Big\{&\omega\in\Omega : \exists \ell,\ell',\, \mathcal{L}(-h,\rho)\le \ell<\ell' \le \mathcal{L}(k-1,\rho);\\ &|\mathcal{Y}^*(\underline{\alpha},\tfrac{\rho\ell}{\epsilon},\tfrac{\rho(\ell+1)}{\epsilon}) - \mathcal{Y}^*(\underline{\alpha},\tfrac{\rho\ell'}{\epsilon},\tfrac{\rho(\ell'+1)}{\epsilon})| \le 2\tilde{\delta}\Big\}.\end{aligned} \tag{5.74}$$

We now prove the following estimate:

Lemma 5.11 . *There exist positive constants $\gamma_0(\beta,\theta)$ and $\rho_0(\beta,\theta)$ such that for all $\gamma\le\gamma_0(\beta,\theta)$, for $0<\rho<\rho_0(\beta,\theta)$, for $\tilde{\delta}=\rho^{2+a}$ with $a>0$, for $\delta^*\gamma<\epsilon\le\epsilon_0(\beta,\theta,\rho)$, where*

$$\epsilon_0(\beta,\theta,\rho) = \tfrac{4(\rho)^{2(2+a)}}{2V^2(\beta,\theta)\log(1944)}, \tag{5.75}$$

and for all $s>0$ we have

$$\begin{aligned}&I\!P\left[\cup_{k,h\ge1}\left(\mathcal{E}(k,h,+)\cap\mathcal{D}(k,h,\rho,\tilde{\delta},+,\epsilon)\right)\right] \le\\ &\frac{2^{16}C_1(\beta,\theta,2\mathcal{F}^*)}{\sqrt{V(\beta,\theta)}}(s+\log 2)\left(\rho^{a/2} + \frac{\sqrt{1+V(\beta,\theta)}(\epsilon\log\frac{C_1(\beta,\theta,2\mathcal{F}^*)}{\epsilon})^{1/4}}{\rho^{3/4}}\right).\end{aligned} \tag{5.76}$$

Proof: By Schwartz inequality

$$I\!P\left[\cup_{k,h\ge1}\mathcal{E}(k,h,+)\cap\mathcal{D}(k,h,\rho,\tilde{\delta},+,\epsilon)\right] \le \sum_{h,k\ge1}\left(I\!P\left[\mathcal{E}(k,h,+)\right]\right)^{1/2}\left(I\!P\left[\mathcal{D}(k,h,\rho,\tilde{\delta},+,\epsilon)\right]\right)^{1/2}. \tag{5.77}$$

Since

$$I\!P\left[\mathcal{E}(k,h,+)\right]^{\frac12} \le 2^{-\frac{(k+h)}{2}} \tag{5.78}$$

will be summable in h,k, it remains to properly estimate the second term into parenthesis in (5.77). From (5.74) we just write

$$I\!P\left[\mathcal{D}(k,h,\rho,\tilde{\delta},+,\epsilon)\right] \le \sum_{\ell=\mathcal{L}(-h,\rho)}^{\mathcal{L}(k-1,\rho)-1}\ \sum_{\ell'=\ell+1}^{\mathcal{L}(k-1,\rho)} I\!P\left[|\mathcal{Y}^*(\underline{\alpha},\tfrac{\rho\ell'}{\epsilon},\tfrac{\rho(\ell'+1)}{\epsilon}) - \mathcal{Y}^*(\underline{\alpha},\tfrac{\rho\ell}{\epsilon},\tfrac{\rho(\ell+1)}{\epsilon})| \le 2\tilde{\delta}\right] \tag{5.79}$$

and estimate each summand on the r.h.s. of (5.79). If $\ell+1<\ell'$ we write:

$$\begin{aligned}&\mathcal{Y}^*(\underline{\alpha},\tfrac{\rho\ell'}{\epsilon},\tfrac{\rho(\ell'+1)}{\epsilon}) - \mathcal{Y}^*(\underline{\alpha},\tfrac{\rho\ell}{\epsilon},\tfrac{\rho(\ell+1)}{\epsilon}) =\\ &\sum_{\alpha=\frac{\rho(\ell+1)}{\epsilon}+1}^{\frac{\rho\ell'}{\epsilon}}\chi(\alpha) + \max_{\frac{\rho\ell'}{\epsilon}+1\le\tilde{\alpha}\le\frac{\rho(\ell'+1)}{\epsilon}}\ \sum_{\alpha=\frac{\rho\ell'}{\epsilon}+1}^{\tilde{\alpha}}\chi(\alpha) + \min_{\frac{\rho\ell}{\epsilon}\le\tilde{\alpha}\le\frac{\rho(\ell+1)}{\epsilon}}\ \sum_{\alpha=\tilde{\alpha}+1}^{\frac{\rho(\ell+1)}{\epsilon}}\chi(\alpha),\end{aligned}$$

and using the independence of the $\chi(\alpha)$ we easily see that:

$$\begin{aligned} I\!\!P\left[|\mathcal{Y}^*(\underline{\alpha},\tfrac{\rho\ell'}{\epsilon},\tfrac{\rho(\ell'+1)}{\epsilon})-\mathcal{Y}^*(\underline{\alpha},\tfrac{\rho\ell}{\epsilon},\tfrac{\rho(\ell+1)}{\epsilon})|\le 2\tilde\delta\right] &\le \sup_x I\!\!P\left[\sum_{\alpha=\frac{\rho(\ell+1)}{\epsilon}+1}^{\frac{\rho\ell'}{\epsilon}}\chi(\alpha)\in[x,x+2\tilde\delta]\right] \\ &\le \frac{4\tilde\delta\sqrt{2\pi}}{V(\beta,\theta)\sqrt{(\ell'-\ell-1)\rho}}. \end{aligned} \tag{5.80}$$

In the last inequality we have used the concentration inequality of Le Cam (e.g. [12], p.407) for the symmetric random variables $\chi(\alpha)$ and assumed $0<\epsilon<\epsilon_0(\beta,\theta,\rho)$ see (5.75). This condition comes from a lower estimate of what Le Cam called $B^2(\tau)$. In our case $B^2(2\tilde\delta)=(\ell'-\ell-1)\frac{\rho}{\epsilon}I\!\!E[1\wedge(\chi(1)/2\tilde\delta)^2]$. A short computation gives

$$I\!\!E[1\wedge(\chi(1)/2\tilde\delta)^2]\ge\frac{I\!\!E[(\chi(1))^2]}{4\tilde\delta^2}\left(1-\frac{I\!\!E[(\chi(1))^2 1\!\!1_{\{|\chi(1)|>4\tilde\delta\}}]}{I\!\!E[(\chi(1))^2]}\right). \tag{5.81}$$

Using (5.2), (5.35), Schwarz inequality, and that $I\!\!P[|\chi(1)|>4\tilde\delta]\le 2e^{-2\tilde\delta^2/(\epsilon V_+^2(\beta,\theta))}$, which follows from (5.36), a short computation shows that for $0<\epsilon<\epsilon_0(\beta,\theta,\rho)$ the last term inside parenthesis in (5.81) is bounded from below by 1/2.

When $\ell'=\ell+1$, we bound the corresponding term on the r.h.s. of (5.79) as:

$$\sup_x I\!\!P\left[\mathcal{Y}^*(\tfrac{\rho}{\epsilon})\in[x,x+2\tilde\delta]\right], \tag{5.82}$$

where $\mathcal{Y}^*(\alpha)\equiv\max_{1\le\tilde\alpha\le\alpha}\mathcal{Y}_{\tilde\alpha}=\mathcal{Y}^*(1,1,\alpha)$ if $\alpha\ge1$, and $\mathcal{Y}_\alpha$ given in (5.3). Putting together (5.72), (5.79), (5.80) and (5.82), we get

$$\begin{aligned} I\!\!P\left[\mathcal{D}(k,h,\rho,\tilde\delta,+,\epsilon)\right] &\le \left(C_1(\beta,\theta,2\mathcal{F}^*)(s+\log 2)\right)^2 2(h+k+1)^2\frac{2\sqrt{2\pi}}{V(\beta,\theta)}\frac{\tilde\delta}{\rho^2} \\ &+\left(C_1(\beta,\theta,2\mathcal{F}^*)(s+\log 2)\right)^2\frac{(h+k+1)}{\rho}\sup_x I\!\!P\left[\mathcal{Y}^*(\tfrac{\rho}{\epsilon})\in[x,x+2\tilde\delta]\right]. \end{aligned} \tag{5.83}$$

The first term on the r.h.s. of (5.83) suggests to take $\tilde\delta=\rho^{2+a}$ with $a>0$. The last term will be estimated in the next Lemma 5.12, cf. (5.84) below.

Recalling (5.77), (5.78), (5.79), (5.83), and using (5.84) a short computation entails (5.76). ■

Lemma 5.12 . *There exist positive constants $\gamma_0(\beta,\theta)$ and $\rho_0(\beta,\theta)$ such that for all $\gamma\le\gamma_0(\beta,\theta)$, for $0<\rho<\rho_0(\beta,\theta)$, for $\tilde\delta=\rho^{2+a}$ with $a>-1/2$, such that for $\delta^*\gamma<\epsilon\le\epsilon_0(\beta,\theta,\rho)$ with $\epsilon_0(\beta,\theta,\rho)$ given in (5.75), we have*

$$\frac{1}{\rho}\sup_x I\!\!P\left[\mathcal{Y}^*(\tfrac{\rho}{\epsilon})\in[x,x+2\tilde\delta]\right]\le\frac{1296}{V(\beta,\theta)}\left(\frac{\tilde\delta+(2+V(\beta,\theta))\sqrt{\epsilon\log\frac{C_1}{\epsilon}}}{\rho^{3/2}}\right) \tag{5.84}$$

where $C_1=C_1(\beta,\theta,2\mathcal{F}^*)$ is given by (5.46).

Proof: Let $\tilde T_x$ be the stopping time given after (5.68). We write

$$I\!P\left[\mathcal{Y}^*(\tfrac{\rho}{\epsilon}) \in [x, x+2\tilde\delta]\right] = I\!P\left[\tilde T_x \le \frac{\rho}{2\epsilon}, \tilde T_{x+2\tilde\delta} > \frac{\rho}{\epsilon}\right] + I\!P\left[\frac{\rho}{2\epsilon} < \tilde T_x < \frac{\rho}{\epsilon} < \tilde T_{x+2\tilde\delta}\right]. \tag{5.85}$$

Observe that for any $\tilde\delta > 0$ we have $\left\{\frac{\rho}{2\epsilon} < \tilde T_x < \frac{\rho}{\epsilon} < \tilde T_{x+2\tilde\delta}\right\} = \left\{\mathcal{Y}^*(\frac{\rho}{2\epsilon}) < x, \max_{\frac{\rho}{2\epsilon}\le\alpha\le\frac{\rho}{\epsilon}} \mathcal{Y}_\alpha \in [x, x+2\tilde\delta]\right\}$. Therefore, if $0 < \epsilon < \epsilon_0(\beta, \theta, \rho)$, we obtain

$$I\!P\left[\frac{\rho}{2\epsilon} < T_x < \frac{\rho}{\epsilon} < T_{x+2\tilde\delta}\right] \le I\!P\left[\max_{\frac{\rho}{2\epsilon}\le\alpha\le\frac{\rho}{\epsilon}} \mathcal{Y}_\alpha \in [x, x+2\tilde\delta]\right] \le \sup_{u\in I\!R} I\!P\left[\mathcal{Y}_{\frac{\rho}{2\epsilon}} \in [u, u+2\tilde\delta]\right] \le \frac{4\tilde\delta\sqrt{2\pi}}{V(\beta,\theta)\sqrt{\rho/2}}. \tag{5.86}$$

In the second inequality in (5.86), we used that the law of $\max_{\frac{\rho}{2\epsilon}\le\alpha\le\frac{\rho}{\epsilon}} \mathcal{Y}_\alpha$ is the convolution of the law of $\mathcal{Y}_{\frac{\rho}{2\epsilon}}$ with another probability (the law of $\mathcal{Y}^*(\frac{\rho}{2\epsilon})$, in this case).

Let us now consider the first summand on the r.h.s. of (5.85). Decomposing according to the value of $\mathcal{Y}_{\tilde T_x}$, $\tilde T_x$ and using the fact the variables $\chi(\cdot)$ are i.i.d. we get

$$I\!P\left[\tilde T_x \le \frac{\rho}{2\epsilon}, \tilde T_{x+2\tilde\delta} > \frac{\rho}{\epsilon}\right] = \sum_{k=0}^{\rho/2\epsilon} \int_x^{x+2\tilde\delta} I\!P\Big[\tilde T_x = k, \mathcal{Y}_k \in dy\Big] I\!P\Big[\tilde T_{x+2\tilde\delta-y} > \frac{\rho}{\epsilon} - k\Big]$$

Since $x - y \le 0$ we can write:

$$I\!P\Big[\tilde T_{x+2\tilde\delta-y} > \frac{\rho}{\epsilon} - k\Big] \le I\!P\Big[\tilde T_{2\tilde\delta} > \frac{\rho}{\epsilon} - k\Big].$$

Integrating in y we then have:

$$I\!P\left[\tilde T_x \le \frac{\rho}{2\epsilon}, \tilde T_{x+2\tilde\delta} > \frac{\rho}{\epsilon}\right] \le I\!P\left[\tilde T_{2\tilde\delta} > \frac{\rho}{2\epsilon}\right], \tag{5.87}$$

and collecting (5.85), (5.86), and (5.87), we get

$$\sup_x I\!P\left[\mathcal{Y}^*(\tfrac{\rho}{\epsilon}) \in [x, x+2\tilde\delta]\right] \le I\!P\Big[\tilde T_{2\tilde\delta} > \frac{\rho}{2\epsilon}\Big] + \frac{4\tilde\delta\sqrt{2\pi}}{V(\beta,\theta)\sqrt{\rho/2}}. \tag{5.88}$$

Now, it is easy to check that

$$I\!P\left[\tilde T_{2\tilde\delta} > \frac{\rho}{2\epsilon}\right] \le I\!P\left[\tilde T^-_{c\sqrt{\rho/2}} \le \tilde T_{2\tilde\delta}\right] + I\!P\left[\tilde T^-_{c\sqrt{\rho/2}} \wedge \tilde T_{2\tilde\delta} \ge \frac{\rho}{2\epsilon}\right], \tag{5.89}$$

where $T^-_{c\sqrt{\rho/2}}$ is the stopping time defined after (5.68) for a constant c to be chosen soon. Then we apply inequalities (5.91) and (5.93) given in the next lemma, with $a = c\sqrt{\rho/2}$, $d = \rho/2$, and $x = 2\tilde\delta$. Collecting all together the estimates for $I\!P\left[\mathcal{Y}^*(\frac{\rho}{\epsilon}) \in [x, x+2\tilde\delta]\right]$, we have:

$$\begin{aligned}\frac{1}{\rho}\sup_x I\!P\left[\mathcal{Y}^*(\tfrac{\rho}{\epsilon}) \in [x, x+2\tilde\delta]\right] &\le \frac{2\tilde\delta + 9V(\beta,\theta)\sqrt{\epsilon\log\frac{C_1}{\epsilon}}}{\rho(2\tilde\delta + c\sqrt{\rho/2})} + \frac{8\sqrt{2}\tilde\delta c}{V^2(\beta,\theta)\rho^{3/2}} + \\ &+ \frac{72}{\rho^{3/2}V^2(\beta,\theta)}\sqrt{\epsilon\log\frac{C_1}{\epsilon}}\left(9(2\tilde\delta + c\sqrt{\rho/2}) + V(\beta,\theta)\sqrt{\epsilon\log\frac{C_1}{\epsilon}}\right)\end{aligned} \tag{5.90}$$

with $C_1 = C_1(\beta, \theta, (2\tilde{\delta}) \vee c\sqrt{\rho/2})$ see (5.46). Taking $c = V(\beta, \theta)$ and assuming that $\rho_0(\beta, \theta)$ is small enough, we have $C_1(\beta, \theta, (2\tilde{\delta}) \vee c\sqrt{\rho/2}) \leq C_1(\beta, \theta, 2\mathcal{F}^*)$, and a short computation entails (5.84). ■

Lemma 5.13 . *For all $x > 0$, $a > 0$, $C_1 = C_1(\beta, \theta, x \vee a)$ as in (5.46), $\epsilon_0(\beta, \theta, x \vee a)$ as in (5.41), and if $\delta^*\gamma < \epsilon \leq \epsilon_0(\beta, \theta, x \vee a)$, we have:*

$$IP\left[\tilde{T}_a^- \leq \tilde{T}_x\right] \leq \frac{x + 9V(\beta, \theta)\sqrt{\epsilon \log \frac{C_1}{\epsilon}}}{x + a}, \tag{5.91}$$

$$IP\left[\tilde{T}_a^- \geq \tilde{T}_x\right] \leq \frac{a + 9V(\beta, \theta)\sqrt{\epsilon \log \frac{C_1}{\epsilon}}}{x + a}, \tag{5.92}$$

$$IP\left[\tilde{T}_a^- \wedge \tilde{T}_x \geq \frac{d}{\epsilon}\right] \leq \frac{4xa}{V^2(\beta,\theta)d} + + \frac{36}{V^2(\beta,\theta)d}\sqrt{\epsilon \log \frac{C_1}{\epsilon}}\left(9(x+a) + V(\beta,\theta)\sqrt{\epsilon \log \frac{C_1}{\epsilon}}\right). \tag{5.93}$$

The proof of the previous lemma is a standard application of (5.38) and (5.42) together with Wald identity applied to the martingales $\mathcal{Y}_\alpha, \alpha \geq 0$ and $(\mathcal{Y}_\alpha)^2 - \epsilon c(\beta, \theta, \gamma/\delta^*)\alpha$, and also the bound (4.59). Details are left out.

To prove (5.29) in Theorem 5.1 we need a classical result on the distribution of the localization of the mimimum or the maximum of a simple random walk. Since their distribution is the same, it is enough to consider the case of maximum. So, recalling (5.73), let us denote $L_{\rho/\epsilon} = \inf\{\alpha > 0 : \mathcal{Y}_\alpha = \mathcal{Y}^*(0, 0, \rho/\epsilon)\}$. Such kind of result was proved by E. Sparre Andersen [33]. Following step by step the very nice computations he did, see Theorem 3 of [33], and using the Berry-Essen theorem to estimate what is there denoted by $IP\{S_n > 0\}$, we can evaluate by the Cauchy integral formula the constant called C at pg. 208, 3 lines before (5.17) of [33]. After simple, however lengthy computations, we obtain the following result.

Proposition 5.14 . *There exists a constant $C(\beta, \theta)$ (related to $V(\beta, \theta)$) and $\rho_0 = \rho_0(\beta, \theta)$ such that for all $0 < \rho < \rho_0$ there exists $\epsilon_0 = \epsilon_0(\beta, \theta)$ such that for all $0 < \epsilon \leq \rho\epsilon_0$, for all $0 < \kappa \leq 1/2$, for all interval $0 < a < a' \leq 1$ such that $a' - a \geq \frac{2\epsilon}{\rho}$,*

$$\begin{aligned} &\left|IP\left[L_{\rho/\epsilon} \in [a\rho/\epsilon, a'\rho/\epsilon]\right] - \frac{\cos(\pi\kappa)}{\pi}\int_{a(\epsilon,\rho)}^{a'(\epsilon,\rho)} \frac{dx}{x^{\frac{1}{2}+\kappa}(1-x)^{\frac{1}{2}-\kappa}}\right| \\ &\leq \frac{1}{\Gamma(\frac{1}{2}-\kappa)}\left(\frac{\epsilon}{\rho}\right)^{\frac{1}{2}+\kappa} + \frac{1}{\Gamma(\frac{1}{2}+\kappa)}\left(\frac{\epsilon}{\rho}\right)^{\frac{1}{2}-\kappa} + \frac{\epsilon}{\rho(a'-a)}\exp\left(8^{\frac{C(\beta,\theta)}{\kappa^2}} 2\log\frac{C(\beta,\theta)}{\kappa^2}\right), \end{aligned} \tag{5.94}$$

where $x(\rho, \epsilon) = (\rho x + \epsilon)(\rho + \epsilon)^{-1}$ for $x = a, a'$

Proof of Theorem 5.1

We start proving (5.24). For any $Q > Q_0 = 4\log 2C_1(\beta, \theta, \mathcal{F}^*)$, if we take Q/ϵ blocks of length ϵ/γ on the right of the origin, then using Lemma 5.8 with $s = \log 2$ and $k = 1 + [Q/(2C_1(\beta, \theta, 2\mathcal{F}^*)\log 2)]$ where $[\cdot]$ is the integer part, with a IP–Probability at least $(1 - 3e^{-Q/2C_1(\beta,\theta,\mathcal{F}^*)})$ there is at least one index i among $1, \ldots, [Q/(2C_1(\beta, \theta, 2\mathcal{F}^*)\log 2)]$ such that $S_i = S_{i+1}$. From Lemma 5.10 with $IP \geq 1 - G(\beta, \theta, \epsilon, f)\log G(\beta, \theta, \epsilon, f)\frac{2}{\log 2}$ with $G(\beta, \theta, \epsilon, f)$ defined in (5.63) we have an elongation there. Therefore the probability of not having any elongation on the right of the origin within Q/ϵ blocks of length ϵ/γ is less than

$$3e^{-\frac{Q}{2C_1(\beta,\theta,\mathcal{F}^*)}} + \frac{2}{\log 2}G(\beta, \theta, \epsilon, f)\log G(\beta, \theta, \epsilon, f), \tag{5.95}$$

which implies (5.24).

The proof of (5.25) is done in a similar way. We first apply Lemma 5.9 with $s = \log 2$ and $L = 1 + [Q/(kc(\beta,\theta,2\mathcal{F}^*)2\log 2)]$ then Lemma 5.10.

To prove (5.27), we recall Lemma 5.11 and the arguments that precede it. Taking $\tilde{\delta}(\rho) = \rho^{2+a}$ and recalling (5.28) we have

$$I\!P\left[\mathcal{P}_2(f,s_1,Q,a_i,b_{i+1},\rho,\tilde{\delta}(\rho))\right] \le 4e^{-s} + (s+\log 2)G_1(\beta,\theta,\tilde{\delta}(\rho),\epsilon). \tag{5.96}$$

Choosing $s = \log 4/(G_1(\beta,\theta,\tilde{\delta}(\rho),\epsilon))$ and taking $\rho_0(\beta,\theta)$ and $\epsilon_0(\beta,\theta,\rho)$ small enough, we get (5.27).

For the proof of (5.26), recalling (5.10) we write

$$I\!P\left[\mathcal{P}_2''(f,Q)\right] \le I\!P\left[\mathcal{P}_2'(f,Q) \cap \left\{\max_{\alpha\in[-Q/\epsilon,Q/\epsilon]} |\chi(\alpha)| \le f\right\}\right] + I\!P\left[\max_{\alpha\in[-Q/\epsilon,Q/\epsilon]} |\chi(\alpha)| > f\right],$$

and taking $\rho' = (9f)^{1/(2+a)}$, we consider the event

$$\tilde{\mathcal{D}}(Q,\rho',\epsilon) \equiv \left\{\exists \ell,\ell',\, -Q/\rho' \le \ell < \ell' \le (Q-1)/\rho'; |\mathcal{Y}^*(\underline{\alpha}, \tfrac{\rho'\ell}{\epsilon}, \tfrac{\rho'(\ell+1)}{\epsilon}) - \mathcal{Y}_*(\underline{\alpha}, \tfrac{\rho'\ell'}{\epsilon}, \tfrac{\rho'(\ell'+1)}{\epsilon}) - 2\mathcal{F}^*| \le 9f\right\},$$

where $\mathcal{Y}_*$ is defined as in (5.73) replacing max by min.

