Continuous interfaces with disorder: Even strong pinning is too weak in 2 dimensions

Christof Külske *and Enza Orlandi†

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Abstract

We consider statistical mechanics models of continuous height effective interfaces in the presence of a delta-pinning of strength ε at height zero. There is a detailed mathematical understanding of the depinning transition in 2 dimensions without disorder. Then the variance of the interface height w.r.t. the Gibbs measure stays bounded uniformly in the volume for $\varepsilon > 0$ and diverges like $|\log \varepsilon|$ for $\varepsilon \downarrow 0$. How does the presence of a quenched disorder term in the Hamiltonian modify this transition? We show that an arbitrarily weak random field term is enough to beat an arbitrarily strong delta-pinning in 2 dimensions and will cause delocalization. The proof is based on a rigorous lower bound for the overlap between local magnetizations and random fields in finite volume. In 2 dimensions it implies growth faster than the volume which is a contradiction to localization. We also derive a simple complementary inequality which shows that in higher dimensions the fraction of pinned sites converges to one with $\varepsilon \uparrow \infty$.

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1 Introduction

1.1 The setup

The study of lattice effective interface models, continuos and discrete, has a long tradition in statistical mechanics [17, 5, 11, 12, 16, 2, 3, 4]. The model we study is given in terms of variables $\varphi_i \in \mathbb{R}$ which, physically speaking, are thought to represent height variables of a random surface at the sites $i \in \mathbb{Z}^d$. Mathematically, they are just continuous unbounded (spin) variables. The model is defined in terms of: a pair potential V, a quenched random term, and a pinning term at interface height zero. More precisely, we are interested in the behavior of the quenched finite-volume Gibbs measures in a finite volume $\Lambda \subset \mathbb{Z}^d$ with fixed boundary condition at height zero, given by

$$\mu_{\varepsilon,\Lambda}[\eta](d\varphi_\Lambda) = \frac{e^{-\frac{1}{4d}\sum_{\langle i,j\rangle\in\Lambda} V(\varphi_i-\varphi_j) - \frac{1}{4d}\sum_{i\in\Lambda, j\in\Lambda^c, |i-j|=1} V(\varphi_i) + \sum_{i\in\Lambda}\eta_i\varphi_i} \prod_{i\in\Lambda}(d\varphi_i + \varepsilon\delta_0(d\varphi_i))}{Z_{\varepsilon,\Lambda}[\eta]}, \tag{1}$$

*University of Groningen, Department of Mathematics and Computing Sciences, Blauwborgje 3, 9747 AC Groningen, The Netherlands kuelske@math.rug.nl, http://www.math.rug.nl/~kuelske/

†Dipartimento di Matematica, Universitá degli Studi Roma Tre, Largo San Leonardo Murialdo, 1, 00146 Roma, ITALY, orlandi@mat.uniroma3.it , http://www.mat.uniroma3.it/users/orlandi/

where the partition function $Z_{\varepsilon,\Lambda}[\eta]$ denotes the normalization constant that turns the last expression into a probability measure. The Dirac-measures at the interface height zero are multiplied with the parameter ε, having the meaning of a coupling strength. The disorder configuration $\eta = (\eta_i)_{i\in\mathbb{R}^d}$ denotes an arbitrary fixed configuration of external fields, modelling a "quenched" (or frozen) random environment. What do we expect for such a model? Recall that the variance of a free massless interface in a finite box diverges like the logarithm of the sidelength when there are no random fields. The corresponding interface is said to be delocalized, while it remains finite for $d \geq 3$. See [17] for a review on the subject. Adding an arbitrarily small pinning ε (without disorder) always localizes the interface uniformly in the volume, with the variance of the field behaving on the scale $|\log\varepsilon|$ when ε tends to zero. Indeed, there is a beautiful and complete mathematical understanding of the model without disorder, in the case of both Gaussian and uniformly elliptic potentials (see [1, 9]) with precise asymptotics as the pinning force tends to zero. These results follow from the analysis of the distribution of pinned sites and the random walk (arising from the random walk representation of the covariance of the φ_i's) with killing at these sites. In this sense there is already a random system that needs to be analyzed even without disorder in the original model.

