Kalikow-type decomposition for multicolor infinite range particle systems

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Abstract

We consider a particle system on $\mathbb{Z}^d$ with compact state space and interactions of infinite range in a high-noise regime. Assuming that the rate of change is continuous and that a Dobrushin-like condition holds, we obtain a Kalikow-type decomposition of the infinite range change rates as a mixture of finite range change rates. As an application of this decomposition we obtain a feasible perfect simulation algorithm to sample from the stationary process.

Key words : Interacting particle systems, infinite range interactions, compact-valued spins, perfect simulation, random Markov chains, Kalikow-type decomposition.

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1 Introduction

In this paper we present a Kalikow-type decomposition for interacting multicolor systems on $\mathbb{Z}^d$ having compact state space and interactions of infinite range. By a Kalikow-type decomposition we mean a representation of the infinite range rates as a countable mixture of local change rates of increasing range. This decomposition extends the notion of random Markov chains to interacting particle systems and has many potential theoretical consequences and applications. As an example we present here a perfect simulation algorithm which based on the decomposition.

We do not assume that the system has a dual, or is attractive, or monotone in any sense. Our system is not spatially homogeneous. The basic assumptions are the continuity of the infinite range change rates together with a high-noise or Dobrushin-like condition (Condition (**??**) : fast decay of the long range influence on the change rate and a certain subcriticality-criterion). This regime has traditionally been studied by perturbation

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methods which rely on sophisticated combinatorial estimations (see for instance Brydges 1984). This is not the approach we follow here. Our approach is probabilistic, based on an explicit construction and gives probabilistic insight into the structure of the stationary law of the process, without combinatorial or complex-analysis techniques. Let us stress that our approach is not an alternative to cluster expansions. It has a different regime of validity and different aims.

Our construction is reminiscent of Harris' graphical representation for particle systems and it is similar in spirit to procedures adopted in Bertein and Galves (1977), Ferrari (1990), Van den Berg and Steif (1999), Ferrari et al. (2002), and Garcia and Marić (2006) among others. However, all these papers only consider particular models, satisfying restrictive assumptions which are not assumed in the present paper. Our approach works for any infinite range continuous interaction under the only assumption of high-noise.

By a perfect simulation algorithm we mean a simulation which samples in a finite window precisely from the stationary law of the infinite process. More precisely, for any finite set of sites F we want to sample the projection of the stationary law on F. Our approach is feasible in the sense that it stops almost surely after a finite number of steps.

There are several techniques for perfect simulation of Markov processes. Among the most popular ones figures Coupling from the Past (CFTP) originally proposed by Propp and Wilson (1996) and applied to several special cases in a vast literature. A good review can be found in Kendall (2005). This kind of technique applies to invariant measures of Markov processes with finite coalescence time. One main point of the CFTP technique is that one has to be able to control the coalescence times uniformly with respect to all possible starting points. This is an issue that becomes particularly difficult in the case of "big" state spaces. The problem of large state spaces can be overcome for processes with certain monotonicity properties or for some specific cases. For example, for spatial point processes there is a vast literature on the subject, we point the works of Kendall (1998), Kendall and Thönnes (1999), Kendall and Møller (2000) among others.

For continuous state spaces, Cai (2005) proposes a non-monotone CFTP but as he points out "the detailed construction of the non-monotone CFTP algorithm is problem specific". Connors and Kendall (2007) show that for a large class for positive recurrent Markov processes it is always possible to perform CFTP, although not always feasible. However, for interacting particle systems with continuous state spaces, it seems to be out of reach to apply CFTP successfully. In general, the continuous case requires more complicated coupling techniques, such as $\varepsilon-$coupling or Nummelin splitting, see for instance Nummelin (1978) and Löcherbach and Loukianova (2008). See also, Murdoch and Green (1998) and Fernández, Ferrari and Grynberg (2007) for some special cases. The aim of the present article is to present an efficient perfect simulation technique which allows to overcome these problems without requiring any duality or monotonicity properties.

We conclude by recalling that the notion of random Markov chains was introduced explicitly in Kalikow (1990) and Bramson and Kalikow (1993) and appeared implicitly in Ferrari et al. (2000) and Comets et al. (2002).

This paper is organized as follows. The model and the Kalikow-type decomposition (Theorem **??**) are presented in Section 2. In Section **??** we present examples where all terms involved in the convex decomposition are explicitly given. In particular, we apply Theorem **??** to the important case of Gibbs measures with infinite range interactions and continuous spin values. In Section **??** we present the perfect simulation algorithm as a

main application of the convex decomposition. In particular, Theorem **??** shows that the proposed algorithm is feasible under a Dobrushin-like condition. The proofs are given in Sections **??** and **??**. We conclude the article with a small section on the impatient user bias.

2 Definitions, notation and convex decomposition

We consider interacting particle systems on $\mathbb{Z}^d$ having compact state space and interactions of infinite range. The elements of the state space are called *colors*. To each site in $\mathbb{Z}^d$ we assign a color. The coloring of the sites changes as time goes by. The rate at which the color of a fixed site i changes from a color a to a new color b is a function of the entire configuration and depends on b.

In what follows, ϱ will be a bounded non-singular non-negative reference measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with compact support $A \in \mathcal{B}(\mathbb{R})$. The initial lowercase letters a, b, $c, \ldots$ will denote elements of A. We endow A with its Borel $\sigma-$algebra $\mathcal{A} = \mathcal{B}(A)$ and denote by $S = A^{\mathbb{Z}^d}$ the configuration space with its product sigma algebra, $\mathcal{S}$. We will call configuration any element of S. Configurations will be denoted by Greek letters $\eta, \zeta, \xi, ...$ A point $i \in \mathbb{Z}^d$ will be called site. We define on $\mathbb{Z}^d$ the L^1 norm, $\|i\| = \sum_{k=1}^{d} |i_k|$. For $k \geq 0$, let the ball of radius k to be denoted as

$$V_i(k) = \{j \in \mathbb{Z}^d; \|j - i\| \leq k\}.$$

As usual, for any $i \in \mathbb{Z}^d$, $\eta(i)$ will denote the value of the configuration η at site i. By extension, for any subset $V \subset \mathbb{Z}^d$, $\eta(V) \in A^V$ will denote the restriction of the configuration η to the set of positions in V. For any η, i and a, we shall denote $\eta^{i,a}$ the modified configuration

$$\eta^{i,a}(j) = \eta(j), \text{ for all } j \neq i, \text{ and } \eta^{i,a}(i) = a.$$

For any $i \in \mathbb{Z}^d$, $\eta \in A^{\mathbb{Z}^d}$, let $a \mapsto c_i(a, \eta)$ be a positive $\mathcal{A} - \mathcal{B}(\mathbb{R}_+)-$measurable function such that, for any $i \in \mathbb{Z}^d$, there exists a constant $\Gamma_i < +\infty$ with

$$c_i(a, \eta) \leq \Gamma_i, \tag{2.1}$$

for every η and $\varrho-$almost all a.

A multicolor system with interactions of infinite range is a Markov process on S whose generator is defined on cylinder functions by

$$\mathcal{G}\, f(\eta) \,=\, \sum_{i \in \mathbb{Z}^d} \int_A \varrho(da) c_i(a, \eta)[f(\eta^{i,a}) - f(\eta)]\,, \tag{2.2}$$

where ϱ is a non-singular arbitrary reference measure.

By Theorem 3.9 of Chapter 1 of Liggett (1985) the following condition implies that $\mathcal{G}$ is the generator of a Feller process (σ_t) on S :

$$\sup_{i \in \mathbb{Z}^d} \sum_{j \neq i} \sup_{\eta} \sup_{b \in A} \{ \int_A \rho(da) |c_i(a, \eta) - c_i(a, \eta^{j,b})| \} < \infty. \tag{2.3}$$

In the following we shall work under conditions stronger than (**??**) ensuring not only that $\mathcal{G}$ is the generator of a unique Feller process, but also the possibility of perfect simulate the stationary process correspondent to this infinitesimal generator. As a byproduct this implies that the system admits the existence of a unique invariant measure μ for the system.

