DYNAMICS AND KINETIC LIMIT FOR A SYSTEM OF NOISELESS d-DIMENSIONAL VICSEK-TYPE PARTICLES

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Abstract. We analyze the continuous time evolution of a d-dimensional system of N self propelled particles subject to a feedback rule inspired by the original Vicsek's one [VCB-JCS]. Interactions among particles are specified by a pairwise potential in such a way that the velocity of any given particle is updated to the weighted average velocity of all those particles interacting with it, which makes the system non-Hamiltonian. The weights are given in terms of the interaction rate function. When the size of the system is fixed, we show the existence of an invariant manifold in the phase space and prove its exponential asymptotic stability. In the kinetic limit we show that the particle density satisfies a Boltzmann-Vlasov equation under suitable conditions on the interaction. We study the qualitative behaviour of the solution and we show that the Boltzmann-Vlasov entropy is strictly decreasing.

1. Introduction

The analysis of the evolution of a network of a large number of coordinated self propelled particles (agents) is a sub discipline of control theory which is has seen a rapid development during the last decade [BDT, W, JLM, CKFL, B-NVR, CS, CHDB]. This is due to its several potential application in understanding the emerging of collective behavior in biological systems (for example fish schools and bird flocks), computer science [R, BDT], engineering [JLM, CS, CHDB], economy [DY] and social sciences [W, CKFL, B-NVR]. To model the particle self-organized behavior one assigns to any particle a simple communication/interaction rule in order for the whole system to dynamically reproduce, in a given regime of the model's parameters, specific phase space patterns.

The emergence of phase space patterns persistent in time described by a large connected cluster of coherently moving particles is called flocking or swarming (also schooling or herd behavior). Basic models of flocking behavior generally follow three simple rules: 1) separation, that is to avoid crowding neighbors (usually modeled by short range repulsion interactions); 2) alignment, i.e. to steer towards average heading of neighbors;

Date: October 21, 2012.

1991 *Mathematics Subject Classification.* 35Q83, 35B40, 35Q82, 82C21, 92D50.

Key words and phrases. Asymptotic analysis, Boltzmann-Vlasov equation, flocking.

M.G. supported by MIUR - PRIN 2011-2013: Campi aleatori, percolazione ed evoluzione stocastica di sistemi con molte componenti.

E.O. supported by MIUR - PRIN 2011-2013: Studio di sistemi disordinati con metodi probabilistici e loro simulazione numerica, and ROMA TRE University.

3) cohesion, i.e. to steer towards the average position of neighbors (usually modeled by long range attraction interactions).

The seminal work in the direction of modeling flocking behavior is the one of Vicsek et al. [VCB-JCS]. They proposed a model of N interacting particles located on a 2-dimensional torus of diameter D. The velocity of each given particle belongs to the unit circle and at each time step its direction is updated at the empirical average of the velocity's directions of all the particles lying in a neighborhood of radius 1 from the given one, including itself, plus a random perturbation. Particles positions are then updated according to their velocity. Computer simulations proved that, when the particle density $\frac{N}{D^2}$ is sufficiently high and the noise intensity sufficiently small, the distribution of the velocities of the particles concentrates around the velocity of the barycenter of the system, although this is not a quantity preserved by the dynamics.

We propose a simple model of continuous time noiseless multi-agent evolution closely inspired to the original Vicsek's one. The particles interact (communicate) with each other trough a pairwise interaction function, which can be chosen to have the shape of an ordinary interaction potential, in such a way that the velocity of any given particle is updated to the weighted average velocity of all those particles communicating with it, with a weight given in terms of the communication rate function. This makes the system non-Hamiltonian. For what concerns flocking behaviour our model takes into account allignment and cohesion, but violates the separation rule since the particles can overlap.

We prove for such model two type of results. First, we analyze the N particle dynamics in $\mathbb{R}^d$. We show that there exists an invariant manifold in the phase space and prove exponential asymptotic stability of the invariant manifold when the initial conditions for particles dynamics are suitably chosen. This implies that the system, under the chosen initial conditions, will reach a state of flocking. Then, we will study the mean-field limit ($N \to \infty$) of the system and prove that the particle density satisfies a Boltzman-Vlasov equation when the particles are confined on a torus and subject to a *short-range potential* of Gaussian type. The same result holds in $\mathbb{R}^d$ when the interaction among particles is given by a suitable regularization of a finite range potential. We further show that the Boltzmann-Vlasov entropy is strictly decreasing. As a consequence, one can argue that, even if the initial distribution of the particles is absolutely continuous w.r.t. Lebesgue measure, the limit density distribution is singular w.r.t. Lebesgue measure.

A continuous time version of Vicsek's model, as well as its stochastic counterpart driven by the Brownian motion, has been proposed in [DM] and the corresponding kinetic equations heuristically derived and studied. In fact, at present time, to our knowledge, a rigorous derivation and analysis of Vicsek's model kinetics, as well as hydrodynamics, is lacking.

Another basic model for flocking is the Cucker-Smale one [CS]. In this and related models [DCBC, AIR] the variation in time of the momentum of a given particle is the weighted sum of the differences between the particle's momentum and those of the other system's components, with weights depending of the relative distances among particles divided by the total number of particles N. It is worth notice that, for all these models, the interaction among two given particles is of order $1/N$, therefore when the size of the system becomes large, particles tend to decorrelate. On the contrary, in the original Vicsek's model, the interaction between a given couple of particles is of order one. Moreover, Cucker-Smale dynamics preserves the velocity of the barycenter, which is not the case for Vicsek's. As a matter of fact, we believe that the order of the interaction with respect to the size of the system is the peculiar feature distinguishing Vicsek's from Cucker-Smale algorithm. Therefore, in our opinion, variants of the Cucker-Smale momenta updating rule taking into account only the differences among the directions of the momenta of the particles, rather than those of the momenta as vectors, are somewhat improperly ascribed to variants of the Vicsek's model [BCC2]. Cuker-Smale and related models have been more deeply investigated in the mathematical literature and their mean-field limit equations rigorously derived and studied in [HT, HL, CFRT, CCR, AIR] in the noiseless case and in [BCC1, BCC2] in the stochastic case driven by Brownian motion. Moreover, the hydrodynamics equations for these models have also been rigorously studied but formally derived [HT, CDP, CCR].

The plan of the paper is the following. In Section 2 we describe the model and set the notations. In Section 3 we analyze the system when the number of particles is fixed. In Section 4 we analyze the system when the number of particles goes to infinity. In the appendix we collect proofs of results used along the previous sections.

Acknowledgements. Enza Orlandi thanks Carlangelo Liverani for useful discussions. Michele Gianfelice thanks Fabio Fagnani and Marco Isopi for interesting discussions on the subject.

2. Description of the model and notation

2.1. **Notations.** Given $x \in \mathbb{R}^d, d \geq 1$, we denote by x^i its i-th component, $i = 1, .., d$, with respect to the canonical basis $(e_1, .., e_d)$. For any $x, y \in \mathbb{R}^d$ we set $x \cdot y := \sum_{i=1}^d x^i y^i$ to be the scalar product between x and y. Hence, we denote by $|x| := \sqrt{x \cdot x}$ the associated Euclidean norm and by $B_r(x) := \{y \in \mathbb{R}^d : |y - x| \leq r\}$ the ball of radius $r > 0$ centered in x and $B_r := B_r(0)$. Furthermore we set $\|x\|_\infty := \max_{i=1,..,d} |x^i|$.

Given an integer $N \geq 2$, let $\mathbf{x} := (x_1, .., x_N) \in \mathbb{R}^{Nd}$. We denote by $\mathbf{x} \cdot \mathbf{y} := \sum_{i=1}^N x_i \cdot y_i$ and $|\mathbf{x}| := \sqrt{\mathbf{x} \cdot \mathbf{y}}$ respectively the scalar product and the Euclidean norm in $\mathbb{R}^{Nd}$. We also denote by $\mathbf{B}_r(\mathbf{x}) := \{\mathbf{y} \in \mathbb{R}^{Nd} : |\mathbf{y} - \mathbf{x}| \leq r\}$ the ball of radius $r > 0$ centered in $\mathbf{x}$.

Partial derivative w.r.t. any component x^i of $x \in \mathbb{R}^d$ will be denoted by ∂_{x^i}, so that ∇_x stands for $(\partial_{x^1}, .., \partial_{x^d})$ while, for any $\mathbf{q} \in \mathbb{R}^{Nd}$, we set $\nabla_{\mathbf{q}} := (\nabla_{q_1}, .., \nabla_{q_N})$.

Moreover, we denote by $\mathcal{L}_n(\mathbb{R})$ the Banach space of linear operators from $\mathbb{R}^n$ to itself and by $\|\cdot\|$ and $\|\cdot\|_\infty$ the operator norm induced by respectively the Euclidean and the supremum norm. In particular $\mathbb{I}_n, \mathbf{0}_n \in \mathcal{L}_n(\mathbb{R})$ denote respectively the identity and the null operator.

2.2. **The model.** Let $N \geq 2$ be an integer. We consider N particles of unitary mass in $\mathbb{R}^d$ evolving according the equations:

$$\begin{cases} \frac{dq_i(t)}{dt} = p_i(t) \\ \frac{dp_i(t)}{dt} = \frac{\sum_{j=1}^N U_R(q_i(t)-q_j(t))(p_j(t)-p_i(t))}{\sum_{j=1}^N U_R(q_i(t)-q_j(t))} \\ q_i(0) = q_i^0 \ ; \ p_i(0) = p_i^0 \end{cases} , \ i = 1, .., N \tag{2.1}$$

where, for $i = 1, .., N$, $(q_i, p_i) \in \mathbb{R}^d \times \mathbb{R}^d$, (q_i^0, p_i^0) are the initial conditions and $U_R(\cdot)$ is the two particles interaction. We assume that $U_R(\cdot)$ is a sufficiently smooth, spherically symmetric positive function, actually a probability density concentrating the mass in the ball of radius R centered at zero, such that $\sup_{x\in\mathbb{R}^d} U_R(x) = U_R(0)$. This implies that the denominator in the second equation of (2.1) is always strictly positive. Other assumptions on U_R will be done in Section 3, where the kinetic limit as $N \to \infty$ will be considered. The vector field in (2.1) is Lipschitz, therefore the existence and the uniqueness of the solution is granted at least for short time. Since the vector field increases at most linearly in $(\mathbf{q}, \mathbf{p})$ the solution exists and it is unique for all $t \geq 0$.

2.2.1. *Communication graph.* We recall some basic definition of graph theory and its applications to Markov chains used in the next section. We refer the reader to basic textbooks such as [B] and [St] for an account on this subject.

A directed graph $\mathcal{G}$ is a ordered pair of sets $(\mathcal{V}, \mathcal{E})$ where $\mathcal{V}$ is a finite set called *set of vertices* and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is called *set of edges* or *bonds*. $\mathcal{G}' = (\mathcal{V}', \mathcal{E}')$ such that $\mathcal{V}' \subseteq \mathcal{V}$ and $\mathcal{E}' \subseteq (\mathcal{V}' \times \mathcal{V}') \cap \mathcal{E}$ is said to be a *subgraph* of $\mathcal{G}$ and this property is denoted by $\mathcal{G}' \subseteq \mathcal{G}$. Two vertices are said to be *adjacents* if belong to the same bond, so that $u, v \in \mathcal{V}$ are adjacents w.r.t. $b \in \mathcal{E}$ if either $b = (u, v)$ or $b = (v, u)$. For any $b \in \mathcal{E}$, let then $\mathcal{V}_b := \{\{u, v\} \subset \mathcal{V} : u \text{ and } v \text{ are adjacent w.r.t. } b\}$. Then, if $b = (u, v)$, b is said *outgoing* from u and *ingoing* in v. Let $\mathcal{E}_v^+ := \{b \in \mathcal{E} : b = (v, u), \ u \in \mathcal{V}\}$ be the set of bonds outgoing from v. We denote by $\mathcal{N}^+(v) := \left(\cup_{b\in\mathcal{E}_v^+} \mathcal{V}_b\right) \subseteq \mathcal{V}$ the *closed outgoing neighborhood* of v and set, for any $\mathcal{W} \subset \mathcal{V}$, $\mathcal{N}^+(\mathcal{W}) := \cup_{v\in\mathcal{W}} \mathcal{N}^+(v)$ to be the closed outgoing neighborhood of $\mathcal{W}$. Given $v \in \mathcal{V}$, we set $\mathcal{N}_1^+(v) := \mathcal{N}^+(v)$ and, for $k \geq 2$, $\mathcal{N}_k^+(v) := \mathcal{N}^+\left(\mathcal{N}_{k-1}^+(v)\right)$ to be the *outgoing k-neighborhood* of v. Given two vertices u and v, u is said to *communicate* with v if there exists $k \geq 1$ such that $u \in \mathcal{N}_k^+(v)$. Therefore, $u, v \in \mathcal{V}$ are said to be *connected* if one communicates with the other. In fact, since if $u \in \mathcal{N}_k^+(v)$ for some $k \geq 1$, then $u \in \mathcal{N}_l^+(v)$, $\forall l > k$, for u and v to be connected there must be $k_1, k_2 \geq 1$ such that $u \in \mathcal{N}_{k_1}^+(v)$ and $v \in \mathcal{N}_{k_2}^+(u)$, that is $u \in \mathcal{N}_{k_1\vee k_2}^+(v)$, $v \in \mathcal{N}_{k_1\vee k_2}^+(u)$. $\mathcal{G}$ is then said to be *strongly connected* if any two distinct vertices are connected. The maximal connected subgraphs of $\mathcal{G}$ are called *components* of $\mathcal{G}$.

An example of directed graph is the one which can be associated to a Markov chain. In this case, $\mathcal{V}$ coincides with the set of states of the chain and, denoting by P the transition matrix associated to the chain $\mathcal{E} = \mathcal{E}(P) := \{(u, v) \in \mathcal{V} \times \mathcal{V} : P_{u,v} > 0\}$. Then, the directed graph associated to the Markov chain with transition matrix P is denoted by $\mathcal{G}(P)$. Hence, the chain and therefore P are said to be *irreducible* if and only if $\mathcal{G}(P)$ is strongly connected.

Definition 2.1. *Given any particles configuration* $\mathbf{q} \in \mathbb{R}^{Nd}$, *we define the* geometric communication graph *of the particle system to be the directed graph* $\mathfrak{G}_R(\mathbf{q}) := (\mathcal{V}(\mathbf{q}), \mathfrak{E}_R(\mathbf{q}))$, *where* $\mathcal{V}(\mathbf{q})$ *is the finite subset of* $\mathbb{R}^d$ *associated to* $\mathbf{q}$ *and*

$$\mathfrak{E}_R(\mathbf{q}) := \{(q, q') \in \mathcal{V}(\mathbf{q}) \times \mathcal{V}(\mathbf{q}) : U_R(q - q') > 0\} \ . \tag{2.2}$$

When considering the particles configuration at a given time t, $\mathbf{q}(t)$, we set $\mathcal{V}_t := \mathcal{V}(\mathbf{q}(t))$ and $\mathfrak{G}_R(t) := \mathfrak{G}_R(\mathbf{q}(t))$.