Simple observations show that $\mathcal{P}_2'(f,Q) \cap \{\max_{\alpha\in[-Q/\epsilon,Q/\epsilon]} |\chi(\alpha)| \le f\} \subseteq \tilde{\mathcal{D}}(Q,\rho',\epsilon)$. Following the arguments leading to (5.83), assuming $0 < \epsilon \le \epsilon_0(\beta,\theta,f) = (9f)^2/(2V^2(\beta,\theta)\log 1944)$, using Lemma 5.12 with $2\tilde{\delta}$ replaced by $9f$ one gets (5.26).

The proof of (5.29) follows from (5.94) estimating the integral in the left hand side of (5.94) by $8(a'-a)^{\frac{1}{2}-\kappa}$ which can be obtained by cutting the interval $[a(\epsilon,\rho), a'(\epsilon,\rho)]$ into two equal pieces. Using (5.94) for $a = 0, a' = \rho$ and a short computation entails (5.29). ■

Proof of Proposition 5.3

To prove (5.33), notice that $\gamma J(\omega) \supset [\epsilon\tau_{-1}, 0] \cup [0, \epsilon\tau_1]$. Therefore, using (5.48) and a short computation one gets

$$I\!P[\gamma|J| \le x] \le 2e^{-\frac{(\mathcal{F}^*)^2}{18xV^2(\beta,\theta)}} \tag{5.97}$$

for $0 < x < (\mathcal{F}^*)^2/(V^2(\beta,\theta)18\log 2)$. (5.34) follows at once, due to (5.52), (5.53), and the fact that $\gamma J(\omega) \subseteq [\epsilon\tau_{i^*_{-2}}, \epsilon\tau_{i^*_2}]$. Therefore (5.56) with $k = 2$ entails

$$I\!P[\gamma|J| \ge x] \le 2I\!P[\epsilon\tau_{2L} \ge \frac{x}{2}] + 2\left(\frac{1}{2^{L-1}} + \left(\frac{3}{4}\right)^{L/2}\right). \tag{5.98}$$

Using now (5.49) with $k = 2L$, $s = \log 2$ one gets $I\!P[\epsilon\tau_{2L} \ge 4LC_1(\beta,\theta,\mathcal{F}^*)\log 2] \le e^{-2L\log 2}$. Taking $L = x/(8C_1(\beta,\theta,\mathcal{F}^*)\log 2)$ one obtains after a short computation (5.34). ■

The following lemma will be useful in the next section; it is in fact an immediate consequence of (5.27) and the proof is omitted .

Lemma 5.15 . *Under the hypothesis of Corollary 5.2 and with the same notations with $I\!P$–probability larger than $1 - \epsilon^{\frac{a}{16(2+a)}}$ we have*

$$\sum_{\alpha=\alpha_1}^{\alpha*_1} \chi(\alpha) \ge \epsilon^{1/4}, \quad \sum_{\alpha=\alpha_0^*}^{\alpha_1} \chi(\alpha) \ge \epsilon^{1/4}, \tag{5.99}$$

provided α_0^* is the beginning and α_1^* is the end of a positive elongation, $\alpha_0^* + \frac{\rho}{\epsilon} < \alpha_1 < \alpha_1^* - \frac{\rho}{\epsilon}$.

6 Proof of Theorems

In this section we prove Theorems 2.1, 2.2, and 2.4. They will be derived from Proposition 6.2 stated and proved below. We will use the following strictly positive finite quantities: $\kappa(\beta,\theta)$ that satisfies (9.25), $\mathcal{F}^*$ defined in (7.5), $V(\beta,\theta)$ in (4.56), $c(\beta,\theta)$ in (4.65) and $c(\beta\theta)$ in (3.55). We denote

$$\alpha(\beta,\theta,\zeta_0) \equiv -\log \frac{\partial g_\beta}{\partial m}(\tilde{m}_{\beta,\theta} - \frac{\zeta_0}{2},\theta) > 0, \tag{6.1}$$

where $g_\beta(m,\theta)$ is defined in (9.8), $\zeta_0 = \zeta_0(\beta,\theta)$ is a small quantity that satisfies requirements written before (7.22). Recalling (7.4), we have $\alpha(\beta,\theta) \ge \alpha(\beta,\theta,\zeta_0)$. The results from Sections 3,4, and 6 require relations among various parameters. For γ_0, d_0, ζ_0 sufficiently small depending on β,θ as stated in Theorem 2.1, $0 < \gamma \le \gamma_0$, $\gamma/\delta^* < d_0$ $1 > \delta > \delta^* > 0$, $\zeta_0 > \zeta_4 > \zeta_1 > \zeta_5 > 8\gamma/\delta^*$, $Q > 1$, $\epsilon > 0$, we assume that the following constraints are satisfied:

The $\mathcal{C}_0$ constraints:

$$\frac{128(1+\theta)}{\kappa(\beta,\theta)}\frac{2(5+\mathcal{F}^*)}{\mathcal{F}^*}\sqrt{\frac{\gamma}{\delta^*}} < \delta\zeta_1^3, \tag{6.2}$$

$$\frac{32}{\kappa(\beta,\theta)}\zeta_1 \le \delta\zeta_4^3, \tag{6.3}$$

$$\left(5184(1+c(\beta\theta))^2\sqrt{\frac{\gamma}{\delta^*}}\right) \vee \left(12\frac{e^3\beta}{c(\beta,\theta)}\frac{(\delta^*)^2}{\gamma}\right)^2 \le \zeta_5, \tag{6.4}$$

$$\frac{512(1+\theta)}{\kappa(\beta,\theta)\alpha(\beta,\theta,\zeta_0)}\sqrt{\frac{\gamma}{\delta^*}}\log\frac{\delta^*}{\gamma} < \delta\zeta_5^3, \tag{6.5}$$

$$\sqrt{\gamma}\log Q \le \frac{1}{12}\sqrt{\frac{12e^3\beta}{c(\beta,\theta)}}, \tag{6.6}$$

$$\frac{\mathcal{F}^*}{32(1+\theta)}\sqrt{\delta^*\gamma} \le \epsilon. \tag{6.7}$$

Remark. The constraints (6.2), (6.3), (6.4), and (6.6) come from Theorem 4.3, where (6.4) was written for ζ_5 replaced by a larger value ζ_1 ; now we impose the stronger restriction (6.4), as it will be needed later. Notice that (6.7) and (6.2) imply that $\epsilon\gamma^{-1} > 2R_1$. (6.5) comes from (7.38) in Corollary 7.5.

Remark 6.1 . Note that in (6.2) one can take $\delta = \delta_1$, in (6.3) $\delta = \delta_4$ and in (6.5) $\delta = \delta_5$, with $\delta_5 = n_5\delta^*$, $\delta_1 = n_1\delta_5$, and $\delta_4 = n_4\delta^*$ for some positive integers that will diverge since $\delta^* \downarrow 0$. This would allow δ_4 to be small without imposing as in Theorem 2.1 that it goes to zero. Since this would introduce new parameters we have decided, for simplification, not to do it.

With the choice of parameters that satisfy the $\mathcal{C}_0$ constraints, we apply Theorem 4.3, Corollary 7.5 with $p = 2 + [(\log Q)/(\log(1/\gamma))]$, Lemma 5.15, and Corollary 5.2 with $k = 5$, to determine measurable sets $\Omega_4 = \Omega_4(\gamma,\delta^*,\Delta_Q,\epsilon,\delta,\zeta_1,\zeta_4)$, $\Omega_{RE} = \Omega_{RE}(\gamma,\delta^*,\, p) \equiv \Omega_{RE}(\gamma,\delta^*,Q)$, Ω_ϵ, and respectively $\mathcal{P}(5,\epsilon,Q)$ such that, calling $\Omega_{51} = \Omega_4 \cap \Omega_{RE} \cap \mathcal{P}(5,\epsilon,Q)^c \cap \Omega_\epsilon$, we have

$$I\!P[\Omega_{51}] \ge 1 - 10e^{-\frac{Q}{10C_1}} - 5\epsilon^{\frac{a}{16(2+a)}} - Q^2\epsilon^{\frac{a}{8+2a}} - Qe^{-\frac{1}{2\epsilon^{3/4}V^2(\beta,\theta)}} - 7\gamma^2, \tag{6.8}$$

when $\delta^*\gamma < \epsilon \le \epsilon_0(\beta,\theta)$ and $a > 0$.

For $\omega \in \mathcal{P}(5,\epsilon,Q)^c$, the origin belongs to an unique elongation $[\alpha^*_{j_0},\alpha^*_{j_0+1}]$ where $j_0 = -1$ or 0, see (5.13) and (5.15), moreover on this set, recalling (5.22), we have,

$$\left[-\frac{\rho}{\gamma},\frac{\rho}{\gamma}\right] \subseteq \left[\frac{\epsilon\alpha^*_{j_0}}{\gamma},\frac{\epsilon\alpha^*_{j_0+1}}{\gamma}\right] \subset \left[-\frac{Q}{\gamma},\frac{Q}{\gamma}\right]. \tag{6.9}$$

We write, for $\eta \in \{-1,+1\}$

$$\Omega^\eta(\epsilon,Q) \equiv \left\{\omega \in \mathcal{P}(5,\epsilon,Q)^c,\ \mathrm{sgn}\left[\frac{\epsilon\alpha^*_{j_0}}{\gamma},\frac{\epsilon\alpha^*_{j_0+1}}{\gamma}\right] = \eta\right\}. \tag{6.10}$$

For concreteness, we take $j_0 = 0$ and we assume that this elongation is positive, that is, we are on $\Omega_{51} \cap \Omega^+(\epsilon,Q)$. We have the following result:

Proposition 6.2 . *If $\mathcal{C}_0$ holds and*

$$8f_1 + 4f_2 + 4f_3 + 32\zeta_5^{\frac{1-z}{2}} + 16\zeta_1 \le \frac{\epsilon^{1/4}}{2} \tag{6.11}$$

where

$$f_1 = 10(1+\theta)\frac{1}{\alpha(\beta,\theta,\zeta_0)}\sqrt{\frac{\gamma}{\delta^*}}\log\frac{\delta^*}{\gamma}, \tag{6.12}$$

$$f_2 = 8V(\beta,\theta)\sqrt{\gamma\log\left(\frac{1}{\gamma}\right)\left(\frac{1}{\alpha(\beta,\theta,\zeta_0)}\log\left(\frac{\delta^*}{\gamma}\right)+R_1\right)} \tag{6.13}$$

with $R_1 = \frac{4(5+\mathcal{F}^*)}{\kappa(\beta,\theta)\delta\zeta_1^3}$,

$$f_3 = 16(1+\theta)R_1\sqrt{\frac{\gamma}{\delta^*}}, \tag{6.14}$$

and $0 < z < 1/2$, there exists Ω_5 such that

$$I\!P[\Omega_5] \ge 1 - 8\gamma^2 - \frac{2\exp(-\frac{\beta^2}{2^6 Q\zeta_5^z c^2(\beta,\theta)})}{1-\exp(-\frac{\beta^2}{2^6 Q\zeta_5^z c^2(\beta,\theta)})} \tag{6.15}$$

and such that on $\Omega_5 \cap \Omega_{51} \cap \Omega^\eta(\epsilon,Q)$,

$$\begin{aligned} &\mu^\omega_{\beta,\theta,\gamma}\left(\exists \ell \in [\frac{\alpha_0^*\epsilon}{\gamma}+\frac{\rho}{\gamma}+R_1,\frac{\alpha_1^*\epsilon}{\gamma}-\frac{\rho}{\gamma}-R_1],\eta^{\delta,\zeta_4}(\ell)\neq\eta\right) \le \\ &\le \left(\frac{3Q}{\gamma^2}\right)^5 e^{-\frac{\beta}{\gamma}\left\{(\frac{\kappa(\beta,\theta)}{4}\delta\zeta_4^3)\wedge\mathcal{F}^*\right\}} + 28R_1^2\left(\frac{2Q}{\gamma}\right)^5 e^{-\frac{\beta}{\gamma}\frac{\epsilon^{1/4}}{5}}\exp\left\{\frac{4Q}{\gamma}e^{-\frac{\beta}{\gamma}\frac{\epsilon^{1/4}}{5}}\right\}, \end{aligned} \tag{6.16}$$

where $\rho \equiv \epsilon^{\frac{1}{4(2+a)}}$.

Remark Recalling (5.21) and Proposition 5.3 the interval $J = [\frac{\alpha_0^*\epsilon}{\gamma},\frac{\alpha_1^*\epsilon}{\gamma}]$ is random, its length being a finite and positive random variable, of order γ^{-1}. On the other hand when choosing the parameters $\rho+\gamma R_1$ will tend to zero.

Proof. We assume that $\eta = +1$, the case $\eta = -1$ being similar. To simplify notation we denote by $N_1 = \frac{1}{\gamma}\alpha_0^*\epsilon$, $N_2 = \frac{1}{\gamma}\alpha_1^*\epsilon$, $I = [N_1 + R_1 + \frac{\rho}{\gamma}, N_2 - R_1 - \frac{\rho}{\gamma}]$, $\eta^{\delta,\zeta_4}(\ell) = \eta(\ell)$ and $B(\ell) = \{\sigma : \eta(\ell) \neq 1\}$ Recalling (4.5), we have that

$$\mu_{\beta,\theta,\gamma}\left(\exists \ell \in I, \eta(\ell) \neq 1)\right) \leq \mu_{\beta,\theta,\gamma}\left(\mathcal{M}_{\delta^*}(\Delta_Q) \setminus \mathcal{A}(\Delta_Q)\right) + \sum_{\ell \in I} \mu_{\beta,\theta,\gamma}\left(B(\ell) \cap \mathcal{A}(\Delta_Q)\right), \tag{6.17}$$

where we denote by $\mathcal{A}(\Delta_Q)^c$ the complement in $\mathcal{M}_{\delta^*}(\Delta_Q)$ of $\mathcal{A}(\Delta_Q)$.

According to Theorem 4.3, for $\omega \in \Omega_{51} \subseteq \Omega_4$ we have

$$\mu_{\beta,\theta,\gamma}\left(\mathcal{M}_{\delta^*}(\Delta_Q) \setminus \mathcal{A}(\Delta_Q)\right) \leq \left(\frac{3Q}{\gamma^2}\right)^5 e^{-\frac{\beta}{\gamma}\left\{\left(\frac{\kappa(\beta,\theta)}{4}\delta\zeta_4^3\right) \wedge \mathcal{F}^*\right\}}. \tag{6.18}$$

To estimate the other term in (6.17) we need to restrict the infinite volume Gibbs measure to a finite volume one. We write

$$\begin{aligned}
&\mu_{\beta,\theta,\gamma}\left(B(\ell) \cap \mathcal{A}(\Delta_Q)\right) \\
&\leq \sum_{\bar\eta_1,\bar\eta_2 \in \{-1,1\}^2} \sum_{\ell_1 = N_1}^{N_1+R_1} \sum_{\ell_2 = N_2 - R_1}^{N_2} \mu_{\beta,\theta,\gamma}\left(\eta^{\delta,\zeta_1}(\ell_1) = \bar\eta_1, \eta^{\delta,\zeta_1}(\ell_2) = \bar\eta_2, B(\ell) \cap \mathcal{A}(\Delta_Q)\right) \\
&+ \mu_{\beta,\theta,\gamma}\left(\eta^{\delta,\zeta_1}(\ell) = 0, \forall \ell \in [N_1, N_1 + R_1]\right) + \mu_{\beta,\theta,\gamma}\left(\eta^{\delta,\zeta_1}(\ell) = 0, \forall \ell \in [N_2 - R_1, N_2]\right).
\end{aligned} \tag{6.19}$$

Using Theorem 4.5, with $p = 2 + [(\log Q)/(\log \gamma^{-1})]$, on $\Omega_{RE} \supset \Omega_{51}$ we have

$$\begin{aligned}
&\mu_{\beta,\theta,\gamma}\left(\forall \ell \in [N_1, N_1 + R_1], \eta^{\delta,\zeta_1}(\ell) = 0\right) + \mu_{\beta,\theta,\gamma}\left(\forall \ell \in [N_2 - R_1, N_2], \eta^{\delta,\zeta_1}(\ell) = 0\right) \\
&\leq \frac{3^4 Q^5}{\gamma^{10}} e^{-\frac{\beta}{\gamma}\left\{\left(\frac{\kappa(\beta,\theta)}{4}\delta\zeta_4^3\right) \wedge \mathcal{F}^*\right\}}
\end{aligned} \tag{6.20}$$

where $R_1 = \frac{4(5+\mathcal{F}^*)}{\kappa(\beta,\theta)\delta\zeta_1^3}$ and we have used the fact that our choice of p entails $Q\gamma^{-1} \leq \gamma^{-p} \leq Q\gamma^{-2}$ to replace $3^4\gamma^{-5p}$ in (4.21) by $3^4 Q^5 \gamma^{-10}$ in (6.20).

Recalling (3.5) and using that $\eta^{\delta,\zeta_1}(\ell_1) = \bar\eta_1$ implies that on the left of ℓ_1

$$\left| E(m^{\delta^*}_{\gamma^{-1}(\ell_1-2,\ell_1-1]}(\sigma), m^{\delta^*}_{\gamma^{-1}(\ell_1-1,\ell_1]}(\sigma')) - E(m^{\delta^*}_{\gamma^{-1}(\ell_1-2,\ell_1-1]}(\sigma), m^{\delta^*}_{T^{\frac{1-\bar\eta}{2}}\beta,\gamma^{-1}(\ell_1-1,\ell_1]}) \right| \leq \zeta_1 \tag{6.21}$$

for σ' such that $\eta^{\delta,\zeta_1}(\ell_1) = \eta^{\delta,\zeta_1}(\ell_1)(\sigma'_{\gamma^{-1}(\ell_1-1,\ell_1]}) = \bar\eta_1$ and similarly on the right of ℓ_2, we get

$$\begin{aligned}
&\mu_{\beta,\theta,\gamma}\left(\eta^{\delta,\zeta_1}(\ell_1) = \bar\eta_1, \eta^{\delta,\zeta_1}(\ell_2) = \bar\eta_2, B(\ell), \mathcal{A}(\Delta_Q)\right) \\
&\leq e^{\frac{\beta}{\gamma}(4\zeta_1 + \delta^*)} \frac{Z^{0,0}_{[\ell_1,\ell_2]}\left(\eta^{\delta,\zeta_1}(\ell_1) = \bar\eta_1, \eta^{\delta,\zeta_1}(\ell_2) = \bar\eta_2, B(\ell), \mathcal{A}([\ell_1,\ell_2])\right)}{Z^{0,0}_{[\ell_1,\ell_2]}\left(\eta^{\delta,\zeta_1}(\ell_1) = \bar\eta_1, \eta^{\delta,\zeta_1}(\ell_2) = \bar\eta_2\right)}.
\end{aligned} \tag{6.22}$$

To get an upper bound for (6.22), we restrict the denominator to profiles that we expect to be typical for the Gibbs measure under the constraint $\eta^{\delta,\zeta_1}(\ell_1) = \bar\eta_1, \eta^{\delta,\zeta_1}(\ell_2) = \bar\eta_2$ given that we are inside a positive elongation. Without the constraints, taking into account only the presence of a positive elongation, the profiles we expect to be typical are of course $\eta^{\delta,\zeta_4} = 1$ for all $\ell \in [\ell_1, \ell_2]$, this is also the case for $(\bar\eta_1, \bar\eta_2) = (+1, +1)$. To take into account the cases $(\bar\eta_1, \bar\eta_2) \neq (+1, +1)$, we leave intervals $[\ell_1, \ell_1 + L_0]$ and/or $[\ell_2 - L_0, \ell_2]$, where L_0 is a positive integer to be chosen later to allow the profiles to change from, say $\eta^{\delta,\zeta_1}(\ell_1) = \bar\eta_1 = -1$

to $\eta^{\delta,\zeta_1}(\ell_1+L_0)=+1$. We actually require the profiles to satisfy $\eta^{\delta,\zeta_5}(\ell_1+L_0)=+1$, with $\zeta_5<\zeta_1$ for a reason that we explain later.