The main result of this article is a Kalikow-type convex decomposition of the change rates. We will prove that the change rate can be decomposed as

$$c_i(a,\eta) = M_i \left[\lambda_i(-1) p_i^{[-1]}(a) + \sum_{k\geq 0} \lambda_i(k) p_i^{[k]}(a|\eta) \right], \tag{2.4}$$

where

- $M_i, i \in \mathbb{Z}^d$ are positive constants,
- for each $i \in \mathbb{Z}^d$, $\{\lambda_i(k), k \geq -1\}$ is a probability distribution,
- for each $i \in \mathbb{Z}^d$, $p_i^{[-1]}(\cdot)$ is a probability density on A with respect to the reference measure ϱ, which does not depend on the configuration,
- for each $k \geq 0$ and for each $\eta \in S$, $p_i^{[k]}(\cdot|\eta(V_i(k))$ is a probability density with respect to the reference measure ϱ, depending only on the local configuration $\eta(V_i(k))$.

For convenience of the presentation we will add additional invisible jumps in (**??**). This is obtained by adding a cemetery Δ to A and defining $A^* := A \cup \{\Delta\}$. Define also

$$\varrho^* := \varrho + \delta_\Delta.$$

Denote

$$M_i := \sup_{\eta \in A^{\mathbb{Z}^d}} \int c_i(a,\eta)\varrho(da). \tag{2.5}$$

Notice that M_i is finite under condition (**??**), and define

$$c_i(\Delta,\eta) := M_i - \int_A c_i(a,\eta)\varrho(da). \tag{2.6}$$

Observe that

$$\inf_\eta c_i(\Delta,\eta) = 0. \tag{2.7}$$

Therefore we can rewrite the generator given by (**??**) as

$$\mathcal{G}\, f(\eta) \;=\; \sum_{i\in\mathbb{Z}^d} \int_{A^*} \varrho^*(da) c_i(a,\eta)[f(\eta^{i,a}) - f(\eta)], \tag{2.8}$$

where, by convention, for any $i \in \mathbb{Z}^d$ and any $\eta \in S = A^{\mathbb{Z}^d}$ we define

$$\eta^{i,\Delta} = \eta.$$

It follows that (**??**) is a representation of the same generator as (**??**).

In order to obtain the decomposition we need the following continuity condition.

Continuity condition. For $\varrho-$almost all a,

$$\sup_{i\in\mathbb{Z}^d}\sup_{\eta,\zeta:\eta(V_i(k))=\zeta(V_i(k))}|c_i(a,\eta)-c_i(a,\zeta)|\to 0\,, \tag{2.9}$$

as $k\to\infty$.

To describe the convex decomposition of the rate function c_i, we have to introduce the following quantities. Define

$$\alpha_i(-1) \;=\; \int_{A^*}\inf_{\zeta\in A^{\mathbb{Z}^d}} c_i(a,\zeta)\varrho^*(da), \tag{2.10}$$

and for any $k\geq 0$,

$$\alpha_i(k) \;=\; \inf_{w\in A^{V_i(k)}}\left(\int_{A^*}\inf_{\zeta:\zeta(V_i(k))=w} c_i(a,\zeta)\varrho^*(da)\right). \tag{2.11}$$

Note that by (**??**)

$$\int_{A^*}\inf_{\zeta\in A^{\mathbb{Z}^d}} c_i(a,\zeta)\varrho^*(da)=\int_{A}\inf_{\zeta\in A^{\mathbb{Z}^d}} c_i(a,\zeta)\varrho(da).$$

Further, by construction, we have that $\alpha_i(k)\leq\alpha_i(k+1)$, for each $k\geq -1$ and

$$M_i=\lim_{k\to\infty}\alpha_i(k). \tag{2.12}$$

To obtain equality (**??**), fix some $w\in A^{V_i(k)}$ and observe that

$$\int_{A^*}\inf_{\zeta:\zeta(V_i(k))=w} c_i(a,\zeta)\varrho^*(da)=\int_{A}\inf_{\zeta:\zeta(V_i(k))=w} c_i(a,\zeta)\varrho(da)+M_i-\sup_{\zeta:\zeta(V_i(k))=w}\int_A c_i(a,\zeta)\varrho(da).$$

But

$$\int_{A}\inf_{\zeta:\zeta(V_i(k))=w} c_i(a,\zeta)\varrho(da)-\sup_{\zeta:\zeta(V_i(k))=w}\int_A c_i(a,\zeta)\varrho(da)\to 0$$

as $k\to\infty$ thanks to condition (**??**).

Hence to each site i we can associate a probability distribution λ_i by

$$\lambda_i(-1) \;=\; \frac{\alpha_i(-1)}{M_i}, \tag{2.13}$$

and for $k\geq 0$

$$\lambda_i(k) \;=\; \frac{\alpha_i(k)\;-\;\alpha_i(k-1)}{M_i}. \tag{2.14}$$

Now we are ready to state the decomposition theorem.

Theorem 1 *Let $(c_i)_{i\in\mathbb{Z}^d}$ be a family of measurable rate functions satisfying conditions (**??**), (**??**) and (**??**). Then, for each site i, for M_i defined by (**??**), and $\lambda_i(\cdot)$ defined by (**??**) and (**??**), there exist*

- $p_i^{[-1]}$ *a probability density with respect to* ϱ *with support* A^*,
- *a family of conditional probability densities* $p_i^{[k]}, k \geq 0$ *on* A^*, *with respect to* ϱ^*, *depending on the local configurations* $\eta(V_i(k)) \in A^{V_i(k)}$

such that for all $a \in A^*$,

$$c_i(a,\eta) \;=\; M_i\, p_i(a|\eta) \tag{2.15}$$

where

$$p_i(a|\eta) = \lambda_i(-1)p_i^{[-1]}(a) + \sum_{k\geq 0} \lambda_i(k) p_i^{[k]}(a|\eta(V_i(k))). \tag{2.16}$$

As a consequence, the infinitesimal generator $\mathcal{G}$ *given by (??) can be rewritten as*

$$\begin{aligned} \mathcal{G}\, f(\eta) \;\;=\;\; & \sum_{i\in\mathbb{Z}^d} M_i \Bigg[\lambda_i(-1) \int_A p_i^{[-1]}(a)[f(\eta^{i,a}) - f(\eta)]\varrho(da) \\ & + \sum_{k\geq 0} \lambda_i(k) \int_{A^*} p_i^{[k]}(a|\eta(V_i(k)))[f(\eta^{i,a}) - f(\eta)]\varrho^*(da)\Bigg]. \end{aligned} \tag{2.17}$$

Note that for $k = -1$, $p_i^{[-1]}(a)$ does not depend on the configuration and $\lambda_i(-1)$ represents the spontaneous self-coloring rate of site i in the process. We will see in the proof that $p_i^{[-1]}$ is defined in such way that $p_i^{[-1]}(\Delta) = 0$ and therefore, the choice $k = -1$ implies always a choice of a real color $a \in A$, not of $a = \Delta$.

The decomposition given in Theorem **??** was designed in such way that the probability of self-coloring is maximized. This is important to speed up the perfect simulation algorithm. Obviously, slight modifications can be employed for different purposes as we will see in Example 2 (Section **??**).

The representation given by (**??**) provides a random finite range description of the time evolution of the process. We start with an initial configuration η at time zero. For each site $i \in \mathbb{Z}^d$, we consider a rate M_i Poisson point process N^i. The Poisson processes corresponding to distinct sites are all independent. If at time t, the Poisson clock associated to site i rings, we choose a range k with probability $\lambda_i(k)$ independently of everything else. And then, we update the value of the configuration at this site by choosing a symbol a with probability $p_i^{[k]}(a|\sigma_t(V_i(k)))\varrho^*(da)$. Choosing the symbol Δ means that we actually keep the current value of the spin.

In Section **??** we give examples of infinite range interacting systems where all terms in the decomposition (**??**) and (**??**) are explicitly computed.