Definition 2.2. *The system is said to have reached a* state of flocking *if there exists* $v \in \mathbb{R}^d$ *such that, for any* $\epsilon > 0, \exists T_\epsilon > 0 : \forall t > T_\epsilon$,

- $p_i(t) \in B_\epsilon(v), \forall i = 1, .., N$;
- *the geometric communication graph* $\mathfrak{G}_R(t)$ *is connected and* $|\mathcal{V}_t| \geq 2$.

Definition 2.3. *The state of the system* $(\mathbf{q}, \mathbf{p})$ *such that* $q_1 = \cdots = q_N, p_1 = \cdots = p_N$ *is called* rendez-vous *state.*

From (2.1) it follows that rendez-vous states belong to the collection of the invariant states for the dynamics.

3. Particle dynamics

In the following we analyze the evolution of N particles according equations (2.1). In this section N is kept fixed, so we omit in the notation to write explicitly the dependence on N. We show that there exists an invariant $(N+1)d$ manifold for evolution (2.1) and prove first its stability, see Corollary 3.5, and then its asymptotic stability, see Theorem 3.11.

3.1. Stability.

We first notice that if the velocities of the particles at time zero are bounded, that is, for all $i = 1, .., N$, $p_i^0 \in B_r$ for some $r > 0$, then they will lie in B_r for later times. In fact we have the following result:

Lemma 3.1. *For any* $i = 1, .., N$, *assume that* $p_i(0) \in B_r$. *Then,* $p_i(t) \in B_r$, *for all* $t > 0$.

Proof. Assume, without loss of generality that $r = 1$ and that there is a t^* such that there is at least one $p_i(t^*)$ such that $|p_i(t^*)| = 1$ and $|p_j(t^*)| \leq 1$ for $j \neq i$. Then

$$\frac{1}{2}\frac{d}{dt}|p_i(t^*)|^2 = \frac{\sum_{j=1}^N U_R(q_i(t^*) - q_j(t^*))\,[p_j(t^*) - p_i(t^*)] \cdot p_i(t^*)}{\sum_{j=1}^N U_R(q_i(t^*) - q_j(t^*))} \leq 0 \ . \tag{3.1}$$

□

Remark 3.2. *The result of Lemma 3.1 holds for any positive smooth interaction* U_R, *regardless of its support. In particular, it holds if* U_R *does not have compact support.*

Remark 3.3. *The only critical point of the system (2.1) is* $(\mathbf{0},\mathbf{0})$. *Moreover, if the particles at initial time have all the same velocity, that is, for* $j=1,..,N,\ p_j^0=v\in\mathbb{R}^d$, *their velocity will remain constant during the evolution and the system describes the motion of* N *non-interacting particles.*

From Remark 3.3 one deduces that the $(N+1)d$ linear manifold

$$\mathcal{I}=\cup_{\{v\in\mathbb{R}^d\}}\mathcal{I}(v) \tag{3.2}$$

where

$$\mathcal{I}(v)=\{(\mathbf{q},\mathbf{p})\in\mathbb{R}^{dN}\times\mathbb{R}^{dN}:p_i=v,i=1,..,N\} \tag{3.3}$$

is invariant for the evolution (2.1). Namely, if the initial data belong to $\mathcal{I}(v)$ the particles evolve independently one from the other with constant velocity v.

The next result shows that if at time $t=0$ the velocity of the N particles is close to its mean velocity vector, then, at any further time t, it will always remain close to the mean initial velocity vector. Let $\Omega\in\mathcal{L}_{Nd}$ be the operator such that

$$\mathbb{R}^{Nd}\ni\mathbf{x}\longmapsto\Omega\mathbf{x}\in\mathbb{R}^{Nd}\ , \tag{3.4}$$

where $\Omega\mathbf{x}$ is the vector in $\mathbb{R}^{Nd}$ whose component are the vectors $(\Omega x)_i=\frac{1}{N}\sum_{j=1}^N x_j\in\mathbb{R}^d$ for $i=1,..,N$. Notice that by definition Ω is the orthogonal projector on $\left\{\mathbf{x}\in\mathbb{R}^{Nd}:x_1=\cdots=x_N\right\}$.

Theorem 3.4. *Let* $w(t,w^0)=(\mathbf{q}(t),\mathbf{p}(t))$ *be the solution of (2.1) at time* t *starting from* $w^0=(\mathbf{q}^0,\mathbf{p}^0)\in\mathbb{R}^{2Nd}$. *Given* $\epsilon>0$, *assume that* $|\mathbf{p}^0-\Omega\mathbf{p}^0|<\epsilon$. *Then*

$$|\mathbf{p}(t)-\Omega\mathbf{p}^0|\le\epsilon,\qquad\forall t\ge 0\ . \tag{3.5}$$

Proof. We proceed as in the proof of Lemma 3.1. Let us denote by $v_i(t)=p_i(t)-(\Omega p^0)_i\in\mathbb{R}^d, i=1,..,N$, and assume that there is a t^* so that there is at least one components $|v_i(t^*)|=\epsilon$ and $|v_j(t^*)|<\epsilon$ for $j\neq i$. We have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|v_i(t)|^2&=v_i(t)\cdot\frac{d}{dt}v_i(t)\\&=\frac{\sum_{j=1}^N U_R(q_i(t^*)-q_j(t^*))\{v_j(t^*)-v_i(t^*)\}\cdot v_i(t^*)}{\sum_{j=1}^N U_R(q_i(t^*)-q_j(t^*))}\le 0\ .\end{aligned} \tag{3.6}$$

□

Note that, for any $w\in\mathbb{R}^{2Nd}$,

$$\operatorname{dist}(w,\mathcal{I})=\inf_{w^0\in\mathcal{I}}|w-w^0|=\inf_{\{\mathbf{p}^0\in\mathbb{R}^{Nd}:w=(\mathbf{q}^0,\mathbf{p}^0)\in\mathcal{I}\}}|\mathbf{p}-\mathbf{p}^0|=|\mathbf{p}-\Omega\mathbf{p}|\ , \tag{3.7}$$

where Ω is the operator defined in (3.4). From Theorem 3.4 one deduces that the invariant manifold $\mathcal{I}$ is stable for the evolution (2.1).

Corollary 3.5. *For any $\epsilon > 0$ let $B(\epsilon, \mathcal{I}) = \left\{w \in \mathbb{R}^{2Nd} : \mathrm{dist}\,(w, \mathcal{I}) \leq \epsilon\right\}$ be a neighborhood of radius ϵ of $\mathcal{I}$. Let $w(t, w^0)$ be the solution of (2.1) at time t starting from $w^0 = (\mathbf{q}^0, \mathbf{p}^0) \in B(\epsilon, \mathcal{I})$. Then*

$$\mathrm{dist}\left(w(t, w^0), \mathcal{I}\right) \leq 2\epsilon, \qquad \forall t > 0 \ . \tag{3.8}$$

Proof. By (3.7) we have

$$\mathrm{dist}\left(w(t, w^0), \mathcal{I}\right) = |\mathbf{p}(t) - \Omega\mathbf{p}(t)| \leq |\mathbf{p}(t) - \Omega\mathbf{p}^0| + |\Omega\mathbf{p}(t) - \Omega\mathbf{p}^0| \ . \tag{3.9}$$

By definition of Ω, see (3.4),

$$|\Omega\mathbf{p}(t) - \Omega\mathbf{p}^0| = |\Omega(\mathbf{p}(t) - \Omega\mathbf{p}^0)| \leq |\mathbf{p}(t) - \Omega\mathbf{p}^0| \ . \tag{3.10}$$

Hence, by Theorem 3.4,

$$\mathrm{dist}\left(w(t, w^0), \mathcal{I}\right) \leq 2|\mathbf{p}(t) - \Omega\mathbf{p}^0| \leq 2\epsilon \ , \qquad \forall t \geq 0 \ . \tag{3.11}$$

□

3.2. **Asymptotic stability.** Next we show a stronger result. Choosing suitably the initial datum, the solution of (2.1) converges exponentially towards the invariant manifold. We show that the N particles will not split into non interacting groups and the velocity of each particle converges exponentially fast to a velocity vector which is the same for all the N particles. In other words, the system will reach a state of flocking as given in Definition 2.2.

We rewrite the non linear system (2.1) as follows:

$$\begin{cases} \begin{pmatrix} \frac{d\mathbf{q}(t)}{dt} \\ \frac{d\mathbf{p}(t)}{dt} \end{pmatrix} = C\left(\mathbf{q}(t)\right) \begin{pmatrix} \mathbf{q}(t) \\ \mathbf{p}(t) \end{pmatrix} \\ \mathbf{q}(0) = \mathbf{q}^0, \mathbf{p}(0) = \mathbf{p}^0 \end{cases} \tag{3.12}$$

where

$$\mathbb{R}^{Nd} \ni \mathbf{q} \longmapsto C\left(\mathbf{q}\right) := \begin{pmatrix} \mathbf{0}_{Nd} & \mathbb{I}_{Nd} \\ \mathbf{0}_{Nd} & L\left(\mathbf{q}\right) \end{pmatrix} \in \mathcal{L}_{2Nd}\left(\mathbb{R}\right) \ , \tag{3.13}$$

$$L\left(\mathbf{q}\right) := A\left(\mathbf{q}\right) - \mathbb{I}_{Nd} \tag{3.14}$$

and $A(\mathbf{q})$ is the linear operator valued function so defined

$$\mathbb{R}^{Nd} \ni \mathbf{q} \longmapsto A(\mathbf{q}) := \begin{bmatrix} a_{1,1}(\mathbf{q})\mathbb{I}_d & a_{1,2}(\mathbf{q})\mathbb{I}_d & & \dots & a_{1,N}(\mathbf{q})\mathbb{I}_d \\ a_{2,1}(\mathbf{q})\mathbb{I}_d & a_{2,2}(\mathbf{q})\mathbb{I}_d & & \dots & a_{2,N}(\mathbf{q})\mathbb{I}_d \\ a_{N,1}(\mathbf{q})\mathbb{I}_d & & \dots & a_{N,N-1}(\mathbf{q})\mathbb{I}_d & a_{N,N}(\mathbf{q})\mathbb{I}_d \end{bmatrix} \in \mathcal{L}_{Nd}\left(\mathbb{R}\right) \tag{3.15}$$

$$a_{i,j}(\mathbf{q}) := \frac{U_R(q_i - q_j)}{\sum_{j=1}^{N} U_R(q_i - q_j)} \ , \qquad j = 1, .., N, \quad i = 1, .., N. \tag{3.16}$$

Remark 3.6. *Notice that for $\mathbf{q} \in \mathbb{R}^{Nd}$*

$$a_{i,j}(\mathbf{q}) = a_{i,j}(\mathbf{q} + \Omega\mathbf{x}), \quad \forall \mathbf{x} \in \mathbb{R}^{Nd}, \qquad j = 1, .., N, \quad i = 1, .., N \tag{3.17}$$

and

$$\sum_{j=1}^{N} a_{i,j}(\mathbf{q}) = 1. \tag{3.18}$$

These two properties are important when studying the spectrum of $C(\mathbf{q})$ *for a fixed value of* $\mathbf{q}$.

3.2.1. *Spectral Analysis of* $C(\mathbf{q})$. Let $\mathbf{q} \in \mathbb{R}^{Nd}$ be fixed. The eigenvalues of $C(\mathbf{q})$ are the roots of the characteristic equation

$$\mathrm{Det}\left[C(\mathbf{q}) - \lambda \mathbb{I}_{2Nd}\right] = (-\lambda)^{Nd} \,\mathrm{Det}\left[L(\mathbf{q}) - \lambda \mathbb{I}_{Nd}\right] = 0. \tag{3.19}$$

We need then to study the spectrum of $L(\mathbf{q})$ and therefore, by (3.14) the spectrum of $A(\mathbf{q})$. To do this it is convenient to introduce the tensor space $\mathbb{R}^N \otimes \mathbb{R}^d$. We denote by $\mathcal{F}$ the isomorphism

$$\mathbb{R}^{Nd} \ni \mathbf{x} \longrightarrow \mathcal{F}(\mathbf{x}) := \sum_{i=1}^{N} \sum_{j=1}^{d} x_i^j e_i \otimes e_j \in \mathbb{R}^N \otimes \mathbb{R}^d \ , \tag{3.20}$$

such that $\mathcal{F}(\mathbf{x})_{i,j} = x_i^j, i = 1, .., N$ and $j = 1, .., d$.

To ease the notation we omit in the following to write the dependence on $\mathbf{q}$ if no confusion arises. We therefore set $A := A(\mathbf{q})$. One obtains immediately that $A : \mathbb{R}^{Nd} \longrightarrow \mathbb{R}^{Nd}$ acts on $\mathbb{R}^N \otimes \mathbb{R}^d$ as follows

$$\tilde{A} \otimes \mathbb{I}_d : \mathbb{R}^N \otimes \mathbb{R}^d \longrightarrow \mathbb{R}^N \otimes \mathbb{R}^d \ , \tag{3.21}$$

where, by (3.16), setting $a_{i,j} := a_{i,j}(\mathbf{q})$,

$$\tilde{A} := \begin{bmatrix} a_{1,1} & a_{1,2} & & \dots & a_{1,N} \\ a_{2,1} & a_{2,2} & & \dots & a_{2,N} \\ a_{N,1} & & \dots & a_{N,N-1} & a_{N,N} \end{bmatrix} . \tag{3.22}$$

Namely, one has that

$$\left(\tilde{A} \otimes \mathbb{I}_d\right) \mathcal{F}(\mathbf{x}) = \mathcal{F}(A\mathbf{x}) \ . \tag{3.23}$$

Furthermore, denoting by $\Sigma(A) \subset \mathbb{C}$ the spectrum of A,

$$\Sigma(A) = \Sigma(\tilde{A} \otimes \mathbb{I}_d) = \Sigma(\tilde{A})\Sigma(\mathbb{I}_d)^1 \ . \tag{3.24}$$

Since the only eigenvalue of $\mathbb{I}_d$ is 1 with multiplicity d, the problem is reduced to study the spectrum of $\tilde{A}$. The matrix $\tilde{A}$ is a (right) stochastic matrix, that is it has non-negative entries and, by (3.18), $\sum_{j=1}^{N} a_{i,j} = 1, \forall i = 1, .., N$. Then, if it is irreducible one can apply the Perron-Frobenius Theorem. We recall that a matrix $D \in \mathcal{L}_n(\mathbb{R})$ with non-negative entries is said to be irreducible if there exists an integer m so that D^m has strictly positive entries.