To proceed on this it is convenient to define: given $N_1\le\ell_1<\ell_2\le N_2$ and $\bar\eta\in\{-1,+1\}$, for $i=1$ and $i=5$

$$\widetilde{\mathcal{R}}_i(\bar\eta,\ell_1,\ell_2)=\Big\{m^{\delta^*}_{[\ell_1,\ell_2]}\colon \eta^{\delta,\zeta_i}(\ell_1)=\bar\eta=\eta^{\delta,\zeta_i}(\ell_2)\Big\},\tag{6.23}$$

$$\mathcal{E}(+1,\ell_1,\ell_2,\bar\eta_1,\bar\eta_2)\equiv\begin{cases}\widetilde{\mathcal{R}}_1(+1,\ell_1,\ell_2)\cap\{\eta^{\delta,\zeta_5}(\ell_1+L_0)=\eta^{\delta,\zeta_5}(\ell_2-L_0)=+1\}\,\text{for}\,\bar\eta_1=-1=\bar\eta_2;\\ \widetilde{\mathcal{R}}_1(+1,\ell_1,\ell_2)\cap\{\eta^{\delta,\zeta_5}(\ell_2-L_0)=+1\}\,\text{for}\,\bar\eta_1=1,\bar\eta_2=-1;\\ \widetilde{\mathcal{R}}_1(+1,\ell_1,\ell_2)\cap\{\eta^{\delta,\zeta_5}(\ell_1+L_0)=+1\}\,\text{for}\,\bar\eta_1=-1,\bar\eta_2=1,\end{cases}\tag{6.24}$$

where the $+1$ on the left hand side is associated to the sign of the elongation, chosen here to be positive. We then estimate the expression in (6.22) as in Section 4 (see (4.39)), to obtain

$$\begin{aligned}&\mu_{\beta,\theta,\gamma}\left(\eta^{\delta,\zeta_1}(\ell_1)=\bar\eta_1,\eta^{\delta,\zeta_1}(\ell_2)=\bar\eta_2,B(\ell),\mathcal{A}(\Delta_Q)\right)\\ &\le e^{\frac{\beta}{\gamma}4(\zeta_1+\zeta_5+2\delta^*)}\frac{Z^{0,0}_{[\ell_1,\ell_2]}\left(\eta^{\delta,\zeta_1}(\ell_1)=\bar\eta_1,\eta^{\delta,\zeta_1}(\ell_2)=\bar\eta_2,B(\ell),\mathcal{A}([\ell_1,\ell_2])\right)}{Z^{0,0}_{[\ell_1,\ell_2]}\big(\mathcal{E}(+1,\ell_1,\ell_2,\bar\eta_1,\bar\eta_2)\big)}\times\\ &\quad\times\frac{Z^{0,m_+}_{[\ell_1,\ell_1+L_0-1]}\big(\eta^{\delta,\zeta_1}(\ell_1)=+1\big)}{Z^{0,m_+}_{[\ell_1,\ell_1+L_0-1]}(\eta^{\delta,\zeta_1}(\ell_1)=\bar\eta_1)}\frac{Z^{m_+,0}_{[\ell_2-L_0+1,\ell_2]}\big(\eta^{\delta,\zeta_1}(\ell_2)=+1\big)}{Z^{m_+,0}_{[\ell_2-L_0+1,\ell_2]}(\eta^{\delta,\zeta_1}(\ell_2)=\bar\eta_2)}.\end{aligned}\tag{6.25}$$

To apply Lemma 7.3 to the last two terms in (6.25), we take

$$L_0=\frac{1}{\alpha(\beta,\theta,\zeta_0)}\log\frac{\delta^*}{\gamma}\ge\frac{1}{\alpha(\beta,\theta)}\log\frac{\delta^*}{8\gamma}.\tag{6.26}$$

Replacing the f_{11} of Lemma 7.3 by f_1 defined in (6.12), since here $\sqrt{\frac{\gamma}{\delta^*}}\ge\delta^*$, we obtain

$$\begin{aligned}\mu_{\beta,\theta,\gamma}\big(\eta^{\delta,\zeta_1}(\ell_1)=\bar\eta_1,\eta^{\delta,\zeta_1}(\ell_2)=\bar\eta_2,B(\ell),\mathcal{A}\big)&\le e^{\frac{\beta}{\gamma}4(\zeta_1+\zeta_5+2\delta^*)}e^{\frac{\beta}{\gamma}(\mathcal{F}^*+2f_1)\left[\frac12(|\bar\eta_1-1|+|\bar\eta_2-1|)\right]}\times\\ &\times\frac{Z^{0,0}_{[\ell_1,\ell_2]}\left(\eta^{\delta,\zeta_1}(\ell_1)=\bar\eta_1,\eta^{\delta,\zeta_1}(\ell_2)=\bar\eta_2,B(\ell),\mathcal{A}([\ell_1,\ell_2])\right)}{Z^{0,0}_{[\ell_1,\ell_2]}\big(\mathcal{E}(+1,\ell_1,\ell_2,\bar\eta_1,\bar\eta_2)\big)}.\end{aligned}\tag{6.27}$$

To treat the last term in (6.27), we make a partition of the set of profiles in $\mathcal{A}([\ell_1,\ell_2])$ distinguishing the profiles according to the number and the location of the changes of phases in $[\ell_1,\ell_2]$.

$$\mathcal{A}([\ell_1,\ell_2])=\cup_{n=0}^{\bar N}\cup_{\{A:|A|=n\}}\mathcal{A}([\ell_1,\ell_2],A,n),\tag{6.28}$$

where $\bar N$ is the number of the $\frac{\epsilon}{\gamma}$ blocks in $[\ell_1,\ell_2]$, i.e.,

$$\bar N=\Big[|\ell_2-\ell_1|\frac{\gamma}{\epsilon}\Big]=\left[\left(\frac{\epsilon}{\gamma}[\alpha_1^*-\alpha_0^*]-2R_1\right)\frac{\gamma}{\epsilon}\right]=\Big[[\alpha_1^*-\alpha_0^*]-2\frac{\gamma}{\epsilon}R_1\Big],\tag{6.29}$$

$[x]$ is the integer part of x, and the first equality follows from (6.7) that entails $\epsilon/\gamma>2R_1$. Moreover in (6.28), $A\subseteq\Big\{\frac{Q_1}{\epsilon}+1,\frac{Q_1}{\epsilon}+2,\dots,\frac{Q_2}{\epsilon}-2,\frac{Q_2}{\epsilon}-1\Big\}$. The integer n represents the cardinality of the set A and therefore the number of $\frac{\epsilon}{\gamma}$ blocks where, in each one of them, there is one and only one interval of length $2R_1$ in which only one change of phases occurs. Recall that in the definition of $\mathcal{A}([\ell_1,\ell_2])$ cf. (4.1) the r_i, $i=1,..,\bar N$ indicate that in $[r_i\frac{\epsilon}{\gamma},(r_i+1)\frac{\epsilon}{\gamma}]$ there is q_i, such that in $[q_i-R_1,q_i+R_1]$ there is only one change of phases and there is no change in $[r_i\frac{\epsilon}{\gamma},(r_i+1)\frac{\epsilon}{\gamma}]\setminus[q_i-R_1,q_i+R_1]$. The notation $\mathcal{A}([\ell_1,\ell_2],A,n)$ is self-explanatory. When there is no ambiguity we denote $\mathcal{A}([\ell_1,\ell_2],A,n)\equiv\mathcal{A}(A,n)$. Going back to (6.17), taking into account (6.20), (6.27) and (6.28) on Ω_{51}, we have that

$$\mu_{\beta,\theta,\gamma}\left(\exists\ell\in I,\eta(\ell)\neq 1\right)\leq 2\left(\frac{3Q}{\gamma^2}\right)^5 e^{-\frac{\beta}{\gamma}\left\{\left(\frac{\kappa(\beta,\theta)}{4}\delta\zeta_4^3\right)\wedge\mathcal{F}^*\right\}}+e^{\frac{4\beta(\zeta_1+\zeta_5+2\delta^*)}{\gamma}}\sum_{n=0}^{\bar N}\mathcal{S}_n, \tag{6.30}$$

where

$$\mathcal{S}_n=e^{\frac{\beta}{\gamma}\frac{1}{2}(|\bar\eta_1-1|+|\bar\eta_2-1|)(\mathcal{F}^*+2f_1)}\times$$
$$\sum_{\ell\in I}\sum_{\bar\eta_1,\bar\eta_2\in\{-1,1\}^2}\sum_{\ell_1=N_1}^{N_1+R_1}\sum_{\ell_2=N_2-R_1}^{N_2}\sum_{A,|A|=n}\frac{Z^{0,0}_{[\ell_1,\ell_2]}\left(\eta^{\delta,\zeta_1}(\ell_1)=\bar\eta_1,\eta^{\delta,\zeta_1}(\ell_2)=\bar\eta_2,\mathcal{A}(A,n),B(\ell)\right)}{Z^{0,0}_{[\ell_1,\ell_2]}\left(\mathcal{E}(+1,\ell_1,\ell_2,\bar\eta_1,\bar\eta_2)\right)}. \tag{6.31}$$

We must estimate $\mathcal{S}_n$ for any n, taking care of the probability subspaces on which we are working. At first sight one could have thought that the presence of n–changes of phases would simplify the analysis, at least for n large, due to the presence of terms proportional to $\exp(-n\frac{\beta}{\gamma}\mathcal{F}^*)$. Unfortunately this is not the right picture since we must control the local contributions of the magnetic field. For $\Delta'\subseteq[\alpha_0^*,\alpha_1^*]$ we only know that $\sum_{\alpha\in\Delta'}\chi(\alpha)\geq-2\left(\mathcal{F}^*-f\right)$. The analysis is therefore more delicate, being summarized in Lemmas 6.3 and 6.4 below.

To complete the estimate of the expression in (6.30) we need to sum up the upper bounds of the $\mathcal{S}_n$, cf. Lemmas 6.3 and 6.4. For this we use the following inequalities that follow from Taylor formula: for all $x>0$,

$$(1+x)^N-\sum_{k=0}^{l}\binom{N}{k}x^k\leq\frac{(xN)^{l+1}}{(l+1)}e^{(N-l-1)x}\leq(xN)^{l+1}e^{Nx}. \tag{6.32}$$

Recall that $\bar N\frac{\epsilon}{\gamma}=\frac{\epsilon}{\gamma}[(\ell_2-\ell_1)\frac{\gamma}{\epsilon}]\leq(\ell_2-\ell_1)\leq\frac{2Q}{\gamma}$; $|I|\leq\frac{2Q}{\gamma}$. To simplify the computations, when necessary, we take half of negative part in the exponential to compensate the positive part. We also use $\zeta_5^{\frac{1-z}{2}}>\zeta_5$. Denote $\Omega_5=\Omega_{51}\cap\Omega_{53}$, with Ω_{53} as in Lemma 6.3. After some easy however lengthy computations, using (6.11), we see that on $\Omega_5\cap\Omega^+(\epsilon,Q)$,

$$\mu_{\beta,\theta,\gamma}\left(\exists\ell\in I,\eta(\ell)\neq 1\right)\leq 2\left(\frac{3Q}{\gamma^2}\right)^5 e^{-\frac{\beta}{\gamma}\left\{\left(\frac{\kappa(\beta,\theta)}{4}\delta\zeta_4^3\right)\wedge\mathcal{F}^*\right\}}+28|R_1|^2\left(\frac{2Q}{\gamma}\right)^5 e^{-\frac{\beta}{\gamma}\frac{\epsilon^{1/4}}{5}}e^{\left\{\frac{4Q}{\gamma}e^{-\frac{\beta}{\gamma}\frac{\epsilon^{1/4}}{5}}\right\}} \tag{6.33}$$

which is (6.16). (6.15) follows from (6.34) since $I\!P[\Omega_{RE}]\geq 1-\gamma^2$. This ends the proof of Proposition 6.2 if we assume Lemmas 6.3 and 6.4. ■

Lemma 6.3 . (n=0) *For f_1 given by (6.12) and f_2 given by (6.13), for $\frac{1}{2}>z>0$, there exists Ω_{53} with*

$$I\!P[\Omega_{53}]\geq 1-4\gamma^2-\frac{2e^{-\frac{\beta^2}{8Q\zeta_5^z c^2(\beta,\theta)}}}{1-e^{-\frac{\beta^2}{8Q\zeta_5^z c^2(\beta,\theta)}}} \tag{6.34}$$

such that on $\Omega^+(\epsilon,Q)\cap\Omega_{53}\cap\Omega_{51}$,

$$\mathcal{S}_0\leq R_1^2|I|e^{\frac{\beta}{\gamma}\left(4f_1+f_2\right)}Ge^{-\frac{\beta}{\gamma}\epsilon^{1/4}}, \tag{6.35}$$

where

$$G = e^{\frac{\beta}{\gamma}(4\zeta_5+2f_1+16\zeta_5^{\frac{1-z}{2}})}\Big(1+e^{-\frac{\beta}{\gamma}\frac{\kappa(\beta,\theta)}{4}\delta\zeta_5^3}\Big). \tag{6.36}$$

Proof. In this case the profiles have no change of phases, therefore we must have $\bar\eta_1 = \bar\eta_2$. If $\bar\eta_1 = \bar\eta_2 = +1$ and we take $|A| = 0$ in (6.28), we have

$$\big\{\eta^{\delta,\zeta_1}(\ell_1) = \bar\eta_1, \eta^{\delta,\zeta_1}(\ell_2) = \bar\eta_2, \mathcal{A}([\ell_1,\ell_2], A, 0), B(\ell)\big\} = \emptyset$$

and there is nothing to prove. So we consider the case $\bar\eta_1 = \bar\eta_2 = -1$. With this choice the set to estimate in (6.31) is

$$\begin{aligned}&\big\{\eta^{\delta,\zeta_1}(\ell_1) = \bar\eta_1, \eta^{\delta,\zeta_1}(\ell_2) = \bar\eta_2, \mathcal{A}([\ell_1,\ell_2], A, 0), B(\ell)\big\}\\ &= \Big\{\eta^{\delta,\zeta_1}(\ell_1) = \eta^{\delta,\zeta_1}(\ell_2) = -1, \forall \tilde\ell \in [\ell_1+1, \ell_2-1], \eta^{\delta,\zeta_4}(\tilde\ell) = -1\Big\} \equiv R_{1,4}(-1,[\ell_1,\ell_2]).\end{aligned} \tag{6.37}$$

To estimate the quotient of the two partition functions in (6.31), we need to extract the contribution of the magnetic field as we did in the proof of Proposition 4.7, see (4.47). If, however, we proceed exactly as it was done there, we should get ζ_4 instead of ζ_5 on the right hand side of (6.34). Since ζ_4 is fixed and Q will be large at the end, such an estimate would be useless. Therefore an extra step is needed. For $\bar\eta = \pm 1$, $\ell_1' < \ell_2'$ such that $\ell_1' - \ell_2' > 4\ell_0 + 8$, $\ell_0 > 0$ to be chosen later, let us denote

$$R_5(\bar\eta, [\ell_1', \ell_2']) \quad = \Big\{m^{\delta^*}_{[\ell_1',\ell_2']} \colon \eta^{\delta,\zeta_5}(\ell) = \bar\eta,\ \forall \ell \in [\ell_1', \ell_2']\Big\} \tag{6.38}$$

and

$$R_{1,4,5}(-1,[\ell_1,\ell_2]) \equiv R_{1,4,5}(-1,[\ell_1,\ell_2])(\ell_0) = R_{1,4}(-1,[\ell_1,\ell_2]) \cap R_5(-1,[\ell_1+\ell_0, \ell_2-\ell_0]). \tag{6.39}$$

Then we write, see (6.31) and (6.37)

$$\frac{Z^{0,0}_{[\ell_1,\ell_2]}(R_{1,4}(-1,[\ell_1,\ell_2]))}{Z^{0,0}_{[\ell_1,\ell_2]}\big(\mathcal{E}(+1,\ell_1,\ell_2,-1,-1)\big)} = \frac{Z^{0,0}_{[\ell_1,\ell_2]}(R_{1,4,5}(-1,[\ell_1,\ell_2]))}{Z^{0,0}_{[\ell_1,\ell_2]}\big(\mathcal{E}(+1,\ell_1,\ell_2,-1,-1)\big)} \times \frac{Z^{0,0}_{[\ell_1,\ell_2]}(R_{1,4}(-1,[\ell_1,\ell_2]))}{Z^{0,0}_{[\ell_1,\ell_2]}(R_{1,4,5}(-1,[\ell_1,\ell_2]))}. \tag{6.40}$$

The choice of ℓ_0 is related to the needed length to go from $\eta^{\delta,\zeta_4}(0) = \eta$ to $\eta^{\delta,\zeta_5}(l_0) = \eta$ knowing that we are within a run of $\eta^{\delta,\zeta_4} = \eta$. It is determined estimating the last term in (6.40) from which we start. Since $R_{1,4,5}(-1,[\ell_1,\ell_2]) \subseteq R_{1,4}(-1,[\ell_1,\ell_2])$ we have

$$1 \le \frac{Z^{0,0}_{[\ell_1,\ell_2]}(R_{1,4}(-1,[\ell_1,\ell_2]))}{Z^{0,0}_{[\ell_1,\ell_2]}(R_{1,4,5}(-1,[\ell_1,\ell_2]))} \le 1 + \frac{Z^{0,0}_{[\ell_1,\ell_2]}(R_{1,4}(-1,[\ell_1,\ell_2]) \cap (R_{1,4,5}(-1,[\ell_1,\ell_2]))^c)}{Z^{0,0}_{[\ell_1,\ell_2]}(R_{1,4,5}(-1,[\ell_1,\ell_2]))} \tag{6.41}$$

From Corollary 7.5 it follows that on $\Omega_{RE} \supset \Omega_{51}$, if

$$\delta\zeta_5^3 > \frac{512(1+\theta)}{\kappa(\beta,\theta)\alpha(\beta,\theta,\zeta_0)}\sqrt{\frac{\gamma}{\delta^*}}\log\frac{\delta^*}{\gamma}, \tag{6.42}$$

where $\alpha(\beta,\theta,\zeta_0)$ is defined in (6.1), and ℓ_0 is chosen* as L_0 defined in (6.26), then

$$\frac{Z^{0,0}_{[\ell_1,\ell_2]}(R_{1,4}(-1,[\ell_1,\ell_2]) \cap (R_{1,4,5}(-1,[\ell_1,\ell_2]))^c)}{Z^{0,0}_{[\ell_1,\ell_2]}(R_{1,4,5}(-1,[\ell_1,\ell_2]))} \le e^{-\frac{\beta}{\gamma}\frac{\kappa(\beta,\theta)}{4}\delta\zeta_5^3}, \tag{6.43}$$

* The L_0 chosen in (6.26) is obtained setting $d = 2$ in Corollary 7.5.

uniformly with respect to $[N_1,N_2]\subseteq[-Q\gamma^{-1},Q\gamma^{-1}]$, $\ell_1\in[N_1,N_1+R_1]$, and $\ell_2\in[N_2-R_1,N_2]$. To treat the first term in the right hand side of (6.40), recalling that, see (6.24),

$$\mathcal{E}(+1,\ell_1,\ell_2,-1,-1)=\widetilde{\mathcal{R}}_1(+1,[\ell_1,\ell_2])\cap\widetilde{\mathcal{R}}_5(+1,[\ell_1+L_0,\ell_2-L_0])$$

we first split the interval $[\ell_1,\ell_2]$ into three intervals $[\ell_1,\ell_1+L_0-1]$, $[\ell_1+L_0,\ell_2-L_0]$ and $[\ell_2-L_0+1,\ell_2]$. On the first and the last interval, we use a block spin representation, the rough estimate Lemma 3.3 with $p=2+[(\log Q)/(\log(1/\gamma))]$, and then the symmetry $m\to Tm$ of the block spin model. Thus, on $\Omega_{RE}=\Omega_{RE}(\gamma,\delta^*,Q)\supset\Omega_{51}$, we get for the first term

$$\begin{aligned}&\frac{Z^{0,m_-}_{[\ell_1,\ell_1+L_0-1]}(\eta^{\delta,\zeta_1}(\ell_1)=-1,\forall\ell\in[\ell_1+1,\ell_1+L_0-1],\,\eta^{\delta,\zeta_4}(\ell)=-1)}{Z^{0,m_+}_{[\ell_1,\ell_1+L_0-1]}(\eta^{\delta,\zeta_1}(\ell_1)=1)}\\&\le e^{\frac{\beta}{\gamma}6(1+\theta)L_0(\delta^*\vee\sqrt{\frac{\gamma}{\delta^*}})}=e^{\frac{\beta}{\gamma}6(1+\theta)\frac{1}{\alpha(\beta,\theta,\zeta_0)}(\log\frac{\delta^*}{\gamma})(\delta^*\vee\sqrt{\frac{\gamma}{\delta^*}})}\le e^{\frac{\beta}{\gamma}f_1}\end{aligned}\tag{6.44}$$

and in the very same way for the other term. Therefore, on $\Omega_{RE}\supset\Omega_{51}$, we have

$$\begin{aligned}\frac{Z^{0,0}_{[\ell_1,\ell_2]}(R_{1,4,5}(-1,[\ell_1,\ell_2])(L_0))}{Z^{0,0}_{[\ell_1,\ell_2]}(\mathcal{E}(+1,\ell_1,\ell_2,-1,-1)))}&\le e^{\frac{\beta}{\gamma}4\zeta_5}e^{\frac{\beta}{\gamma}2f_1}\frac{Z^{0,0}_{[\ell_1+L_0,\ell_2-L_0]}(R_5(-1,[\ell_1+L_0,\ell_2-L_0]))}{Z^{0,0}_{[\ell_1+L_0,\ell_2-L_0]}(R_5(1,[\ell_1+L_0,\ell_2-L_0]))}\\&=e^{\frac{\beta}{\gamma}4\zeta_5}e^{\frac{\beta}{\gamma}2f_1}e^{-\beta\Delta\mathcal{G}(m^{\delta^*}_{\beta,[\ell_1+L_0,\ell_2-L_0]})}\frac{Z_{-1,0}([\ell_1+L_0,\ell_2-L_0])}{Z_{+1,0,}([\ell_1+L_0,\ell_2-L_0])}\end{aligned}\tag{6.45}$$

where $\Delta\mathcal{G}(m^{\delta^*}_{\beta,[\ell_1+L_0,\ell_2-L_0]})=\sum_{x\in\mathcal{C}_{\delta^*}([\ell_1+L_0,\ell_2-L_0])}X(x)$ and the remaining term is defined in (4.48) with $R(\eta)$ replaced by $R_5(+,[\ell_1+L_0,\ell_2-L_0])$. The equality in (6.45) is obtained by extracting the main contribution of the random field as we did in (4.47).