3 Proof of Theorem ??

Put for any $a \in A^*$,

$$\begin{aligned} c_i^{[-1]}(a) &= \inf_{\zeta} c_i(a,\zeta), \\ \Delta_i^{[-1]}(a) &= c_i^{[-1]}(a), \\ c_i^{[0]}(a|\eta(i)) &= \inf_{\zeta:\zeta(i)=\eta(i)} c_i(a,\zeta), \\ \Delta_i^{[0]}(a|\eta(i)) &= c_i^{[0]}(a|\eta(i)) - c_i^{[-1]}(a). \end{aligned}$$

For any $k \geq 1$, define

$$c_i^{[k]}(a|\eta(V_i(k))) = \inf_{\zeta:\zeta(V_i(k))=\eta(V_i(k))} c_i(a,\zeta),$$

$$\Delta_i^{[k]}(a|\eta(V_i(k))) = c_i^{[k]}(a|\eta(V_i(k))) - c_i^{[k-1]}(a|\eta(V_i(k-1))).$$

Then we have that for any $a \in A$,

$$c_i(a,\eta) = \sum_{j=-1}^{k} \Delta_i^{[j]}(a|\eta(V_i(j))) + \left[c_i(a,\eta) - c_i^{[k]}(a|\eta(V_i(k)))\right]. \tag{3.18}$$

Note that

$$c_i^{[-1]}(\Delta) = \inf_{\eta} c_i(\Delta,\eta) = M_i - \sup_{\eta} \int_A c_i(a,\eta)\varrho(da) = 0.$$

Therefore, for $a = \Delta$ decomposition (**??**) starts with $j = 0$,

$$c_i(\Delta,\eta) = \sum_{j=0}^{k} \Delta_i^{[j]}(\Delta|\eta(V_i(j))) + \left[c_i(\Delta,\eta) - c_i^{[k]}(\Delta|\eta(V_i(k)))\right].$$

By continuity of $c_i(a,\eta)$ for every fixed η, we have for ϱ^*- almost all $a \in A^*$ that

$$c_i^{[k]}(a|\eta(V_i(k))) \to c_i(a,\eta) \text{ as } k \to \infty.$$

Hence for ϱ^*- almost all a and all η,

$$\sum_{j=-1}^{\infty} \Delta_i^{[j]}(a|\eta(V_i(j))) = c_i(a,\eta).$$

Taking into account (**??**) and (**??**)

$$M_i\lambda_i(-1) = \int_A \Delta_i^{[-1]}(a)\varrho(da).$$

Hence we can define

$$p_i^{[-1]}(a) = \frac{\Delta_i^{[-1]}(a)}{M_i\lambda_i(-1)}$$

and

$$p_i^{[-1]}(\Delta) = 0.$$

Hence, $p_i^{[-1]}(a)$ is a probability density with respect to ϱ^*. Now, for $k \geq 0$, put

$$\tilde{\lambda}_i(k, \eta(V_i(k))) = \frac{1}{M_i} \int_{A^*} \Delta_i^{[k]}(a|\eta(V_i(k)))\varrho^*(da), \tag{3.19}$$

and for any i, k such that $\tilde{\lambda}_i(k, \eta(V_i(k))) > 0$, we define

$$\tilde{p}_i^{[k]}(a|\eta(V_i(k))) = \frac{\Delta_i^{[k]}(a|\eta(V_i(k)))}{M_i\, \tilde{\lambda}_i(k, \eta(V_i(k)))}.$$

For i, k such that $\tilde{\lambda}_i(k, \eta(V_i(k))) = 0$, define $\tilde{p}_i^{[k]}(a|\eta(V_i(k)))$ in an arbitrary fixed way. Hence for ϱ^*- almost all $a \in A^*$,

$$c_i(a, \eta) = M_i \left[\lambda_i(-1) p_i^{[-1]}(a) + \sum_{k=0}^{\infty} \tilde{\lambda}_i(k, \eta(V_i(k))) \tilde{p}_i^{[k]}(a|\eta(V_i(k))) \right]. \tag{3.20}$$

In (**??**) the factors $\tilde{\lambda}_i(k, \eta(V_i(k))), k \geq 0$, still depend on $\eta(V_i(k))$. To obtain the decomposition as in the theorem, we must rewrite it as follows.

For any i, take M_i as in (**??**) and the sequences $\alpha_i(k), \lambda_i(k), k \geq -1$, as defined in (**??**) and (**??**), respectively. Define the new quantities

$$\alpha_i(k, \eta(V_i(k))) = M_i \sum_{l \leq k} \tilde{\lambda}_i(l, \eta(V_i(l))).$$

Finally, for any $k \geq 0$, we define the conditional finite range probability densities by

$$\begin{aligned}
&p_i^{[k]}(a|\eta(V_i(k))) = \\
&\sum_{-1=l' \leq l}^{k-1} 1_{\{\alpha_i(l'-1, \eta(V_i(l'-1))) < \alpha_i(k-1) \leq \alpha_i(l', \eta(V_i(l')))\}} 1_{\{\alpha_i(l, \eta(V_i(l))) < \alpha_i(k) \leq \alpha_i(l+1, \eta(V_i(l+1)))\}} \\
&\qquad \left[\frac{\alpha_i(l', \eta(V_i(l'))) - \alpha_i(k-1)}{M_i\, \lambda_i(k)} \tilde{p}_i^{[l']}(a|\eta(V_i(l'))) \right. \\
&\qquad + \sum_{m=l'+1}^{l} \frac{\tilde{\lambda}_i(m, \eta(V_i(m)))}{\lambda_i(k)} \tilde{p}_i^{[m]}(a|\eta(V_i(m))) \\
&\qquad \left. + \frac{\alpha_i(k) - \alpha_i(l, \eta(V_i(l)))}{M_i\, \lambda_i(k)} \tilde{p}_i^{[l+1]}(a|\eta(V_i(l+1))) \right].
\end{aligned}$$

The desired decomposition follows from a straightforward computation. •

4 Examples

In this section we show that the decomposition presented in Theorem **??** can be effectively implemented in several interesting Gibbsian systems with compact-valued spins. In all the examples we take $A = [-1, 1]$.

Definition 1 *A pairwise potential is a collection* $\{J(i,j),(i,j)\in\mathbb{Z}^d\times\mathbb{Z}^d\}$ *of real numbers which satisfies*

$$J(i,i)=0,\qquad \sup_{i\in\mathbb{Z}^d}\sum_{j\in\mathbb{Z}^d}|J(i,j)|<\infty. \tag{4.21}$$

In what follows we use the notation

$$\Sigma_i=\sum_{j\in\mathbb{Z}^d}|J(i,j)|.$$

For any $i\in\mathbb{Z}^d$, let $\eta(i)$ be the value of the spin at site i in the configuration $\eta\in S$.

Definition 2 *A probability measure* μ *on* $(S,\mathcal{S})$ *is said to be a Gibbs state relative to the potential* $\{J(i,j)\}$ *if for all* $i\in\mathbb{Z}^d$*, a version of the conditional probability density of* $\eta(i)$*, given* $\eta(j), j\neq i$*, is given by*

$$\mu(\eta(i)=a|\eta(j)\ \textit{for all}\ j\neq i\})=\frac{\exp\left(a\sum_{j\neq i}J(i,j)\eta(j)\right)}{Z^\eta},$$

where

$$Z^\eta=\int_A\exp\left(a\sum_{j\neq i}J(i,j)\eta(j)\right)\varrho(da).$$

In the following we consider the interaction $J^\beta=\beta J$, where β is a positive parameter. The associated Gibbs measure will be denoted μ without indicating explicitly the dependence on β. Now, put

$$c_i(a,\eta)=e^{\beta a\sum_{j\in\mathbb{Z}^d}J(i,j)\eta(j)}. \tag{4.22}$$

Then, by construction, the process (σ_t) with generator (**??**) and this choice of change rates is reversible with respect to the Gibbs state μ corresponding to the potential $J_\beta(i,j)=\beta J(i,j)$. It is immediate to see that condition (**??**) implies the continuity condition (**??**).

We now give the explicit decomposition in two special cases.