[1]If $Z := \{z_1, .., z_n\}$ and $W := \{w_1, .., w_m\}$ are two discrete subsets of $\mathbb{C}$ we denote by

$$ZW := \{z_i w_j \in \mathbb{C} : i = 1, .., n \ ; \ j = 1, .., m\} .$$

The existence of $m \in \mathbb{N}$ such that $\tilde{A}^m(\mathbf{q})$ has strictly positive entries is related to have the graph $\mathcal{G}\left(\tilde{A}(\mathbf{q})\right)$ associated to $\tilde{A}(\mathbf{q})$ strongly connected. In fact, $\mathcal{G}(\tilde{A}(\mathbf{q}))$ is strongly connected if and only if $\tilde{A}(\mathbf{q})$ is irreducible. Clearly this property is also shared by $\mathfrak{G}_R(\mathbf{q})$ which is nothing else but a geometric representation of $\mathcal{G}(\tilde{A}(\mathbf{q}))$ in $\mathbb{R}^d$. Therefore, it follows:

Lemma 3.7. *For any $\mathbf{q} \in \mathbb{R}^{Nd}$, let $\tilde{A}(\mathbf{q})$ be irreducible. Then 1 is the maximum eigenvalue and all the other eigenvalues $\lambda(\mathbf{q}) \in \mathbb{C}$ are strictly smaller in absolute value of 1, i.e. $|\lambda(\mathbf{q})| < 1$. The eigenspace associate to the eigenvalue 1 is one dimensional. Furthermore, there exists an eigenvector $\eta \in \mathbb{R}^N$ associated to 1 with strictly positive entries, i.e. $\tilde{A}(\mathbf{q})\eta = \eta$ and $\eta_i > 0$ for $i = 1, .., N$. Moreover, $\eta_i = \frac{1}{\sqrt{N}}$ for $i = 1, .., N$ and there are no other positive eigenvectors except multiples of η.*

Proof. Because for any $\mathbf{q} \in \mathbb{R}^{Nd}$, $\|\tilde{A}(\mathbf{q})\|_\infty \leq \max_{i=1,..,N} \sum_{j=1}^N a_{i,j}(\mathbf{q}) = 1$, we have that the maximum eigenvalue is 1 and any other eigenvalue $\lambda(\mathbf{q}) \in \mathbb{C}$ is strictly smaller in absolute value of 1. By Perron Frobenius Theorem the maximum eigenvalue is simple and the associated positive eigenvector is η with $\eta_i = \frac{1}{\sqrt{N}}$ for $i = 1, .., N$. $\square$

Lemma 3.8. *For any $\mathbf{q} \in \mathbb{R}^{Nd}$, so that $\tilde{A}(\mathbf{q})$ is irreducible, let $A(\mathbf{q})$ be the matrix as in (3.15). We have that $1 \in \Sigma(A(\mathbf{q}))$ is the maximum eigenvalue. The associated eigenspace is the d-dimensional manifold $\{\mathbf{p} \in \mathbb{R}^{Nd} : p_i = v,\ i = 1, .., N; v \in \mathbb{R}^d\}$. All the other eigenvalues $\lambda(\mathbf{q}) \in \Sigma(A(\mathbf{q}))$ are such that $|\lambda(\mathbf{q})| < 1$.*

Proof. It is an immediate consequence of (3.24) and Lemma 3.7. $\square$

We have finally the following result.

Theorem 3.9. *For any $\mathbf{q} \in \mathbb{R}^{Nd}$, so that $\tilde{A}(\mathbf{q})$ is irreducible, let $C(\mathbf{q})$ be defined in (3.13). We have that $0 \in \Sigma(C(\mathbf{q}))$. The $(N+1)d$ dimensional manifold $\mathcal{I}$ defined in (3.2) is the eigenspace associated to the eigenvalue 0. All the other eigenvalues of $C(\mathbf{q})$ have real part strictly negative.*

Proof. From (3.19) and Lemma 3.8 we deduce that $0 \in \Sigma(C(\mathbf{q}))$ and all other eigenvalues have real part strictly negative. It is immediate to see that the algebraic multiplicity of 0 is $Nd + d$. The $(N+1)\,d$-dimensional manifold $\mathcal{I}$ defined in (3.2) is the associated eigenspace. Namely, if $w \in \mathcal{I}$ then $C(\mathbf{q})\,w \in \mathcal{I}$. From this one deduces that $\mathcal{I}$ is an eigenspace for the matrix $C(\mathbf{q})$. Moreover, since the kernel of $C^2(\mathbf{q})$ is $\mathcal{I}$, we get that $\mathcal{I}$ is the eigenspace associated to the eigenvalue 0. $\square$

We denote by $\alpha(\mathbf{q})$ the spectral gap of the matrix $C(\mathbf{q})$, that is

$$\alpha(\mathbf{q}) := \min\left\{|\mathrm{Re}(\lambda(\mathbf{q}))| : \lambda(\mathbf{q}) \in \Sigma(C(\mathbf{q})),\ \mathrm{Re}(\lambda(\mathbf{q})) < 0\right\}\ . \tag{3.25}$$

Let $\mathbf{q} \in \mathbb{R}^{Nd}$ so that $\tilde{A}(\mathbf{q})$ is irreducible. By Theorem 3.9, $\mathcal{I}$ is the eigenspace associated to the 0 eigenvalue of $C(\mathbf{q})$ for any $\mathbf{q}$. We can therefore decompose $\mathbb{R}^{2Nd}$ as follows:

$$\mathbb{R}^{2Nd} = W(\mathbf{q}) \oplus \mathcal{I} \tag{3.26}$$

in such a way that $W(\mathbf{q})$ and $\mathcal{I}$ are eigenspaces of $C(\mathbf{q})$ and denote by $\Pi(\mathbf{q})$ the projection operator

$$\Pi(\mathbf{q}) : \mathbb{R}^{2Nd} \to W(\mathbf{q}) \ . \tag{3.27}$$

3.3. **Asymptotic Analysis.** Let $w^0 = (\mathbf{q}^0, \mathbf{p}^0) \in \mathcal{I}$ be such that $\tilde{A}(\mathbf{q}^0)$ is irreducible and let us set, for any $r > 0$ and $\epsilon > 0$,

$$\tilde{\mathcal{B}}(r, \epsilon, w^0) := \{w = (\mathbf{q}, \mathbf{p}) \in \mathbb{R}^{2Nd} : |\mathbf{q} - \mathbf{q}^0| \leq r \ ; \ |\mathbf{p} - \mathbf{p}^0| \leq \epsilon\} \ . \tag{3.28}$$

Denote by r_0 the biggest value of r such that, for any $w = (\mathbf{q}, \mathbf{p}) \in \tilde{\mathcal{B}}(r_0, \epsilon, w^0)$, $\tilde{A}(\mathbf{q})$ is still irreducible and $\alpha(\mathbf{q}) \geq \frac{1}{2}\alpha(\mathbf{q}^0)$. We set

$$\mathcal{B}(r_0, \epsilon, w^0) := \left\{ w = (\mathbf{q}, \mathbf{p}) \in \tilde{B}(r_0, \epsilon, w^0) : \alpha(\mathbf{q}) \geq \frac{1}{2}\alpha(\mathbf{q}^0) \right\} \ . \tag{3.29}$$

The existence of r_0 is granted since, by assumption, $\tilde{A}(\mathbf{q}^0)$ is irreducible and U_R is smooth. To apply the spectral results obtained for $C(\mathbf{q})$, ($\mathbf{q}$ fixed), to the nonlinear system (3.12) we write

$$C(\mathbf{q}(t)) = C(\mathbf{q}(0)) + \Gamma(\mathbf{q}(t)) \ , \tag{3.30}$$

where

$$\Gamma(\mathbf{q}(t)) := \begin{pmatrix} \mathbf{0}_{Nd} & \mathbf{0}_{Nd} \\ \mathbf{0}_{Nd} & B\left(\mathbf{q}\left(t\right)\right) \end{pmatrix} \ , \tag{3.31}$$

and

$$B(\mathbf{q}(t)) := A(\mathbf{q}(t)) - A(\mathbf{q}(0)) \ . \tag{3.32}$$

Next we estimate the norm of $B(\mathbf{q}(t))$.

Lemma 3.10. *Let $(\mathbf{q}(t), \mathbf{p}(t))$ be the solution of (3.12) starting from the initial data $(\mathbf{q}^0, \mathbf{p}^0)$. Then, setting*

$$b_{i,j}(\mathbf{q}(t)) := a_{i,j}(\mathbf{q}(t)) - a_{i,j}(\mathbf{q}^0) \quad i, j = 1, .., N \ , \tag{3.33}$$

we have

$$\|B(\mathbf{q}(t))\| \leq 2N \frac{\sup_{x \in \mathbb{R}^d} |\nabla U(x)|}{U_R(0)} \sup_{i,k \in \{1,..,N\}} \left| -(q_i^0 - q_k^0) + q_i(t) - q_k(t) \right| \ . \tag{3.34}$$

We defer the proof of this result to the appendix.

Theorem 3.11. *Let $w^0 = (\mathbf{q}^0, \mathbf{p}^0) \in \mathcal{I}$ and assume that $\tilde{A}(\mathbf{q}^0)$ is irreducible. There exist three positive constants $r_0 = r_0(w_0), T = T(w^0)$ and $\epsilon_0 = \epsilon_0(N, w^0)$ such that, for any initial datum $w^1 \in \mathcal{B}(r_0, \epsilon_0, w^0) \subset \mathbb{R}^{2Nd}$,*

$$\mathrm{dist}\left(w(t, w^1), \mathcal{I}\right) \leq \epsilon_0 e^{-t\frac{\log 2}{T}} \ , \tag{3.35}$$

where $w(t, w^1)$ is the solution at time t of (3.12) starting from w^1.

Proof. For any $s > 0$, we define

$$\tilde{\mathcal{Q}}(s, w^0) := \{w = (\mathbf{q}, \mathbf{p}) \in \mathbb{R}^{2Nd} : |[\mathbb{I}_{Nd} - \Omega]\left(\mathbf{q} - \mathbf{q}^0\right)| \leq s\} \ , \tag{3.36}$$

where Ω is the operator defined in (3.4). Denote by s_0 the largest value of s such that, for any $w = (\mathbf{q}, \mathbf{p}) \in \tilde{\mathcal{Q}}(s_0, w^0)$, $\tilde{A}(\mathbf{q})$ is still irreducible and $\alpha(\mathbf{q}) \geq \frac{1}{4}\alpha(\mathbf{q}^0)$. Such a value s_0 exists since $\tilde{A}(\mathbf{q}^0)$ is irreducible and U_R is smooth. Let us set

$$\mathcal{Q}(s_0, w^0) := \left\{w = (\mathbf{q}, \mathbf{p}) \in \tilde{\mathcal{Q}}(s_0, w^0) : \alpha(\mathbf{q}) \geq \frac{1}{4}\alpha(\mathbf{q}^0)\right\} \ . \tag{3.37}$$

We have that

$$\mathcal{B}(r_0, \epsilon, w^0) \subset \mathcal{Q}(s_0, w^0) \ , \qquad \forall \epsilon > 0 \ . \tag{3.38}$$

Namely we have that $s_0 \geq r_0$ since requirement (3.29) is stronger than (3.37) and

$$|[\mathbb{I}_{Nd} - \Omega](\mathbf{q} - \mathbf{q}^0)| \leq |\mathbf{q} - \mathbf{q}^0| \leq r_0 \ . \tag{3.39}$$

Let $w(t, w^1) = (\mathbf{q}(t, w^1), \mathbf{p}(t, w^1))$ be the solution of system (3.12) starting from an initial datum $w^1 \in \mathcal{B}(r_0, \epsilon, w^0)$ and let $t^*(w_1) > 0$ be the first exit time of $w(t, w^1)$ from $\mathcal{Q}(s_0, w^0)$. If $w(t, w^1) \in \mathcal{Q}(s_0, w^0)$ for all $t \geq 0$, then we set $t^*(w^1) = \infty$. Next we analyze the solution for $t < t^*(w^1)$ and we will show that $t^*(w^1) = \infty$ for any initial datum $w^1 \in \mathcal{B}(r_0, \epsilon, w^0)$, provided that ϵ in (3.28) is suitably chosen. Let us define

$$\xi(t) := \Pi(\mathbf{q}(t, w^1))w(t, w^1) \ , \tag{3.40}$$

$$\chi(t) := \left(\mathbb{I}_{2Nd} - \Pi(\mathbf{q}(t, w^1)\right) w(t, w^1) \ , \quad t < t^*(w^1) \ . \tag{3.41}$$

By construction $\chi(t) \in \mathcal{I}, \xi(t) \in W(\mathbf{q}(t, w^1))$. We then have

$$\begin{aligned} \frac{d}{dt}\xi(t) &= \left(\frac{d}{dt}\Pi(\mathbf{q}(t))\right) w(t, w^1) + \Pi(\mathbf{q}(t))\frac{d}{dt}w(t, w^1) \\ &= \left(\frac{d}{dt}\Pi(\mathbf{q}(t))\right) w(t, w^1) + \Pi(\mathbf{q}(t))C(\mathbf{q}(t))w(t, w^1) \ . \end{aligned} \tag{3.42}$$

Taking into account that $w(t, w^1) = \xi(t) + \chi(t)$ we get

$$\frac{d}{dt}\xi(t) = \left(\frac{d}{dt}\Pi(\mathbf{q}(t))\right) \xi(t) + \left(\frac{d}{dt}\Pi(\mathbf{q}(t))\right) \chi(t) + \Pi(\mathbf{q}(t))C(\mathbf{q}(t))\xi(t) \ . \tag{3.43}$$

Since for any given $w \in \mathcal{I}$, by the definition $\Pi(\mathbf{q}(t))$, we have $\frac{d}{dt}\Pi(\mathbf{q}(t))w = 0$ and $C(\mathbf{q}(t))$ and $\Pi(\mathbf{q}(t))$ commute, we obtain

$$\frac{d}{dt}\xi(t) = \left(\frac{d}{dt}\Pi(\mathbf{q}(t))\right) \xi(t) + C(\mathbf{q}(t))\xi(t) \ . \tag{3.44}$$

Setting

$$C(\mathbf{q}(t)) = C(\mathbf{q}(0)) + \Gamma(\mathbf{q}(t)) \ , \tag{3.45}$$

where $\Gamma(\mathbf{q}(t))$ is defined in (3.31), we get

$$\frac{d}{dt}\xi(t) = \left(\frac{d}{dt}\Pi(\mathbf{q}(t))\right) \xi(t) + C(\mathbf{q}(0))\xi(t) + \Gamma(\mathbf{q}(t))\xi(t) \ . \tag{3.46}$$

By the formula of variation of constants:

$$\xi(t)=e^{C(\mathbf{q}(0))t}\xi(0)+\int_0^t e^{C(\mathbf{q}(0))(t-s)}\left\{\left(\frac{d}{ds}\Pi(\mathbf{q}(s))\right)\xi(s)+\Gamma(\mathbf{q}(s))\xi(s)\right\}ds. \tag{3.47}$$

Performing the exponential of the matrix $C(\mathbf{q}(0))$ one needs to take into account that, because of the possible presence of Jordan blocks, powers of t might appear. We control such terms paying $e^{-\frac{1}{2}\alpha(\mathbf{q}(0))t}$ and multiplying the remaining exponential by a constant $D(C(\mathbf{q}(0)))$ which depends only on $C(\mathbf{q}(0))$. Since $\mathbf{q}(0)\in\mathbf{B}_{r_0}(\mathbf{q}^0)$, which is a compact set in $\mathbb{R}^{Nd}$, we denote by $D_0:=\sup_{\mathbf{q}\in\mathbf{B}_{r_0}(\mathbf{q}^0)}D(C(\mathbf{q}))$, which depends only on $C(\mathbf{q}^0)$ and r_0. Therefore, we get

$$|\xi(t)|\leq D_0e^{-\frac{1}{2}\alpha(\mathbf{q}(0))t}|\xi(0)|+D_0\int_0^t e^{-\frac{1}{2}\alpha(\mathbf{q}(0))(t-s)}\left\{|\left(\frac{d}{ds}\Pi(\mathbf{q}(s))\right)\xi(s)|+|\Gamma(\mathbf{q}(s))\xi(s)|\right\}ds\,. \tag{3.48}$$