To estimate the last term in (6.45), we use Lemma 4.8 with ζ_4 replaced by ζ_5, $a=\zeta_5^{\frac{1-z}{2}}$, for some $0<z<1/2$. Using (6.4), this entails that on a subset Ω_{54}, with

$$I\!P[\Omega_{54}]\ge 1-\frac{2e^{-\frac{\beta^2}{8Q\zeta_5^z c^2(\beta,\theta)}}}{1-e^{-\frac{\beta^2}{8Q\zeta_5^z c^2(\beta,\theta)}}}\tag{6.46}$$

we have

$$\max_{[\ell_1,\ell_2]\subset[-Q\gamma^{-1},Q\gamma^{-1}]}\frac{Z_{-1,0}([\ell_1+L_0,\ell_2-L_0])}{Z_{+1,0,}([\ell_1+L_0,\ell_2-L_0])}\le e^{\frac{\beta}{\gamma}16\zeta_5^{\frac{1-z}{2}}}.\tag{6.47}$$

Some care is necessary to estimate the contribution of the first factor of the r.h.s. of (6.45). By definition, on $\Omega^+(\epsilon,Q)$, we have $\Delta^+\mathcal{G}(m^{\delta^*}_{\beta,[\alpha_0^*,\alpha_1^*]})\ge 2\mathcal{F}^*+f\equiv 2\mathcal{F}^*+\epsilon^{1/4}$. However the random contribution we extracted in (6.45) is merely $\Delta^+\mathcal{G}(m^{\delta^*}_{\beta,[\ell_1+L_0,\ell_2-L_0]})$, with $\ell_1\in[N_1,N_1+R_1]$, $\ell_2\in[N_2-R_1,N_2]$. It is easy to check that there exists a subset Ω_{55}, that depends on (γ,δ^*,Q) with $I\!P[\Omega_{55}]\ge 1-8\gamma^2$, such that on Ω_{55}, uniformly with respect to $[N_1,N_2]\subseteq[-Q\gamma^{-1},Q\gamma^{-1}]$, and $\ell_1\in[N_1,N_1+R_1]$, $\ell_2\in[N_2-R_1,N_2]$, we have

$$e^{-\beta\Delta\mathcal{G}(m^{\delta^*}_{\beta,[\ell_1+L_0,\ell_2-L_0]})}\le e^{-\frac{\beta}{\gamma}\left(2\mathcal{F}^*+\epsilon^{1/4}-f_2\right)},\tag{6.48}$$

where f_2 is given in (6.13). Collecting (6.44), (6.45) and (6.47), on $\Omega^+(Q,f)\cap\Omega_{RE}\cap\Omega_{54}\cap\Omega_{55}$, we have

$$\frac{Z^{0,0}_{[\ell_1,\ell_2]}(R_{1,4,5}(-1,[\ell_1,\ell_2]))}{Z^{0,0}_{[\ell_1,\ell_2]}(\mathcal{E}(+1,\ell_1,\ell_2,\bar\eta_1,\bar\eta_2))}\le e^{+\frac{\beta}{\gamma}\left(4\zeta_5+2f_1+16\zeta_5^{\frac{1-z}{2}}\right)}e^{-\frac{\beta}{\gamma}\left(2\mathcal{F}^*+\epsilon^{1/4}-f_2\right)}\tag{6.49}$$

Now, collecting (6.40), (6.41), (6.43) and (6.49), and calling $\Omega_{53}=\Omega_{54}\cap\Omega_{55}$, on $\Omega^+(Q,f)\cap\Omega_{53}\cap\Omega_{51}$ we have

$$\frac{Z^{0,0}_{[\ell_1,\ell_2]}\left(\eta^{\delta,\zeta_1}(\ell_1)=\bar\eta_1,\eta^{\delta,\zeta_1}(\ell_2)=\bar\eta_2,\mathcal{A}([\ell_1,\ell_2],A,0),B(\ell)\right)}{Z^{0,0}_{[\ell_1,\ell_2]}\big(\mathcal{E}(+1,[\ell_1,\ell_2],\bar\eta_1,\bar\eta_2)\big)}\le Ge^{-\frac{\beta}{\gamma}\left(2\mathcal{F}^*+\epsilon^{1/4}-f_2\right)} \tag{6.50}$$

from which we get easily (6.35). ■

Lemma 6.4 . **($n\ge 1$)** *On $\Omega(54)\cap\Omega_{51}\cap\Omega^+(\epsilon,Q)\cap\Omega_\epsilon$, we have*

$$\mathcal{S}_1\le R_1^2|I|[\ell_2-\ell_1]e^{\frac{\beta}{\gamma}(2f_1+f_2+f_3+4\zeta_1)}Ge^{-\frac{\beta}{\gamma}\epsilon^{1/4}}, \tag{6.51}$$

$$\mathcal{S}_n\le R_1^2|I|\binom{\bar N}{n}\left(\frac{\epsilon}{\gamma}\right)^n e^{n\frac{\beta}{\gamma}(f_3+4\zeta_1)}G^{\frac{n}{2}}e^{-\frac{\beta}{\gamma}\frac{n}{2}\epsilon^{1/4}},\quad n\ even\quad \bar\eta_1=\bar\eta_2=1, \tag{6.52}$$

$$\mathcal{S}_n\le R_1^2|I|e^{\frac{\beta}{\gamma}(4f_1+2f_2)}\binom{\bar N}{n}\left(\frac{\epsilon}{\gamma}\right)^n G^{\frac{n}{2}+1}e^{n\frac{\beta}{\gamma}(f_3+4\zeta_1)}e^{-\frac{\beta}{\gamma}\left(\epsilon^{1/4}[3-n]^++\frac{n-2}{2}\epsilon^{1/4}\right)},\quad n\ge 2\quad \bar\eta_1=\bar\eta_2=-1, \tag{6.53}$$

$$\mathcal{S}_n\le R_1^2|I|e^{\frac{\beta}{\gamma}2f_1}\binom{\bar N}{n}\left(\frac{\epsilon}{\gamma}\right)^n e^{n\frac{\beta}{\gamma}[f_3+4\zeta_1]}G^{\frac{n+1}{2}}e^{-\frac{\beta}{\gamma}\left(\frac{n-1}{2}\epsilon^{1/4}\right)},\quad n>1\quad odd, \tag{6.54}$$

where f_1 is defined in (6.12) f_2 in (6.13), f_3 in (6.14) and G in (6.36).

Proof. We prove explicitly the case $n=1$. The $n>1$ can be done similarly following the general strategy outlined later. When $n=1$ the magnetization profiles have only one change of phases and are therefore compatible only with boundary conditions $\bar\eta_1\neq\bar\eta_2$. Suppose that $\bar\eta_1=-\bar\eta_2=1$. The reverse case is done similarly. Denote by r_1 the index of the $\frac{\epsilon}{\gamma}$ block in which the change of phases occurs. When $[r_1\frac{\epsilon}{\gamma}-R_1,(r_1+1)\frac{\epsilon}{\gamma}+R_1)]\subseteq[N_2-R_1-\frac{\rho}{\gamma},\ell_2]$ we have $\left\{\eta^{\delta,\zeta_1}(\ell_1)=1,\eta^{\delta,\zeta_1}(\ell_2)=-1,\mathcal{A}_{[\ell_1,\ell_2]}(A,1),B(\ell)\right\}=\emptyset$ since $\ell\in[N_1+R_1+\frac{\rho}{\gamma},N_2-R_1-\frac{\rho}{\gamma}]$. Therefore we may assume that $[r_1\frac{\epsilon}{\gamma}-R_1,(r_1+1)\frac{\epsilon}{\gamma}+R_1)]\subseteq[\ell_1,N_2-R_1-\frac{\rho}{\gamma}]$. We split the interval $[\ell_1,\ell_2]$ into three adjacent intervals $[\ell_1,q_1-R_1]$, $[q_1-R_1+1,q_1+R_1-1]$ and $[q_1+R_1,\ell_2]$, assuming that the change of phases happens in the interval $[q_1-R_1,q_1+R_1]$. Recalling Definition 4.1 in Section 4, one has $\eta^{\delta,\zeta_1}(\tilde\ell)$ is equal to $+1$ for $\tilde\ell=\ell_1$ and for $\tilde\ell=q_1-R_1$ while it is equal to -1 for $\tilde\ell=q_1+R_1$. We associate the interactions between the intervals to the middle interval. Suitably restricting the denominator we get

$$\begin{aligned}&\frac{Z^{0,0}_{[\ell_1,\ell_2]}\left(\eta^{\delta,\zeta_1}(\ell_1)=+1,\eta^{\delta,\zeta_1}(\ell_2)=-1,\mathcal{A}_{[\ell_1,\ell_2]}(A,1),B(\ell)\right)}{Z^{0,0}_{[\ell_1,\ell_2]}\big(\widetilde{\mathcal{R}}_1(+1,\ell_1,\ell_2)\cap\{\eta^{\delta,\zeta_5}(\ell_2-L_0)=+1\}\big)}\le e^{\frac{\beta}{\gamma}4\zeta_1}\times\\ &\frac{Z^{0,0}_{[\ell_1,q_1-R_1]}\big(R_{1,4}(+1,[\ell_1,q_1-R_1])\big)}{Z^{0,0}_{[\ell_1,q_1-R_1]}\big(\tilde{\mathcal{R}}_1(+1,\ell_1,q_1-R_1)\big)}\times\frac{Z^{m_+,m_-}_{[q_1-R_1+1,q_1+R_1-1]}}{Z^{m_+,m_+}_{[q_1-R_1+1,q_1+R_1-1]}\big(\widetilde{\mathcal{R}}_1(+1,q_1-R_1+1,q_1+R_1-1)\big)}\times\\ &\frac{Z^{0,0}_{[q_1+R_1,\ell_2]}\big(R_{1,4}(-1,[q_1+R_1,\ell_2])\big)}{Z^{0,0}_{[q_1+R_1,\ell_2]}\big(\widetilde{\mathcal{R}}_1(+1,q_1+R_1,\ell_2)\cap\{\eta^{\delta,\zeta_5}(\ell_2-L_0)=+1\}\big)}.\end{aligned} \tag{6.55}$$

Since $R_{1,4}(+1,[\ell_1,q_1-R_1])\subseteq\tilde{\mathcal{R}}_1(+1,\ell_1,q_1-R_1)$, see (6.37) and (6.23), the first ratio on the right hand side of (6.55) is smaller than 1. The second ratio in (6.55) is treated in a similar way as in the proof of Lemma 7.3. However, since the volume we are considering is $[q_1-R_1+1,q_1+R_1-1]$, the error terms that come from the block spin approximation and the rough estimates, see Lemma 3.3, are $e^{\frac{\beta}{\gamma}f_3}$ with f_3 given in (6.14). Therefore, on $\Omega_{RE} \supset \Omega_{51}$, uniformly with respect to the position of the change of phases in the interval $[-Q\gamma^{-1}, Q\gamma^{-1}]$, we have

$$\frac{Z^{m_+,m_-}_{[q_1-R_1+1,q_1+R_1-1]}}{Z^{m_+,m_+}_{[q_1-R_1+1,q_1+R_1-1]}\big(\widetilde{\mathcal{R}}_1(+1,q_1-R_1+1,q_1+R_1-1)\big)} \le e^{-\frac{\beta}{\gamma}(\mathcal{F}^*-f_3)}. \tag{6.56}$$

It remains to treat the last ratio in (6.55). We claim that on $\Omega_{54}\cap\Omega_{RE}\supset\Omega_{53}$, see just before (6.50), we have

$$\begin{aligned}&\frac{Z^{0,0}_{[q_1+R_1,\ell_2]}\big(R_{1,4}(-1,[q_1+R_1,\ell_2])\big)}{Z^{0,0}_{[q_1+R_1,\ell_2]}\big(\widetilde{\mathcal{R}}_1(+1,q_1+R_1,\ell_2)\cap\{\eta^{\delta,\zeta_5}(\ell_2-L_0)=+1\}\big)}\\&\le e^{\frac{\beta}{\gamma}4\zeta_5}e^{\frac{\beta}{\gamma}2f_1}e^{\frac{\beta}{\gamma}16\zeta_5^{\frac{1-z}{2}}}(1+e^{-\frac{\beta}{\gamma}\frac{\kappa(\beta,\theta)}{4}\delta\zeta_5^3})e^{-\beta\Delta\mathcal{G}(m^{\delta^*}_{\beta,[q_1+R_1+L_0+1,\ell_2-L_0-1]})}=Ge^{-\beta\Delta^+\mathcal{G}(m^{\delta^*}_{\beta,[q_1+R_1+L_0+1,\ell_2-L_0-1]})}\end{aligned}\tag{6.57}$$

where G is defined in (6.36).

Let us explain where those terms come from: We have written the ratio on the left hand side of (6.57) as a product of two ratios in the very same way as in (6.40). The second ratio gives the term $(1+e^{-\frac{\beta}{\gamma}\frac{\kappa(\beta,\theta)}{4}\delta\zeta_5^3})$ as in (6.41) and (6.43), and this occurs on Ω_{RE}. The first ratio was treated by first splitting the volume $[q_1+R_1,\ell_2]$ in three intervals $[q_1+R_1,q_1+R_1+L_0]$, $[q_1+R_1+L_0+1,\ell_2-L_0-1]$, and $[\ell_2-L_0,\ell_2]$. The first and the last intervals give us the term $\exp(\frac{\beta}{\gamma}2f_1)$ that comes from the rough estimates, and therefore occurs on Ω_{RE}. There is also a term $\exp(\frac{\beta}{\gamma}4\zeta_5)$ that comes from the interactions between the intervals. We remain with a term similar to the left hand side of (6.45) but in the volume $[q_1+R_1+L_0+1,\ell_2-L_0-1]$. It give us the term $\exp(\frac{\beta}{\gamma}16\zeta_5^{\frac{1-z}{2}})$ and the last term in (6.57) and this occurs on Ω_{54}. Collecting (6.55), (6.56), and (6.57), we have, on $\Omega_{51}\cap\Omega_{53}$

$$\begin{aligned}\mathcal{S}_1 &\le \sum_{\ell_1=N_1}^{N_1+R_1}\sum_{\ell_2=N_2-R_1}^{N_2}\sum_{r_1}^{*}\sum_{q_1=\frac{r_1}{\gamma}\epsilon}^{\frac{r_1+1}{\gamma}\epsilon}\sum_{\ell=q_1-R_1}^{\ell_2-\frac{\rho}{\gamma}} e^{\frac{\beta}{\gamma}(\mathcal{F}^*+2f_1)}e^{-\frac{\beta}{\gamma}(\mathcal{F}^*-f_3)}e^{\frac{\beta}{\gamma}4\zeta_1}\times\\&\quad\times Ge^{-\beta\Delta^+\mathcal{G}(m^{\delta^*}_{\beta,[q_1+R_1+L_0+1,\ell_2-L_0-1]})}\\&\le e^{\frac{\beta}{\gamma}(2f_1+f_3+4\zeta_1)}G\sum_{\ell_1=N_1}^{N_1+R_1}\sum_{\ell_2=N_2-R_1}^{N_2}\sum_{r_1}^{*}\sum_{q_1=\frac{r_1}{\gamma}\epsilon}^{\frac{r_1+1}{\gamma}\epsilon}e^{-\beta\Delta^+\mathcal{G}(m^{\delta^*}_{\beta,[q_1+R_1+L_0+1,\ell_2-L_0-1]})}.\end{aligned}\tag{6.58}$$

By $\sum^*_{r_1}$ we denote the sum over blocks of length $\frac{\epsilon}{\gamma}$ contained in the interval $[\ell_1, N_2-R_1-\frac{\rho}{\gamma}]$, so that $\sum^*_{r_1}1\le[\ell_2-\ell_1]\frac{\gamma}{\epsilon}\le\frac{2Q}{\epsilon}$. The contribution of the magnetic field in (6.58) is estimated using Lemma 5.15 and therefore occurs on Ω_ϵ. By definition, for any value under consideration of $q_1\in[\frac{r_1}{\gamma}\epsilon,\frac{r_1+1}{\gamma}\epsilon]$ and r_1, we in fact have that

$$\begin{aligned}\Delta\mathcal{G}(m^{\delta^*}_{\beta,[q_1+R_1+L_0+1,\ell_2-L_0-1]}) &= \sum_{x\in\mathcal{C}_{\delta^*}([q_1+R_1+L_0+1,\ell_2-L_0-1])}X(x)\\&=\frac{1}{\gamma}\sum_{\alpha=\frac{q_1+R_1+L_0+1}{\epsilon}\gamma}^{\alpha_1^*}\chi(\alpha)-\sum_{x\in\mathcal{C}_{\delta^*}(\ell_2-L_0,N_2)}X(x).\end{aligned}\tag{6.59}$$

The point is that α_1^* is the end of a positive elongation, that is a maximum, and by construction $|\alpha_1^* - \frac{q_1+R_1+L_0+1}{\epsilon}\gamma| \geq \frac{\rho}{\epsilon}$, recall $\rho = \epsilon^{\frac{1}{4(2+a)}}$. Therefore recalling Lemma 5.15 on $\Omega^+(Q,f) \cap \Omega_\epsilon \cap \Omega_{53}$ we have

$$\Delta\mathcal{G}(m^{\delta^*}_{\beta,[q_1+R_1+1,\ell_2]}) \geq \frac{1}{\gamma}\left(\epsilon^{1/4} - f_2\right). \tag{6.60}$$

This entails that on $\Omega_{51} \cap \Omega_{53} \cap \Omega^+(Q,f)$, see (6.58),

$$\mathcal{S}_1 \leq R_1^2 |I| [\ell_2 - \ell_1] e^{\frac{\beta}{\gamma}(2f_1+f_2+f_3+4\zeta_1)} G e^{-\frac{\beta}{\gamma}\epsilon^{1/4}}. \tag{6.61}$$

Remark: The fact that $N_2 = \alpha_1^*\epsilon/\gamma$ is a maximum is crucial here. In the case $\bar\eta_1 = -1, \bar\eta_2 = +1$, it would be essential that α_0^* is a minimum.

General strategy. The estimate of the terms with $n > 1$ in (6.30) is a simple modification of what we did in the cases $n = 0$ and $n = 1$. Let us summarize the general strategy:

a) Similarly to (6.55), if n changes occur we bound the ratio of two constrained partition functions by the product of ratios over the n intervals $[q_i - R_1, q_i + R_1]$ where the changes occur, a factor $e^{\frac{\beta}{\gamma}4n\zeta_1}$ and a product of ratios over the intervals with no change of phases.

b) The contribution of a ratio corresponding to a change of phases is estimated by $e^{\frac{\beta}{\gamma}[\mathcal{F}^* - f_3]}$ where f_3 is given in (6.14), as we did in (6.56). This holds on Ω_{RE} since a rough estimate is used and therefore on Ω_{51}.

c) The contribution of a ratio over an interval, say $\mathcal{J}$, where there is no change of phases is bounded by 1 when the profile is ζ_4-near m_β, that is for a run of $\eta^{\delta,\zeta_4} = +1$. If, instead, the profile gives a run of $\eta^{\delta,\zeta_4} = -1$, as in (6.57), then the corresponding ratio is bounded from above by

$$e^{\frac{\beta}{\gamma}(4\zeta_5 + 2f_1 + 16\zeta_5^{\frac{1-z}{2}})}(1 + e^{-\frac{\beta}{\gamma}\frac{\kappa(\beta,\theta)}{4}\delta\zeta_5^3}) e^{-\frac{\beta}{\gamma}\Delta\mathcal{G}(m^{\delta^*}_{\beta,\mathcal{J}})} = G e^{-\frac{\beta}{\gamma}\Delta\mathcal{G}(m^{\delta^*}_{\beta,\mathcal{J}})}, \tag{6.62}$$

on $\Omega_{51} \cap \Omega_{53}$.

d) The contribution of $\Delta\mathcal{G}(m^{\delta^*}_{\beta,\mathcal{J}})$ in (6.62) depends whether $\mathcal{J}$ is between two consecutive changes of phases or not, with $\mathcal{J}$ being located at an extreme of I. In the first case we use $\sum_{\alpha\in\Delta'}\chi(\alpha) \geq -(2\mathcal{F}^* - f) \equiv -\left(2\mathcal{F}^* - \epsilon^{1/4}\right)$ which holds on $\Omega^+(\epsilon, Q)$. In the second case, if the length of $\mathcal{J}$ is larger than $\frac{\rho}{\epsilon} \equiv \epsilon^{-\frac{5+4a}{8+4a}}$, we apply Lemma 5.15 as in (6.60), on Ω_ϵ. This gives $\Delta^+\mathcal{G}(m^{\delta^*}_\beta, \mathcal{J}) \geq \gamma[\epsilon^{1/4} - f_2]$. Otherwise, we use the fact that

$$\inf_{\ell_1 \leq t \leq \ell_2} \sum_{\alpha=t}^{\alpha_1^*} \chi(\alpha) \geq 0 \qquad\qquad \inf_{\ell_1 \leq t \leq \ell_2} \sum_{\alpha=\alpha_0^*}^{t} \chi(\alpha) \geq 0, \tag{6.63}$$

since α_1^* is the location of a maximum and α_0^* the location of a minimum.

e) At least there are two factors in (6.51), (6.53), and (6.54) that come from

$$\sum_{r_1,\dots r_n} 1 \leq \binom{\bar N}{n} \qquad\qquad \sum_{q_1,\dots q_n} 1 \leq \left(\frac{\epsilon}{\gamma}\right)^n. \tag{6.64}$$

■

Proof of Theorem 2.1 The proof of Theorem 2.1 is a consequence of Proposition 6.2 and of the next choice of parameters. Take $g(\cdot)$ such that $g(x)$ is increasing, $g(x) \geq 1$ diverges as $x \uparrow \infty$, $x^{-1}g(x) \leq 1$ and

$$x^{-1}g^{38}(x) \downarrow 0, \tag{6.65}$$

$$\epsilon^{1/4} = \frac{5}{g(\frac{\delta^*}{\gamma})}, \tag{6.66}$$

$$Q = \exp\left(\frac{\log g(\frac{\delta^*}{\gamma})}{\log\log g(\frac{\delta^*}{\gamma})} \right), \tag{6.67}$$

$$\zeta_5 = \frac{1}{2^{18} c^6(\beta,\theta)} \frac{1}{g^3(\frac{\delta^*}{\gamma})}, \quad z = \frac{1}{3}, \tag{6.68}$$

$$\zeta_1 = \frac{1}{160 g(\frac{\delta^*}{\gamma})} \quad \text{and} \quad \delta = \frac{1}{5(g(\frac{\delta^*}{\gamma}))^{1/2}}. \tag{6.69}$$

First we have to check that the $\mathcal{C}_0$ constraints are satisfied if the parameters are chosen as above. (6.2) is immediate from (6.69) and (6.65). (6.3) is just (2.29) with the choice in (6.69). (6.4) is just (2.30) with (6.68) and (6.65). (6.5) is immediate from (6.68) and (6.65). (6.6) is immediate from (6.67), (6.65), $\delta^* < 1$ and $\gamma/\delta < d_0$, by taking d_0 small enough. (6.7) follows from (6.65) by taking γ_0 and d_0 small enough. It is immediate to check that (6.11) holds and also that (6.16) implies (2.20) after easy simplifications. It remains to check (2.17). Notice that (6.68) gives $2^6 Q \zeta_5^z c^2(\beta,\theta) = Q/g$, and taking d_0 small enough we have $e^{-\beta^2/(2^6 Q\zeta_5^z c^2(\beta,\theta))} \le e^{-\beta^2\sqrt{g}}$. It is then easy to check that with our choice of Q the leading term in (6.8) is $5\epsilon^{\frac{a}{16(2+a)}}$ from which we easily get (2.17).