Example 1. Take $\varrho(da)=\frac{1}{2}da$, $a\in[-1,1]$, and nearest neighborhood interactions

$$c_i(a,\eta)=\exp\left(\beta a\sum_j J(i,j)\eta(j)\right),$$

where

$$J(i,j)=0\text{ if }j\notin V_i(1)\setminus\{i\}.$$

In other terms, in this case we are considering only nearest neighbor interactions. Then we have

$$M_i=\frac{1}{2\beta\Sigma_i}\left(e^{\beta\Sigma_i}-e^{-\beta\Sigma_i}\right),$$

$$\alpha_i(-1)=\alpha_i(0)=\frac{1}{\beta\Sigma_i}\left[1-\exp\left(-\beta\Sigma_i\right)\right],$$

$$\alpha_i(1) = M_i,$$
$$p_i^{[-1]}(a) = \frac{\exp(-\beta|a|\Sigma_i)}{\alpha_i(-1)},$$
$$p_i^{[-1]}(\Delta) = 0,$$
$$p_i^{[1]}(a|\eta) = \frac{c_i(a,\eta) - \exp(-\beta|a|\Sigma_i)}{M_i - \alpha_i(-1)}$$

and

$$p_i^{[1]}(\Delta|\eta) = \frac{M_i - \int_E c_i(a,\eta)\varrho(da)}{M_i - \alpha_i(-1)}.$$

To check the above expressions we start by calculating the constant M_i. Notice that

$$\int c_i(a,\eta)\varrho(da) = \frac{1}{2\beta\sum_j J(i,j)\eta(j)}\left(e^{\beta\sum_j J(i,j)\eta(j)} - e^{-\beta\sum_j J(i,j)\eta(j)}\right).$$

Maximizing the above expression with respect to η yields

$$M_i = \sup_i \int c_i(a,\eta)\varrho(da) = \frac{1}{2\beta\Sigma_i}\left(e^{\beta\Sigma_i} - e^{-\beta\Sigma_i}\right).$$

We are now going to calculate the coefficients $\alpha_i(k)$. Since $c_i(a,\eta)$ does not depend on $\eta(i)$, we have that $\alpha_i(-1) = \alpha_i(0)$. Moreover, due to the nearest neighborhood interaction, $\alpha_i(k) = \alpha_i(1) = M_i$ for all $k \geq 1$. So we only have to evaluate $\alpha_i(-1)$. First observe that

$$\inf_\eta c_i(a,\eta) = \exp\left(-\beta|a|\Sigma_i\right).$$

Integrating this with respect to $\varrho(da)$ yields

$$\alpha_i(-1) = \int\left(\inf_\eta c_i(a,\eta)\right)\varrho(da) = \frac{1}{\beta\Sigma_i}\left[1 - \exp\left(-\beta\Sigma_i\right)\right]$$

and thus

$$\lambda_i(-1) = 2\,\frac{1 - \exp\left(-\beta\Sigma_i\right)}{\exp(\beta\Sigma_i) - \exp(-\beta\Sigma_i)}.$$

Note that $\lambda_i(-1) \leq 1$ and evidently, $\lambda_i(-1) \geq 0$. Finally note that, in this special case,

$$\begin{aligned} M_i\tilde{\lambda}_i(1,\eta(V_i(1))) &= \int_{A^*}\Delta_i^{[1]}(a,\eta)\varrho^*(da) \\ &= \left(\int_E c_i(a,\eta)\varrho(da) - \int_A c_i^{[-1]}(a)\varrho(da)\right) + c_i(\Delta,\eta) \\ &= M_i - \alpha_i(-1) = M_i(1-\lambda_i(-1)), \end{aligned}$$

which does not depend on η, so

$$\tilde{\lambda}_i(1,\eta) = \lambda_i(1) \text{ and } \tilde{p}_i^{[1]}(a|\eta) = p_i^{[1]}(a|\eta).$$

In particular, this yields

$$p_i^{[-1]}(a) = \frac{\exp(-\beta|a|\Sigma_i)}{\alpha_i(-1)},$$

$$p_i^{[-1]}(\Delta) = 0,$$
$$p_i^{[1]}(a|\eta) = \frac{c_i(a,\eta) - \exp(-\beta|a|\Sigma_i)}{M_i\lambda_i(1)}$$

and

$$p_i^{[1]}(\Delta|\eta) = \frac{M_i - \int_E c_i(a,\eta)\varrho(da)}{M_i\lambda_i(1)}.$$

Example 2. The following example is a Gibbsian time evolution with infinite range interaction. The decomposition we present here is inspired by the one presented in Galves et al. (2010) in the case of two colors systems. In Galves et al. (2010), for coupling reasons, it was convenient to give a slightly different decomposition. The goal there was to be able to couple together the infinite range Gibbsian system with the finite range Gibbsian system obtained by truncating the potential interaction. For the sake of the readers, we recall here their decomposition in a more general case by adding an external field. Let

$$\varrho(da) = \delta_1(da) + \delta_{-1}(da).$$

Let $\{h_i, i \in \mathbb{Z}^d\}$ be a collection of real numbers, representing the external field, and put

$$c_i(\eta) = c_i(-\eta(i),\eta) = \exp\left(-\beta\sum_j J(i,j)\eta(i)\eta(j) - \beta h_i\eta(i)\right).$$

Then the decomposition (**??**) holds with

$$M_i = 2e^{\beta\sum_j |J(i,j)| + \beta|h_i|},$$

$$\lambda_i(-1) = \exp\left(-2\beta\left[\sum_j |J(i,j)| + |h_i|\right]\right),$$

$$\lambda_i(0) = \exp\left(-2\beta\sum_j |J(i,j)|\right) - \lambda_i(-1),$$

$$\lambda_i(1) = e^{-\beta\sum_{j:\|i-j\|>1}|J(i,j)|} - e^{-2\beta\sum_j |J(i,j)|}$$

and for $k \geq 2$,

$$\lambda_i(k) = e^{-\beta\sum_{j:\|i-j\|>k}|J(i,j)|} - e^{-\beta\sum_{j:\|i-j\|\geq k}|J(i,j)|}.$$

Moreover, for all $\eta \in S$ and for any $k \geq 2$, we define the update probabilities

$$p_i^{[k]}(-\eta(i)|\eta) = \frac{1}{M_i}e^{-\beta h_i\eta(i)}e^{-\beta\sum_{j:\|i-j\|<k}J(i,j)\eta(i)\eta(j)} \frac{e^{-\beta\sum_{j:\|i-j\|=k}J(i,j)\eta(i)\eta(j)} - e^{-\beta\sum_{j:\|i-j\|=k}|J(i,j)|}}{1 - e^{-\beta\sum_{j:\|i-j\|=k}|J(i,j)|}},$$

for $k = 1$,

$$p_i^{[1]}(-\eta(i)|\eta) = \frac{1}{M_i}e^{-\beta h_i\eta(i)}\frac{e^{-\beta\sum_{j:\|i-j\|\leq 1}J(i,j)\eta(i)\eta(j)} - e^{-\beta\sum_{j:\|i-j\|\leq 1}|J(i,j)|}}{1 - e^{-2\beta\sum_{j:\|i-j\|\leq 1}|J(i,j)|}e^{-\beta\sum_{j:\|i-j\|>1}|J(i,j)|}},$$

and for $k = 0$,

$$p_i^{[0]}(-\eta(i)|\eta) = \frac{1}{2} e^{-\beta|h_i|} \frac{e^{-\beta h_i \eta(i)} - e^{-\beta|h_i|}}{1 - e^{-\beta|h_i|}}.$$

To obtain a probability measure on A, we define for all $k \geq 0$,

$$p_i^{[k]}(\eta(i)|\eta) = 1 - p_i^{[k]}(-\eta(i)|\eta).$$

Finally, for $k = -1$, we define

$$p_i^{[0]}(1) = p_i^{[0]}(-1) = \frac{1}{2}. \tag{4.23}$$

Example 3. We specialize Example 2 in the case where the spin distribution ϱ is symmetric and the external field $h = 0$.

Define for any $i \in \mathbb{Z}^d$ and any $k \geq -1$,

$$S_i^{>k} := \sum_{j:\|i-j\|>k} |J(i,j)|, \; S_i^{\leq k} := \sum_{j:\|i-j\|\leq k} |J(i,j)|.$$

Note that $\Sigma_i = S_i^{>-1}$.

Then

$$M_i = \int_0^1 \left(e^{a\beta\Sigma_i} + e^{-a\beta\Sigma_i} \right) \varrho(da). \tag{4.24}$$

Moreover,

$$\alpha_i(-1) = 2 \int_0^1 e^{-a\beta\Sigma_i} \varrho(da) \tag{4.25}$$

and

$$\alpha_i(k) = M_i + \int_0^1 e^{a\beta S_i^{\leq k}} e^{-a\beta S_i^{>k}} \varrho(da) - \int_0^1 e^{a\beta\Sigma_i} \varrho(da). \tag{4.26}$$

Finally,

$$\lambda_i(-1) = 2 \frac{\int_0^1 e^{-a\beta\Sigma_i} \varrho(da)}{\int_0^1 \left(e^{a\beta\Sigma_i} + e^{-a\beta\Sigma_i} \right) \varrho(da)} \tag{4.27}$$

and

$$\lambda_i(k) = \frac{\int_0^1 e^{a\beta S_i^{\leq k-1}} e^{-a\beta S_i^{>k}} \left(e^{a\beta \sum_{j:\|j-i\|=k} |J(i,j)|} - e^{-a\beta \sum_{j:\|j-i\|=k} |J(i,j)|} \right) \varrho(da)}{\int_0^1 \left(e^{a\beta\Sigma_i} + e^{-a\beta\Sigma_i} \right) \varrho(da)}. \tag{4.28}$$

Expression (**??**) follows from the definition of M_i in (**??**) together with

$$\int_{-1}^1 c_i(a,\eta) \varrho(da) = \int_0^1 \left(e^{\beta a \sum_j J(i,j)\eta(j)} + e^{-\beta a \sum_j J(i,j)\eta(j)} \right) \varrho(da),$$

by symmetry of the measure ϱ. Maximizing this expression with respect to η yields (**??**).