What do we expect if we turn on randomness in the model and add the η_i's ? Let us review first what we know about the same model without a pinning force. In $d = 2$ we recently proved the deterministic lower bound $\mu_{\Lambda_N}[\eta](|\varphi_0| \geq t\sqrt{\log L}) \geq c\exp(-ct^2)$ uniformly over all fixed disorder configuration η, for general potentials V (assuming not too slow growth at infinity) [15]. So, it is not possible to stabilize an interface by cleverly choosing a random field configuration (one could think e.g. that this might be possible with a staggered field). As this result holds at any arbitrary fixed configuration here we don't need any assumptions on the distribution of random fields. This result clearly excludes the existence of an infinite-volume Gibbs measure describing a two dimensional interface in infinite volume in the presence of random fields. In another paper [10] the question of existence of *gradient Gibbs measures* (Gibbs distributions of the increments of the interface) in infinite volume was raised. Note that while interface states may not exist in the infinite volume such gradient states may very well exist, as the example of the two-dimensional Gaussian free field shows, by computation. (For existence beyond the Gaussian case which is far less trivial, see [12, 13].) It was proved in [10] that there are no such gradient Gibbs measures in the random model in dimension $d = 2$.

Now, turn to the full model in $d = 2$. In view of the localization taking place at any positive pinning force ε without disorder, a natural guess might be that with disorder at least at very large ε there would be pinning. However, we show as a result of the present paper that this is not the case, somewhat to our own surprise, and an arbitrarily strong pinning does not suffice to keep the interface bounded.

1.2 Main results

Delocalization in $d = 2$ - superextensivity of the overlap

Denote by $\Lambda_L \subset \mathbb{Z}^2$ the square of sidelength $2L + 1$ centered at the origin. Denote by $\mathbb{E}(\cdot)$ the mean with respect to the disorder η.

In this subsection we consider the disorder average of the overlap between local magnetizations and random fields in Λ_L showing that it grows faster than the volume. This in particular implies that in two dimensions there is never pinning, for arbitrarily weak random field and arbitrarily large pinning forces ε. Here is the result.

Theorem 1.1 *Assume that* $V(t) = V(-t)$ *is a twice differentiable potential,* $\sup_t V''(t) \leq 1$*,* $\liminf_{|t|\uparrow\infty} \frac{\log V(t)}{\log |t|} > 1$*, and let* η_i *be symmetrically distributed, i.i.d. with finite second moment. Let* $d = 2$*. Then there is a constant* $a > 0$*, independent of the distribution of the random fields and the pinning strength* $\varepsilon \geq 0$*, such that*

$$\liminf_{L\uparrow\infty} \frac{1}{L^2 \log L} \sum_{i\in\Lambda_L} \mathbb{E}\Big(\eta_i \mu_{\varepsilon,\Lambda_L}[\eta](\varphi_i)\Big) \geq a\,\mathbb{E}(\eta_0^2). \tag{2}$$

Note that the growth condition on V includes the quadratic case and ensures the finiteness of the integrals appearing in (1) for all arbitrarily fixed choices of η, even at $\varepsilon = 0$.

Generalizations to interactions that are non-nearest neighbor are obvious; all results go through e.g. for finite range and we skip them in this presentation for the sake of simplicity. We like to exhibit the case of Gaussian random fields (and not necessarily Gaussian potential V) since the bound acquires a form that looks even more striking because it becomes independent of the size of the variance of the η_i's (as long as this is strictly positive).

Corollary 1.2 *Let us assume that the random fields* η_i *have an i.i.d. Gaussian distribution with mean zero and strictly positive variance of arbitrary size.*

Then, with the same constant a *as above, we have the bound*

$$\liminf_{L\uparrow\infty} \frac{1}{L^2 \log L} \sum_{i\in\Lambda_L} \mathbb{E}\Big(\mu_{\varepsilon,\Lambda_L}[\eta](\varphi_i^2) - \mu_{\varepsilon,\Lambda_L}[\eta](\varphi_i)^2\Big) \geq a > 0 \tag{3}$$

for any $0 \leq \varepsilon < \infty$*.*

The equation (3) follows from (2) by partial integration w.r.t. the Gaussian disorder average (transforming the overlap into the variance of the φ_i's).

Note that, even in the unpinned case of $\varepsilon = 0$, Theorem 1.1 is not entirely trivial in the case of general potentials V. Here it provides an alternative simple way to see the delocalization in the presence of random fields (while the explicit lower bound on the tails of [15] provides more information).

Lower bound on overlap in $d \geq 3$

The analogue of Theorem 1.1 for higher dimensions is the following.