We then set

$$I(\omega) = \left[\frac{\epsilon\alpha_0^*}{\gamma} + \frac{\rho}{\gamma} + R_1, \frac{\epsilon\alpha_1^*}{\gamma} - \frac{\rho}{\gamma} - R_1 \right]$$

and $\tau(\omega) = +1$ if $\omega \in \Omega^+(\epsilon, Q) \cap \Omega_5$ and $\tau(\omega) = -1$ if $\omega \in \Omega^-(\epsilon, Q) \cap \Omega_5$. The estimates (2.18) and (2.19) are immediate consequences of Proposition 5.3. The proof of (2.21) is an immediate consequence of (2.20), since J contains at least one interval of length larger than $2\rho/\gamma$ that is within one of the two adjacent elongations that, by construction, have the opposite sign. ■

Proof of Theorem 2.4 Since the proof follows from arguments similar to the ones we already used, we will sketch it. It is enough to consider two consecutive elongations

$$\begin{aligned} I_0 &= \left[\frac{\alpha_0^*\epsilon}{\gamma} + R_1 + \frac{\rho}{\gamma}, \frac{\alpha_1^*\epsilon}{\gamma} - R_1 - \frac{\rho}{\gamma} \right] \\ I_1 &= \left[\frac{\alpha_1^*\epsilon}{\gamma} + R_1 + \frac{\rho}{\gamma}, \frac{\alpha_2^*\epsilon}{\gamma} - R_1 - \frac{\rho}{\gamma} \right] \end{aligned} \tag{6.70}$$

with $\text{sgn} I_0 = +1$ and $\text{sgn} I_1 = -1$. The main point is to estimate $\mu_{\beta,\theta,\gamma}(\mathcal{C}_{0,1})$, where

$$\mathcal{C}_{01} \equiv \mathcal{W}_1^c \left([\frac{\alpha_1^*\epsilon}{\gamma} - R_1 - \frac{\rho}{\gamma}, \frac{\alpha_1^*\epsilon}{\gamma} + R_1 + \frac{\rho}{\gamma}], R_2, \zeta_4 \right) \cap \mathcal{A}(\Delta_{2Q}) \tag{6.71}$$

and $\mathcal{W}_1$ is defined in Definition 2.3. Using Theorem 4.5, we get

$$\begin{aligned} \mu_{\beta,\theta,\gamma}[\mathcal{C}_{01}] \le \sum_{\bar\eta_1,\bar\eta_2 \in \{-1,+1\}} \sum_{\ell_1 = \frac{\epsilon\alpha_1^*}{\gamma} - R_1}^{\frac{\epsilon\alpha_1^*}{\gamma}} \sum_{\ell_2 = \frac{\epsilon\alpha_1^*}{\gamma}}^{\frac{\epsilon\alpha_1^*}{\gamma} + R_1} \mu_{\beta,\theta,\gamma}[\mathcal{C}_{01} \cap \{\eta^{\delta,\zeta_1}(\ell_1) = \bar\eta_1, \eta^{\delta,\zeta_1}(\ell_2) = \bar\eta_2\}] + \\ + 3^4 \left(\frac{2Q}{\gamma^2} \right)^5 e^{-\frac{\beta}{\gamma}\left((\frac{\kappa}{4}\delta\zeta_4^3) \wedge \mathcal{F}^* \right)}, \end{aligned} \tag{6.72}$$

where Q is defined in (6.67). To study $\mu_{\beta,\theta,\gamma}[\mathcal{C}_{01} \cap \{\eta^{\delta,\zeta_1}(\ell_1) = \bar{\eta}_1, \eta^{\delta,\zeta_1}(\ell_2) = \bar{\eta}_2\}]$, we decompose the event in a way similar to (6.28). Consider first the case $\bar{\eta}_1 = +1$. To be able to use (6.52) where there is a positive elongation, we need to have another $\eta^{\delta,\zeta_1}(\ell) = +1$ for ℓ on the left of $\frac{\alpha_1^* \epsilon}{\gamma} - R_1 - \frac{\rho}{\gamma}$ instead of the $\eta^{\delta,\zeta_4}(\ell) = 1$ that is present by Theorem 2.1. Using Theorem 4.5, we will find such an ℓ in the interval $[\frac{\alpha_1^* \epsilon}{\gamma} - 2R_1 - \frac{\rho}{\gamma}, \frac{\alpha_1^* \epsilon}{\gamma} - R_1 - \frac{\rho}{\gamma}]$, and we apply (6.52) in the interval $[\ell, \ell_1] \subset [\frac{\alpha_1^* \epsilon}{\gamma} - 2R_1 - \frac{\rho}{\gamma}, \frac{\alpha_1^* \epsilon}{\gamma}]$. As a consequence, on $\Omega_{51} \cap \Omega_{53}$, the Gibbs–probability to have an even number of changes of phases $n \geq 2$ within $[\frac{\alpha_1^* \epsilon}{\gamma} - 2R_1 - \frac{\rho}{\gamma}, \frac{\alpha_1^* \epsilon}{\gamma} - R_1]$ is bounded from above by

$$56 R_1^2 (\frac{2Q}{\gamma})^5 e^{-\frac{\beta}{\gamma} \frac{\epsilon^{1/4}}{4}} e^{\left\{ \frac{Q}{\gamma} e^{-\frac{\beta}{\gamma} \frac{\epsilon^{1/4}}{4}} \right\}}. \tag{6.73}$$

Consider now the case $\bar{\eta}_1 = -1$. Thus, within the interval $[\frac{\epsilon \alpha_1^*}{\gamma} - R_1 - \frac{\rho}{\gamma}, \frac{\epsilon \alpha_1^*}{\gamma}]$ the profile makes an odd number of changes of phases. When $n > 1$, we can apply (6.54) and we get that the contribution of these terms is also bounded from above by (6.73).

So, on the left of α_1^*, there are two cases left from the previous analysis: no change of phases when $\bar{\eta}_1 = +1$ or a single change of phases when $\bar{\eta}_1 = -1$.

The same arguments apply on the right of α_1^* and therefore we can have at most one change of phases on the left of α_1^* and at most one change of phases on its right. Now we show that to have simultaneously one change of phases on the right of α_1^* and one on its left has a very small Gibbs–probability. It only remains to consider the case $\bar{\eta}_1 = -1, \bar{\eta}_2 = +1$. Since $\eta^{\delta,\zeta_4}(\frac{\alpha_1^* \epsilon}{\gamma} + R_1 + \frac{\rho}{\gamma}) = -1$ the profile in $\mathcal{C}_{01}$ makes two changes of phases on the right of ℓ_1 but since we are on $\mathcal{A}(\Delta_{2Q})$ this means that there exists an $\ell \in [\ell_1, \frac{\alpha_1^* \epsilon}{\gamma} + R_1 + \frac{\rho}{\gamma}]$ with $\ell - \ell_1 \geq \epsilon/\gamma$ such that $\eta^{\delta,\zeta_1}(\ell) = +1$. That is within the negative elongation that occurs on the left of α_1^*, we have $\eta^{\delta,\zeta_1}(\ell_2) = +1, \eta^{\delta,\zeta_1}(\ell) = +1$. By using the very same argument as in (6.52), taking care that here with the same notations as in (6.45), we will merely use

$$\frac{Z^{0,0}_{[\ell_2 + L_0, \ell - L_0]}(R_5(+1, [\ell_2 + L_0, \ell - L_0]))}{Z^{0,0}_{[\ell_2 + L_0, \ell - L_0]}(R_5(-1, [\ell_2 + L_0, \ell - L_0]))} = e^{+\beta \Delta \mathcal{G}(m^{\delta^*}_{\beta, [\ell_2 + L_0, \ell - L_0]})} \frac{Z_{+1,0}([\ell_2 + L_0, \ell - L_0])}{Z_{-1,0}([\ell_2 + L_0, \ell - L_0])}, \tag{6.74}$$

and since we are within a negative effective elongation we have

$$\gamma \Delta \mathcal{G}(m^{\delta^*}_{\beta, [\ell_2 + L_0, \ell - L_0]}) \leq 2\mathcal{F}^* - \epsilon^{1/4}. \tag{6.75}$$

As in (6.52), the $2\mathcal{F}^*$ cancels with the contributions of the two changes of phases and we get a contribution which is bounded from above by (6.73).

Therefore we are left with the three cases $\bar{\eta}_1 = -1, \bar{\eta}_2 = -1$, $\bar{\eta}_1 = +1, \bar{\eta}_2 = +1$, and $\bar{\eta}_1 = +1, \bar{\eta}_2 = -1$ that belong to $\mathcal{W}_1\left([\frac{\alpha_1^* \epsilon}{\gamma} - R_1 - \frac{\rho}{\gamma} \frac{\alpha_1^* \epsilon}{\gamma} + R_1 + \frac{\rho}{\gamma}], R_2, \zeta_4\right)$. This ends the proof of Theorem 2.4. ■

7 Functional

We introduce the so called "excess free energy functional" $\mathcal{F}(m)$, $m \in \mathcal{T}$:

$$\begin{aligned} \mathcal{F}(m) &= \mathcal{F}(m_1, m_2) \\ &= \frac{1}{4} \int \int J(r - r') \left[\tilde{m}(r) - \tilde{m}(r')\right]^2 dr dr' + \int \left[f_{\beta,\theta}(m_1(r), m_2(r)) - f_{\beta,\theta}(m_{\beta,1}, m_{\beta,2})\right] dr, \end{aligned} \tag{7.1}$$

with $f_{\beta,\theta}(m_1, m_2)$ given by (9.6) and $\tilde{m}(r) = (m_1(r) + m_2(r))/2$. The functional $\mathcal{F}$ is well defined and non-negative, although it may take the value $+\infty$. Clearly, the absolute minimum of $\mathcal{F}$ is attained at the functions constantly equal to the minimizers of $f_{\beta,\theta}$. $\mathcal{F}$ represents the continuum approximation of the deterministic contribution to the free energy of the system (cf. (3.15)) subtracted by $f_{\beta,\theta}(m_\beta)$, the free energy of the homogeneous phases. Notice that $\mathcal{F}$ is invariant under the T-transformation, defined in (2.14). It has been proven in [14] that under the condition $m_1(0)+m_2(0)=0$, there exists a unique minimizer $\bar m=(\bar m_1,\bar m_2)$, of $\mathcal{F}$ over the set

$$\mathcal{M}_\infty=\{(m_1,m_2)\in\mathcal{T};\liminf_{r\to-\infty} m_i(r)<0<\liminf_{r\to+\infty} m_i(r), i=1,2\}. \tag{7.2}$$

Without the condition $m_1(0)+m_2(0)=0$, there is a continuum of minimizers, all other minimizers are translates of $\bar m$. The minimizer $\bar m(\cdot)$ is infinitely differentiable. Furthermore, there exists positive constant c depending only on β and θ such that

$$\begin{aligned}\|\bar m(r)-m_\beta\|_1\le ce^{-\alpha|r|}, &\quad \text{if} \quad r>0;\\ \|\bar m(r)-Tm_\beta\|_1\le ce^{-\alpha|r|}, &\quad \text{if} \quad r<0,\end{aligned} \tag{7.3}$$

where $\alpha=\alpha(\beta,\theta)>0$ is given by (recall (9.13)):

$$e^{-\alpha(\beta,\theta)}=\frac{\partial g_\beta}{\partial m}(\tilde m_\beta,\theta). \tag{7.4}$$

Since $\mathcal{F}$ is invariant by the T-transformation, see (2.14), interchanging $r\to\infty$ and $r\to-\infty$ in (7.2) there exists one other family of minimizers obtained translating $T\bar m$. We denote

$$\mathcal{F}^*=\mathcal{F}(\bar m)=\mathcal{F}(T\bar m)>0. \tag{7.5}$$

The functional $\mathcal{F}$ that enters in the above decomposition into a deterministic and a stochastic part, $\mathcal{F}+\gamma\mathcal{G}$, is merely a finite volume version of (7.1); however (7.3) and $\mathcal{F}^*$ will play a crucial role here.

In this section we prove some estimates needed in Section 5, based on results on a finite volume version of the excess free energy functional, $\mathcal{F}(\cdot)$, see (7.1). They are adaptation to our case from results in [16] and [9]. More care is needed here, since the profiles belong to $\mathcal{T}\subseteq L^\infty(I\!R,[-1,+1])\times L^\infty(I\!R,[-1,+1])$ instead of $L^\infty(I\!R,[-1,+1])$ and the norm involved, see (7.7), is stronger than the L^∞ norm used in [16] and [9].

• I: Minimizers in finite volume

As in Section 2, $\mathcal{D}_\delta$ denotes the partition of $I\!R$ into the intervals $((\ell-1)\delta,\ell\delta]$, $\ell\in Z\!\!Z$, for $\delta>0$ rational. In particular, if $\delta=n\delta'$, $n\in I\!N$, then $\mathcal{D}_\delta$ is coarser than $\mathcal{D}_{\delta'}$. For $r\in I\!R$, we denote by $D^\delta(r)$ the interval of $\mathcal{D}_\delta$ that contains r. A function $f(\cdot)$ is $\mathcal{D}_\delta$–measurable if it is constant on each interval of $\mathcal{D}_\delta$. In terms of the notation of Section 2, we have $D^\delta(r)=\tilde A_\delta([r/\delta]+1)$, where $[x]$ denotes the integer part of x. We define for $m=(m_1,m_2)\in\mathcal{T}$, see (2.12),

$$m_i^\delta(r)=\frac{1}{\delta}\int_{D^\delta(r)} m_i(s)ds \qquad\qquad i=1,2. \tag{7.6}$$

By definition, the functions $m_i^\delta(\cdot), i=1,2$, are constant on each $D^\delta(r)$. Definition (2.15) is extended to functions in $\mathcal{T}$, and, with an abuse of notation, we denote $\eta^{\delta,\zeta}(\ell)$, $\ell\in I\!N$,

$$\eta^{\delta,\zeta}(\ell)=\begin{cases}+1 & \text{if } \forall_{u\in(\ell-1,\ell]},\ \frac{1}{\delta}\int_{D^\delta(u)} ds\|m^{\delta^*}(s)-m_\beta\|_1\le\zeta;\\ -1 & \text{if } \forall_{u\in(\ell-1,\ell]},\ \frac{1}{\delta}\int_{D^\delta(u)} ds\|m^{\delta^*}(s)-Tm_\beta\|_1\le\zeta;\\ 0 & \text{otherwise.}\end{cases} \tag{7.7}$$

If $m^{\delta^*}(x) = m^{\delta^*}(x,\sigma)$ for $x \in \mathcal{C}_{\delta^*}(I)$, see Section 2 before (2.11), and we identify it with an element of $\mathcal{T}$, piecewise constant on each $((x-1)\delta^*, x\delta^*]$, and take $\delta = k\delta^*$, then (7.7) coincides with (2.15). Given $L_0 \in I\!N$, $\delta > \delta^* > 0, \zeta > 0$ and $\eta \in \{-1,+1\}$ we set

$$\mathcal{V}_{\delta,\zeta,L_0}(\eta) = \left\{m = (m_1, m_2)\colon (\eta m_1, \eta m_2) \in \mathcal{M}_\infty,\ \eta^{\delta,\zeta}(0) = -\eta, \eta^{\delta,\zeta}(L_0) = \eta\right\}, \tag{7.8}$$

where $\mathcal{M}_\infty$ was defined in (7.2).

Lemma 7.1 . *Let $(\beta,\theta) \in \mathcal{E}$. There exist $\delta_0 = \delta_0(\beta,\theta) > 0$, $\zeta_0 = \zeta_0(\beta,\theta) > 0$ such that for all $0 < \delta \le \delta_0$ and $0 < \zeta \le \zeta_0$, for all integers $L_0 \ge \frac{2}{\alpha(\beta,\theta)} \log 1/\zeta$, with $\lambda(\beta,\theta)$ given in (7.4) we have*

$$\inf_{m \in \mathcal{V}_{\delta,\zeta,L_0}(+1)} \mathcal{F}(m) = \mathcal{F}^* = \inf_{m \in \mathcal{V}_{\delta,\zeta,L_0}(-1)} \mathcal{F}(m), \tag{7.9}$$

where $\mathcal{F}^*$ is defined in (7.5). The infimun in the first (last) term of (7.9) is a minimum, attained at a suitable translate of $\bar m$ ($T\bar m$, respectively).

Lemma 7.1 follows from the variational result proven in [14] once we show that the set $\mathcal{V}_{\delta,\zeta,L_0}(+1)$ ($\mathcal{V}_{\delta,\zeta,L_0}(-1)$) contains a suitable translate of $\bar m$ ($T\bar m$, respectively). Due to the T-invariance of the functional $\mathcal{F}$ it suffices to check the first. This is easily obtained. Namely, from the exponential decay properties of $\bar m$, see (7.3), $\|\bar m(r) - m_\beta\|_1 \le \zeta$ for $r \ge \frac{1}{\alpha(\beta,\theta)} \log c/\zeta$ and $\|\bar m(r) - Tm_\beta\|_1 \le \zeta$ for $r \le -\frac{1}{\alpha(\beta,\theta)} \log c/\zeta$. Taking into account the definition (7.8) we can take $L_0 \ge \frac{2}{\alpha(\beta,\theta)} \log c/\zeta$ and find a translate of $\bar m$ in the set $\mathcal{V}_{\delta,\zeta,L_0}(+1)$.

For any interval $I \subseteq I\!R$ and $m = (m_1, m_2) \in \mathcal{T}$, we denote by $m_I \equiv m 1\!\!I_I$ the function that coincides with m on I and vanishes outside I. We define

$$\mathcal{F}^0(m_I) \equiv \int_I \left(f_{\beta,\theta}(m(r)) - f_{\beta,\theta}(m_\beta)\right) dr + \frac{1}{4}\int_I dr \int_I dr' J(r-r')\left[\tilde m(r) - \tilde m(r')\right]^2, \tag{7.10}$$

where $f_{\beta,\theta}$ is defined in (9.6) and $\tilde m = \frac{m_1+m_2}{2}$. For a given $\underline{m} \in \mathcal{T}$, we denote

$$\mathcal{F}(m_I|\underline{m}_{\partial I}) \equiv \mathcal{F}^0(m_I) + \frac{1}{2}\int_I dr \int_{I^c} dr' J(r-r')\left[\tilde m(r) - \underline{\tilde m}(r')\right]^2. \tag{7.11}$$

Both functionals are positive and well defined for all $I \subset I\!R$, however they could be infinite if I is unbounded. Observe that when $m_I \equiv m_\beta$ (or $m_I \equiv Tm_\beta$) then $\mathcal{F}^0(m_I)$ reaches its minimum value $\mathcal{F}^0(m_\beta) = \mathcal{F}^0(Tm_\beta) = 0$ in I. The same holds for $\mathcal{F}(m_I|\underline{m}_{\partial I})$ when $\underline{m}_{\partial I} \equiv m_\beta$ (or $\underline{m}_{\partial I} \equiv Tm_\beta$). When the boundary conditions $m_{\partial I}$ are different from m_β (or Tm_β) but are suitably close to them we will prove that the minimizer exists and it decays exponentially fast to m_β (or Tm_β) with the distance from the boundaries of I. The value of the functional at the minimizer will be, therefore, close to the null value. For all $\eta \in \{-1,+1\}$, we denote

$$\mathcal{M}(\zeta,\delta,\eta) = \left\{m = (m_1, m_2) \in \mathcal{T};\ \eta^{\delta,\zeta}(\ell) = \eta,\ \forall \ell \in Z\!\!Z\right\}, \tag{7.12}$$

$$\mathcal{A}(\zeta,\delta,\eta) = \left\{m = (m_1, m_2) \in \mathcal{T};\ \bar\eta^{\delta,\zeta}(\ell) = \eta,\ \forall \ell \in Z\!\!Z\right\}, \tag{7.13}$$

where $\eta^{\delta,\zeta}(\cdot)$ was defined in (7.7) and

$$\bar\eta^{\delta,\zeta}(\ell) = \begin{cases} +1 & \text{if } \forall_{u \in (\ell-1,\ell]}\ \|m^\delta(u) - m_\beta\|_1 \le \zeta; \\ -1 & \text{if } \forall_{u \in (\ell-1,\ell]}\ \|m^\delta(u) - Tm_\beta\|_1 \le \zeta; \\ 0 & \text{otherwise.} \end{cases} \tag{7.14}$$

Using $\|m^\delta(u) - m_\beta\|_1 \le \delta^{-1}\int_{D^\delta(u)} ds \|m^{\delta^*}(s) - m_\beta\|_1$, it is easy to see that $\mathcal{M}(\zeta,\delta,\eta) \subseteq \mathcal{A}(\zeta,\delta,\eta)$. We denote by $\mathcal{M}_I(\zeta,\delta,\eta) = \{m 1\!\!\mathrm{I}_I \text{ for } m \in \mathcal{M}(\zeta,\delta,\eta)\}$ and in a similar way $\mathcal{A}_I(\zeta,\delta,\eta)$.

Theorem 7.2 . *For $(\beta,\theta) \in \mathcal{E}$ there exists $0 < \zeta_0 = \zeta_0(\beta,\theta) < 1$ and, for $0 < \zeta \le \zeta_0$, there exists $\delta_0 = \delta_0(\zeta) > 0$, such that for any $0 < \delta \le \delta_0$, given a $\mathcal{D}_\delta$–measurable interval I and boundary conditions $\underline{m}_{\partial I} \in \mathcal{M}_{\partial I}(\zeta,\delta,+1)$ there exists an unique $\psi = (\psi_1,\psi_2)$ in $\mathcal{M}_I(\zeta,\delta,+1)$ such that*

$$\inf_{m_I \in \mathcal{M}_I(\zeta,\delta,+)} \mathcal{F}(m_I|\underline{m}_{\partial I}) = \mathcal{F}(\psi|\underline{m}_{\partial I}). \tag{7.15}$$

The minimizer ψ is a continuous function with uniformly bounded first derivative in the interior of I, $\lim_{r\uparrow\partial^+ I}\psi(r)$ and $\lim_{r\downarrow\partial^- I}\psi(r)$ exist, with the further property that

$$|\psi_1(r) - m_{\beta,1}| + |\psi_2(r) - m_{\beta,2}| \le \zeta \qquad \forall r \in I \tag{7.16}$$

$$|\psi_1(r) - m_{\beta,1}| + |\psi_2(r) - m_{\beta,2}| \le \zeta e^{-\alpha(\beta,\theta,\zeta_0)[2d(r,\partial I)]} \qquad \forall r \in I \quad \text{such that} \quad d(r,\partial I) \ge \frac{1}{2}, \tag{7.17}$$

where $d(r,\partial I)$ denotes the distance from r, to the closure of ∂I, $[\cdot]$ refers to the integer part, and $\alpha(\beta,\theta,\zeta_0)$ is defined in (6.1).
Remark: An analogous result, changing m_β to Tm_β, holds for $\eta = -1$.

Proof: Since $\mathcal{M}_I(\zeta,\delta,1) \subseteq \mathcal{A}_I(\zeta,\delta,1)$, we first prove that the infimum of $\mathcal{F}(\cdot|\underline{m}_{\partial I})$ over $\mathcal{A}_I(\zeta,\delta,1)$, a priori smaller than the one in (7.15), is reached at a unique $\psi \in \mathcal{A}_I(\zeta,\delta,1)$. Then we prove that ψ can be taken continuous and that it verifies (7.16). This implies that $\psi \in \mathcal{M}_I(\zeta,\delta,1)$, and therefore (7.15) holds. The proof that the minimizer of $\mathcal{F}(\cdot|\underline{m}_{\partial I})$ over $\mathcal{A}_I(\zeta,\delta,1)$ exists is obtained dynamically. We study a system of integral differential equations for which $\mathcal{F}(\cdot|\underline{m}_{\partial I})$ is decreasing along its solutions:

$$\begin{aligned} \frac{\partial m_1}{\partial t} &= -m_1 + \tanh\{\beta\,(J\star\tilde m + \theta + J\star\underline{\tilde m}_{\partial I})\}; \\ \frac{\partial m_2}{\partial t} &= -m_2 + \tanh\{\beta\,(J\star\tilde m - \theta + J\star\underline{\tilde m}_{\partial I})\}. \end{aligned} \tag{7.18}$$

Here $\star$ is the usual convolution.