Equation (**??**) is an immediate consequence of the definition (**??**) since

$$\inf_{\eta} c_i(a,\eta) = e^{-\beta|a|\Sigma_i}.$$

Concerning (**??**), note first that

$$\inf_{\zeta:\zeta(V_i(k))=w} c_i(a,\zeta) = e^{a\beta\sum_{j:\|j-i\|\le k} J(i,j)w(j)} e^{-|a|\beta S_i^{>k}}.$$

Integrating this with respect to ϱ yields

$$\int_{-1}^{1} \inf_{\zeta:\zeta(V_i(k))=w} c_i(a,\zeta)\varrho(da) = \int_0^1 e^{-a\beta S_i^{>k}} \left(e^{a\beta\sum_{j:\|j-i\|\le k} J(i,j)w(j)} + e^{-a\beta\sum_{j:\|j-i\|\le k} J(i,j)w(j)} \right) \varrho(da),$$

by symmetry of ϱ. Moreover, by definition of $c_i(\Delta,\eta)$,

$$\begin{aligned}\inf_{\zeta:\zeta(V_i(k))=w} c_i(\Delta,\zeta) &= M_i - \sup_{\zeta:\zeta(V_i(k))=w} \int c_i(a,\zeta)\varrho(da) \\ &= M_i - \sup_{\zeta:\zeta(V_i(k))=w} \int_0^1 \left(e^{a\beta\sum_j J(i,j)\zeta(j)} + e^{-a\beta\sum_j J(i,j)\zeta(j)} \right) \varrho(da).\end{aligned}$$

Maximizing the expression with respect to ζ under the integral yields

$$\inf_{\zeta:\zeta(V_i(k))=w} c_i(\Delta,\zeta) = M_i - \int_0^1 \left(e^{a\beta(|\sum_{j:\|j-i\|\le k} J(i,j)w(j)| + S_i^{>k})} + e^{-a\beta(|\sum_{j:\|j-i\|\le k} J(i,j)w(j)| + S_i^{>k})} \right) \varrho(da).$$

Putting things together we conclude that

$$\int_{-1}^{1} \inf_{\zeta:\zeta(V_i(k))=w} c_i(a,\zeta)\varrho^*(da) = M_i + \int_0^1 e^{a\beta|\sum_{j:\|j-i\|\le k} J(i,j)w(j)|} \left(e^{-a\beta S_i^{>k}} - e^{a\beta S_i^{>k}} \right) \varrho(da).$$

Taking finally the infimum with respect to w, and noticing that $e^{-a\beta S_i^{>k}} - e^{a\beta S_i^{>k}} < 0$, we obtain that

$$\alpha_i(k) = M_i + \int_0^1 e^{a\beta S_i^{\le k}} e^{-a\beta S_i^{>k}} \varrho(da) - \int_0^1 e^{a\beta\Sigma_i} \varrho(da).$$

As a consequence we obtain that

$$\lambda_i(k) = \frac{\int_0^1 e^{a\beta S_i^{\le k-1}} e^{-a\beta S_i^{>k}} \left(e^{a\beta\sum_{j:\|j-i\|=k} |J(i,j)|} - e^{-a\beta\sum_{j:\|j-i\|=k} |J(i,j)|} \right) \varrho(da)}{\int_0^1 \left(e^{a\beta\Sigma_i} + e^{-a\beta\Sigma_i} \right) \varrho(da)}.$$

5 Perfect simulation

The goal of this section is to give an application of the Kalikow-type decomposition given by Theorem **??**. This application is a perfect simulation algorithm for the invariant measure of an interacting multicolor system. We assume that the interaction rates are continuous and satisfy a Dobrushin-like condition. The basis of the algorithm is the convex decomposition given in Theorem **??**. First of all, the Proposition **??** gives a sufficient condition for exponential ergodicity which is based on the construction of a dominating branching process.

From now on we will denote by (σ_t^η) (and (σ_t^μ)) the multicolor system having generator $\mathcal{G}$ given by (**??**) with a fixed initial configuration η (a random configuration chosen with probability distribution μ).

Proposition 1 *Let $(c_i)_{i\in\mathbb{Z}^d}$ be a family of rate functions satisfying the conditions of Theorem **??**. Furthermore, assume that*

$$\underline{M} = \inf_{i\in\mathbb{Z}^d} M_i > 0 \tag{5.29}$$

and

$$\sup_{i\in\mathbb{Z}^d} \sum_{k\geq 0} |V_i(k)|\lambda_i(k) = \gamma < 1. \tag{5.30}$$

Then, the process (σ_t) admits a unique invariant probability measure μ. Moreover, for any finite set of sites $F \subset \mathbb{Z}^d$, for any $T > 0$ and any initial configuration η, there exists a coupling between the process (σ_t^η) and the stationary process (σ_t^μ) such that

$$P(\sigma_T^\eta(F) \neq \sigma_T^\mu(F)) \leq |F| e^{-\underline{M}(1-\gamma)T}.$$

Let us compare the above proposition to known results in the literature on particle systems.

1. Condition (**??**) is stronger than Liggett's existence condition (**??**) which does not imply the uniqueness of the invariant measure.

2. Condition (3.3) of Liggett (2000), page 22, is equivalent to

$$\sup_i \sup_\eta \int c_i(a,\eta)\varrho(da) = \sup_i M_i < \infty,$$

which is implied by conditions (**??**) and (**??**). Moreover, it can be easily seen that the quantity M appearing in Equation (3.8) of Liggett (2000), page 26, can be upper bounded by

$$\sup_{i\in\mathbb{Z}^d} M_i \sum_{k\geq 0} \lambda_i(k)|V_i(k)|.$$

Since $\sup_i M_i < \infty$, condition $\sup_{i\in\mathbb{Z}^d} \sum_{k\geq 0} |V_i(k)|\lambda_i(k) < \infty$ implies Condition (3.8) of Liggett (2000).

3. Condition (**??**) is a high-noise condition which implies a Dobrushin-like condition, see Dobrushin (1968). It is a sufficient condition ensuring that there is no phase transition.

4. For infinite range Gibbs measures, in the situation of Example 3, a sufficient condition for (**??**) is

$$\sup_{i\in\mathbb{Z}^d}\sum_k |V_i(k)|\left(\sum_{j:\|j-i\|=k}|J(i,j)|\right)<\infty$$

and $\beta<\beta_c$, where β_c is solution of

$$2\beta\sum_{k\geq 1}\left(|V_i(k)|\sum_{j:\|j-i\|=k}|J(i,j)|\right)=1.$$

This follows from (**??**), using that $1-e^{-x}\leq x$ for $x\geq 0$.

We are now in position to present the perfect simulation scheme. Suppose we want to sample the configuration at site i under μ. In a first step, we determine the set of sites whose spins influence the spin at site i under equilibrium. We call this set of sites *ancestors* of i and this stage *backward sketch procedure*. First, we climb up from time 0 using a reverse time Poisson point process with rate M_i. We stop when the last Poisson clock before time 0 rings. At that time, we choose a range k with probability $\lambda_i(k)$. If $k=-1$, we decide the value of the spin using the law $p_i^{[-1]}d\varrho$, independently of everything else. If k is different from -1, we restart the above procedure from every site $j\in V_i(k)$. The procedure stops once each site involved has chosen range -1. When this occurs, we can start the second stage, in which we go back to the future assigning spins to all sites visited during the first stage. We call this procedure forward spin assignment procedure. This is done from the past to the future by using the update probability densities $p_i^{[k]}$ starting at the sites which ended the first procedure by choosing range -1. For each one of these sites a spin is chosen according to $p^{[-1]}d\varrho$. The values obtained in this way enter successively in the choice of the values of the spins depending on a neighborhood of range greater or equal to 0.

We now give the precise form of the algorithm. Fix a finite set $F\subset\mathbb{Z}^d$. The following variables will be used.