Next we estimate $\left\|\frac{d}{dt}\Pi(\mathbf{q}(t))\right\|$. Let $\Pi(\mathbf{q}(t))=\{\pi_{i,j}(\mathbf{q}(t))\}_{i,j=1,..,,N}$, we then have

$$\begin{aligned}\frac{d}{dt}\Pi(\mathbf{q}(t))&=\left(\nabla_{\mathbf{q}(t)}\pi_{i,j}(\mathbf{q}(t))\cdot\mathbf{p}(t)\right)_{i,j=1,..,,N}\\&=\left(\nabla_{\mathbf{q}(t)}\pi_{i,j}(\mathbf{q}(t))\cdot[\mathbf{p}(t)-\Omega\mathbf{p}(t)]\right)_{i,j=1,..,,N}\ .\end{aligned} \tag{3.49}$$

The last equality holds since, by (3.17), $\Pi(\mathbf{q}(t))=\Pi(\mathbf{q}(0)),\forall t\in\mathbb{R}$, when $\mathbf{q}(t)$ is the evolution given by the flow on the invariant manifold, i.e. when $\mathbf{p}(t)=\Omega\mathbf{p}(t)$. We get by Corollary 3.5

$$\begin{aligned}\left\|\frac{d}{dt}\Pi(\mathbf{q}(t))\right\|&\leq\sup_{\{\mathbf{q}\in\mathbb{R}^{Nd}\,:\,w=(\mathbf{q},\mathbf{p})\in\mathcal{Q}(s_0,w^0)\}}\sup_{i,j=1,..,N}|\nabla_q\pi_{i,j}(\mathbf{q})|\,|\mathbf{p}(t)-\Omega\mathbf{p}(t)|\\&\leq D'(s_0)\epsilon\ ,\qquad\forall t\in[0,t^*(w^1))\ ,\end{aligned} \tag{3.50}$$

where $D'(s_0)>0$. Furthermore, by (3.31) and Lemma 3.10, we have

$$\|\Gamma(\mathbf{q}(t))\|=\|B(\mathbf{q}(t))\|\leq 2\frac{N}{U_R(0)}\sup_{x\in\mathbb{R}^d}|\nabla U(x)|\max_{1\leq i,k\leq N}\left|-(q_i^1-q_k^1)+q_i(t)-q_k(t)\right| \tag{3.51}$$

and, by Theorem 3.4, for $i,k=1,..,N$,

$$\begin{aligned}\left|q_i(t)-q_k(t)-(q_i^1-q_k^1)\right|&=\int_0^t|p_i(s')-p_k(s')|\,ds'\\&=\int_0^t\left|p_i(s')-p_i^0+p_k^0-p_k(s')\right|ds'\leq 2\epsilon t,\end{aligned} \tag{3.52}$$

where $p_i^0=p_k^0$ since $(\mathbf{q}^0,\mathbf{p}^0)\in\mathcal{I}$. Thus, setting $D_1:=2\frac{\sup_{x\in\mathbb{R}^d}|\nabla U(x)|}{U_R(0)},\forall t\in[0,t^*(w^1))$ we obtain

$$|\xi(t)|\leq D_0e^{-\frac{1}{2}\alpha(\mathbf{q}(0))t}|\xi(0)|+D_0\epsilon\int_0^t e^{-\frac{1}{2}\alpha(\mathbf{q}(0))(t-s)}\left\{[D'(s_0)+2ND_1s]\,|\xi(s)|\right\}ds\ . \tag{3.53}$$

Given $K \geq \max\{D'(s_0), 2D_1\}$, take $T \in (0, t^*(w^1))$. A suitable choice of T will be done later. Then,

$$|\xi(t)| \leq D_0 e^{-\frac{1}{2}\alpha(\mathbf{q}(0))t}|\xi(0)| + \epsilon D_0 K\{1+TN\}\int_0^t e^{-\frac{1}{2}\alpha(\mathbf{q}(0))(t-s)}|\xi(s)|ds, \qquad \forall t \in [0,T]. \tag{3.54}$$

By the Gronwall's inequality, which we recall in appendix (see Lemma 5.1), we get

$$|\xi(t)| \leq D_0|\xi(0)|e^{-t\left[\frac{1}{2}\alpha(\mathbf{q}(0))-\epsilon\delta\right]} \leq D_0|\xi(0)|e^{-t\left[\frac{1}{8}\alpha(\mathbf{q}^0)-\epsilon\delta\right]}, \quad \forall t \in [0,T], \tag{3.55}$$

where we made use of (3.37) and set $\delta := D_0 K\{1+NT\}$. Let us choose ϵ such that

$$\frac{1}{16}\alpha(\mathbf{q}^0) \geq \epsilon D_0 K\{1+NT\}. \tag{3.56}$$

Then,

$$|\xi(t)| \leq D_0|\xi(0)|e^{-t\frac{1}{16}\alpha(\mathbf{q}^0)}, \quad \forall t \in [0,T]. \tag{3.57}$$

Since

$$\begin{aligned} \operatorname{dist}\big(w(t,w^1),\mathcal{I}\big) &= \inf_{\tilde{w}\in\mathcal{I}}|w(t,w^1)-\tilde{w}| = \inf_{\tilde{w}\in\mathcal{I}}|\chi(t)+\xi(t)-\tilde{w}| \\ &\leq |\xi(t)| + \inf_{\tilde{w}\in\mathcal{I}}|\chi(t)-\tilde{w}| = |\xi(t)|, \end{aligned} \tag{3.58}$$

we have

$$\operatorname{dist}\big(w(t,w^1),\mathcal{I}\big) \leq D_0|\xi(0)|e^{-t\frac{1}{16}\alpha(\mathbf{q}^0)}, \quad \forall t \in [0,T]. \tag{3.59}$$

Then, recalling that $\operatorname{dist}(w(t,w^1),\mathcal{I}) = |\mathbf{p}(t) - \Omega\mathbf{p}(t)|$, and, since for $w^1 \in \mathcal{B}(r_0,\epsilon,w^0)$,

$$|\xi(0)| \leq D(s_0)\operatorname{dist}\big(w^1,\mathcal{I}\big) \leq D(s_0)\epsilon, \tag{3.60}$$

we have

$$|\mathbf{p}(t)-\Omega\mathbf{p}(t)| \leq D_0 D(s_0)\epsilon e^{-\frac{1}{16}\alpha(\mathbf{q}^0)t} \leq \epsilon(D_0 D(s_0)\vee 1)e^{-\frac{1}{16}\alpha(\mathbf{q}^0)t}, \quad \forall t \in [0,T]. \tag{3.61}$$

From this we get

$$\begin{aligned} |[\mathbb{I}_{Nd}-\Omega]\big(\mathbf{q}(t)-\mathbf{q}^0\big)| &\leq |[\mathbb{I}_{Nd}-\Omega]\big(\mathbf{q}(0)-\mathbf{q}^0\big)| + \int_0^t |[\mathbb{I}_{Nd}-\Omega]\,\mathbf{p}(s)|ds \\ &\leq |\mathbf{q}(0)-\mathbf{q}^0| + (D_0 D(s_0)\vee 1)\frac{16}{\alpha(\mathbf{q}^0)}\epsilon(1-e^{-\frac{1}{16}\alpha(\mathbf{q}^0)t}) \\ &\leq |\mathbf{q}(0)-\mathbf{q}^0| + (D_0 D(s_0)\vee 1))\frac{16}{\alpha(\mathbf{q}^0)}\epsilon. \end{aligned} \tag{3.62}$$

Let us choose ϵ such that

$$r_0 + (D_0 D(s_0)\vee 1))\frac{16}{\alpha(\mathbf{q}^0)}\epsilon \leq \frac{1}{2}s_0, \tag{3.63}$$

and denote this chosen value by $\tilde{\epsilon}_1$. Now we first choose T such that

$$(D_0 D(s_0)\vee 1)e^{-\frac{1}{16}\alpha(\mathbf{q}^0)T} = \frac{1}{2}\wedge\frac{1}{2D_0} \tag{3.64}$$

and denote this chosen value by T_0, then we choose $\tilde{\epsilon}_2$ in such a way that (3.56) holds with T replaced by T_0. We then set

$$\epsilon_0 := \min\{\tilde{\epsilon}_1, \tilde{\epsilon}_2\} \ . \tag{3.65}$$

Notice that, by (3.63),

$$\tilde{\epsilon}_1 \frac{16}{\alpha(\mathbf{q}^0)} \leq r_0 + (D_0 D(s_0) \vee 1)) \frac{16}{\alpha(\mathbf{q}^0)} \epsilon \leq \frac{1}{2} s_0 \tag{3.66}$$

so, since $\alpha(\mathbf{q}^0) < 1$,

$$\epsilon_0 \leq \tilde{\epsilon}_1 \leq \frac{1}{32} s_0 \ . \tag{3.67}$$

We remark that the choice of T_0 depends only on $w^0 \in \mathcal{I}$, while the choice of ϵ_0 depends on $w^0 \in \mathcal{I}$, on T_0 and N. Therefore, at time T_0 we have

$$|[\mathbb{I}_{Nd} - \Omega] \left(\mathbf{q}(T_0) - \mathbf{q}^0\right)| \leq \frac{1}{2} s_0 \tag{3.68}$$

and

$$\operatorname{dist}\left(w(T_0, w^1), \mathcal{I}\right) = |\mathbf{p}(T_0) - \Omega \mathbf{p}(T_0)| \leq \frac{\epsilon_0}{2} \ . \tag{3.69}$$

We can then repeat the previous argument for the solution of the system (2.1) starting at time T_0 from the initial datum $(\mathbf{q}(T_0), \mathbf{p}(T_0))$. We need to recall that $\alpha(\mathbf{q}(T_0)) \geq \frac{1}{4}\alpha(\mathbf{q}^0)$. In a similar way we can show that that for $t \in [T_0, 2T_0]$,

$$|\mathbf{p}(t) - \Omega \mathbf{p}(t)| \leq D_0 |\xi(T_0)| e^{-(t-T_0)\frac{1}{16}\alpha(\mathbf{q}^0)} \ , \quad \forall t \in [T_0, 2T_0] \ . \tag{3.70}$$

Therefore, by (3.64), we have

$$\operatorname{dist}\left(w(2T_0, w(T_0)), \mathcal{I}\right) = |\mathbf{p}(2T_0) - \Omega \mathbf{p}(2T_0)| \leq \frac{\epsilon_0}{2^2 (D_0 D(s_0) \vee 1)} \leq \frac{\epsilon_0}{2^2} \ , \tag{3.71}$$

and, by (3.70),

$$\begin{aligned} |[\mathbb{I}_{Nd} - \Omega] \left(\mathbf{q}(t) - \mathbf{q}^0\right)| &\leq |[\mathbb{I}_{Nd} - \Omega] \left(\mathbf{q}(T_0) - \mathbf{q}^0\right)| + \int_{T_0}^{t} |[\mathbb{I}_{Nd} - \Omega] \mathbf{p}(s)| ds \\ &\leq |[\mathbb{I}_{Nd} - \Omega] \left(\mathbf{q}(T_0) - \mathbf{q}^0\right)| + D_0 |\xi(T_0)| \int_{T_0}^{t} e^{-(s-T_0)\frac{1}{16}\alpha(\mathbf{q}^0)} ds \\ &\leq \frac{1}{2} s_0 + \frac{1}{4} s_0 \ , \end{aligned} \tag{3.72}$$

the last inequality being a consequence of (3.63) and (3.64). Thus, at time $T_1 = 2T_0$

$$|[\mathbb{I}_{Nd} - \Omega] \left(\mathbf{q}(T_1) - \mathbf{q}^0\right)| \leq \frac{1}{2} s_0 + \frac{s_0}{4} \ . \tag{3.73}$$

Hence, we have that $(\mathbf{q}(T_1), \mathbf{p}(T_1)) \in \mathcal{Q}(s_0, w^0)$. Iterating this procedure m times we get

$$\operatorname{dist}\left(w(T_m, w^1), \mathcal{I}\right) = |\mathbf{p}(T_m) - \Omega \mathbf{p}(T_m)| \leq \frac{\epsilon_0}{2^{m+1}} \ , \tag{3.74}$$

and

$$|[\mathbb{I}_{Nd} - \Omega]\left(\mathbf{q}(T_m) - \mathbf{q}^0\right)| \le s_0 \sum_{k=0}^{m} \frac{1}{2^{k+1}} \ . \tag{3.75}$$

Since $\sum_{k\ge 1} \frac{1}{2^m} = \frac{1}{2}$ we obtain the thesis of the theorem. □

4. Kinetic limit: Boltzmann-Vlasov equation

We would like to study system (2.1) when the number of particles N goes to infinity and to derive the kinetic equation for the density $f_t(x,v)$ of particles at x with velocity v at time t. Let $(q_j(t), p_j(t)), j = 1,..,N$ be the solution of the system (2.1) for some initial datum (q_j^0, p_j^0), $\|p_j^0\| \le 1, j = 1,..,N$ and

$$\mu_t^N(dx,dv) = \frac{1}{N}\sum_{j=1}^{N} \delta(q_j(t) - x)\delta(p_j(t) - v)dxdv \ , \tag{4.1}$$

the empirical measure. By Lemma 3.1 and Remark 3.2 $\mu_t^N(dx,dv)$ has support on $\mathbb{R}^d \times B_1$. Given a smooth function g on $\mathbb{R}^d \times B_1$, we denote by

$$\mu_t^N(g) = \int_{\mathbb{R}^d\times B_1} g(x,v)\mu_t^N(dx,dv) \ . \tag{4.2}$$

Writing the second equation of (2.1) in term of μ_t^N we get

$$\begin{aligned} \frac{dp_i(t)}{dt} &= \frac{\sum_{j=1}^N U_R(q_i(t) - q_j(t))\,(p_j(t) - p_i\,(t))}{\sum_{j=1}^N U_R(q_i(t) - q_j(t))} \\ &= \frac{\int_{\mathbb{R}^d\times B_1} U_R(q_i(t) - y)\,(u - p_i\,(t))\,\mu_t^N(dy,du)}{\int_{\mathbb{R}^d\times B_1} U_R(q_i(t) - y)\mu_t^N(dy,du)} \\ &=: M(q_i(t), p_i(t), \mu_t^N) \ . \end{aligned} \tag{4.3}$$

Therefore, the evolution of μ_t^N is given by

$$\frac{\partial(\mu_t^N(g))}{\partial t} = \mu_t^N(v\cdot\nabla_x g) + \mu_t^N(M\left(\cdot,\cdot,\mu_t^N\right)\cdot\nabla_v g) \ . \tag{4.4}$$