Theorem 1.3 *Let* $d \geq 3$ *and let* $\varepsilon \geq 0$ *be arbitrary and assume the same conditions on* V *and* η_i *as in Theorem 1.1.*

There are positive constants $B_1, B_2 < \infty$*, independent of the distribution of the random fields and the pinning strength* $\varepsilon \geq 0$*, such that*

$$\liminf_{L\uparrow\infty} \frac{1}{L^d} \sum_{i\in\Lambda_L} \mathbb{E}\Big(\eta_i \mu_{\varepsilon,\Lambda_L}[\eta](\varphi_i)\Big) \geq \frac{\mathbb{E}(\eta_0^2)\,(-\Delta^{-1})_{0,0}}{2} - \log(B_1 + B_2\varepsilon), \tag{4}$$

where the positive constant $(-\Delta^{-1})_{0,0}$ *is the diagonal element of the inverse of the infinite-volume lattice Laplace operator whose existence is guaranteed in* $d \geq 3$*.*

We recall that the infinite-volume lattice Laplace operator $\Delta = (\Delta_{i,j})_{i,j\in\mathbb{Z}^d}$ is the matrix with entries $\Delta_{i,j} = -1$ if $i = j$, $\Delta_{i,j} = \frac{1}{2d}$ if i and j are nearest neighbors and zero otherwise. The existence of its inverse in $d \geq 3$ follows e.g. from the random walk expansion [6, 14].

Lower bound on the pinned volume in $d \geq 3$

We complement the previous lower bounds on the overlaps which are depinning-type of results by a pinning-type result. It is a lower bound on the disorder average of the quenched Gibbs-expectation of the fraction of pinned sites. Contrary to before, when we needed an upper bound on the interaction potential V, now we assume a lower bound on V.

Theorem 1.4 *Let $d \geq 3$. Assume that $\inf_t V''(t) = c_- > 0$ and let η_i be symmetrically distributed, i.i.d. with finite second moment.*

Then there exist dimension-dependent constants $C_1, C_2 > 0$, independent of the distribution of the disorder, such that, for all $\varepsilon > 1$ and for all volumes Λ, the disorder average of the fraction of pinned sites obeys the estimate

$$\frac{1}{|\Lambda|}\sum_{i\in\Lambda}\mathbb{E}\Big(\mu_{\varepsilon,\Lambda}[\eta](\varphi_i = 0)\Big) \geq 1 - \frac{C_1 + C_2\mathbb{E}(\eta_0^2)}{\log\varepsilon}. \tag{5}$$

This shows pinning for the large ε regime in the "thermodynamic sense" that the fraction of pinned sites can be made arbitrarily close to one, uniformly in the volume. As usual this result does not allow to make statement about the Gibbs measure itself.

The proofs follows from "thermodynamic reasoning". The first "depinning-type" result follows from taking the log of the partition function and differentiating and integrating back w.r.t. the coupling strength of the random fields. Exploiting the linear form of the random fields, convexity, comparison of non-Gaussian with the Gaussian partition functions, and asymptotics of Green's functions the results follow, see Chapter 2.

2 Proof of Depinning-type results

The estimates in formulas (2), (3), and (4) are immediate consequences of the next fixed-disorder estimate stated in Proposition 2.1. To formulate it, we denote by $\Delta_\Lambda = (\Delta_{\Lambda;i,j})_{i,j\in\Lambda}$ the Dirichlet Laplacian in the volume $\Lambda\subset\mathbb{Z}^d$, that is the matrix with entries $\Delta_{\Lambda;i,j} = -1$ if $i = j$, $\Delta_{\Lambda;i,j} = \frac{1}{2d}$ if i and j are nearest neighbors in Λ and zero otherwise. Λ may be finite or infinite here.

For convenience of the reader we recall that the matrix inverse has a well-known probabilistic interpretation, see for instance [6, 14]. Indeed the matrix $(-\Delta_\Lambda)^{-1}_{i,j}, i, j \in \Lambda$ is the Green's function of the simple random walk X killed as it exits Λ:

$$(-\Delta_\Lambda)^{-1}_{i,j} := \sum_{n\geq 0}\mathbb{P}_i[X_n = j, \tau_\Lambda > n] \quad i \in \Lambda,$$

with $\tau_\Lambda := \min\{n : X_n \notin \Lambda\}$. So, the matrix element $(-\Delta_\Lambda)^{-1}_{i,j}$ is the mean expected number of visits to $j \in \Lambda$ starting from $i \in \Lambda$ up to the killing time τ_Λ. It is well known that if $\Lambda \neq \mathbb{Z}^d$ then $(-\Delta_\Lambda)^{-1}_{i,j}$ is finite in all dimensions. When $\Lambda = \mathbb{Z}^d$, $(-\Delta_{\mathbb{Z}^d})^{-1}_{i,j}$ is finite for $d \geq 3$, see [14] for further details. We denote $(-\Delta)^{-1}_{i,j} := (-\Delta_{\mathbb{Z}^d})^{-1}_{i,j}$. With this in mind we have the following proposition.