- N is an auxiliary variables taking values in the set of non-negative integers $\{0,1,2,\ldots\}$
- $N_{STOP}^{(F)}$ is a counter taking values in the set of non-negative integers $\{0,1,2,\ldots\}$
- I is a variable taking values in $\mathbb{Z}^d$
- K is a variable taking values in $\{-1,0,1,\ldots\}$
- B is an array of elements of $\mathbb{Z}^d\times\{-1,0,1,\ldots\}$
- C is a variable taking values in the set of finite subsets of $\mathbb{Z}^d$
- W is an auxiliary variable taking values in A^*
- σ is a function from $\mathbb{Z}^d$ to A^*

Algorithm 1 Backward sketch procedure

1. *Input:* F; *Output:* $N_{STOP}^{(F)}$, B

2. $N \leftarrow 0$, $N^{(F)}_{STOP} \leftarrow 0$, $B \leftarrow \emptyset$, $C \leftarrow \{F\}$
3. WHILE $C \neq \emptyset$
4. $N \leftarrow N + 1$
5. Choose randomly a position $I \in C$ and an integer $K \geq -1$ according to the probability distribution

$$P(I = i, K = k) = \frac{M_i \lambda_i(k)}{\sum_{j \in C} \sum_{l \geq -1} M_j \lambda_j(l)}$$

6. IF $K = -1$, $C \leftarrow C \setminus \{I\}$
7. ELSE $C \leftarrow C \cup B_I(K)$
8. ENDIF
9. $B(N) \leftarrow (I, K)$
10. ENDWHILE
11. $N^{(F)}_{STOP} \leftarrow N$
12. RETURN $N^{(F)}_{STOP}$, B.

Now we use the following Forward spin assignment procedure to sample from the invariant measure μ. Recall that the choice of Δ in (**??**) implies that the system does not change its colors. This explains Step 9 in Algorithm 2.

Algorithm 2 Forward spin assignment procedure

1. *Input:* $N^{(F)}_{STOP}$, B; *Output:* $\{(i, \sigma(i)) : i \in F\}$
2. $N \leftarrow N^{(F)}_{STOP}$
3. $\sigma(j) \leftarrow \Delta$ for all $j \in \mathbb{Z}^d$
4. WHILE $N \geq 1$
5. $(I, K) \leftarrow B(N)$.
6. IF $K = -1$ choose W randomly in A according to the probability distribution

$$p_I^{[-1]} d\varrho$$

7. ELSE choose W randomly in A^* according to the probability distribution

$$p_I^{[K]}(\cdot|\sigma) d\varrho^*$$

8. ENDIF
9. IF $W \neq \Delta$ put $\sigma(I) \leftarrow W$

10. ENDIF

11. $N \leftarrow N - 1$

12. ENDWHILE

13. RETURN $\{(i, \sigma(i)) : i \in F\}$

The next theorem summarizes the properties of Algorithms 1 and 2.

Theorem 2 *Suppose that the sub-criticality condition (**??**) holds. The Algorithm 1 stops almost surely after a finite number of steps and*

$$\sup_{i \in \mathbb{Z}^d} P(N_{STOP}^{(F)} > N) \leq |F| \gamma^N, \tag{5.31}$$

*where γ is given in (**??**). The law of the set $\{(i, \sigma(i)) : i \in F\}$ printed at the end of Algorithm 1 and 2 is the projection on A^F of the unique invariant measure μ of the process.*

The proofs of Proposition **??** and Theorem **??** will be given in the next section.

6 Proofs of Proposition ?? and Theorem ??

The proofs rely on the notion of *black and white time-reverse sketch process* that we will introduce now. The black and white time-reverse sketch process gives the mathematically precise description of the backward black and white Algorithm 1 given above.

We start by introducing some more notation. For each $i \in \mathbb{Z}^d$, denote by $\ldots T_{-2}^i < T_{-1}^i < T_0^i < 0 < T_1^i < T_2^i < \ldots$ the occurrence times of the rate M_i Poisson point process N^i on the real line. The Poisson point processes associated to different sites are independent. To each point T_n^i associate an independent mark K_n^i according to the probability distribution $(\lambda_i(k))_{k \geq -1}$. As usual, we identify the Poisson point processes and the counting measures through the formula

$$N^i[s,t] = \sum_{n \in \mathbb{Z}} \mathbf{1}_{\{s \leq T_n^i \leq t\}}.$$

It follows from this identification that for any $t > 0$ we have $T_{N^i(0,t]}^i \leq t < T_{N^i(0,t]+1}^i$, and for any $t \leq 0$, $T_{-N^i(t,0]}^i \leq t < T_{-N^i(t,0]+1}^i$.

For each $i \in \mathbb{Z}^d$ and $t \in \mathbb{R}$ we define the time-reverse point process starting at time t, associated to site i,

$$\begin{aligned} \tilde{T}_n^{(i,t)} &= t - T_{N^i(0,t]-n+1}^i, \quad t \geq 0, \\ \tilde{T}_n^{(i,t)} &= t - T_{-N^i(t,0]-n+1}^i, \quad t < 0. \end{aligned} \tag{6.32}$$

We also define the associated marks

$$\begin{aligned} \tilde{K}_n^{(i,t)} &= K_{N^i(0,t]-n+1}^i, \quad t \geq 0, \\ \tilde{K}_n^{(i,t)} &= K_{-N^i(t,0]-n+1}^i, \quad t < 0. \end{aligned} \tag{6.33}$$

For each site $i \in \mathbb{Z}^d$, $k \geq -1$, the reversed k-marked Poisson point process returning from time t is defined as

$$\tilde{N}^{(i,t,k)}[s,u] \;=\; \sum_n \mathbf{1}_{\{s \leq \tilde{T}_n^{(i,t)} \leq u\}} \mathbf{1}_{\{\tilde{K}_n^{(i,t)} = k\}}. \tag{6.34}$$

To define the black and white time-reverse sketch process we need to introduce a family of transformations $\{\pi^{(i,k)}, i \in \mathbb{Z}^d, k \geq -1\}$ on the set of finite subsets of $\mathbb{Z}^d$, $\mathcal{F}(\mathbb{Z}^d)$, defined as follows. For any unitary set $\{j\}$,

$$\pi^{(i,k)}(\{j\}) \;=\; \left\{ \begin{array}{ll} V_i(k), & \text{if } j = i \\ \{j\}, & \text{otherwise} \end{array} \right\}. \tag{6.35}$$

Notice that for $k = -1$, $\pi^{(i,k)}(\{i\}) = \emptyset$. For any set finite set $F \subset \mathbb{Z}^d$, we define similarly

$$\pi^{(i,k)}(F) \;=\; \cup_{j \in F} \pi^{(i,k)}(\{j\}). \tag{6.36}$$

The black and white time-reverse sketch process starting at site i at time t will be denoted by $(C_s^{(i,t)})_{s \geq 0}$. $C_s^{(i,t)}$ is the set of sites at time s whose colors affect the color of site i at time t. We call this set $C_s^{(i,t)}$ *set of ancestors of* i *at time* s *before time* t. The evolution of this process is defined through the following equation: $C_0^{(i,t)} := \{i\}$, and

$$f(C_s^{(i,t)}) \;=\; f(C_0^{(i,t)}) \;+\; \sum_{k \geq -1} \sum_{j \in \mathbb{Z}^d} \int_0^s [f(\pi^{(j,k)}(C_{u-}^{(i,t)})) - f(C_{u-}^{(i,t)})]\, \tilde{N}^{(j,t,k)}(du), \tag{6.37}$$

where $f : \mathcal{F}(\mathbb{Z}^d) \to \mathbb{R}$ is any bounded cylindrical function. This family of equations characterizes completely the time evolution $\{C_s^{(i,t)}, s \geq 0\}$. For any finite set $F \subset \mathbb{Z}^d$ define

$$C_s^{(F,t)} \;=\; \cup_{i \in F} C_s^{(i,t)}.$$

The following proposition summarizes the properties of the family of processes defined above.

Proposition 2 *For any finite set* $F \subset \mathbb{Z}^d$, $\{C_s^{(F,t)}, s \geq 0\}$ *is a Markov jump process having as infinitesimal generator*

$$Lf(C) \;=\; M_i \sum_{i \in C} \sum_{k \geq 0} \lambda_i(k)[f(C \cup V_i(k)) - f(C)] + \lambda_i(-1)[f(C \setminus \{i\}) - f(C)], \tag{6.38}$$

where f *is any bounded cylindrical function.*

Proof The proof follows in a standard way from the construction (**??**). •

If we are interested in simulating from the invariant measure of the process, then we will start the black and white time-reverse sketch process at time $t = 0$, if however we wish to construct the process at time t, we shall start the black and white time-reverse sketch process at that time t precisely.