Therefore the minimizers of $\mathcal{F}(\cdot|\underline{m}_{\partial I})$ correspond to stationary solutions of(7.18), i.e:

$$\begin{aligned} \psi_1 &= \tanh\left\{\beta\left(J\star\tilde\psi + \theta + J\star\underline{\tilde m}_{\partial I}\right)\right\}; \\ \psi_2 &= \tanh\left\{\beta\left(J\star\tilde\psi - \theta + J\star\underline{\tilde m}_{\partial I}\right)\right\}. \end{aligned} \tag{7.19}$$

This method has been already applied to characterize the minimum of the infinite volume functional (7.1), see [14] and reference therein. To show (7.16) set $\tilde\psi = \frac{1}{2}(\psi_1+\psi_2)$ so that, from (7.19),

$$\tilde\psi = \frac{1}{2}\tanh\left\{\beta\left(J\star\tilde\psi + \theta + J\star\underline{\tilde m}_{\partial I}\right)\right\} + \frac{1}{2}\tanh\left\{\beta\left(J\star\tilde\psi - \theta + J\star\underline{\tilde m}_{\partial I}\right)\right\}. \tag{7.20}$$

Since, see (9.8), $g_\beta(s,\theta) < s$ when $s > \tilde m_\beta$ and $g_\beta(s,\theta) > s$ when $0 \le s < \tilde m_\beta$, it is easy to see that for $0 < \zeta \le \tilde m_\beta$ there exists $\delta_0(\zeta)$ such that for $\delta \le \delta_0(\zeta)$, $|\tilde\psi(r) - \tilde m_\beta| \le \frac{\zeta}{2}$ for $r \in I$. (7.16) is then easily derived, once we observe that

$$\begin{aligned} |\psi_1(r) - m_{\beta,1}| &= |\tanh\beta[J\star(\tilde\psi + \underline{\tilde m}_{\partial I})(r) + \theta] - \tanh\beta[\tilde m_\beta + \theta]| \\ &= \left|\int_0^1 ds\beta(1-\tanh^2\beta[sJ\star(\tilde\psi+\underline{\tilde m}_{\partial I})(r) + (1-s)\tilde m_\beta + \theta]\left[J\star(\tilde\psi+\underline{\tilde m}_{\partial I})(r) - \tilde m_\beta\right]\right|. \end{aligned} \tag{7.21}$$

Replacing $\underline{\tilde{m}}_{\partial I}$ by $\tilde{\underline{m}^\delta}_{\partial I}(r)$, we obtain

$$|\psi_1(r)-m_{\beta,1}| \le \beta \left[1-\tanh^2 \beta\{\tilde m_\beta - \frac{\zeta}{2} - \delta + \theta\}\right] \left|(J\star(\tilde\psi + \tilde{\underline{m}^\delta}_{\partial I})(r) + \delta - \tilde m_\beta (J \star 1\!\!1_{I\cup \delta I})(r)\right|$$
$$\le \beta \left[1-\tanh^2 \beta\{\tilde m_\beta - \frac{\zeta}{2} - \delta + \theta\}\right]\left(\frac{\zeta}{2}+\delta\right).$$

Doing something similar for the other component we obtain

$$|\psi_1(r)-m_{\beta,1}| + |\psi_2(r)-m_{\beta,2}| \le e^{-\alpha(\beta,\theta,\zeta+2\delta)}[\zeta+2\delta],$$

where we set $\alpha(\beta,\theta,\zeta) = -\log \frac{\partial g_\beta}{\partial m}(\tilde m_{\beta,\theta} - \frac{\zeta}{2},\theta)$, g_β being defined in (9.8). By the smoothness of g_β, since (9.13), there exists $\zeta_0 = \zeta_0(\beta,\theta)$ so that for $\zeta \le \zeta_0(\beta,\theta)$ and δ small enough (depending on ζ) we have $e^{-\alpha(\zeta+2\delta)}[\zeta+2\delta] \le \zeta$. To get (7.17) we first show that $\tilde\psi$ solution of (7.20) has the following property

$$|\tilde\psi(r) - \tilde m_\beta| \le \frac{\zeta}{2} e^{-\alpha(\beta,\theta,\zeta_0)[2d(r,\partial I)]} \qquad \text{if} \qquad d(r,I^c) \ge \frac{1}{2}, \tag{7.22}$$

where $[x]$ is the integer part of x. Since $\tilde m_\beta$ is a solution of (9.8), we have:

$$\left|\tilde\psi_I(r) - \tilde m_\beta\right| \le e^{-\alpha(\beta,\theta,\zeta)} \left|J\star\tilde\psi_I(r) - \tilde m_\beta\right| + e^{-\alpha(\beta,\theta,\zeta)} J\star|\underline{\tilde m}_{\partial I}|(r), \quad \forall r\in I. \tag{7.23}$$

Notice that $(J\star|\underline{\tilde m}_{\partial I}|)(r) = 0$ for $r\in I,\, d(r,\partial I)\ge \frac12$ and, since $J(r) = 1\!\!1_{\{|r|\le 1/2\}}$, if r is such that $d(r,\partial I) > N_0/2$ for some $N_0 \in I\!\!N$, we have $(J^{\star N_0}\star|\underline{m}_{\partial I}|)(r) = 0$. Therefore, iterating (7.23) N_0–times, for r such that $(N_0+1)/2 \ge d(r,\partial I) > N_0/2$, we see that

$$\left|\tilde\psi_I(r) - \tilde m_\beta\right| \le e^{-N_0\alpha(\beta,\theta,\zeta)} \left|\tilde J\star\psi_I(r) - \tilde m_\beta\right| \le e^{-N_0\alpha(\beta,\theta,\zeta)}\frac{\zeta}{2}. \tag{7.24}$$

Since $e^{-\alpha(\beta,\theta,\zeta)} < 1$ for $\zeta\le\zeta_0$, we obtain (7.22). Since $d(r,\partial I)\ge\frac12$ implies that $(J\star\underline{\tilde m}_{\partial I})(r) = 0$, from (7.21) and (7.22), and doing similarly for the other component, we obtain that

$$|\psi_1(r)-m_{\beta,1}| + |\psi_2(r)-m_{\beta,2}| \le e^{-\alpha(\beta,\theta,\zeta)}\zeta e^{-\alpha(\beta,\theta,\zeta_0)[2d(r,\partial I)]} \le \zeta e^{-\alpha(\beta,\theta,\zeta)[2d(r,\partial I)]}. \tag{7.25}$$

□

• **II: Surface tension.**

Lemma 7.3 . *Given $(\beta,\theta)\in\mathcal{E}$, there exist $\gamma_0 = \gamma_0(\beta,\theta) > 0$, $d_0 = d_0(\beta,\theta) > 0$, $1 > \zeta_0 = \zeta_0(\beta,\theta) > 0$ such that for all $0<\gamma\le\gamma_0$, all $\delta^* > 0$ with $\gamma/\delta^* \le d_0$, and all positive integer p satisfying*

$$(1+p)\delta^*\log\frac{1}{\gamma} \le \frac{1}{12} \tag{7.26}$$

there exists $\Omega_{RE} = \Omega_{RE}(\gamma,\delta^*,p)$ with $I\!\!P[\Omega_{RE}] \ge 1-\gamma^2$ such that for any $\omega\in\Omega_{RE}$, any $1>\delta>\delta^*>0$, and any $\zeta_0 > \zeta_1 > 8\gamma/\delta^*$, if $L_0 = \frac{d}{\alpha(\beta,\theta)}\log(\frac{\delta^*}{8\gamma})$ for some $d\ge 2$ and $\alpha(\beta,\theta)$ defined in (7.4), we then we have, uniformly with respect to the choice of $[\ell_1,\ell_1+L_0-1]$ and $[\ell_2-L_0+1,\ell_2]$ inside $[-\gamma^{-p},\gamma^{-p}]$:

$$\frac{Z^{m_+,0}_{[\ell_2-L_0+1,\ell_2]}\big(\eta^{\delta,\zeta_1}(\ell_2)=+1\big)}{Z^{m_+,0}_{[\ell_2-L_0+1,\ell_2]}(\eta^{\delta,\zeta_1}(\ell_2)=\bar\eta_2)}\,\frac{Z^{0,m_+}_{[\ell_1,\ell_1+L_0-1]}\big(\eta^{\delta,\zeta_1}(\ell_1)=+1\big)}{Z^{0,m_+}_{[\ell_1,\ell_1+L_0-1]}(\eta^{\delta,\zeta_1}(\ell_1)=\bar\eta_1)} \le e^{\frac{\beta}{\gamma}\left(\mathcal{F}^*+f_{11}\right)\left[\frac12(|\bar\eta_1-1|+|\bar\eta_2-1|)\right]}, \tag{7.27}$$

where $\mathcal{F}^*$ is defined in (7.5) and

$$f_{11} \equiv 10(1+\theta)(\delta^* \vee \sqrt{\frac{\gamma}{\delta^*}}) d \log \frac{\delta^*}{8\gamma}. \tag{7.28}$$

Proof: We start estimating $\frac{Z^{0,m_+}_{[\ell_1,L_0-1]}(\eta^{\delta,\zeta_1}(\ell_1)=\bar\eta_1)}{Z^{0,m_+}_{[\ell_1,L_0-1]}\left(\eta^{\delta,\zeta_1}(\ell_1)=+1\right)}$ from below. When $\bar\eta_1 = +1$, the previous quantity is equal to 1 and there is nothing to prove. We then suppose that $\bar\eta_1 = -1$ and to simplify notation we set $\ell_1 = 0$. We perform a block spin transformation as in Section 4 and use Lemma 3.1. For the random terms we use the rough estimate, Lemma 3.3, obtaining for $\omega \in \Omega_{RE}$,

$$\begin{aligned} Z^{0,m_+}_{[0,L_0-1]}(\eta^{\delta,\zeta_1}(0) = \bar\eta_1) \geq\; & e^{-\frac{\beta}{\gamma}L_0\left(\delta^* + c\frac{\gamma}{\delta^*}\log\frac{\gamma}{\delta^*} + 4\theta\sqrt{\frac{\gamma}{\delta^*}}\right)} e^{-\frac{\beta}{\gamma}\left[\tilde{\mathcal{F}}(\widehat{m}^{\delta^*}_{[0,L_0-1]}|\underline{m}^{\delta^*}_{\partial[0,L_0-1]})\right]}\times \\ & \times e^{+\frac{\beta}{\gamma}\left[\frac{\delta^*}{2}\sum_{y\in\mathcal{C}_{\delta^*}(\partial[0,L_0-1])}\left[\underline{\tilde{m}}^{\delta^*}(y)\right]^2\sum_{x\in\mathcal{C}_{\delta^*}([0,L_0-1])}J_{\delta^*}(x-y)\right]}, \end{aligned} \tag{7.29}$$

where $\underline{m}^{\delta^*}_{\partial[0,L_0-1]}$ is the profile associated to the chosen boundary conditions, i.e., $\underline{m}^{\delta^*}_{\partial^-[0,L_0-1]} = 0$, $\underline{m}^{\delta^*}_{\partial^+[0,L_0-1]} = m^{\delta^*}_\beta$ and $\widehat{m}^{\delta^*}_{[0,L_0-1]} \in \mathcal{M}^- \equiv \mathcal{M}_{\delta^*}([0,L_0-1]) \cap \{\eta^{\delta,\zeta_1}(0) = -1\}$ will be suitable chosen in the following. In a similar way, we estimate the denominator by

$$\begin{aligned} Z^{0,m_+}_{[0,L_0-1]}\left(\eta^{\delta,\zeta_1}(0) = 1\right) \leq\; & e^{\frac{\beta}{\gamma}L_0\left(\delta^*+4\theta\sqrt{\frac{\gamma}{\delta^*}}\right)} \times e^{\frac{1}{\gamma}[L_0\frac{\gamma}{\delta^*}\log\frac{\delta^*}{\gamma}]} \times e^{\frac{\beta}{\gamma}[L_0 c\frac{\gamma}{\delta^*}\log\frac{\delta^*}{\gamma}]}\times \\ & \times e^{-\frac{\beta}{\gamma}\left[\inf_{\{m^{\delta^*}\in\mathcal{M}^+\}}\tilde{\mathcal{F}}(m^{\delta^*}|\underline{m}^{\delta^*}_{\partial[0,L_0-1]})\right]}\times \\ & \times e^{+\frac{\beta}{\gamma}\left[\frac{\delta^*}{2}\sum_{y\in\mathcal{C}_{\delta^*}(\partial[0,L_0-1])}\left[\underline{\tilde{m}}^{\delta^*}(y)\right]^2\sum_{x\in\mathcal{C}_{\delta^*}([0,L_0-1])}J_{\delta^*}(x-y)\right]}. \end{aligned} \tag{7.30}$$

The term $e^{\frac{1}{\gamma}[L_0\frac{\gamma}{\delta^*}\log\frac{\delta^*}{\gamma}]}$ comes from counting the number of configurations of $m^{\delta^*} \in \mathcal{M}_{\delta^*}([0,L_0-1])$. The infimum in (7.30) is over the set $\mathcal{M}^+ \equiv \mathcal{M}_{\delta^*}([0,L_0-1]) \cap \left\{\eta^{\delta,\zeta_1}(0) = 1\right\}$ and it is attained on the configuration $\{m^{\delta^*}(x) = m^{\delta^*}_\beta, \forall x \in \mathcal{C}_{\delta^*}([0,L_0-1])\}$, since the boundary conditions are at one side zero and at the other side already equal to $m^{\delta^*}_\beta$. We need only that $\zeta_1 > 8\gamma/\delta^*$ to be sure that $\|m^{\delta^*}_\beta - m_\beta\|_1 \leq \zeta_1$ entails that the configuration constantly equal to $m^{\delta^*}_\beta$ belongs to $\mathcal{M}^+$. Taking in account (7.29), (7.30) we obtain

$$\begin{aligned} \frac{Z^{0,m_+}_{[0,L_0-1]}(\eta^{\delta,\zeta_1}(0) = \bar\eta_1)}{Z^{0,m_+}_{[0,L_0-1]}\left(\eta^{\delta,\zeta_1}(0) = +1\right)} \geq\; & e^{-\frac{\beta}{\gamma}\frac{1}{2}|\bar\eta_1-1|\left[2L_0\left(\delta^*+(1+c)\frac{\gamma}{\delta^*}\log\frac{\delta^*}{\gamma}+4\theta\sqrt{\frac{\gamma}{\delta^*}}\right)\right]} \\ & \times e^{-\frac{\beta}{\gamma}\frac{1}{2}|\bar\eta_1-1|\left[\tilde{\mathcal{F}}(\widehat{m}^{\delta^*}_{[0,L_0-1]}|\underline{m}^{\delta^*}_{\partial[0,L_0-1]})-\tilde{\mathcal{F}}(m^{\delta^*}_\beta|\underline{m}^{\delta^*}_{\partial[0,L_0-1]})\right]}. \end{aligned} \tag{7.31}$$

The exponent in the last line of (7.31) can be written as

$$\begin{aligned} \left[\tilde{\mathcal{F}}(\widehat{m}^{\delta^*}_{[0,L_0-1]}|\underline{m}^{\delta^*}_{\partial[0,L_0-1]}) - \tilde{\mathcal{F}}(m^{\delta^*}_\beta|\underline{m}^{\delta^*}_{\partial[0,L_0-1]})\right] =\; & \mathcal{F}^0(\widehat{m}^{\delta^*}_{[0,L_0-1]}) + [f(m_\beta) - f(m^{\delta^*}_\beta)][L_0-1] + \\ + \frac{\delta^*}{2}\sum_{x\in\mathcal{C}_{\delta^*}([0,L_0-1])}\sum_{y\in\mathcal{C}_{\delta^*}(\partial[0,L_0-1])} & J_{\delta^*}(x-y)\left[\tilde{\widehat{m}}^{\delta^*}(x) - \underline{\tilde{m}}^{\delta^*}_{\partial[0,L_0-1]}(y)\right]^2 \\ - \frac{\delta^*}{2}\sum_{x\in\mathcal{C}_{\delta^*}([0,L_0-1])}\sum_{y\in\mathcal{C}_{\delta^*}(\partial[0,L_0-1])} & J_{\delta^*}(x-y)\left[\tilde{m}^{\delta^*}_\beta(x) - \underline{\tilde{m}}^{\delta^*}_{\partial[0,L_0-1]}(y)\right]^2 \end{aligned} \tag{7.32}$$

where $\mathcal{F}^0$ is the functional defined in (7.10). We take $\zeta = 8\frac{\gamma}{\delta^*}$ in Lemma 7.1, assuming that γ/δ^* is smaller that the ζ_0 there, $L_0 = \frac{d}{\alpha(\beta,\theta)} \log \frac{\delta^*}{8\gamma}$ with $d \geq 2$, and $\alpha(\beta,\theta)$ defined in (7.4). Then, Lemma 7.1 says that a suitable translate of $\bar{m}$ belongs to $\mathcal{V}_{\delta,\zeta,L_0}$, see (7.8), provided and $0 < \delta < \delta_0$. By an abuse of notation we always denote such translate by $\bar{m}$. Since $\mathcal{M}^- \subseteq \mathcal{V}_{\delta,\zeta,L_0}$, we can choose $\widehat{m}^{\delta^*} \in \mathcal{M}^-$ such that $\|\widehat{m}^{\delta^*}(r) - \bar{m}(r)\|_1 \leq 8\gamma/\delta^*$ for all $r \in [0, L_0 - 1]$, where $\bar{m}$ is the previous chosen minimizer. An easy computation gives

$$|f(m_\beta) - f(m_\beta^{\delta^*})||[L_0 - 1] + |\mathcal{F}^0(\widehat{m}^{\delta^*}_{[0,L_0-1]}) - \mathcal{F}^0(\bar{m}_{[0,L_0-1]})| \leq 8L_0(1+\theta)\sqrt{\frac{\gamma}{\delta^*}}. \tag{7.33}$$

Since $\widehat{m}^{\delta^*} \in \mathcal{V}_{\delta,\zeta,L_0}$ and $\zeta = 8\frac{\gamma}{\delta^*}$, the difference of the last two sums in (7.32) is bounded from above by $64\frac{\gamma}{\delta^*} < \sqrt{\frac{\gamma}{\delta^*}}$ and $\frac{\gamma}{\delta^*}$ is small enough. Since $\mathcal{F}^0(\bar{m}_I) \leq \mathcal{F}^*$ we obtain

$$\frac{Z^{0,m_+}_{[\ell_1,\ell_1+L_0-1]}(\eta^{\delta,\zeta_1}(\ell_1) = -1)}{Z^{0,m_+}_{[\ell_1,\ell_1+L_0-1]}(\eta^{\delta,\zeta_1}(\ell_1) = 1)} \geq e^{-\frac{\beta}{\gamma}[\mathcal{F}^* + 10(1+\theta)(\delta^* \vee \sqrt{\frac{\gamma}{\delta^*}})L_0)]}. \tag{7.34}$$

Repeating similar arguments for the term with $\bar{\eta}_2$ we end the proof. □

• **III: Shrinking of the typical profiles.**

Theorem 7.4 . *Given $(\beta,\theta) \in \mathcal{E}$, there exist $0 < \gamma_0 = \gamma_0(\beta,\theta) < 1$, $0 < d_0 = d_0(\beta,\theta) < 1$ and $0 < \zeta_0 = \zeta_0(\beta,\theta) < 1$, such that for all $0 < \gamma \leq \gamma_0$, $\gamma/\delta^* \leq d_0$, for all $p \in I\!N$ verifying the condition*

$$(1+p)\delta^* \log \frac{1}{\gamma} \leq \frac{1}{12}, \tag{7.35}$$

there exists $\Omega_{RE} = \Omega_{RE}(\gamma,\delta^*,p)$ with $I\!P\big[\Omega_{RE}\big] \geq 1 - \gamma^2$ such that for any $\omega \in \Omega_{RE}$, $\bar{\eta} \in \{-1,+1\}$, $\ell_0 \in I\!N$, δ, ζ_4, ζ_5 with $1 > \delta > \delta^* > 0$, and any $\zeta_0 \geq \zeta_4 > \zeta_1 > \zeta_5 \geq 8\gamma/\delta^*$, we have

$$\begin{aligned} &\sup_{\Delta_L \subseteq [-\gamma^{-p},\gamma^{-p}]} \mu_{\beta,\theta,\gamma}\Big(R_{1,4}(\bar{\eta},[\ell_1,\ell_2]) \cap (R_{1,4,5}(\bar{\eta},[\ell_1,\ell_2])(\ell_0))^c\Big) \\ &\leq \frac{2}{\gamma^p} e^{-\frac{\beta}{\gamma}\Big\{\frac{\kappa(\beta,\theta)}{2}\delta\zeta_5^3 - 2\zeta_4 e^{-\alpha(\beta,\theta,\zeta_0)[2\ell_0]} - 12(1+\theta)(4\ell_0+10)[\delta^* \vee \sqrt{\frac{\gamma}{\delta^*}}]\Big\}}, \end{aligned} \tag{7.36}$$

where $R_{1,4,5}(\bar{\eta},[\ell_1,\ell_2])(\ell_0)$ is defined in (6.39), and $R_{1,4}(\bar{\eta},[\ell_1,\ell_2])$ in (6.37), $\kappa(\beta,\theta) > 0$ satisfies (9.25), $\alpha(\beta,\theta,\zeta_0)$ is defined in (6.1) and $\Delta_L = [\ell_1,\ell_2]$ is an interval of length $L \geq 4\ell_0 + 10$. Moreover

$$\sup_{\Delta_L \subseteq [-\gamma^{-p},\gamma^{-p}]} \frac{Z^{0,0}_{[\ell_1,\ell_2]}(R_{1,4}(\bar{\eta},[\ell_1,\ell_2]) \cap (R_{1,4,5}(\bar{\eta},[\ell_1,\ell_2])(\ell_0))^c)}{Z^{0,0}_{[\ell_1,\ell_2]}(R_{1,4,5}(\bar{\eta},[\ell_1,\ell_2]))} \tag{7.37}$$

satisfies the same estimates as (7.36).

Remark: Note the crucial fact that the last term in the exponent on the right hand side of (7.36) is proportional to $4\ell_0 + 10$ and not to L.

The following corollary is an immediate consequence of Theorem 7.4. Its proof consists essentially in choosing an appropriate ℓ_0 in (7.36), see (7.40), and taking in account that, under (7.39) and $\delta^* > \gamma$, we have $\delta^* \vee \sqrt{\frac{\gamma}{\delta^*}} = \sqrt{\frac{\gamma}{\delta^*}}$.

Corollary 7.5 . *Under the same hypothesis of Theorem 7.4 with the further requirements*

$$\delta\zeta_5^3 > \frac{512(1+\theta)}{\kappa(\beta,\theta)\alpha(\beta,\theta,\zeta_0)}\sqrt{\frac{\gamma}{\delta^*}} \log \frac{\delta^*}{\gamma}, \tag{7.38}$$

$$\frac{(\delta^*)^2}{\gamma} \le \frac{1}{6e^3\beta} \tag{7.39}$$

where $\kappa(\beta,\theta) > 0$ satisfies (9.25) and $\alpha(\beta,\theta,\zeta_0)$ is defined in (6.1).

$$\ell_0 = \frac{d}{2\alpha(\beta,\theta,\zeta_0)} \log\frac{\delta^*}{\gamma} \qquad d > 1, \tag{7.40}$$

then for $\omega \in \Omega_{RE}$ and $\bar\eta \in \{-1,+1\}$, we have

$$\sup_{\Delta_L \subseteq [-\gamma^{-p},\gamma^{-p}]} \mu_{\beta,\theta,\gamma}\Big((R_{1,4}(\bar\eta,[\ell_1,\ell_2]) \cap (R_{1,4,5}(\bar\eta,[\ell_1,\ell_2])(\ell_0))^c \Big) \le e^{-\frac{\beta}{\gamma}\frac{\kappa(\beta,\theta)}{4}\delta_5\zeta_5^3} \tag{7.41}$$

where Δ_L is an interval of length $L \ge 4\ell_0 + 10$. Moreover (7.37) satisfies the same estimates as (7.41).