In the equation (4.4), N is fixed. To study the limit as $N \to \infty$ we assume that at $t = 0$

$$\mu_0^N(dx,dv) = \frac{1}{N}\sum_{j=1}^{N} \delta(q_j^0 - x)\delta(p_j^0 - v)dxdv \overset{w}{\Longrightarrow} \mu_0(dx,dv) \ , \tag{4.5}$$

where the convergence is weakly as measures, i.e., for every bounded and continuous function g,

$$\lim_{N\to\infty} \int g(x,v)\mu_0^N(dx,dv) = \int g(x,v)\mu_0(dx,dv) \ .$$

We want to show that if (4.5) holds at time $t = 0$, then

$$\mu_t^N(dx,dv) \overset{w}{\Longrightarrow} \mu_t(dx,dv) \ , \tag{4.6}$$

where μ_t is the measure solution of the following equation

$$\frac{\partial(\mu_t(g))}{\partial t} = \mu_t(v \cdot \nabla_x g) + \mu_t(M(\cdot,\cdot,\mu_t) \cdot \nabla_v g) \ , \tag{4.7}$$

which is the formal limit of (4.4). Furthermore, if

$$\mu_0(dx,dv) = f_0(x,v)dxdv \ , \tag{4.8}$$

i.e. the initial measure is absolutely continuous with respect to Lebesgue measure, then we want to show that

$$\mu_t(dx,dv) = f_t(x,v)dxdv \ , \tag{4.9}$$

where f_t is the weak solution of

$$\frac{\partial}{\partial t} f_t(x,v) + v \cdot \nabla_x f_t(x,v) + \nabla_v \cdot [M(x,v,f_t)f_t(x,v)] = 0 \ . \tag{4.10}$$

If one assumes that $\mu_t(dx,dv) = f_t(x,v)dxdv$, equation (4.10) is obtained integrating by parts in (4.7)

$$\begin{aligned}\int_{\mathbb{R}^d\times B_1} \frac{\partial}{\partial t}\mu_t(dx,dv)g(x,v) = &\int_{\mathbb{R}^d\times B_1} v \cdot \nabla_x g(x,v)\mu_t(dx,dv) \\ &+ \int_{\mathbb{R}^d\times B_1} M(x,v,\mu_t) \cdot [\nabla_v g(x,v)]\,\mu_t(dx,dv) \ .\end{aligned} \tag{4.11}$$

Therefore,

$$\begin{aligned}\int_{\mathbb{R}^d\times B_1} \frac{\partial}{\partial t} f_t(x,v)g(x,v)dxdv = &-\int_{\mathbb{R}^d\times B_1} v \cdot \nabla_x f_t(x,v)g(x,v)dxdv \\ &- \int_{\mathbb{R}^d\times B_1} g(x,v)\nabla_v \cdot [(M(x,v,f_t))\,f_t(x,v)]\,dxdv \ ,\end{aligned} \tag{4.12}$$

where, for $F(v) \in \mathbb{R}^d$, we denote by $\nabla_v \cdot F(v) = \sum_{i=1}^d \frac{\partial F_i(v)}{\partial v_i} = div(F(v))$. This holds for any test function $g(\cdot,\cdot)$; therefore equation (4.10) holds. This is a sort of Boltzmann-Vlasov equation where the collision kernel is replaced by $M(x,v,f_t)$. When $M(x,v,f_t) = 0$ the equation (4.10) becomes linear and describes the motion of particles moving independently from each other with the same velocity.

We prove rigorously the previous argument under some assumptions over the interaction U_R and the configurations space. We first show in Theorem 4.5 the existence in a suitable space of measure of the evolution (4.7). Then we show in Corollary 4.7 that if (4.5) holds, then (4.6) holds and it is the measure solution of (4.7). Furthermore, in Theorem 4.8, we show that if (4.8) holds then (4.9) holds and f_t is the weak solution of (4.10). Requiring more regularity to the initial distribution f_0 and to the interaction U_R we proved that f_t is the strong solution of (4.10).

The results are shown adapting to our context the method reported in Spohn's book (1991) [S] (see also Neunzert (1984) [N] and Dobrushin (1979) [D]) and some classical tools of dynamical systems.

In Subsection 4.1 we present some qualitative properties of the solution of (4.7). More accurate analysis of the qualitative behavior of this solution, in special way its long time behavior, is behind the aim of this paper.

We start defining the space of measures and the metric we will be using. We will consider measures with finite total mass on $(X \times B_1, \mathcal{B}(X \times B_1))$ where the symbol X stands either for $\mathbb{R}^d$ or for the torus of linear size $D > 0, \mathcal{T}_D$ and $\mathcal{B}(X \times B_1)$ is the Borel σalgebra on $X \times B_1$. We will denote by $\mathcal{M}$ the space of such measures and, without loss of generality, we will take into account only probability measures. Notice that we will be using the same notations either to denote the space of probability measure on $\mathcal{T}_D \times B_1$ or the space of probability measure on $\mathbb{R}^d \times B_1$, unless we will have the need to distinguish between the two configuration space and therefore use the notation $\mathcal{M}(X \times B_1)$. The bounded Lipschitz distance[2], $d_{b\mathcal{L}}$, between two measures μ and ν in $\mathcal{M}$ is given by

$$d_{b\mathcal{L}}(\mu, \nu) = \sup_{g \in \mathcal{D}} \left| \int g(x,v)\mu(dx,dv) - \int g(x,v)\nu(dx,dv) \right| \, , \tag{4.13}$$

where

$$\mathcal{D} := \left\{ g \mid g : X \times B_1 \to [0,1], \; |g(x,v) - g(y,w)| \leq \sqrt{|v-w|^2 + |x-y|^2} \right\} \, . \tag{4.14}$$

For any smooth function g we consider the evolution (4.7) in $\mathcal{M}$. To prove existence and uniqueness of the solution of equation (4.7), an important assumption we make is that $M(\cdot,\cdot,\nu)$ is Lipschitz continuous in x and v for any $\nu \in \mathcal{M}$. We are then forced to consider only those interactions U_R for which not only $M(\cdot,\cdot,\nu)$ can be well defined for all $(x,v) \in X \times B_1$ and for any $\nu \in \mathcal{M}$, but is also Lipschitz continuous in x and v for any $\nu \in \mathcal{M}$. As a matter of fact, it is not clear even how to define the quantity $M(x,v,\mu_t^N)$ for $(x,v) \in X \times B_1$ regardless of how smooth one can take the interaction U_R. The problem is caused by the presence of the denominator in (4.3). When x is in the support of μ_t^N the denominator is always positive, actually bigger than $\frac{1}{N}$ and then $M(x,v,\mu_t^N)$ is well defined. When x is not in the support of μ_t^N it might happen that $\int_{\mathbb{R}^d \times B_1} U_R(x-y)\mu_t^N(dy,du) = 0$, therefore it is not clear how to define $M(x,v,\mu_t^N)$. To overcome this problem we consider two classes of interaction U_R. The first one is the collection of smooth interactions U_R such that $(U_R \star \nu)(x) > 0$ for all $x \in X$. In this case we define

$$M(x,v,\nu) = \left(\frac{\int_{X \times B_1} U_R(x-y)u\nu(dy,du)}{\int_{X \times B_1} U_R(x-y)\nu(dy,du)} \right) - v =: A(x,\nu) - v \, . \tag{4.15}$$

[2]The bounded Lipschitz distance is identical to the Kantorich-Rubinstein (Vaserstein) distance. The metric $d_{b\mathcal{L}}$ generates the weak $*$-topology on $\mathcal{M}$. For a sequence μ^N,

$$\lim_{N \to \infty} d_{b\mathcal{L}}(\mu^N, \nu) = 0$$

is equivalent to

$$\lim_{N \to \infty} \int g(x,v)\mu^N(dx,dv) = \int g(x,v)\nu(dx,dv)$$

for all bounded and continuous functions g.

The second one is the class of smooth interactions U_R with compact support. In this case, we fix $\epsilon > 0$ and define

$$M^{\epsilon}(x,v,\nu) := \left(\frac{\int_{X\times B_1} U_R(x-y)(u-v)\nu(dy,du)}{\int_{X\times B_1} U_R(x-y)\nu(dy,du)+\epsilon} \right) . \tag{4.16}$$

Hence, in the case U_R has compact support, we modify the interaction term in such a way that when $\int_{X\times B_1} U_R(x-y)\nu(dy,du) = 0$ then $M^{\epsilon}(x,v,\nu) = 0$, when $\int_{X\times B_1} U_R(x-y)\nu(dy,du) > \epsilon$ then $M^{\epsilon}(x,v,\nu) = M(x,v,\nu)+O(\epsilon)$, when $\epsilon > \int_{X\times B_1} U_R(x-y)\nu(dy,du) > 0$ then $M^{\epsilon}(x,v,\nu)$ is a large perturbation of $M(x,v,\nu)$. Notice that if one considers the system (2.1) replacing U_R with the modified interaction, the one in (4.16), then Lemma 3.1 continues to hold. This is the reason why we kept the velocity in the ball B_1.

It is easy to see that for any measure ν on $\mathbb{R}^d \times B_1$ we have

$$\sup_{(x,v)\in\mathbb{R}^d\times B_1} |M(x,v,\nu)| \leq 2 , \qquad \sup_{(x,v)\in\mathbb{R}^d\times B_1} |M^{\epsilon}(x,v,\nu)| \leq 2 . \tag{4.17}$$

The Lipschitz continuity of $M(\cdot,\cdot,\nu)$ with respect to v follows from the linearity of $M(\cdot,\cdot,\nu)$ as a function of v. The Lipschitz continuity of $M(\cdot,\cdot,\nu)$ with respect to x does not hold in general even if one takes a smooth interaction U_R. The problem is created by the presence of the denominator in $M(\cdot,v,\nu)$. The denominator of the derivative in $x_i, i = 1,..,d$, of any component of the vector $A(\cdot,\nu)$ might be very small while the numerator, because of the presence of the derivative of U_R, might be not of the same order. Hence, one can certainly control the gradient of $A(\cdot,\nu)$ if the gradient of U_R is of the same order of U_R. We have then the following lemma.

Lemma 4.1. *Let $\nu \in \mathcal{M}$ and let K be a positive constant such that* $\sup_{x\in\mathbb{R}^d} |\nabla \log U_R(x)| \leq K$. *Then*

$$|A_i(x,\nu) - A_i(z,\nu)| \leq L\,|x-z| \ , \qquad x \in X, z \in X,\ i = 1,..,d, \qquad L = 2K \ . \tag{4.18}$$

Proof. $\forall i = 1,..,d$, we have

$$\begin{aligned} \nabla_x A_i(x,\nu) =& \frac{\int_{X\times B_1} u_i \nabla_x U_R(x-y)\nu(dy,du)(U_R\star\nu)(x) - \int_{X\times B_1} U_R(x-y)u_i\nu(dy,du)\nabla_x(U_R\star\nu)(x)}{[(U_R\star\nu)(x)]^2} \\ =& \frac{\int_{(X\times B_1)^2} \nu(dy,du)\nu(dy',du')u_i\left[\nabla_x U_R(x-y)U_R(x-y') - U_R(x-y)\nabla_x U_R(x-y')\right]}{[(U_R\star\nu)(x)]^2}, \end{aligned} \tag{4.19}$$

Taking into account that $|u| \leq 1$ we have

$$\begin{aligned}
|\nabla_x A_i(x,\nu)| &\leq \frac{\int_{(X\times B_1)^2} \nu(dy,du)\nu(dy',du')\left|\nabla_x U_R(x-y)U_R(x-y') - U_R(x-y)\nabla_x U_R(x-y')\right|}{[(U_R \star \nu)(x)]^2} \\
&\leq 2\frac{\int_{(X\times B_1)^2} \nu(dy,du)\nu(dy',du')\left|\nabla_x U_R(x-y)\right| U_R(x-y')}{[(U_R \star \nu)(x)]^2} \\
&= 2\frac{\int_{(X\times B_1)^2} \nu(dy,du)\nu(dy',du')\left|\nabla_x U_R(x-y)\right| \frac{U_R(x-y)}{U_R(x-y)} U_R(x-y')}{[(U_R \star \nu)(x)]^2} \\
&\leq 2\sup_y \frac{|\nabla_x U_R(x-y)|}{U_R(x-y)} \leq 2K \ .
\end{aligned} \tag{4.20}$$

□

Remark 4.2. *The assumption about the boundedness of* $\sup_{x\in\mathbb{R}^d}|\nabla \log U_R(x)|$ *is quite strong. As a matter of fact, an interaction* U_R *verifying this assumption should decay for* $|x|$ *large as* $e^{-\frac{|x|}{R}}$. *Interactions with compact support do not satisfy this assumption as well as the interaction* $\mathbb{R}^d \ni x \longmapsto U_R(x) = \frac{1}{(2\pi R^2)^{\frac{d}{2}}} e^{-\frac{|x|^2}{2R^2}} \in \mathbb{R}_+$ *considered next.*

Lemma 4.3. *Let* $\mathcal{T}_D$ *be a* d *dimensional torus of linear size* $D > 0$,

$$\mathbb{R}^d \ni x \longmapsto U_R(x) = \frac{1}{(2\pi R^2)^{\frac{d}{2}}} e^{-\frac{|x|^2}{2R^2}} \in \mathbb{R}_+ \tag{4.21}$$

and V_R *be the corresponding interaction defined on* $\mathcal{T}_D$ *through the periodization of* U_R

$$\mathbb{R}^d \ni x \longmapsto \bar{V}_R(x) = \sum_{n\in\mathbb{Z}^d} U_R(x+nD) \in \mathbb{R}_+ \ . \tag{4.22}$$

For $\nu \in \mathcal{M}$ *and* $A(\cdot,\nu)$ *as defined in (4.15), with* U_R *replaced by* V_R *we have*

$$|A_i(x,\nu) - A_i(z,\nu)| \leq L\,|x-z| \ , \qquad x,z \in \mathcal{T}_D, \quad i = 1,..,d, \qquad L = \frac{D}{R^2} \ . \tag{4.23}$$

Proof. Let us write

$$\begin{aligned}
|A_i(x,\mu) - A_i(z,\mu)| &= \left|\int_0^1 ds \frac{d}{ds} A_i(sx + (1-s)z,\mu)\right| \\
&\leq \sup_{s\in[0,1]} \left|\frac{d}{ds} A_i(sx+(1-s)z,\mu)\right| \ , \qquad i = 1,..,d,
\end{aligned} \tag{4.24}$$

and set $x_0 = sx + (1-s)z$ and $(V_R \star \mu)(x) = \int_{\mathcal{T}_D \times B_1} V_R(x-y)\mu(dy,du)$, we obtain

$$\frac{d}{ds}A_i(x_0,\mu) = \frac{(x-z)}{R}\frac{\int_{\mathcal{T}_D \times B_1} \frac{-(x_0-y)}{R} V_R(x_0-y)u_i\mu(dy,du)(V_R \star \mu)(x_0)}{[(V_R\star\mu)(x_0)]^2} \tag{4.25}$$

$$- \frac{(x-z)}{R}\frac{\int_{\mathcal{T}_D \times B_1} V_R(x_0-y')u_i'\mu(dy',du')\int_{\mathcal{T}_D \times B_1} \frac{-(x_0-y)}{R} V_R(x_0-y)\mu(dy,du)}{[(V_R\star\mu)(x_0)]^2} \tag{4.26}$$

$$= \frac{(x-z)}{R} C_i(x_0,R,\mu)\ , \tag{4.27}$$

where we denote by

$$\begin{aligned} C_i(x_0,R,\mu) :=& \frac{1}{[(V_R\star\mu)(x_0)]^2}\times \\ & \left[\int_{\mathcal{T}_D \times B_1} \frac{-(x_0-y)}{R} V_R(x_0-y)u_i\mu(dy,du)(V_R\star\mu)(x_0) \right. \\ & \left. - \int_{\mathcal{T}_D \times B_1} V_R(x_0-y')u_i'\mu(dy',du')\int_{\mathcal{T}_D \times B_1} \frac{-(x_0-y)}{R} V_R(x_0)\mu(dy,du)\right]\ . \end{aligned} \tag{4.28}$$