6.1 Backwards oriented percolation and sub-criticality

For the algorithm to be successful it is crucial to show that $\bigcup_{s\geq 0} C_s^{(i,t)}$, the set of ancestors of any site i, is finite with probability one. Formally, let

$$T_{STOP}^{(i)} = \inf\{s : C_s^{(i,0)} = \emptyset\}$$

be the relaxation time. We introduce the sequence of successive jump times $\tilde{T}_n^{(i)}, n \geq 1$, of processes $N^{(j,k)}$ whose jumps occur in (**??**), for $t = 0$. Let $\tilde{T}_1^{(i)} = T_1^{(i,0)}$ and define successively for $n \geq 2$

$$\tilde{T}_n^{(i)} = \inf\{t > \tilde{T}_{n-1}^{(i)} : \exists j \in C^{(i,0)}_{\tilde{T}_{n-1}^{(i)}}, \exists k : N^{(j,k)}(]\tilde{T}_{n-1}^{(i)}, t]) = 1\}. \tag{6.39}$$

We write $\tilde{K}_n^{(i)}$ for the associated marks. Now we put

$$\mathbf{C}_n^{(i)} = C^{(i,0)}_{\tilde{T}_n^{(i)}} \tag{6.40}$$

and

$$N_{STOP}^{(i)} = \inf\{n : \mathbf{C}_n^{(i)} = \emptyset\}.$$

This is the number of steps of the backward sketch process – and it is exactly the number of steps of Algorithm 1. For the perfect simulation algorithm to be successful, it is crucial to show that both relaxation time $T_{STOP}^{(i)}$ and the number of steps $N_{STOP}^{(i)}$ are finite. To this aim we start estimating the volume of the set $C_s^{(F,t)} = \cup_{i\in F} C_s^{(i,t)}$ where F is a bounded set of $\mathbb{Z}^d$.

Lemma 1

$$E(|C_s^{(F,t)}|) \leq |F| e^{-\underline{M}(1-\gamma)s}, \tag{6.41}$$

where $\underline{M}$ *is defined in* (**??**) *and* γ *in* (**??**).

Proof Fix some $N \in \mathbb{N}$. Let $L_s^i = |C_s^{(i,t)}|$ and

$$T_N = \inf\{t : L_t^i \geq N\}.$$

Then by (**??**),

$$\begin{aligned} L^i_{s\wedge T_N} &\leq 1 + \sum_{k\geq 1}\sum_{j\in\mathbb{Z}^d} \int_0^{s\wedge T_N} [|V_j(k)| - 1] 1_{\{j\in C_{u-}^{(i,t)}\}} \tilde{N}^{(j,t,k)}(du) \\ &\quad - \sum_{j\in\mathbb{Z}^d} \int_0^{s\wedge T_N} 1_{\{j\in C_{u-}^{(i,t)}\}} \tilde{N}^{(j,t,-1)}(du). \end{aligned} \tag{6.42}$$

Recall that that $\underline{M} = \inf_{i\in\mathbb{Z}^d} M_i > 0$. Passing to expectation and using that, by condition (**??**),

$$M_j \left((\sum_{k\geq 1} \lambda_j(k)[|V_j(k)| - 1]) - \lambda_j(-1) \right) \leq -\underline{M}(1-\gamma) < 0,$$

this yields

$$\begin{aligned} E(L^i_{s\wedge T_N}) &\leq 1+\sum_{j\in\mathbb{Z}^d} M_j \left((\sum_{k\geq 1} \lambda_j(k)[|V_j(k)|-1]) - \lambda_j(-1) \right) \\ &\qquad \times E\int_0^{s\wedge T_N} 1_{\{j\in C^{(i,t)}_{u-}\}} du \\ &\leq 1-\underline{M}(1-\gamma)\, E\int_0^{s\wedge T_N} L^i_u du. \end{aligned} \tag{6.43}$$

Letting $N\to\infty$, we thus get that

$$E(L^i_s) \leq 1-\underline{M}(1-\gamma)\int_0^s E(L^i_u)du,$$

and Gronwall's lemma yields

$$E(L^i_s) \leq e^{-\underline{M}(1-\gamma)s}. \tag{6.44}$$

Hence, since $|C^{(F,t)}_s| \leq \sum_{i\in F}|C^{(i,t)}_s| = \sum_{i\in F} L^i_s$,

$$E(|C^{(F,t)}_s|) \leq |F|e^{-\underline{M}(1-\gamma)s.} \tag{6.45}$$

•

6.2 Proof of Proposition ??

The proof of Proposition **??** is an immediate consequence of the following Lemma.

Lemma 2 *Fix a time $t>0$, some finite set of sites $F\subset\mathbb{Z}^d$ and two initial configurations η and $\zeta\in A^{\mathbb{Z}^d}$. Then there exists a coupling of the two processes $(\sigma^\eta_s)_s$ and $(\sigma^\zeta_s)_s$ such that*

$$P(\sigma^\eta_t(F)\neq\sigma^\zeta_t(F)) \leq |F|e^{-\underline{M}(1-\gamma)t}.$$

From this lemma, it follows immediately that μ is the unique invariant measure of the process and that the convergence towards the invariant measure takes place exponentially fast.

Proof of Lemma ??. We use a slight modification of Algorithm 1 and 2 in order to construct σ^η_t. The modification is defined as follows. Let T_{STOP} and T be variables taking values in $(0,\infty)$. Replace Steps $1-3$ of Algorithm 1 by

1. Input: F; Output: $N^{(F)}_{STOP}, B, C$.

2. $N\leftarrow 0$, $N^1_{STOP}\leftarrow 0$, $B\leftarrow\emptyset$, $C\leftarrow F$, $T_{STOP}\leftarrow 0$

3. WHILE $T_{STOP}<t$ and $C\neq\emptyset$

3'. Choose a time $T\in(0,+\infty)$ randomly according to the exponential distribution with parameter $\sum_{j\in C} M_j$. Update

$$T_{STOP}\leftarrow T_{STOP}+T.$$

Finally replace Step 12 of Algorithm 1 by

12. RETURN $N_{STOP}^{(F)}, B, C$.

In this modified version, we stop the algorithm after time t, hence the output set C might not be empty. The output C is exactly the set $C_t^{(F,t)}$, the set of sites at time 0 whose colors influence the colors of sites in F at time t.

Concerning Algorithm 2, replace Step 1 of Algorithm 2 by

1. Input: $N_{STOP}^{(F)}, B, C$; Output: $\{(i, \sigma(i)) : i \in F\}$.

and Step 3 by

3. $\sigma(j) \leftarrow \eta(j)$ for all $j \in C$; $\sigma(j) \leftarrow \Delta$ for all $j \in \mathbb{Z}^d \setminus C$.

Then the law of the set $\{(i, \sigma(i)) : i \in F\}$ printed at the end of the modified Algorithm 2 is the law of $\sigma_t^\eta(F)$. Now, in order to realize the coupling, we use the same realizations of T, I and K for the construction of σ_t^η and σ_t^ζ. Write L_s for the cardinal of $C_s^{(F,t)}$. Clearly, both realizations of σ_t^η and σ_t^ζ do not depend on the initial configuration η, ζ respectively if and only if the output C of Algorithm 1 is void. Thus, by Lemma **??**

$$\begin{aligned} P(\sigma_t^\eta(F) \neq \sigma_t^\zeta(F)) &\leq P(T_{STOP} \geq t) \\ &= P(L_t \geq 1) \\ &\leq E(L_t) \leq |F| e^{-\underline{M}(1-\gamma)t}. \end{aligned}$$

This concludes the proof of Lemma **??**. •

6.3 Proof of Theorem ??

We only have to prove (**??**). Let

$$L_n^{(i)} = |\mathbf{C}_n^{(i)}|$$

be the cardinal of the set $\mathbf{C}_n^{(i)}$ after n steps of the algorithm (recall (**??**)). Then due to our assumptions, $L_n^{(i)}$ can be compared to a multi-type branching process Z_n having offspring mean which is bounded by γ at each step, such that $L_n^{(i)} \leq Z_n$ for all n. Thus,

$$P(N_{STOP}^{(i)} > n) = P(L_n^{(i)} > 0) = P(L_n^{(i)} \geq 1) \leq P(Z_n \geq 1) \leq E(Z_n) = \gamma^n.$$

When starting with the initial set F instead of the singleton $\{i\}$, then the above estimates remain true by multiplying with $|F|$, due to the independence properties of the branching process.