Proposition 2.1 *Assume that* $\sup_t V''(t) \leq 1$, $\liminf_{|t|\uparrow\infty} \frac{\log V(t)}{\log |t|} > 1$, *and let* η_i *be symmetrically distributed, i.i.d. with finite second moment.*

For any dimension d, *there are constants* $C_{nG,d} < \infty$ *and* $c_{G,d} > 0$ *such that, for all fixed configurations of local fields* η, *we have*

$$\frac{1}{2}\sum_{i,j\in\Lambda} (-\Delta_\Lambda)^{-1}_{i,j}\eta_i\eta_j - |\Lambda| \log \frac{C_{nG,d}+\varepsilon}{c_{G,d}} \leq \sum_{i\in\Lambda} \eta_i \mu_{\varepsilon,\Lambda}[\eta](\varphi_i). \tag{6}$$

Proof: Let us see what comes out when we differentiate and integrate back the free energy in finite volume w.r.t. strength of the random fields,

$$\frac{d}{dh}\log Z_{\varepsilon,\Lambda}[h\eta] = \sum_{i\in\Lambda} \eta_i \mu_{\varepsilon,\Lambda}[h\eta](\varphi_i). \tag{7}$$

At every fixed η, this quantity is a monotone function of h. This is seen by another differentiation w.r.t. h which produces the variance of the overlap q between local magnetizations and random fields w.r.t. to the Gibbs measures, that is

$$\begin{aligned} \left(\frac{d}{dh}\right)^2 \log Z_{\varepsilon,\Lambda}[h\eta] = \mu_{\varepsilon,\Lambda}[h\eta](q^2) - \big(\mu_{\varepsilon,\Lambda}[h\eta](q)\big)^2 \geq 0 \text{ where} \\ q = \sum_{i\in\Lambda} \eta_i\varphi_i. \end{aligned} \tag{8}$$

As a consequence we have

$$\log \frac{Z_{\varepsilon,\Lambda}[\eta]}{Z_{\varepsilon,\Lambda}[0]} = \sum_{i\in\Lambda}\int_0^1 dh\ \eta_i\mu_{\varepsilon,\Lambda}[h\eta](\varphi_i) \leq \sum_{i\in\Lambda} \eta_i\ \mu_{\varepsilon,\Lambda}[\eta](\varphi_i). \tag{9}$$

Dropping the pinning term and retaining only the terms in the partition function corresponding to the Lebesgue measure we have the lower bound

$$Z_{\varepsilon,\Lambda}[\eta] \geq Z_{\varepsilon=0,\Lambda}[\eta]. \tag{10}$$

We may assume that $V(0) = 0$ since a constant drops out of the Gibbs-expectation, without changing the hypothesis over V. Then, from the upper bound $\sup_t V''(t) \leq 1$, follows $V(t) \leq \frac{t^2}{2}$. Applying this estimate to the exponent of the partition function we immediately obtain

$$Z_{\varepsilon=0,\Lambda}[\eta] \geq Z^{\text{Gauss}}_{\varepsilon=0,\Lambda}[\eta], \tag{11}$$

where we have denoted by $Z^{\text{Gauss}}_{\varepsilon=0,\Lambda}[\eta]$ the Gaussian partition function with potential $V(t) = \frac{t^2}{2}$. This Gaussian partition function can be easily computed: Using the linearity of the random external field η, the Gaussian integral produces the quadratic form of the covariance matrix $(-\Delta_\Lambda)^{-1}$ and the partition function without disorder:

$$Z^{\text{Gauss}}_{\varepsilon=0,\Lambda}[\eta] = \exp\Big(\frac{1}{2}\sum_{i,j\in\Lambda}(-\Delta_\Lambda)^{-1}_{i,j}\eta_i\eta_j\Big) Z^{\text{Gauss}}_{\varepsilon=0,\Lambda}[0]. \tag{12}$$