Finally, rewriting

$$C_i(x_0,R,\mu) = \frac{\int \frac{(y'-y)}{R} V_R(x_0-y')V_R(x_0-y)\int u_i'\mu(dy',du')\mu(dy,du)}{\int V_R(x_0-y')V_R(x_0-y)\int \mu_t(dy',du')\mu(dy,du)} \tag{4.29}$$

and recalling that $|u| \le 1$, we obtain

$$|C(x_0,R,\mu)| \le \frac{\int_{\mathcal{T}_D \times B_1} \frac{|y'-y|}{R} V_R(x_0-y')V_R(x_0-y)\int_{\mathcal{T}_D \times B_1} \mu_t(dy',du')\mu(dy,du)}{\int_{\mathcal{T}_D \times B_1} V_R(x_0-y')V_R(x_0-y)\int_{\mathcal{T}_D \times B_1} \mu(dy',du')\mu_t(dy,du)} \le \frac{D}{R}\ , \tag{4.30}$$

since in the torus $|y'-y| \le D$ and the result follows by (4.25). □

$M^\epsilon(\cdot,v,\nu)$ is easily seen to be Lipschitz continuous in X. In fact we have:

Lemma 4.4. *Let $\nu \in \mathcal{M}, \epsilon > 0, U_R(\cdot)$ a smooth interaction whose support contained in a ball of radius R such that $\sup_{x\in B_R}|\nabla U_R(x)| \le 1$ and $M^\epsilon(\cdot,v,\nu)$ as in (4.16). Then $M^\epsilon(\cdot,v,\nu)$ is Lipschitz continuous in X.*

$$|M_i^\epsilon(x,v,\nu) - M_i^\epsilon(y,v,\nu)| \le L\,|x-y|\ , \qquad x,y \in X, i = 1,..,d, \qquad L = \frac{2}{\epsilon}. \tag{4.31}$$

To prove the existence of the solution of (4.7) we prescribe a curve $t \to \mu_t \in \mathcal{M}$ weakly continuous in t and we consider the following non-autonomous system of ordinary differential equations:

$$\begin{cases} \frac{d}{dt}x(t) = v(t) \\ \frac{d}{dt}v(t) = M(x(t),v(t),\mu_t) \end{cases} . \tag{4.32}$$

Under the assumption that $M(\cdot,\cdot,\mu_t)$ is Lipschitz continuous in $X \times B_1$ there exists an unique global solution of (4.32) for any given initial datum. The corresponding time dependent two parameters flow is denoted by $T_{t,s}[\mu_\cdot]$. Under this time dependent flow any initial measure evolves as

$$\nu_t = \nu_0 \circ T_{0,t}[\mu_\cdot] \ , \tag{4.33}$$

where $T_{0,t}[\mu_\cdot]$ is the push forward of the measure ν_0. For any test function g we have that

$$\nu_t(g) = \nu_0(g \circ T_{t,0}[\mu_\cdot]) \ , \tag{4.34}$$

where $T_{t,0}[\mu_\cdot]$ is the pull back acting over the test functions. By the existence and uniqueness of the solution of (4.32) for any initial datum, the inverse flow $(T_{t,s}[\mu_\cdot])^{-1}$ is well defined. The equation for the evolution of ν_t, easily derived, is

$$\begin{aligned}\frac{\partial(\nu_t(g))}{\partial t} &= \frac{\partial(\nu_0(g \circ T_{t,0}[\mu_\cdot]))}{\partial t} = \nu_0((v\nabla_x g) \circ T_{t,0}[\mu_\cdot]) + \nu_0((M(x,v,\mu_t)\nabla_v g) \circ T_{t,0}[\mu_\cdot]) \\ &= \nu_t(v \cdot \nabla_x g) + \nu_t(M(x,v,\mu_t) \cdot \nabla_v g) \ .\end{aligned} \tag{4.35}$$

One immediately realizes that proving the existence and uniqueness of the solution of (4.7) is equivalent to prove the existence of a fixed point for the time dependent flow $\mu_t = \mu_0 \circ T_{0,t}[\mu_\cdot]$. This is the content of the next theorem.

Theorem 4.5. *Let U_R be as in Lemma 4.1 or as in Lemma 4.3 and let $M(\cdot,\cdot,\nu)$ be defined as in (4.15) for any $\nu \in \mathcal{M}(\mathcal{T}_D \times B_1)$. The equation (4.7) has an unique solution in the space $\mathcal{M}(\mathcal{T}_D \times B_1)$ if $\mu_0 \in \mathcal{M}(\mathcal{T}_D \times B_1)$. Furthermore, take two solutions of (4.7), μ_t starting at $\mu_0 = \mu$ and ν_t starting at $\nu_0 = \nu$ then in the bounded Lipschitz distance*

$$d_{b\mathcal{L}}(\nu_t, \mu_t) \le e^{ct} d_{b\mathcal{L}}(\mu, \nu) \ , \tag{4.36}$$

where c is a constant which depends on the Lipschitz constant of $M(\cdot,\cdot,\nu)$ and on $\inf_{x\in\mathcal{T}_D} U_R(x) =: a > 0$.[3]

The proof is obtained adapting the method explained in Chapter 5 of [S] to our context. The main difference between the case considered here and the one presented in Section 5 of [S] is that, in our case, the dependence of $M(\cdot,\cdot,\nu)$ from ν is not linear. To overcome this problem we assume that the interaction U_R is such that $\inf_{x\in\mathcal{T}_D} U_R(x) = a > 0$. This is the case for interactions as in Lemma 4.1 but defined on $\mathcal{T}_D \times B_1$ via periodization or as in Lemma 4.3. To facilitate the reader we report the proof of Theorem 4.5 in the Appendix.

Remark 4.6. *Theorem 4.5 does not hold in $\mathbb{R}^d \times B_1$ when U_R satisfies Lemma 4.1. Although in this case U_R is globally Lipschitz continuous in $\mathbb{R}^d$, we are not able to show that $M(x,v,\cdot)$ when $x \in \mathbb{R}^d$, is Lipschitz continuous with respect to $\nu \in \mathcal{M}$ in the $d_{b\mathcal{L}}$ metric. The theorem applies with obvious modification if we take an interaction U_R with compact support and define M^ϵ as in (4.16). In such case the theorem holds either for*

[3]Notice that for U_R as in Lemma 4.1 or as in Lemma 4.3 $(U_R \star \nu)(x) > a$ for all $x \in \mathcal{T}_D$, where $a = \inf_{x\in\mathcal{T}_D} U_R(x) > 0$.

the system defined on $\mathcal{T}_D \times B_1$ or on $\mathbb{R}^d \times B_1$. The constant c in the statement of Theorem 4.5 will then depend on ϵ, the lower bound of the denominator of M^ϵ.

Corollary 4.7. *Let $\mu \in \mathcal{M}$ and $(\mathbf{q}_N^0, \mathbf{p}_N^0) \in (\mathcal{T}_D \times B_1)^N$ be a sequences of particle configurations, so that*

$$\lim_{N\to\infty} d_{b\mathcal{L}}(\mu^N, \mu) = 0 \ , \tag{4.37}$$

where μ^N is the empirical measure. Let $(\mathbf{q}_N(t), \mathbf{p}_N(t)) \in (\mathcal{T}_D \times B_1)^N$ be the solution of (2.1) with initial datum $(\mathbf{q}_N^0, \mathbf{p}_N^0)$ and with U_R chosen as in Lemma 4.1 in $\mathcal{T}_D \times B_1$ or as in Lemma 4.3. Then there exists $\mu_t \in \mathcal{M}$ such that

$$\lim_{N\to\infty} d_{b\mathcal{L}}(\mu_t^N, \mu_t) = 0 \ , \tag{4.38}$$

and μ_t solves equation (4.7).

The proof is immediate from (4.36). The validity of Corollary 4.7 for smooth compact potential U_R and for the *local mean velocity increment* M^ϵ is immediate as well.

Theorem 4.8. *Let $M(\cdot,\cdot,\mu)$ be as in (4.15) and assume that $M(\cdot,\cdot,\mu) \in C^1(X\times B_1)$ for $\mu \in \mathcal{M}$. If $\mu_0(dx,dv) = f_0(x,v)dxdv$, then $\mu_t(dx,dv) = f_t(x,v)dxdv$ and f_t is the weak solution of (4.10). Furthermore, if $f_0 \in C^k(X\times B_1), k \geq 1$, and $M(\cdot,\cdot,\mu) \in C^k(X\times B_1)$ for $\mu \in \mathcal{M}$, then $f_t \in C^k(X \times B_1)$.*

Proof. We start showing that for any given weakly continuous curve $t \to \mu_t \in \mathcal{M}$, if $\nu_0(dx,dv) = q_0(x,v)dxdv$, i.e. absolutely continuous with respect to the Lebesgue measure, then $\nu_t(dx,dv) = q_t(x,v)dxdv$, where

$$\frac{\partial}{\partial t} q_t(x,v) + \nabla_x q_t(x,v)\cdot v + \nabla_v \cdot [M(x,v,\mu_t)q_t(x,v)] = 0 \ , \tag{4.39}$$

and, if $q_0 \in C^k(X \times B_1)$ and $M(\cdot,v,\mu) \in C^k(X \times B_1)$ for any $\mu \in \mathcal{M}$, then $q_t \in C^k(X \times B_1)$. Note that (4.39) corresponds to a linearization of (4.10) since $M(x,v,\mu_t)$ does not depend on $q.$ once μ_t is given. In Theorem 4.5 we proved that the fixed point equation $\mu_t = \mu_0 \circ T_{0,t}[\mu.]$ holds. Therefore, by this result and the validity of (4.39), one immediately obtains that μ_t has density and the thesis of the theorem is proven. We are then left with the proof of (4.39). Let us set $w = (x,v) \in X \times B_1$. For any test function g we obtain

$$\begin{aligned}\nu_t(g) &= \nu_0 \circ T_{0,t}[\mu.](g) = \nu_0(g \circ T_{t,0}[\mu.]) \\ &= \int_{X\times B_1} \nu_0(dw)(g\circ T_{t,0}[\mu.])(w) = \int_{X\times B_1} q_0(w)\left(g\circ T_{t,0}[\mu.]\right)(w)dw \\ &= \int_{X\times B_1} q_0(w) \circ (T_{t,0}[\mu.])^{-1} \mathcal{J}(w,\mu_t) g(w)dw\end{aligned} \tag{4.40}$$

where $\mathcal{J}(w,\mu_t) = Det\,[\partial.(T_{t,0})[\mu.])^{-1}(w)]$ is the Jacobian of the flow $(T_{t,0}[\mu.])^{-1}$ computed in w. Since the divergence of the vector field $(v(s), M(x,v,\mu_s))$ is given by

$$\sum_{i=1}^d \left[\frac{\partial v_i}{\partial x_i} + \frac{M(x,v,\mu_s)}{\partial v_i}\right] = -d \ , \tag{4.41}$$

by Liouville Theorem (see [A1] or [A2]) for any weakly continuous curve $t \to \mu_t \in \mathcal{M}$, we have

$$Det\left[\partial.(T_{t,0})[\mu.])\right](w) = e^{-dt} \ , \qquad \forall w \in X \times B_1 \ , \tag{4.42}$$

hence,

$$\mathcal{J}(w, \mu_t) = e^{dt} \ , \qquad \forall w \in X \times B_1 \ . \tag{4.43}$$

Then, from (4.40) and (4.43), we obtain

$$\nu_t(g) = \int_{X \times B_1} e^{dt}\left(q_0 \circ (T_{t,0}[\mu.])^{-1}\right)(w) g(w) dw = \int_{X \times B_1} q_t(w) g(w) dw \ , \tag{4.44}$$

where we denote by

$$q_t(w) := e^{dt}\left(q_0 \circ (T_{t,0}[\mu.])^{-1}\right)(w) \ . \tag{4.45}$$

Notice that $q_t(w)$ is weakly continuous in time, since $\mu.$ is weakly continuous. Furthermore, if, for $k \geq 1, M(\cdot, \cdot, \mu) \in C^k(X \times B_1), \mu \in \mathcal{M}$ and $q_0 \in C^k(X \times B_1)$, then $q_t \in C^k(X \times B_1)$. Writing $e^{-dt}(q_t \circ T_{t,0}[\mu.])(w) = q_0(w)$ and differentiating with respect to t we get

$$\begin{aligned} \frac{\partial}{\partial t}\left(e^{-dt} q_t \circ (T_{t,0}[\mu.])(w)\right) = -de^{-dt}(q_t \circ T_{t,0}[\mu.])(w) + \\ + e^{-dt}\frac{\partial}{\partial t}(q_t \circ T_{t,0}[\mu.])(w) + e^{-dt}\nabla_x (q_t \circ T_{t,0}[\mu.])(w) \cdot v \circ T_{t,0}[\mu.] + \\ + e^{-dt}\nabla_v\left((q_t \circ T_{t,0}[\mu.])(w)\right) \cdot (M(\cdot, \cdot, \mu_t) \circ T_{t,0}[\mu.])(w) = 0 \ . \end{aligned} \tag{4.46}$$

Multiplying both members of the previous identity for e^{dt} and applying to $(T_{t,0}[\mu.])^{-1} w$ we obtain

$$-dq_t(x,v) + \frac{\partial}{\partial t} q_t(x,v) + \nabla_x q_t(x,v) \cdot v + \nabla_v q_t(x,v) \cdot M(x,v,\mu_t) = 0 \ . \tag{4.47}$$

Notice that this last equation is linear in $q.$ since $\mu.$ is given. Therefore, the equation for f_t is

$$\frac{\partial}{\partial t} f_t(x,v) - df_t(x,v) + \nabla_x f_t(x,v) \cdot v + M(x,v,f_t) \cdot \nabla_v f_t(x,v) = 0 \tag{4.48}$$

which corresponds to (4.10). $\square$

Remark 4.9. *Theorem 4.8 can be proven when U_R is a smooth function with compact support and the* local mean velocity increment *is M^ϵ for $\epsilon > 0$ as defined in (4.16). In this case the theorem holds either in $\mathbb{R}^d \times B_1$ or in $\mathcal{T}_D \times B_1$. The only difference in the computations done in Theorem 4.8 is that*

$$\nabla \cdot M^\epsilon(x,v,\mu_s) = -d\frac{(U_R \star \mu_s)(x)}{(U_R \star \mu_s)(x) + \epsilon} \equiv -dh_s(x) \equiv -dh(\mu_s)(x) \ . \tag{4.49}$$

Then, for any weakly continuous curve $t \to \mu_t \in \mathcal{M}$ and any $w \in X \times B_1$, by Liouville Theorem, we have

$$\mathrm{Det}\left[\partial.(T_{t,0}[\mu.])\right](w) = e^{-d\int_0^t ds (h_s \circ T_{s,0}[\mu.])(w)} \ , \tag{4.50}$$

therefore

$$\mathcal{J}(w,\mu_t) = \mathrm{Det}\left[\partial.[(T_{t,0}[\mu.])^{-1}]\right] = e^{d\int_0^t ds (h_s \circ T_{s,t}[\mu.])(w)} \ . \tag{4.51}$$