7 Impatient user bias

Perfect simulation procedures, very often cannot be run until the algorithm stops either by limitations of time or limitations of buffer. In this section we give upper bounds for the probability of these two types of errors.

According to our construction, the perfect simulation algorithm of μ presented in this article is a function $F : [0,1]^{\mathbb{N}\times\mathbb{Z}^d}$ to S such that, if $(U_n)_n = (U_n(i), i \in \mathbb{Z}^d)_n$ is a sequence of i.i.d. families, indexed by $\mathbb{Z}^d$, of uniform in $[0,1]$ random variables, then for any site $i \in \mathbb{Z}^d$, there exists a stopping time $N^{(i)}_{STOP}$, such that F depends only on the first $N^{(i)}_{STOP}$ families of $(U_n)_n$, i.e. for any measurable $B \in \mathcal{A}$,

$$P\left[F((U_1(j))_j, \ldots, (U_{N^{(i)}_{STOP}}(j))_j)(i) \in B\right] = \mu(\sigma(i) \in B).$$

Note that $N^{(i)}_{STOP}$ is **not** the number of uniform random variables that have to be simulated in order to sample from μ, this number will in general be considerably larger. $N^{(i)}_{STOP}$ is the number of steps of the backward sketch procedure.

A first kind of "impatient user bias" occurs whenever the user, for reasons independent of the algorithm, has to stop the algorithm after, say N steps maximal. In this case, we do not sample from μ, but instead sample from

$$P\left[F((U_1(j))_j, \ldots, (U_{N^{(i)}_{STOP}}(j))_j)(i) \in B | N^{(i)}_{STOP} \leq N\right].$$

By Proposition 6.2 of Fill (1998), compare also to section 6 of Ferrari et al. (2002), the error made above can be bounded by

$$\frac{P(N^{(i)}_{STOP} > N)}{1 - P(N^{(i)}_{STOP} > N)} \leq \frac{\gamma^N}{1-\gamma^N},$$

see Theorem **??** above.

At each step of the backward sketch procedure, a range of order k is chosen, where k is, in general, not bounded from above. In practical situations, however, a user will be limited in the choice of the interaction range and will restrict the simulation to the choice of ranges bounded by a a certain upper bound L that he decided to fix in advance. More precisely, writing

$$T^{(i)}_L := \inf\{\tilde{T}^{(i)}_n : \tilde{K}^{(i)}_n > L\},$$

the use will therefore sample from the measure

$$P\left[F((U_1(j))_j, \ldots, (U_{N^{(i)}_{STOP}}(j))_j)(i) \in B | \{N^{(i)}_{STOP} \leq N\} \cap \{T^{(i)}_L > T^{i)}_{STOP}\}\right].$$

In order to control the error made induced by this "space-time impatient user bias", we have to control

$$P(T^{(i)}_L \leq T^{i)}_{STOP}).$$

Using arguments similar to Lemma 2 of Galves et al. (2010), this can be bounded by

$$P(T^{(i)}_L \leq T^{i)}_{STOP}) \leq \sup_{i\in\mathbb{Z}^d}\left(\frac{M_i - \alpha_i(L)}{M_i}\right)\frac{1}{1-\gamma}.$$

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References

[1] Bertein, F.; Galves, A. Une classe de systèmes de particules stable par association. *Z. Wahrscheinlichkeitstheor. Verw. Geb.* **41**, 73-85 (1977).

[2] Brydges, D. C. A short course on cluster expansions. *Phnomnes critiques, systmes alatoires, thories de jauge*, Part I, II (Les Houches, 1984), 129–183, North-Holland, Amsterdam, 1986.

[3] Cai, Y., A non-monotone CFTP perfect simulation method. *Statist. Sinica*, **15**(4), 927–943 (2005).

[4] Van den Berg, J.; Steif, J.E. On the existence and nonexistence of finitary codings for a class of random fields. *Ann. Probab.* **27**, No.3, 1501-1522 (1999).

[5] Bramson, M.; Kalikow, S. Nonuniqueness in g-functions. *Isr. J. Math.* **84**, No. 1-2, 153-160 (1993).

[6] Comets, F.; Fernández, R.; Ferrari, P.A. Processes with long memory: Regenerative construction and perfect simulation. *Ann. Appl. Probab.* **12**, No.3, 921-943 (2002).

[7] Connor, S.B, Kendall, W.S. Perfect Simulation for a Class of Positive Recurrent Markov Chains. *Ann. Appl. Probab.* **17** (3), 781-808, also Correction. Perfect simulation for a class of positive recurrent Markov chains. *Ann. Appl. Probab.* **17**(5-6), 1808-1810 (2007).

[8] Dobrushin, R. L., *The problem of uniqueness of Gibbs random field and the problem of phase transition.* Funct. Anal. Appl. 2, 302-312, (1968).

[9] Fernández, P.J., Ferrari, P.A., Grynberg, S.P., Perfectly random sampling of truncated multinormal distributions. *Adv. in Appl. Probab.*, **39**(4), 973-990 (2007).

[10] Ferrari, P.A.; Maass, A; Martínez, S; Ney, P. Cesàro mean distribution of group automata starting from measures with summable decay. *Ergodic Theory Dynam. Systems* **20**, no. 6, 1657–1670 (2000).

[11] Ferrari, P.A. Ergodicity for spin systems with stirrings. *Ann. Probab.* **18**, No.4, 1523-1538 (1990).

[12] Ferrari, P. A.; Fernández, R.; Garcia, N. L. Perfect simulation for interacting point processes, loss networks and Ising models. *Stochastic Processes Appl.* **102**, No. 1, 63-88 (2002).

[13] Fill, J.A. An interruptible algorithm for perfect sampling via Markov chains. *Ann. Appl. Probab.* **8** 131162 (1998).

[14] Galves, A.; Löcherbach, E.; Orlandi, V. Perfect simulation of infinite range Gibbs measures and coupling with their finite range approximations. *J. Stat. Phys.* **138**, 476-495 (2010).

[15] Garcia, N. L.; Marić, N. Existence and perfect simulation of one-dimensional loss networks. *Stochastic Processes Appl.* **116**, No. 12, 1920-1931 (2006).

[16] Häggström, O.; Steif, J. E. Propp-Wilson algorithms and finitary codings for high noise Markov random fields. *Comb. Probab. Comput.* **9**, No.5, 425-439 (2000).

[17] Kalikow, S. Random Markov processes and uniform martingales. *Isr. J. Math.* **71**, No.1, 33-54 (1990).

[18] Kendall, W.S., Perfect simulation for the area-interaction point process. In: *Probability Towards 2000*, edited by L. Accardi and C.C. Heyde. Lecture Notes in Statistics, volume 128, New York; Springer-Verlag, 218-234 (1998).

[19] Kendall, W. S., Notes on perfect simulation, pp. 93-146 in *Markov Chain Monte Carlo: Innovations and Applications* (Lecture Note Series, Vol. 7), edited by W. S. Kendall, F. Liang and J. Wang. Institute for Mathematical Science, National University of Singapore, Singapore. (2005)

[20] Kendall, W.S., Thönnes, E. Perfect Simulation in Stochastic Geometry. *Pattern Recognition* **32**(9), 1569-1586 (1999).

[21] Kendall, W.S., Møller, J. Perfect simulation using dominating processes on ordered state spaces, with application to locally stable point processes. *Advances in Applied Probability*, **32** (3), 844-865 (2000).

[22] Liggett, T. *Interacting Particle Systems.* Springer, Berlin, 2000.

[23] Löcherbach, E., Loukianova, D.. On Nummelin splitting for continuous time Harris recurrent Markov processes and application to kernel estimation for multi-dimensional diffusions. *Stoch. Proc. Appl.* **118**, No. 8, 1301–1321 (2008).

[24] Murdoch, D. J., Green, P. J., Exact sampling from a continuous state space. *Scand. J. Statist.* **25**(3), 483–502 (1998).

[25] Nummelin, E. A splitting technique for Harris recurrent Markov chains. *Z. Wahrscheinlichkeitstheorie Verw. Geb.* **43**, 309–318 (1978).