Proof of Theorem 7.4 Given an interval $\Delta_L \equiv [\ell_1,\ell_2]$, with $\ell_2 - \ell_1 = L > 4\ell_0 + 10$ for some ℓ_0 to be chosen later, $\ell \in [\ell_1 + 2\ell_0, \ell_2 - 2\ell_0]$, $\bar\eta = \pm 1$, we denote

$$\mathcal{E}_{\bar\eta}(\ell) \equiv \Big\{ m^{\delta^*}(x), x \in \mathcal{C}_{\delta^*}(\Delta_L) : \eta^{\delta,\zeta_5}(\ell) = 0, \eta^{\delta,\zeta_4}(\ell') = \bar\eta \quad \forall \ell' \in [\ell - 2\ell_0 - 5, \ell + 2\ell_0 + 5] \Big\}. \tag{7.42}$$

Since

$$R_{1,4}(\bar\eta,[\ell_1,\ell_2]) \cap (R_{1,4,5}(\bar\eta,[\ell_1,\ell_2]))^c \subset \cup_{\ell=\ell_1+2\ell_0}^{\ell_2-2\ell_0} \mathcal{E}_{\bar\eta}(\ell) \tag{7.43}$$

it is enough to estimate $\mu_{\beta,\theta,\gamma}\big(\mathcal{E}_{\bar\eta}(\ell)\big)$ and we assume $\bar\eta = +1$. After an easy computation, calling $I = [\ell - 2\ell_0 - 5, \ell + 2\ell_0 + 5]$, for $\omega \in \Omega_{RE}$, introduced in Lemma 3.3, for all $\ell \in [-\gamma^{-p},\gamma^{-p}]$, we obtain

$$\begin{aligned} \mu_{\beta,\theta,\gamma}\big(\mathcal{E}_1(\ell)\big) \le & \frac{1}{Z_{\beta,\theta,\gamma,\Lambda}} \\ & \sum_{\sigma_{\Lambda\setminus\gamma^{-1}I}} e^{-\beta H(\sigma_{\Lambda\setminus\gamma^{-1}I})} \mathbb{1}_{\{\eta^{\delta,\zeta_4}(\ell-2\ell_0-5)=1\}}(\sigma_{\gamma^{-1}\partial I}) \mathbb{1}_{\{\eta^{\delta,\zeta_4}(\ell+2\ell_0+5)=1\}}(\sigma_{\gamma^{-1}\partial I}) Z^{\sigma_{\gamma^{-1}\partial I}}_{\beta,\theta,\gamma,\gamma^{-1}I} \\ & \times \frac{e^{-\frac{\beta}{\gamma}\left\{ \inf_{\mathcal{E}_1(\ell)} \widetilde{\mathcal{F}}\left(m_I^{\delta^*} | m_{\partial I}^{\delta^*}(\sigma)\right) - 8(1+\theta)(4\ell_0+10)[\delta^* \vee \sqrt{\frac{\gamma}{\delta^*}}]\right\}}}{e^{-\frac{\beta}{\gamma}\left\{ \widetilde{\mathcal{F}}\left(\overline{m}_I^{\delta^*} | m_{\partial I}^{\delta^*}(\sigma)\right)\right\}}}, \end{aligned} \tag{7.44}$$

where $\widetilde{\mathcal{F}}$ is given in (3.17) and $\overline{m}_I^{\delta^*}$ is a fixed profile. This inequality is obtained as follows: writing $\mu_{\beta,\theta,\gamma}(\mathcal{E}_1(\ell))$ as a sum of the expression in (2.4) over the configurations in $\sigma_\Lambda \in \mathcal{E}_1(\ell)$ we multiply and divide by $Z^{\sigma_{\Lambda\setminus\gamma^{-1}I}}_{\beta,\theta,\gamma,I}$, inside the sum over $\sigma_{\gamma^{-1}I}$, perform a block spin transformation in the volume $\gamma^{-1}I$ and roughly estimate the magnetic field applying Lemma 3.3. This last two steps are done in the numerator and the denominator and they produce an error term $8(1+\theta)(4\ell_0+10)[\delta^* \vee \sqrt{\frac{\gamma}{\delta^*}}]$. We get an upper bound restricting in the denominator the sum over all profiles to the single one $\overline{m}_I^{\delta^*}$. Notice the important fact that the term

$$\frac{\delta^*}{2} \sum_{y \in \mathcal{C}_{\delta^*}(\partial I)} \big[\tilde m^{\delta^*}(y,\sigma)\big]^2 \sum_{x \in \mathcal{C}_{\delta^*}(I)} J_{\delta^*}(x-y) \tag{7.45}$$

in (3.18) cancels out in the formula (7.44), since it is present both in the numerator and in the denominator. We can subtract from the two $\widetilde{\mathcal{F}}$ in (7.44) the quantity $f(m^\beta)|I|$ obtaining $\mathcal{F}\big(\,\cdot\,|m_{\partial I}^{\delta^*}(\sigma)\big)$ instead of $\widetilde{\mathcal{F}}\big(\,\cdot\,|m_{\partial I}^{\delta^*}(\sigma)\big)$. Therefore to prove Theorem 7.4, it remains to prove that we can choose $\overline{m}_I^{\delta^*}$ in such a way that

$$\inf_{m_I^{\delta^*} \in \mathcal{E}_1(\ell)} \mathcal{F}\big(m_I^{\delta^*} \,|\, m_{\partial I}^{\delta^*}\big) \ge \frac{\kappa(\beta,\theta)}{2}\delta\zeta_5^3 - 2\zeta_4 e^{-\alpha(\beta,\theta,\zeta_0)2\ell_0} - 4(4\ell_0+10)(1+\theta)\sqrt{\frac{\gamma}{\delta^*}} + \mathcal{F}\big(\overline{m}_I^{\delta^*} \,|\, m_{\partial I}^{\delta^*}\big) \tag{7.46}$$

uniformly with respect to $m^{\delta^*}_{\partial I} \in R_{1,4}(+1,[\ell_1,\ell_2])$. In fact the terms in the second line of (7.44) will be bounded by $Z_{\beta,\theta,\gamma,\Lambda}$ uniformly in Λ and we get (7.36). It is rather delicate to prove (7.46).

Using (7.10) and (7.11), and splitting $I = I^- \cup (\ell-1,\ell] \cup I^+$ where $I^- \equiv (\ell-2\ell_0-5,\ell-1]$ and $I^+ \equiv (\ell,\ell+2\ell_0+5]$, we get that for all $m^{\delta^*}_I \in \mathcal{E}_1(\ell)$

$$\begin{aligned}\mathcal{F}(m^{\delta^*}_I|m^{\delta^*}_{\partial I}) \geq & \inf_{m^{\delta^*}_{I^-}\in\mathcal{M}_{I^-}(\zeta_4,\delta,+1)} \mathcal{F}(m^{\delta^*}_{I^-}|\, m^{\delta^*}_{\partial^- I}, m^{\delta^*}_{(\ell-1,\ell]}) + \mathcal{F}^0(m^{\delta^*}_{(\ell-1,\ell]}) \\ & + \inf_{m^{\delta^*}_{I^+}\in\mathcal{M}_{I^+}(\zeta_4,\delta,+1)} \mathcal{F}(m^{\delta^*}_{I^+}|\, m^{\delta^*}_{(\ell-1,\ell]}, m^{\delta^*}_{\partial^+ I}),\end{aligned} \tag{7.47}$$

where $m^{\delta^*}_{(\ell-1,\ell]} \equiv \{m^{\delta^*}(x), x\in\mathcal{C}_{\delta^*}((\ell-1,\ell])\}$. Since $m^{\delta^*}_{\partial^\pm I}$ belongs to $\mathcal{M}_{\partial^\pm I}(\zeta_4,\delta,+1)$ and $m^{\delta^*}_{(\ell-1,\ell]}$ to $\mathcal{M}_{(\ell-1,\ell]}(\zeta_4,\delta,+1)$, using Theorem 7.2, there exist unique minimizers $\psi^1_{I^+}\in\mathcal{M}_{I^+}(\zeta_4,\delta,+1)$ and $\psi^2_{I^-}\in\mathcal{M}_{I^-}(\zeta_4,\delta,+1)$ such that

$$\mathcal{F}(m^{\delta^*}_I|\, m^{\delta^*}_{\partial I}) \geq \mathcal{F}(\psi^1_{I^-}|\, m^{\delta^*}_{\partial^- I}(\sigma), m^{\delta^*}_{(\ell-1,\ell]}) + \mathcal{F}^0(m^{\delta^*}_{(\ell-1,\ell]}) + \mathcal{F}(\psi^2_{I^+}|\, m^{\delta^*}_{\partial^+ I}(\sigma), m^{\delta^*}_{(\ell-1,\ell]}), \tag{7.48}$$

for any fixed boundary condition and any $m^{\delta^*}_I \in \mathcal{E}_1(\ell)$. By (9.25)

$$\mathcal{F}^0(m^{\delta^*}_{(\ell-1,\ell]}) \geq \frac{\kappa(\beta,\theta)}{2}\zeta_5^3\delta_5. \tag{7.49}$$

Denote by $I^-_1 = (\ell-2\ell_0-5,\ell-\ell_0-3]$, $I^-_1\subseteq I^-$. By the positivity property of the functional, see (7.11),

$$\mathcal{F}(\psi^1_{I^-}|\, m^{\delta^*}_{\partial^- I}(\sigma), m^{\delta^*}_{(\ell-1,\ell]}) \geq \mathcal{F}(\psi^1_{I^-_1}|\, m^{\delta^*}_{\partial^- I}(\sigma), \psi^1_{(\ell-\ell_0-3,\ell-\ell_0-2]}).$$

Applying (7.17) of Theorem 7.2 we have that

$$\mathcal{F}(\psi^1_{I^-_1}|\, m^{\delta^*}_{\partial^- I}(\sigma), \psi^1_{(\ell-\ell_0-3,\ell-\ell_0-2]}) \geq \mathcal{F}(\psi^1_{I^-_1}|\, m^{\delta^*}_{\partial^- I}(\sigma), m_\beta 1\!\!1_{(\ell-\ell_0-3,\ell-\ell_0-2]}) - \zeta_4 e^{-\alpha(\beta,\theta,\zeta_0)[2\ell_0]}.$$

Doing the same computations for $\mathcal{F}(\psi^2_{I^+}|\, m^{\delta^*}_{\partial^+ I}(\sigma), m^{\delta^*}_{(\ell-1,\ell]})$ and setting $I^+_2 = (\ell+\ell_0+3,\ell+2\ell_0+5]$, we obtain

$$\begin{aligned}&\mathcal{F}(\psi^1_{I^-}|\, m^{\delta^*}_{\partial^- I}(\sigma), m^{\delta^*}_{(\ell-1,\ell]}) + \mathcal{F}(\psi^2_{I^+}|\, m^{\delta^*}_{\partial^+ I}(\sigma), m^{\delta^*}_{(\ell-1,\ell]}) \\ &\geq \mathcal{F}(\psi^1_{I^-_1}|\, m^{\delta^*}_{\partial^- I}(\sigma), m_\beta 1\!\!1_{(\ell-\ell_0-3,\ell-\ell_0-1]}) + \mathcal{F}(\psi^2_{I^+{}_2}|\, m^{\delta^*}_{\partial^+ I}(\sigma), m_\beta 1\!\!1_{(\ell+\ell_0+1,\ell+\ell_0+3]}) - 2\zeta_4 e^{-\alpha(\beta,\theta,\zeta_0)[2\ell_0]} \\ &= \mathcal{F}(\psi^3_I|\, m^{\delta^*}_{\partial^- I}(\sigma), m^{\delta^*}_{\partial^+ I}(\sigma)) - 2\zeta_4 e^{-\alpha(\beta,\theta,\zeta_0)[2\ell_0]},\end{aligned} \tag{7.50}$$

where we set $\psi^3_I = \psi^1_{I^-_1} + m_\beta 1\!\!1_{(\ell-\ell_0-3,\ell+\ell_0+3]} + \psi^2_{I^+_2}$. In the last equality in (7.50) we use that $\mathcal{F}^0(m_\beta)=0$.

By Theorem 7.2, there exists an unique $\psi^*_I\in\mathcal{M}_I(\zeta_4,\delta,+1)$ such that

$$\inf_{\psi_I\in\mathcal{M}_I(\zeta_4,\delta,+1)} \mathcal{F}(\psi_I|m^{\delta^*}_{\partial I}) \equiv \mathcal{F}(\psi^*_I|m^{\delta^*}_{\partial I}). \tag{7.51}$$

Therefore, since $\psi^3_I\in\mathcal{M}_I(\zeta_4,\delta,+1)$, we have

$$\mathcal{F}(\psi^3_I|\, m^{\delta^*}_{\partial^- I}(\sigma), m^{\delta^*}_{\partial^+ I}(\sigma)) \geq \mathcal{F}(\psi^*_I|\, m^{\delta^*}_{\partial^- I}(\sigma), m^{\delta^*}_{\partial^+ I}(\sigma)). \tag{7.52}$$

Then, from (7.48), (7.49), (7.50), (7.52) we obtain

$$\inf_{m^{\delta^*}_I\in\mathcal{E}_1(\ell)} \mathcal{F}\big(m^{\delta^*}_I|\, m^{\delta^*}_{\partial I}\big) \geq \mathcal{F}(\psi^*_I|\, m^{\delta^*}_{\partial^- I}(\sigma), m^{\delta^*}_{\partial^+ I}(\sigma)) + \frac{\kappa(\beta,\theta)}{2}\zeta_5^3\delta_5 - 2\zeta_4 e^{-\alpha(\beta,\theta,\zeta_0)[2\ell_0]}. \tag{7.53}$$

Choosing for $\overline{m}_I^{\delta^*}$ a $\mathcal{D}^{\delta^*}$–measurable approximation of ψ_I^* with values in $\mathcal{M}_{\delta^*}(I)$, see (2.11), we get

$$\mathcal{F}(\psi_I^*|m_{\partial I}^{\delta^*}) \geq \mathcal{F}^{\delta^*}\left(\overline{m}_I^{\delta^*}\,|\,m_{\partial I}^{\delta^*}\right) - 4(4\ell_0+10)(1+\theta)\left(\delta^* \vee \sqrt{\frac{\gamma}{\delta^*}}\right). \tag{7.54}$$

Collecting (7.53) and (7.54) we get (7.46). ■

8 Appendix: The cluster expansion

In this section we prove Lemma 3.4 of Section 4. We will write $V(m_I^{\delta^*},h)$, defined in (3.16), as an absolute convergent series and then estimate its Lipschitz norm.

To state the result we need some preliminary definitions. Let $I \subseteq I\!R$ be a bounded, $\mathcal{D}_{\delta^*}-$ measurable interval, $\mathcal{A}(I)$ the set of blocks $A(x)$, $x \in \mathcal{C}_{\delta^*}(I)$. We denote by $\lambda = (A, A')$ a pair of different blocks belonging to $\mathcal{A}(I)$ and by $\bar\lambda = A \cup A'$ its support. We define a *graph* g in $\mathcal{A}(I)$ as any collection of pairs of different blocks $g = \{\lambda_1, \lambda_2, .., \lambda_m\}$, with $0 \leq m \leq \frac{|\mathcal{A}(I)|}{2}(|\mathcal{A}(I)|-1)$, such that $\lambda_s \neq \lambda_t$ for all $s \neq t$. A graph g will be said to be connected if, for any pair B and C of disjoint subsets of $\mathcal{A}(I)$ such that $B\cup C = \cup_{s=1}^m \bar\lambda_s$, there is a $\lambda_s \in g$ such that $\bar\lambda_s \cap B \neq \emptyset$ and $\bar\lambda_s \cap C \neq \emptyset$. Given a graph $g = \{\lambda_1, \lambda_2, .., \lambda_m\}$, $\lambda_1, \lambda_2, .., \lambda_m$ are called *links* of the graph g and the blocks $A(x)$ belonging to $\cup_{s=1}^m \bar\lambda_s$ are called vertices of

g. We denote $\mathcal{G}_{\mathcal{A}(I)}$ the set of all connected graphs of $\mathcal{A}(I)$. A connected *tree graph* τ (or simply a tree graph) is a connected graph with m vertices and $m-1$ links. We denote by $\mathcal{T}_{\mathcal{A}(I)}$ the set of all tree graphs in $\mathcal{A}(I)$. Given a tree graph τ the incidence number of the vertex $A(x)$, denoted by $d_{A(x)}$, is the number of links λ_s in τ such that $A(x)\cap\bar\lambda_s \neq \emptyset$. In the following we denote by a *polymer* R a subset of blocks of $\mathcal{A}(I)$, by $\mathcal{C}_{\delta^*}(R) = \{x \in \mathcal{C}_{\delta^*}(I)\,\text{such that}\,A(x)\in R\}$ and $m_R^{\delta^*} = \{m^{\delta^*}(x); x \in \mathcal{C}_{\delta^*}(R)\}$. We have the following Theorem.

Theorem 8.1 . *For all $\beta > 0$, $h \in \Omega$, for any bounded interval $I \subseteq I\!R$, for $\delta^* > 0$, $\frac{(\delta^*)^2}{\gamma} < \frac{1}{6e^3\beta}$, $V(m_I^{\delta^*},h)$ can be written as an absolutely convergent series:*

$$V(m_I^{\delta^*},h) = \frac{1}{\beta}\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{R_1,R_2,\ldots,R_n,|R_\ell|\geq 2}\Phi^T(R_1,R_2,\ldots,R_n)\prod_{\ell=1}^{n}\rho(R_\ell), \tag{8.1}$$

where $\Phi^T(R_1,R_2,\ldots,R_n)$ are the Ursell coefficients, see (8.10), and $\rho(R_\ell)$ is given by

$$\rho(R_\ell) = \rho(R_\ell,h) = I\!E_{m^{\delta^*}_{(R_\ell)}}\left[\sum_{g\in\mathcal{G}_{R_\ell}}\prod_{(x,y)\in g,x\neq y}\left[e^{\beta U(\sigma_{A(x)},\sigma_{A(y)})}-1\right]\right]. \tag{8.2}$$

$\mathcal{G}_R$ is the set of the connected graphs in R and x is a short notation for $A(x)$. (So $(x,y) \in g$ is a short notation for $(A(x),A(y)) \in g$.) Moreover

$$\left|V(m_I^{\delta^*},h)\right| \leq |\mathcal{C}_{\delta^*}(I)|\frac{1}{\beta}\frac{S}{1-S}, \tag{8.3}$$

where

$$S = \sup_h \sup_{x\in\mathcal{C}_{\delta^*}(I)}\sum_{R:x\in R}e^{|R|}\rho(R) < 6e^3\beta\frac{(\delta^*)^2}{\gamma} < 1 \tag{8.4}$$

and

$$\sup_{I\subseteq Z\!\!\!Z}\sup_{i\in I}\|\partial_i V_I\|_\infty \leq \frac{S}{1-S}\frac{1}{\beta}. \tag{8.5}$$

Proof: The proof is obtained via a standard tool of Statistical Mechanics, the so called cluster expansion, see [11] and bibliography therein. This expansion is done in three steps:
(1) express the $\log V$ as a formal series,
(2) establish sufficient conditions for the series to converge absolutely,
(3) control that under the hypothesis of Theorem 8.1 these conditions are indeed satisfied.
We start with the following identity

$$\begin{aligned} &I\!\!E_{m_I^{\delta^*}}[\prod_{x\neq y} e^{\beta U(\sigma_{A(x)},\sigma_{A(y)})}] = I\!\!E_{m_I^{\delta^*}} \prod_{x\neq y}[e^{\beta U(\sigma_{A(x)},\sigma_{A(y)})} - 1 + 1] \\ &= 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{R_1,R_2,..,R_n,|R_\ell|\geq 2} e^{-\tilde{U}(R_1,..R_n)} \prod_{\ell=1}^{n} \rho(R_\ell), \end{aligned} \tag{8.6}$$

where

$$\tilde{U}(R_1,\dots,R_n) = \sum_{1\leq \ell,s\leq n} \tilde{U}(R_\ell,R_s), \tag{8.7}$$

$$\tilde{U}(R_\ell,R_s) = \begin{cases} 0, & \text{if } R_\ell\cap R_s = \emptyset; \\ \infty, & \text{if } R_\ell \cap R_s \neq \emptyset. \end{cases} \tag{8.8}$$

and $\rho(R_\ell)$ is given in (8.2). Since $|\mathcal{A}(I)| < \infty$ the number of terms contributing to (8.6) is finite. We have that the log of the right hand side of (8.6) can be written as a formal expansion

$$\begin{aligned} \beta V(m_I^{\delta^*}) &= \log\left[1 + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{R_1,R_2,\dots,R_n,|R_\ell|\geq 2} e^{-\tilde{U}(R_1,\dots,R_n)} \prod_{\ell=1}^{n} \rho(R_\ell)\right] \\ &= \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{R_1,R_2,\dots,R_n,|R_\ell|\geq 2} \Phi^T(R_1,R_2,\dots,R_n) \prod_{\ell=1}^{n} \rho(R_\ell), \end{aligned} \tag{8.9}$$

where $\Phi^T(R_1,R_2,\dots,R_n)$ are the Ursell coefficients

$$\Phi^T(R_1,R_2,\dots,R_n) = \begin{cases} \displaystyle\sum_{g\in\mathcal{G}_{R_1,\dots,R_n}} \prod_{(\ell,s)\in g,\ell\neq s} \left[e^{-\tilde{U}(R_\ell,R_s)} - 1\right], \text{ if } n\geq 2; \\ 1, \text{ if } n = 1. \end{cases} \tag{8.10}$$

Observe that $\Phi^T(R_1,R_2,\dots,R_n) = 0$ if $g\in\mathcal{G}_{R_1,\dots R_n}$ is not connected.
We must now prove that the formal series (8.9) actually converges. Fix $x\in C_{\delta^*}(I)$ and a polymer R, such that $A(x)\in R$. Recall that $\Phi^T(R) = 1$, when $n = 1$. Then, (8.9) can be written as

$$\beta V(m_I^{\delta^*},h) = \sum_{x\in C_{\delta^*}(I)} \sum_{R,x\in R,|R|\geq 2} \rho(R)\left[1 + \sum_{n\geq 2} \frac{1}{n!} B_n(R)\right], \tag{8.11}$$

where

$$B_n(R) = \sum_{R_2,\dots,R_n,|R_\ell|\geq 2} \prod_{\ell=2}^{n} \rho(R_\ell)\Phi^T(R,R_2,\dots,R_n). \tag{8.12}$$

From the definition of $\Phi^T(R, R_2, \ldots, R_n)$ we see that $B_n(R)$ can be written as

$$B_n(R) = \sum_{g \in \mathcal{G}_{R,R_2,\ldots,R_n}} \sum_{f \subseteq g} (-1)^{|f|} \sum_{R_2,\ldots,R_n, |R_\ell| \geq 2} \prod_{\ell=2}^{n} \rho(R_\ell), \tag{8.13}$$

where $f \subseteq g$ means that every link of f is also a link of g. Recall that, from Rota inequality, see [31],

$$\left| \sum_{f \subseteq g} (-1)^{|f|} \right| \leq N(g),$$

where $N(g)$ is the number of connected tree graphs in g. Setting $\mathcal{T}_{R,R_2,\ldots,R_n} \equiv \mathcal{T}_n$, we have that

$$\sum_{g \in \mathcal{G}_{R_1,\ldots,R_n}} = \sum_{\tau \in \mathcal{T}_n} \sum_{g : \tau \in g} \frac{1}{N(g)}$$

and then we can express

$$B_n(R) = \sum_{\tau \in \mathcal{T}_n} w(\tau) \tag{8.14}$$

where

$$w(\tau) = \sum_{R_2,\ldots,R_n, |R_\ell| \geq 2, \tau \in g(R,R_2,\ldots,R_n)} \prod_{\ell=2}^{n} \rho(R_i). \tag{8.15}$$