Then, from (4.40) and (4.51), we obtain

$$\begin{aligned}\nu_t(g) &= \int_{\mathcal{T}_D\times B_1} e^{d\int_0^t ds(h_s\circ T_{s,t}[\mu.])(w)} \left(q_0\circ (T_{t,0}[\mu.])^{-1}\right)(w) g(w)dw \\ &= \int_{\mathcal{T}_D\times B_1} q_t(w)g(w)dw \ ,\end{aligned} \tag{4.52}$$

where we denoted by

$$q_t(w) := e^{d\int_0^t ds(h_s\circ T_{s,t}[\mu.])(w)} \left(q_0\circ (T_{t,0}[\mu.])^{-1}\right)(w) \ . \tag{4.53}$$

Notice that $q_t(w)$ is weakly continuous in time, since $\mu.$ is weakly continuous. Furthermore, if $M(\cdot,\cdot,\mu)\in C^k(X\times B_1), \mu\in\mathcal{M}$ and $q_0\in C^k(X\times B_1)$, then $q_t\in C^k(X\times B_1)$. Writing

$$e^{-d\int_0^t ds(h_s\circ T_{s,0}[\mu.])(w)}(q_t\circ T_{t,0}[\mu.])(w) = q_0(w) \tag{4.54}$$

and differentiating with respect to t we get

$$\begin{aligned}&\frac{\partial}{\partial t}\left(e^{-d\int_0^t ds(h_s\circ T_{s,0}[\mu.])(w)}(q_t\circ (T_{t,0}[\mu.])(w)\right) = -d(h_t\circ T_{t,0}[\mu.])(w)e^{-d\int_0^t ds(h_s\circ T_{s,0}[\mu.])(w)}(q_t\circ T_{t,0}[\mu.])(w) \\ &+e^{-d\int_0^t ds(h_s\circ (T_{s,0}[\mu.])(w)}\frac{\partial}{\partial t}(q_t\circ T_{t,0}[\mu.])(w) + e^{-d\int_0^t ds(h_s\circ T_{s,0}[\mu.])(w)}\nabla_x q_t(w)\circ T_{t,0}[\mu.]\cdot v\circ T_{t,0}[\mu.] \\ &\qquad + e^{-d\int_0^t ds(h_s(\circ T_{s,0}[\mu.])(w)}\nabla_v\left((q_t\circ T_{t,0}[\mu.])(w)\right)\cdot (M(\cdot,\cdot,\mu_t)\circ T_{t,0}[\mu.])(w) = 0 \ .\end{aligned}$$

Multiplying by $e^{d\int_0^t ds(h_s\circ(T_{s,0}[\mu.])(w)}$ and applying to $(T_{t,0}[\mu.])^{-1}w$ we obtain

$$-dh(\mu_t)(x)q_t(x,v) + \frac{\partial}{\partial t}q_t(x,v) + \nabla_x q_t(x,v)\cdot v + \nabla_v q_t(x,v)\cdot M(x,v,\mu_t) = 0, \tag{4.55}$$

which is linear in $q.$ since $\mu.$ is given. Therefore the equation for f_t is

$$\frac{\partial}{\partial t}f_t(x,v) - dh(f_t)(x)f_t(x,v) + \nabla_x f_t(x,v)\cdot v + M(x,v,f_t)\cdot\nabla_v f_t(x,v) = 0 \ , \tag{4.56}$$

which corresponds to (4.10) taking into account the definition of h in (4.49).

4.1. **Qualitative behaviour of the solution of (4.7).**

Lemma 4.10. *Let $t\to\mu_t\in\mathcal{M}$ be the solution of (4.7) with initial datum μ_0. We have*

$$\mu_t(x) = \mu_0(x) + \int_0^t \mu_s(v)ds \ , \tag{4.57}$$

$$\int_{X\times B_1} |v|^2\,\mu_t(dx,dv) \le \int_{X\times B_1}|v|^2\,\mu_0(dx,dv) \ . \tag{4.58}$$

Proof. By (4.7) we have

$$\frac{d}{dt}\int_{X\times B_1} x_i\mu_t(dx,dv) = \mu_t(v_i)\ , \qquad i=1,..,d\ , \tag{4.59}$$

which imply (4.57). To obtain (4.58), again from equation (4.7) we get

$$\begin{aligned}\frac{d}{dt}\int_{X\times B_1} |v|^2\,\mu_t(dx,dv) &= \frac{d}{dt}\mu_t(|v|^2) \\ &= \mu_t(v\cdot\nabla_x|v|^2) + \mu_t(M(\cdot,\cdot,\mu_t)\cdot\nabla_v|v|^2) \\ &= 2\sum_{i=1}^d \mu_t(M_i(\cdot,\cdot,\mu_t)v_i) \le 0\ .\end{aligned} \tag{4.60}$$

Namely, for $i=1,..,d$, when $M_i(\cdot,\cdot,\mu_t)\neq 0$, we obtain

$$\begin{aligned}&\mu_t(M_i(\cdot,\cdot,\mu_t)v_i) = \int_{X\times B_1}\mu_t(dx,dv)M_i(x,v,\mu_t)v_i \\ &= \int_{X\times B_1}\mu_t(dx,dv)\left(\frac{\int_{X\times B_1}U_R(x-y)\,(v_iu_i-v_i^2)\,\mu_t(dy,du)}{\int_{X\times B_1}U_R(x-y)\mu_t(dy,du)}\right) \\ &= \int_{(X\times B_1)^2}\mu_t(dx,dv)\mu_t(dy,du)\frac{U_R(x-y)v_iu_i}{\int_{X\times B_1}U_R(x-y)\mu_t(dy,du)} - \int_{X\times B_1}\mu_t(dx,dv)v_i^2 \le 0\ ,\end{aligned} \tag{4.61}$$

by Schwartz inequality. □

Jensen inequality and (4.58) imply the boundedness of $\left|\int_{X\times B_1} v\mu_t(dx,dv)\right|$. Anyway, we are not able to show more interesting estimates about the mean velocity $\int_{X\times B_1} v\mu_t(dx,dv)$.

Lemma 4.11. *Take M as in (4.15). The equation (4.10) is not time reversible, i.e. invariant under simultaneous reflection $t\to -t$ and $v\to -v$.*

Proof. Let us set $b_t(x,v) := f_{-t}(x,-v)$. We have that $\frac{\partial}{\partial t}b_t(x,v) = -\frac{\partial}{\partial t}f_{-t}(x,-v)$ and $\frac{\partial b_t(x,v)}{\partial v_i} = -\frac{\partial f_{-t}(x,-v)}{\partial v_i}$. Therefore, by (4.10),

$$\frac{\partial}{\partial t}b_t(x,v) = -\frac{\partial}{\partial t}f_{-t}(x,-v) = -v\cdot\nabla_x f_{-t}(x,-v) - df_{-t}(x,-v) + \sum_{i=1}^d M_i(x,-v,f_{-t})\frac{\partial f_{-t}(x,-v)}{\partial v_i} \tag{4.62}$$

and

$$\begin{aligned}M(x,-v,f_{-t}) &= \left(\frac{\int_{\mathbb{R}^d\times\mathbb{R}^d}U_R(x-y)\,(u+v)\,f_{-t}(y,u)dydu}{\int_{X\times B_1}U_R(x-y)f_{-t}(y,u)dydu}\right) \\ &= \left(\frac{\int_{X\times B_1}U_R(x-y)\,(-u+v)\,b_t(y,u)dydu}{\int_{X\times B_1}U_R(x-y)b_t(y,u)dydu}\right) = -M(x,v,b_t)\ .\end{aligned} \tag{4.63}$$

Thus, the equation for $b_t(x,v)$ is

$$\frac{\partial}{\partial t}b_t(x,v)+v\cdot\nabla_x b_t(x,v)+db_t(x,v)-\sum_{i=1}^{d}M_i(x,v,b_t)\frac{\partial b_t(x,v)}{\partial v_i}=0$$

which differs from equation (4.10). □

Remark 4.12. *Lemma 4.11 also holds when U_R has compact support and M is replaced by M^ϵ.*

Let f_t be the solution at time t of the equation (4.10). We denote by $H(f_t)$ the Boltzmann-Vlasov entropy

$$H(f_t):=-\int_{X\times B_1}f_t(x,v)\ln(f_t(x,v))dxdv\ . \tag{4.64}$$

In the next lemma we show that $H(f_t)$ is a decreasing function of the time.

Lemma 4.13. *Let $f_\cdot$ be the solution of (4.10) with M chosen as in (4.15), then*

$$\frac{d}{dt}H(f_t)=-d\ . \tag{4.65}$$

Let $f^\epsilon_\cdot$ be the solution of (4.10) with M replaced by M^ϵ chosen as in (4.16), then

$$\frac{d}{dt}H(f^\epsilon_t)=-d\int_{X\times B_1}h_t(x)f^\epsilon_t(x,v)dxdv\ , \tag{4.66}$$

where, as in (4.49), $h_t=\frac{(U_R\star f^\epsilon_t)}{(U_R\star f^\epsilon_t)+\epsilon}$.

Proof. We start showing (4.65). The proof of (4.66) is similar and we will only outline the differences.

$$\begin{aligned}\frac{d}{dt}H(f_t)&=-\int_{X\times B_1}\frac{\partial f_t}{\partial t}(x,v)\left[\ln(f_t(x,v))+1\right]dxdv \qquad (4.67)\\ &=\int_{X\times B_1}\left[\ln f_t(x,v)+1\right]\left[v\cdot\nabla_x f_t(x,v)+\nabla_v\cdot M(x,v,t)f_t(x,v)\right]dxdv\ .\end{aligned}$$

Integrating by part the last term in (4.67) we get

$$\begin{aligned}\frac{d}{dt}H(f_t)&=-\int_{X\times B_1}\nabla_x\left[\ln f_t(x,v)+1\right]\cdot vf_t(x,v)dxdv \qquad (4.68)\\ &\quad-\int_{X\times B_1}\nabla_v\left[\ln f_t(x,v)+1\right]\cdot\left[M(x,v,f_t)f_t(x,v)\right]dxdv\\ &=-\int_{X\times B_1}\nabla_x f_t(x,v)\cdot vdxdv-\int_{X\times B_1}\nabla_v f_t(x,v)\cdot\left[M(x,v,f_t)\right]dxdv\ .\end{aligned}$$

The first integral gives zero contribution since $\int_{X\times B_1} f_t(x,v)dxdv = 1$ for all $t>0$, i.e. $f_t \in L^1(X\times B_1)$. For the second term notice that $\nabla_v \cdot [M(x,v,f_t)] = -d$, therefore

$$\begin{aligned}\int_{X\times B_1} \nabla_v f_t(x,v)\cdot[M(x,v,f_t)]\,dxdv &= -\int_{X\times B_1} f_t(x,v)\nabla_v\cdot[M(x,v,f_t)]\,dxdv \\ &= d\int_{X\times B_1} f_t(x,v)dxdv = d\ .\end{aligned} \tag{4.69}$$

We then obtain (4.65). To get (4.66) we proceed in the same way. We need only to modify (4.69) as

$$\begin{aligned}\int_{X\times B_1} \nabla_v f^\epsilon_t(x,v)\cdot[M^\epsilon(x,v,f_t)]\,dxdv &= -\int_{X\times B_1} f^\epsilon_t(x,v)\nabla_v\cdot[M(x,v,f^\epsilon_t)]\,dxdv \\ &= d\int_{X\times B_1} h_t(x) f^\epsilon_t(x,v)dxdv\ .\end{aligned} \tag{4.70}$$

□

By the above lemma, if $h_t(\cdot)\neq 0$ for t large enough then

$$\lim_{t\to\infty} H(f_t) = -\infty\ .$$

From this we can deduce that even starting at time $t=0$ from a measure which is absolutely continuous with respect to Lebesgue measure in $X\times B_1$, having therefore finite Boltzmann-Vlasov entropy, at infinity the asymptotic measure is singular with respect to the Lebesgue one.

5. Appendix

5.1. **Proof of Lemma 3.10.** We remark that (3.15) and (3.18) imply

$$\sum_{j=1}^N b_{i,j}\left(\mathbf{q}\left(t\right)\right) = 0\ ,\quad i=1,..,N\ . \tag{5.1}$$

We write $a_{i,j}(\mathbf{q}(t))$ defined in (3.16) as

$$a_{i,j}(\mathbf{q}(t)) = a_{i,j}(\mathbf{q}^0) + \int_0^1 ds\frac{d}{ds}a_{i,j}((1-s)\mathbf{q}^0 + s\mathbf{q}(t)). \tag{5.2}$$

We can therefore decompose $A(\mathbf{q}(t))$ as

$$A(\mathbf{q}(t)) = A(\mathbf{q}^0) + B(\mathbf{q}(t))\ , \tag{5.3}$$

where $B(\mathbf{q}(t)) = \{b_{i,j}(\mathbf{q}(t))\mathbb{I}_d\}_{i,j=1,..,N}$ and

$$b_{i,j}(\mathbf{q}(t)) = \int_0^1 ds\frac{d}{ds}a_{i,j}((1-s)\mathbf{q}^0_N + s\mathbf{q}_N(t)). \tag{5.4}$$

Making use of the isomorphism $\mathcal{F}$ introduced in (3.23), it is enough to consider the matrix $\tilde{B}\left(\mathbf{q}\left(t\right)\right):\mathbb{R}^{N}\longrightarrow\mathbb{R}^{N}$ since

$$\|B(\mathbf{q}(t))\| \leq \left\|\tilde{B}\left(\mathbf{q}\left(t\right)\right)\right\|_{\infty} = \max_{i=1,..,N}\sum_{j=1}^{N}\left|b_{i,j}(\mathbf{q}(t))\right| \ . \tag{5.5}$$

Therefore, setting

$$x_{i,j}(s,t) := (1-s)(q_i^0 - q_j^0) + s\left(q_i(t) - q_j(t)\right) \ , \quad i,j = 1,..,N \ , \tag{5.6}$$

we have

$$\begin{aligned}
\frac{d}{ds}a_{i,j}((1-s)\mathbf{q}^0 + s\mathbf{q}(t)) &= \frac{d}{ds}\left(\frac{U_R((1-s)(q_i^0-q_j^0)+s(q_i(t)-q_j(t)))}{\sum_{k=1}^N U_R((1-s)(q_i^0-q_k^0)+s(q_i(t)-q_k(t)))}\right) \\
&= \frac{d}{ds}\left(\frac{U_R(x_{i,j}(s,t))}{\sum_{k=1}^N U_R(x_{i,k}(s,t))}\right) \\
&= \frac{\nabla U_R(x_{i,j}(s,t))\cdot\left[-(q_i^0-q_j^0)+q_i(t)-q_j(t)\right]}{\sum_{k=1}^N U_R(x_{i,k}(s,t))} \\
&- \frac{U_R(x_{i,j}(s,t))\sum_{k=1}^N \nabla U_R(x_{i,k}(s,t))\cdot\left[-(q_i^0-q_k^0)+q_i(t)-q_k(t)\right]}{\left(\sum_{k=1}^N U_R(x_{i,k}(s,t))\right)^2} \ .
\end{aligned} \tag{5.7}$$

Hence, since by definition of $U_R, \sup_{x\in\mathbb{R}^d}|U_R(x)| \leq U_R\left(0\right), \forall i = 1,..,N,$

$$\begin{aligned}
\sum_{j=1}^N |b_{i,j}(\mathbf{q}(t))| &\leq 2\frac{\sum_{j=1}^N |\nabla U_R(x_{i,j}(s,t))|\left|-(q_i^0-q_j^0)+q_i(t)-q_j(t)\right|}{\sum_{k=1}^N U_R(x_{i,k}(s,t))} \\
&\leq 2\frac{N}{U_R\left(0\right)}\sup_{x\in\mathbb{R}^d}|\nabla U_R(x)|\max_{i,j\in\{1,..,N\}}\left|-(q_i^0-q_j^0)+q_i(t)-q_j(t)\right| \ .
\end{aligned} \tag{5.8}$$

5.2. **Proof of Theorem 4.5.** We follow [S] Theorem 5.1 and divide the proof in two steps.

Step 1 We start proving (4.36). Assume that ν_t and μ_t solve (5.28). We have, by the triangular inequality, that

$$\begin{aligned}
d_{b\mathcal{L}}(\nu_t,\mu_t) &= d_{b\mathcal{L}}(\nu_0\circ T_{0,t}[\nu.],\mu_0\circ T_{0,t}[\mu.]) \\
&\leq d_{b\mathcal{L}}(\mu_0\circ T_{0,t}[\nu.],\mu_0\circ T_{0,t}[\mu.]) + d_{b\mathcal{L}}(\mu_0\circ T_{0,t}[\nu.],\nu_0\circ T_{0,t}[\nu.]) \ .
\end{aligned} \tag{5.9}$$

Denote by $w=(x,v), V(\mu.)_s(w) = (v(s), A(x(s),\mu_s) - v(s))$ the vector field on the right hand side of (4.32). The second term can be bounded as

$$\begin{aligned}
d_{b\mathcal{L}}(\mu_0\circ T_{0,t}[\nu.],\nu_0\circ T_{0,t}[\nu.]) &= e^{Lt}\sup_{f\in\mathcal{D}}\left|\int_{\mathcal{T}_D\times B_1}[d\mu_0 - d\nu_0]\left(e^{-Lt}f\circ T_{t,0}[\nu.]\right)\right| \\
&\leq e^{Lt}d_{b\mathcal{L}}(\mu_0,\nu_0)
\end{aligned} \tag{5.10}$$

where L is the Lipschitz constant of $V(\mu_\cdot)_s(\cdot)$. Notice that the Lipschitz bound of $V(\mu_\cdot)_s(\cdot)$ can be easily derived from the Lipschitz bound of $A(\cdot,\mu_\cdot)$. We get (5.10) if we can show, since $f \in \mathcal{D}$, that $e^{-Lt} f \circ T_{t,0}[\nu_\cdot]$ is Lipschitz continuous with constant one and therefore it belongs to $\mathcal{D}$. Let $w(t) = (x(t), v(t))$ be the solution of (4.32) with initial condition $w_0 = (x_0, v_0)$ and let $\tilde{w}(t)$ be the solution of (4.32) with initial condition $\tilde{w}_0 = (\tilde{x}_0, \tilde{v}_0)$, then we need to show that

$$|f(w(t)) - f(\tilde{w}(t))| \le C(t)|w_0 - \tilde{w}_0| \ , \tag{5.11}$$

with $C(t) \le e^{Lt}$. Writing

$$w(t) = w_0 + \int_0^t V(\mu_\cdot)_s(w(s)) \tag{5.12}$$

and

$$\tilde{w}(t) = \tilde{w}_0 + \int_0^t V(\mu_\cdot)_s(\tilde{w}(s)) \ , \tag{5.13}$$

since $f \in \mathcal{D}$, we have

$$|f(w(t)) - f(\tilde{w}(t))| \le |w(t) - \tilde{w}(t)| \ . \tag{5.14}$$

Furthermore,

$$\begin{aligned} |w(t) - \tilde{w}(t)| &\le |w_0 - \tilde{w}_0| + \int_0^t |V(\mu_\cdot)_s(w(s)) - V(\mu_\cdot)_s(\tilde{w}(s))| ds \\ &\le |w_0 - \tilde{w}_0| + L \int_0^t |w(s) - \tilde{w}(s)| ds \ . \end{aligned} \tag{5.15}$$

By the Gronwall's inequality

$$|w(t) - \tilde{w}(t)| \le e^{Lt}|w_0 - \tilde{w}_0| \tag{5.16}$$

proving $e^{-Lt} f \circ T_{t,0}[\nu_\cdot] \in \mathcal{D}$ and so (5.10). We are then left with the estimate the other term in (5.9) which, since $f \in \mathcal{D}$,

$$\begin{aligned} d_{b\mathcal{L}}(\mu_0 \circ T_{0,t}[\nu_\cdot], \mu_0 \circ T_{0,t}[\mu_\cdot]) &= \sup_{f \in \mathcal{D}} \left| \int_{\mathcal{T}_D \times B_1} d\mu_0 \left\{ f \circ T_{t,0}[\nu_\cdot] - f \circ T_{t,0}[\mu_\cdot] \right\} \right| \\ &\le \int_{\mathcal{T}_D \times B_1} \mu_0\,(dw)\, |T_{t,0}[\nu_\cdot] w - T_{t,0}[\mu_\cdot] w| =: \lambda(t) \end{aligned} \tag{5.17}$$

where $T_{t,0}[\nu.]$ and $T_{t,0}[\mu.]$ are both solutions of the equation (4.32) with the same initial conditions but with different vector fields. We have

$$\begin{aligned}\lambda(t) &= \int_{\mathcal{T}_D\times B_1} \mu_0(dw)\,|\{T_{t,0}[\nu.]w - T_{t,0}[\mu.]w|\} \\ &= \int_{\mathcal{T}_D\times B_1} \mu_0(dw)\left|\int_0^t dsV(\nu.)_s(T_{s,0}[\nu.]w) - \int_0^t dsV(\mu.)_s(T_{s,0}[\mu.]w)\right| \\ &\leq \int_{\mathcal{T}_D\times B_1} \mu_0(dw)\left|\int_0^t ds\,\{V(\nu.)_s(T_{s,0}[\nu.]w) - V(\nu.)_s(T_{s,0}[\mu.]w)\}\right| \\ &+ \int_{\mathcal{T}_D\times B_1} \mu_0(dw)\left|\int_0^t ds\,\{V(\mu.)_s(T_{s,0}[\mu.]w) - V(\nu.)_s(T_{s,0}[\mu.]w)\}\right| \ .\end{aligned} \tag{5.18}$$

The first term of (5.18) can be estimated by the Lipschtiz property of the vector field

$$\begin{aligned}&\int_{\mathcal{T}_D\times B_1} \mu_0(dw)\left|\int_0^t ds\,\{V(\nu.)_s(T_{s,0}[\nu.]w) - V(\nu.)_s(T_{s,0}[\mu.]w)\}\right| \\ &\leq L\int_{\mathcal{T}_D\times B_1} \mu_0(dw)\int_0^t ds\,|T_{s,0}[\nu.]w - T_{s,0}[\mu.]w|| \\ &= L\int_0^t ds\int_{\mathcal{T}_D\times B_1} \mu_0(dw)\,|T_{s,0}[\nu.]w - T_{s,0}[\mu.]w|| = L\int_0^t \lambda(s)ds \ .\end{aligned} \tag{5.19}$$

For the second term of (5.18) we have

$$\begin{aligned}&\int_{\mathcal{T}_D\times B_1} \mu_0(dw)\left|\int_0^t ds\,\{V(\mu.)_s(T_{s,0}[\mu.]w) - V(\nu.)_s(T_{s,0}[\mu.]w)\}\right| \\ &\leq \int_{\mathcal{T}_D\times B_1} \mu_0(dw)\int_0^t ds\,|V(\mu.)_s\,(T_{s,0}[\mu.]w) - V(\nu.)_s\,(T_{s,0}[\mu.]w)| \\ &= \int_0^t ds\int_{\mathcal{T}_D\times B_1} \mu_0(dw)\,|V(\mu.)_s(T_{s,0}[\mu.]w) - V(\nu.)_s(T_{s,0}[\mu.]w)| \\ &= \int_0^t ds\int_{\mathcal{T}_D\times B_1} \mu_s(dw)\,|V(\mu.)_s(w) - V(\nu.)_s(w)| \ .\end{aligned} \tag{5.20}$$

But,

$$\begin{aligned}|V(\mu.)_s(w) - V(\nu.)_s(w)| &\leq |A(x,\mu_s) - A(x,\nu_s)| \\ &\leq \left|\frac{\int_{\mathcal{T}_D\times B_1} U_R(x-y)u\mu_s(dy,du) - \int_{\mathcal{T}_D\times B_1} U_R(x-y)u\nu_s(dy,du)}{\int_{\mathcal{T}_D\times B_1} U_R(x-y)\mu_s(dy,du)}\right| \\ &+ \left|\frac{\int_{\mathcal{T}_D\times B_1} U_R(x-y)\mu_s(dy,du) - \int_{\mathcal{T}_D\times B_1} U_R(x-y)\nu_s(dy,du)}{\int_{\mathcal{T}_D\times B_1} U_R(x-y)\mu_s(dy,du)}\right|\end{aligned} \tag{5.21}$$

Since for any measure $\nu \in \mathcal{M}$, $\int_{\mathcal{T}_D \times B_1} U_R(x-y)\nu_s(dy,du) \geq \inf_{x\in\mathcal{T}_D} U_R(x) = a$, we have

$$\left| \frac{\int_{\mathcal{T}_D \times B_1} U_R(x-y)\mu_s(dy,du) - \int_{\mathcal{T}_D \times B_1} U_R(x-y)\nu_s(dy,du)}{\int_{\mathcal{T}_D \times B_1} U_R(x-y)\mu_s(dy,du)} \right| \leq \frac{\sup_{x\in\mathcal{T}_D} |\nabla U_R(x)| + \sup_{x\in\mathcal{T}_D} U_R(x)}{a} d_{b\mathcal{L}}(\mu_s,\nu_s) \tag{5.22}$$

and

$$\begin{aligned} &\left| \frac{\int_{\mathcal{T}_D \times B_1} U_R(x-y)u\mu_s(dy,du) - \int_{\mathcal{T}_D \times B_1} U_R(x-y)u\nu_s(dy,du)}{\int_{\mathcal{T}_D \times B_1} U_R(x-y)\mu_s(dy,du)} \right| \\ &\leq \sum_{i=1}^d \left| \frac{\int_{\mathcal{T}_D \times B_1} U_R(x-y)u_i\mu_s(dy,du) - \int_{\mathcal{T}_D \times B_1} U_R(x-y)u_i\nu_s(dy,du)}{\int_{\mathcal{T}_D \times B_1} U_R(x-y)\mu_s(dy,du)} \right| \\ &\leq d\frac{\sup_{x\in\mathcal{T}_D} |\nabla U_R(x)| + \sup_{x\in\mathcal{T}_D} U_R(x)}{a} d_{b\mathcal{L}}(\mu_s,\nu_s) \ . \end{aligned} \tag{5.23}$$

Therefore,

$$|V(\mu_\cdot)_s(w) - V(\nu_\cdot)_s(w)| \leq 2d\frac{\sup_{x\in\mathcal{T}_D} |\nabla U_R(x)| + \sup_{x\in\mathcal{T}_D} U_R(x)}{a} d_{b\mathcal{L}}(\mu_s,\nu_s) = \frac{c_0}{a} d_{b\mathcal{L}}(\mu_s,\nu_s) \ , \tag{5.24}$$

where we have set $c_0 := 2d(\sup_{x\in\mathcal{T}_D} |\nabla U_R(x)| + \sup_{x\in\mathcal{T}_D} U_R(x))$. It is essential that $a > 0$. This is the case for interactions considered in the Lemmata 4.1 and 4.3 once the system is confined on the torus $\mathcal{T}_D$[4]. Thus, by (5.18), (5.19), (5.20) and (5.21) we have that

$$\lambda(t) \leq L\int_0^t \lambda(s)ds + \frac{c_0}{a}\int_0^t d_{b\mathcal{L}}(\mu_s,\nu_s)ds \ . \tag{5.25}$$

Hence, since by (5.18) $\lambda(0) = 0$ we obtain

$$\lambda(t) \leq \frac{c_0}{a}\int_0^t e^{L(t-s)} d_{b\mathcal{L}}(\mu_s,\nu_s)ds \ . \tag{5.26}$$

Taking in account (5.9), (5.10), (5.17) and (5.26) we get

$$d_{b\mathcal{L}}(\nu_t,\mu_t) \leq e^{Lt} d_{b\mathcal{L}}(\mu_0,\nu_0) + \frac{c_0}{a}\int_0^t e^{L(t-s)} d_{b\mathcal{L}}(\mu_s,\nu_s)ds \ . \tag{5.27}$$

Applying the Gronwall's lemma we get bound (4.36).

Step 2 To prove the existence of a solution for the fixed point equation

$$\mu_t = \mu_0 \circ T_{0,t}[\mu_\cdot] \ , \tag{5.28}$$

[4]In the case where U_R is with compact support and M is replaced by M^ϵ we have that

$$\inf_{x\in X}(U_R \star \nu)(x) + \epsilon \geq \inf_{x\in X} U_R(x) + \epsilon \geq \epsilon.$$

In this case X can be either $\mathbb{R}^d$ or $\mathcal{T}_D$.

we use the Banach fixed point theorem. Let μ be the initial condition. To every curve $t \to \mu_t, \mu_0 = \mu$ we obtain the solution curve

$$t \to \mu \circ T_{0,t}[\mu.] \tag{5.29}$$

Let us denote this map $\mathcal{F} : C_{\mathcal{M}} \to C_{\mathcal{M}}$, where $C_{\mathcal{M}}$ is the space of weakly continuous function $[0, T] \to \mathcal{M}$ with $\mu_0 = \mu$. We equip $C_{\mathcal{M}}$ with the metric

$$d_\alpha(\mu(\cdot), \nu(\cdot)) = \sup_{t\in[0,T]} \left[e^{-\alpha t} d_{b\mathcal{L}}(\nu_t, \mu_t)\right] \ , \tag{5.30}$$

for some $\alpha > 0$ which will be suitably chosen. Since $(\mathcal{M}, d_{b\mathcal{L}})$ is a complete metric space, so is $(C_{\mathcal{M}}, d_\alpha)$. Now from Step 1 we have

$$d_{b\mathcal{L}}(\nu_t, \mu_t) = d_{b\mathcal{L}}(\mathcal{F}(\mu(\cdot))(t), \mathcal{F}(\nu(\cdot))(t)) \le \frac{c_0}{a} \int_0^t e^{L(t-s)} d_{b\mathcal{L}}(\mu_s, \nu_s) ds \tag{5.31}$$

and therefore

$$d_\alpha(\mathcal{F}(\mu(\cdot))(t), \mathcal{F}(\nu(\cdot))(t)) \le \frac{c_0}{a(\alpha - L)} d_\alpha(\mu(\cdot), \nu(\cdot)) \tag{5.32}$$

for $\alpha > L$. By a suitable choice of α this proves that $\mathcal{F}$ is a contraction. □

We recall the integral form of the Gronwall's inequality.

Lemma 5.1. *Let L and T positive numbers, f and η in $C([0, T], \mathbb{R})$. If, for all $t \in [0, T]$,*

$$\eta(t) \le L \int_0^t \eta(s) ds + f(t) \ , \tag{5.33}$$

then

$$\eta(t) \le f(t) + L \int_0^t e^{L(t-s)} f(s) ds \ . \tag{5.34}$$

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