For any fixed set R' we have the bound

$$\sum_{R, R \cap R' \neq \emptyset} \leq |R'| \sup_{x \in R'} \sum_{R : x \in R}$$

then

$$w(\tau) \leq |R|^{d_1} \prod_{\ell=2}^{n} \left[\sup_{x \in \mathcal{C}_{\delta^*}(I)} \sum_{R_\ell : x \in R_\ell} |R_\ell|^{d_i - 1} |\rho(R_\ell)| \right], \tag{8.16}$$

where d_ℓ is the incidence number of the vertex ℓ in the tree τ. Using Caley formula [11], we get

$$\begin{aligned}
B_n(R) &= \sum_{\tau \in \mathcal{T}_n} w(\tau) \\
&\leq \sum_{d_1,\ldots,d_n} |R|^{d_1} \frac{(n-2)!}{\prod_{\ell=1}^{n} (d_\ell - 1)} \prod_{\ell=2}^{n} \left[\sup_{x \in \mathcal{C}_{\delta^*}(I)} \sum_{R_i : x \in R_\ell} |R_\ell|^{d_\ell - 1} |\rho(R_\ell)| \right] \\
&\leq (n-1)! \left[\sum_{d_1=1}^{\infty} \frac{|R|^{d_1}}{d_1!} \right] \prod_{\ell=2}^{n} \left[\sup_{x \in C_{\delta^*}(I)} \sum_{R_\ell : x \in R_\ell} \sum_{d_\ell=1}^{\infty} |\rho(R_\ell)| \frac{|R|^{d_\ell - 1}}{(d_\ell - 1)!} \right] \\
&\leq (n-1)! \left(e^{|R|} - 1 \right) \prod_{\ell=2}^{n} \left[\sup_{x \in C_{\delta^*}(I)} \sum_{R_\ell : x \in R_\ell} |\rho(R_\ell)| e^{|R_\ell|} \right] \leq (n-1)! e^{|R|} S^{n-1},
\end{aligned} \tag{8.17}$$

where in the second inequality we used that $n - 1 \geq d_1$ to obtain the factor $\frac{1}{d_1!}$ and in the last inequality we set

$$S = \sup_h \sup_{x \in \mathcal{C}_{\delta^*}(I)} \sum_{R : x \in R} e^{|R|} \rho(R). \tag{8.18}$$

Thus, under the condition that $S<1$ we obtain

$$\sum_{R,|R|\ge 2,x\in R}|\rho(R)|\left[1+\sum_{n\ge 2}\frac{1}{n!}B_n(R)\right]\le\sum_{R,|R|\ge 2,x\in R}|\rho(R)|\left[1+e^{|R|}\sum_{n\ge 2}\frac{1}{n}S^{n-1}\right]$$
$$\le\sum_{R,|R|\ge 2,x\in R}|\rho(R)|e^{|R|}\left[1+\frac{S}{1-S}\right]=\frac{S}{1-S}. \tag{8.19}$$

Therefore, recalling (8.11), we obtain (8.3). The important remark to prove (8.5) is that to obtain the Lipschitz norm we make the difference of two absolute convergent series having the only difference in one site i. We then obtain

$$V(m^{\delta^*}_{\tilde I_{12}},h)-V(m^{\delta^*}_{\tilde I_{12}},h^i)=$$
$$\frac{1}{\beta}\quad\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{R_1,R_2,\dots,R_n,|R_\ell|\ge 2}\Phi^T(R_1,R_2,\dots,R_n)\left[\prod_{\ell=1}^{n}\rho(R_\ell,h)-\prod_{\ell=1}^{n}\rho(R_\ell,h^i)\right]$$
$$\le\frac{2}{\beta}\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{R_1,R_2,\dots,R_n,|R_l|\ge 2\exists l:i\in R_l}\left|\Phi^T(R_1,R_2,\dots,R_n)\right|\prod_{\ell=1}^{n}\sup_h|\rho(R_\ell,h)|\,. \tag{8.20}$$

Following the same strategy used above we obtain (8.5). Next we show that S, see (8.18), satisfies (8.4). Taking into account (3.20) and setting $\Phi(x,y)=1\!\!1_{\{\frac12-\delta^*\le\delta^*|x-y|\le\frac12+\delta^*\}}\left(\frac{\beta(\delta^*)^2}{\gamma}\right)$ we obtain that if g is a connected graph with support R, then:

$$\sup_h I\!\!E_{m^{\delta^*}_R}\left[\prod_{(x,y)\in g,x\ne y}\left[e^{\beta U(\sigma_{A(x)},\sigma_{A(y)})}-1\right]\right]\le\prod_{(x,y)\in g,x\ne y}\Phi(x,y). \tag{8.21}$$

In the last estimate we used (3.19). From (8.18) we have that

$$S\equiv\sup_h\sup_{x\in C_{\delta^*}(I)}\sum_{R:x\in R}|\rho(R)|e^{|R|}$$
$$\le\sup_{x\in C_{\delta^*}(I)}\sum_{R:x\in R}\sum_{g\in\mathcal{G}_R}e^{|R|}\prod_{(z,y)\in g,z\ne y}\Phi(z,y). \tag{8.22}$$

An essential fact to prove (8.4) is that $\Phi(z,y)\ne 0$ only when $\frac12-\delta^*\le\delta^*|z-y|\le\frac12+\delta^*$, i.e., the block $A(z)$ interacts only with three blocks, the $A(y)$ block which is at distance $\frac{1}{2\delta^*}$ from it and the two blocks, to the left and to the right of $A(y)^*$. Therefore for any fixed polymer R, $x\in R$, $|R|=\ell$, the number of graphs that contribute to the sum in (8.22) is at most $3^{(\ell-1)}$. Namely, $\ell-1$ is the number of links connecting the ℓ vertices of the graph and 3 is the maximum number of links that a vertex can have with the others, since

* This depends on the particular choice of the potential, $1\!\!1_{|x|\le\frac12}$. For general potential, always with support $\{x:|x|\le\frac12\}$ this will be not true. In that case $\Phi(z,y)\ne 0$ when $\delta^*|z-y|\le\frac12$, therefore the block $A(z)$ will interact with $\frac{1}{\delta^*}$ blocks. Nevertheless this will not cause problems to get (8.4). Namely in this case the function Φ, using Taylor formula to estimate the potential, becomes $\Phi(x,y)=1\!\!1_{\{\delta^*|x-y|\le\frac12\}}\left(\frac{\beta(\delta^*)^2}{\gamma}c\delta^*\right)$, where c is a positive constant depending on the potential. Performing the sums in (8.23) we should replace 3 with $\frac{1}{\delta^*}$. The result will be similar. The only difference is given by the presence of the constant c.

$\Phi(z,y)\neq 0$ only when $\frac{1}{2}-\delta^*\le \delta^*|z-y|\le \frac{1}{2}+\delta^*$. Since Φ is translational invariant we can assume $x=0$. Then from (8.22) we obtain that

$$\begin{aligned} S &\le \sum_{R:0\in R}\sum_{g\in\mathcal{G}_R} e^{|R|}\prod_{(z,y)\in g,z\neq y}\Phi(z,y)=\sum_{\ell\ge 2}\sum_{R,0\in R,|R|=\ell}\sum_{g\in\mathcal{G}_R} e^{|R|}\prod_{(z,y)\in g,z\neq y}\Phi(z,y)\\ &\le \sum_{\ell\ge 2}\ell 3^{(\ell-1)}e^{\ell}[\frac{\beta}{\gamma}(\delta^*)^2]^{\ell-1}<3\left[\frac{e^3}{1-3e^2\frac{\beta}{\gamma}(\delta^*)^2}\right]\frac{\beta}{\gamma}(\delta^*)^2\le 6e^3\frac{\beta}{\gamma}(\delta^*)^2. \end{aligned} \tag{8.23}$$

□

9 Appendix: The random field Curie Weiss model

The Random Field Curie–Weiss model is defined as follows: Let $(\Omega,\mathcal{A},I\!P)$ be a probability space on which we have defined $h\equiv\{h_i\}_{i\in I\!N}$, a family of independent identicaly distributed Bernoulli random variable with $I\!P[h_i=+1]=I\!P[h_i=-1]=1/2$. The configuration space is $\mathcal{S}\equiv\{-1,+1\}^{I\!N}$. For $N\in I\!N$, we denote $\mathcal{S}_N\equiv\{-1,+1\}^N$. Given $\sigma\in\mathcal{S}_N$ and a realization of the random field, the Hamiltonian is the volume $\{1,\dots,N\}$ is

$$\begin{aligned} H_N(\sigma)(\omega) &= -\frac{1}{2N}\sum_{i,j=1}^N\sigma_i\sigma_j-\theta\sum_{i=1}^N h_i(\omega)\sigma_i\\ &= -\frac{N}{2}\left(\frac{1}{N}\sum_{i=1}^N\sigma_i\right)^2-N\theta\left(\frac{1}{N}\sum_{i=1}^N h_i(\omega)\sigma_i\right). \end{aligned} \tag{9.1}$$

Using a partition of $\{1,\dots,N\}$ similar to the one done for the volume $A(x)$ before (2.8), calling $N^{\pm},\lambda,D$ what was called $B^{\pm}(x),\lambda(x),D(x)$, we define

$$m_N(\pm,\sigma)=\frac{2}{N}\sum_{i\in N^{\pm}}\sigma_i \tag{9.2}$$

and as in (2.9) and (2.10) one has

$$\frac{1}{N}\sum_{i=1}^N\sigma_i=\frac{1}{2}(m_N(+,\sigma)+m_N(-,\sigma)) \tag{9.3}$$

and

$$\frac{1}{N}\sum_{i=1}^N h_i\sigma_i=\frac{1}{2}(m_N(+,\sigma)-m_N(-,\sigma))+\lambda\frac{2}{N}\sum_{i\in D}\sigma_i \tag{9.4}$$

The “ canonical” free energy is defined as follows: For $m_1,m_2\in[-1,+1]^2$

$$f_{\beta,\theta}(m_1,m_2)=\lim_{\epsilon\downarrow 0}\lim_{N\uparrow\infty}-\frac{1}{\beta N}\log\sum_{\sigma\in\mathcal{S}_N}1\!\!1_{\{|m_N(+,\sigma)-m_1|\le\epsilon,|m_N(-,\sigma)-m_2|\le\epsilon\}}e^{-\beta H_N(\sigma)} \tag{9.5}$$

It is not too difficult to check that the above limit exists uniformly with respect to $m_1,m_2\in]-1,+1[^2$ $I\!P$–almost surely and that

$$f_{\beta,\theta}(m_1,m_2)=-\frac{(m_1+m_2)^2}{8}-\frac{\theta}{2}(m_1-m_2)+\frac{1}{2\beta}(\mathcal{I}(m_1)+\mathcal{I}(m_2)), \tag{9.6}$$

where $\mathcal{I}(m) = \frac{(1+m)}{2}\log\left(\frac{1+m}{2}\right) + \frac{(1-m)}{2}\log\left(\frac{1-m}{2}\right)$.

Differentiating (9.6) we see that $(m_1, m_2) \in]-1,1[^2$ is a critical point of $f_{\beta,\theta}(\cdot,\cdot)$ if and only if

$$\begin{aligned} m_1 &= \tanh(\beta(m_1+m_2)/2+\beta\theta) \\ m_2 &= \tanh(\beta(m_1+m_2)/2-\beta\theta). \end{aligned} \tag{9.7}$$

The sum of the two equations in (9.7) is closed with respect to $m = (m_1+m_2)/2$

$$m = g_\beta(m,\theta) \equiv \frac{1}{2}\tanh\beta(m+\theta) + \frac{1}{2}\tanh\beta(\ m-\theta). \tag{9.8}$$

The needed results for the Random Field Curie Weiss model are collected in the following lemma.

Lemma 9.1 . *If $0 \le \beta \le 1$ then for all θ, $m = 0$ is the only solution of*

$$m = g_\beta(m,\theta). \tag{9.9}$$

If $\beta > 1$, let

$$\theta_{1,c}(\beta) = \frac{1}{\beta}\,\text{arctanh}\,(1-\frac{1}{\beta})^{1/2} \tag{9.10}$$

and $\mathcal{E}$, the region defined by

$$0 < \theta < \theta_{1,c}(\beta) \quad \text{for} \quad 1 < \beta < \frac{3}{2} \tag{9.11}$$

or

$$0 < \theta \le \theta_{1,c}(\beta) \quad \text{for} \quad \beta \ge \frac{3}{2}. \tag{9.12}$$

Then, $(\beta,\theta) \in \mathcal{E}$ is necessary and sufficient for the existence of only one strictly positive solution $\tilde{m}_\beta$ of (9.9) that satisfies

$$\frac{\partial g_\beta}{\partial m}(\tilde{m}_\beta,\theta) = \frac{\beta}{2\cosh^2\beta(\tilde{m}_\beta+\theta)} + \frac{\beta}{2\cosh^2\beta(\tilde{m}_\beta-\theta)} < 1. \tag{9.13}$$

Proof: Note that $m = 0$ is a solution of (9.9) for all θ and β positive. It is immediate to verify that $m = 0$ is indeed the only solution of (9.9) when $0 \le \beta \le 1$. Indeed $\frac{\partial g_\beta}{\partial m}(m,\theta) \le \beta\left[1-(1/2)\tanh^2(\beta\theta)\right] < 1$ for all $m \in I\!R$ and $\theta > 0$.

To treat the case $\beta > 1$, we introduce the variables $x = \tanh\beta m$ and $y = \tanh\beta\theta$, and notice that

$$\frac{\partial g_\beta}{\partial m}(m=0,\theta) = \beta(1-y^2). \tag{9.14}$$

Recalling (9.10), we have

$$\begin{aligned} 0 < \theta < \theta_{1,c}(\beta) &\Longleftrightarrow \frac{\partial g_\beta}{\partial m}(m=0,\theta) > 1, \\ \theta_{1,c}(\beta) < \theta &\Longleftrightarrow \frac{\partial g_\beta}{\partial m}(m=0,\theta) < 1. \end{aligned} \tag{9.15}$$

On the other hand,

$$\begin{aligned} \frac{\partial^2 g_\beta}{\partial m^2}(m,\theta) &= -\beta^2\left(\frac{\tanh\beta(m+\theta)}{\cosh^2\beta(m+\theta)} + \frac{\tanh\beta(m-\theta)}{\cosh^2\beta(m-\theta)}\right), \\ &= -\frac{2\beta^2 x(1-x^2)(1-y^2)}{(1-x^2y^2)^3}\left[1-3y^2+x^2(3y^2-y^4)\right], \end{aligned} \tag{9.16}$$

so that, for $m > 0$

$$\frac{\partial^2 g_\beta}{\partial m^2}(m,\theta) < 0 \Longleftrightarrow x^2 > \frac{3y^2-1}{3y^2-y^4}, \quad \frac{\partial^2 g_\beta}{\partial m^2}(m,\theta) > 0 \Longleftrightarrow x^2 < \frac{3y^2-1}{3y^2-y^4}. \tag{9.17}$$

Therefore, calling

$$\theta_{2,c}(\beta) = \frac{1}{\beta}\,\mathrm{arctanh}\,\frac{1}{\sqrt{3}} \tag{9.18}$$

one sees that

$$0 < \theta \le \theta_{2,c}(\beta) \Rightarrow \frac{\partial^2 g_\beta}{\partial m^2}(m,\theta) < 0,\ \forall m > 0 \tag{9.19}$$

while, using $(3y^2-1)/(3y^2-y^4) \le 1$ with equality only when $y = 1$, it is easy to check that for $\theta > \theta_{2,c}(\beta)$ there exists an unique $m_2^* = m_2^*(\beta,\theta) > 0$ such that $\tanh^2(\beta m_2^*) = (3y^2-1)/(3y^2-y^4)$ and we have

$$\frac{\partial^2 g_\beta}{\partial m^2}(m,\theta) > 0 \text{ for } m < m_2^*, \quad \frac{\partial^2 g_\beta}{\partial m^2}(m,\theta) < 0 \text{ for } m > m_2^*. \tag{9.20}$$

Another fact to be used is that for all $m > 0, \theta > 0$

$$\frac{\partial g_\beta}{\partial \theta}(m,\theta) = \frac{-4\beta xy}{(1-x^2y^2)^2}(1-x^2)(1-y^2) < 0 \tag{9.21}$$

and the function $\theta \in (0,\infty) \mapsto g_\beta(m,\theta)$ is strictly decreasing for all $0 < m < +\infty$.

Now we can consider the various cases:

• $1 < \beta < \frac{3}{2}$ (that is $\theta_{1,c}(\beta) < \theta_{2,c}(\beta)$).

If $0 < \theta < \theta_{1,c}(\beta)$, then $\frac{\partial g_\beta}{\partial m}(m = 0,\theta) > 1$ and using (9.19) $g_\beta(m,\theta)$ is a strictly concave function of m. Therefore there is one and only one strictly positive solution to (9.9), say $\tilde{m}_\beta$. It satisfies (9.13) since by strict concavity we have

$$g_\beta(m,\theta) < g_\beta(m_0,\theta) + \frac{\partial g_\beta}{\partial m}(m_0,\theta)(m - m_0) \tag{9.22}$$

Taking $m = 0$ and $m_0 = \tilde{m}_\beta$ in (9.22) it is immediate to get (9.13).

If $\theta_{1,c}(\beta) \le \theta \le \theta_{2,c}(\beta)$, then $\frac{\partial g_\beta}{\partial m}(m = 0,\theta) \le 1$ and by concavity there is no strictly positive solution to (9.9).

• $\frac{3}{2} < \beta$ (that is $\theta_{2,c}(\beta) < \theta_{1,c}(\beta)$).

If $0 < \theta < \theta_{2,c}$, then $\frac{\partial g_\beta}{\partial m}(m = 0,\theta) > 1$ and $g_\beta(m,\theta)$ is a strictly concave function of m therefore there is one and only one strictly positive solution of (9.9), it satisfies (9.13) by the same argument as before.

If $\theta_{2,c} < \theta \le \theta_{1,c}$, recalling (9.15) and (9.20), for $0 < m < m_2^*$, $g_\beta(m,\theta)$ is a strictly convex function of m and therefore, see (9.14)

$$g_\beta(m,\theta) > \beta(1 - \tanh^2(\beta\theta))m \ge m, \tag{9.23}$$

so there is no strictly positive solution of (9.9) in this region. On the other hand for $m > m_2^*$, $g_\beta(m,\theta)$ is strictly concave and since $\lim_{m\uparrow\infty} g_\beta(m,\theta) = 1$ there is one and only one strictly positive solution of (9.9). It satisfies (9.13) by the same argument as before.

It is not difficult to check that when $\theta > \theta_{1,c}$ but $\theta - \theta_{1,c}$ is small enough there exist two strictly positive solutions of (9.9), say $0 < \tilde{m}_1^* < \tilde{m}_3^*$. The heuristic argument is the following: using Taylor formula one has $g_\beta(m,\theta) \approx m[\beta(1-y^2)+\frac{m^2}{3}\beta^3(3y^2-1)]$. Therefore it remains to solve $0 = m - g_\beta(m,\theta) \approx m[1-\beta(1-y^2)-\frac{m^2}{3}\beta^3(3y^2-1)]$, and $m^* = (3[1-\beta(1-y^2)]/(\beta^3(3y^2-1)))^{1/2}$ is the strictly positive solution of this equation.

To make it rigorous, call $m_1(\lambda) = [(3\lambda\epsilon)/(\beta^3(3y^2-1))]^{1/2}$, and take $\epsilon_0 = 2^3\beta(3y^2-1)/(3y^2)$, it can be checked that $g_\beta(m_1(2^{-4}),\theta) < m_1(2^{-4})$ for all $0<\epsilon<\epsilon_0$. On the other hand, taking $0<\epsilon\le\epsilon_1$ for some suitably small ϵ_1 one can check that $g_\beta(m_1(2),\theta) > m_1(2)$ and $m_1(2) < m_2^*$. This implies that there exists a strictly positive solution of (9.9) that satisfies $0 < m_1(2^{-4}) < m_1^* \le m_1(2) < m_2^*$. Moreover by convexity, $\frac{\partial g_\beta}{\partial m}(m_1^*) > 1$. The existence and uniqueness of $m_3^* > m_2^*$ follows from the fact that $g_\beta(m,\theta)$ is a concave function of m for $m > m_2^*$, that $g_\beta(m_2^*,\theta) > m_2^*$, that $\frac{\partial g_\beta}{\partial m}(m_2^*) > 1$, and that $\lim_{m\uparrow\infty} g_\beta(m,\theta) = 1$. Moreover by concavity $\frac{\partial g_\beta}{\partial m}(m_3^*) < 1$.

If $m_{\beta,\theta}$ is a solution of (9.9), we have

$$\frac{\partial m_{\beta,\theta}}{\partial\theta} = \frac{\partial g_\beta}{\partial\theta}(m_{\beta,\theta},\theta) + \frac{\partial g_\beta}{\partial m}(m_{\beta\theta},\theta)\frac{\partial m_{\beta,\theta}}{\partial\theta} \tag{9.24}$$

from which it follows that m_1^* is an increasing function of θ and m_3^* a decreasing function of θ.

Now using (9.21) one sees that there exists a unique $\theta_{3,c} > \theta_{1,c}$ such that for $\theta = \theta_{3,c}$ there exists a unique strictly positive solution $\tilde m_\beta^*$ of (9.9), however by continuity $\frac{\partial g_\beta}{\partial m}(\tilde m_\beta^*,\theta_{3,c}) = 1$. This ends the proof of the Lemma. ■

In the region $\mathcal{E}$, $f_{\beta,\theta}(m_1,m_2)$ has exactly three critical points, two points of minima around which $f_{\beta,\theta}(\,\cdot\,)$ is quadratic and a local maximum. Moreover there exists a strictly positive constant $\kappa(\beta,\theta)$ so that for each $m\in[-1,+1]^2$

$$f_{\beta,\theta}(m) - f_{\beta,\theta}(m_\beta) \ge \kappa(\beta,\theta)\min\{\|m-m_\beta\|_1^2, \|m-Tm_\beta\|_1^2\}, \tag{9.25}$$

where $\|\cdot\|_1$ is the ℓ^1 norm in $I\!R^2$ and $m_\beta = (m_{\beta,1}, m_{\beta,2})$ with

$$\begin{aligned} m_{\beta,1} &= \tanh(\beta\tilde m_\beta + \beta\theta) \\ m_{\beta,2} &= \tanh(\beta\tilde m_\beta - \beta\theta), \end{aligned} \tag{9.26}$$

where $\tilde m_\beta$ is the unique, strictly positive solution of (9.9) and $Tm_\beta = (-m_{\beta,2}, -m_{\beta,1})$.

Remark: Note that for $1<\beta<3/2$, as $\theta\uparrow\theta_{1,c}$ we have $\kappa(\beta,\theta)\downarrow 0$. We stress that in $\mathcal{E}$, we have always $\kappa(\beta,\theta)>0$. Since we work in the whole region $\mathcal{E}$, care of the $\kappa(\beta,\theta)$ dependence has been taken into account when writing the constraints, see (2.29) for example.

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