The Gaussian partition function in zero field is given by a determinant, $Z^{\text{Gauss}}_{\varepsilon=0,\Lambda}[0] = (2\pi)^{\frac{|\Lambda|}{2}}\sqrt{\det(-\Delta_\Lambda)^{-1}} = (2\pi)^{\frac{|\Lambda|}{2}} \exp -\frac{1}{2}\mathrm{Tr}\log(-\Delta_\Lambda)$ which, by expansion of the logarithm, again has a standard random walk representation

$$
\begin{aligned}
\mathrm{Tr}\log(-\Delta_\Lambda) &= -\sum_{i\in\Lambda}\sum_{n\geq 0}\frac{1}{n}\mathbb{P}_i[X_n = i, \tau_\Lambda > n] \\
&\geq -|\Lambda|\sum_{n\geq 0}\frac{1}{n}\mathbb{P}_0[X_n = 0].
\end{aligned}
\tag{13}
$$

The equality is well known: it is a probabilistic rewriting of formula (6.18) of [7]. The inequality is obtained by dropping the killing outside the volume, and provides for us the uniformity in the volume Λ. Indeed, the quantity on the r.h.s. is finite even in all dimensions $d \geq 1$ (see [14]) which proves that there is a constant such that $Z^{\text{Gauss}}_{\varepsilon=0,\Lambda}[0] \geq c^{|\Lambda|}_{\mathrm{G},d}$. Using this, along with (12) we have arrived at

$$
Z_{\varepsilon,\Lambda}[\eta] \geq \exp\Big(\frac{1}{2}\sum_{i,j\in\Lambda}(-\Delta_\Lambda)^{-1}_{i,j}\eta_i\eta_j\Big)c^{|\Lambda|}_{\mathrm{G},d}.
$$

This is the first ingredient to bound the l.h.s. of (9) from below.

Let us now explain how to get an upper bound on $Z_{\varepsilon,\Lambda}[0]$. We expand the integration measure in $Z_{\varepsilon,\Lambda}[0]$ as a sum $\prod_{i\in\Lambda}(d\varphi_i+\varepsilon\delta_0(d\varphi_i)) = \sum_{A\subset\Lambda}\varepsilon^{|A|}\prod_{i\in A}\delta_0(d\varphi_i))\prod_{j\in\Lambda\setminus A}d\varphi_j$. This has been a standard trick in all treatments of models with delta-pinning, see e.g. [1]. For our purposes we use that the lower bound on $V(t)$ implies that $\int d\varphi_i \exp\Big(-\frac{V(\varphi_i)}{4d}\Big) =: C_{\mathrm{nG},d}$ is finite. Let us choose a spanning tree for $\Lambda\subset\mathbb{Z}^d$. Let us call the edge set E. Let us single out one vertex at the boundary of Λ and call it i_0. Then, dropping all the terms in the exponent appearing in the partition function that are not corresponding to the tree edges $(i,j)\in E$, we may bound

$$
\begin{aligned}
Z_{\varepsilon=0,\Lambda}[0] &= \int e^{-\frac{1}{4d}\sum_{\langle i,j\rangle\in\Lambda}V(\varphi_i-\varphi_j)-\frac{1}{4d}\sum_{i\in\Lambda,j\in\Lambda^c,|i-j|=1}V(\varphi_i)}\prod_{i\in E}d\varphi_i \\
&\leq \int e^{-\frac{1}{4d}\sum_{(i,j)\in E}V(\varphi_i-\varphi_j)-\frac{1}{4d}V(\varphi_{i_0})}\prod_{i\in\Lambda}d\varphi_i \\
&= C^{|\Lambda|}_{\mathrm{nG},d}.
\end{aligned}
\tag{14}
$$

This is the desired upper bound. These two ingredients give

$$
\begin{aligned}
Z_{\varepsilon,\Lambda}[0] &= \sum_{A\subset\Lambda}\varepsilon^{|A|}Z_{\varepsilon=0,\Lambda\setminus A}[0] \\
&\leq \sum_{A\subset\Lambda}\varepsilon^{|A|}C^{|\Lambda\setminus A|}_{\mathrm{nG},d} = (C_{\mathrm{nG},d}+\varepsilon)^{|\Lambda|}.
\end{aligned}
\tag{15}
$$

So the desired estimate on the overlap (6) follows from (9),(12) and (15). $\square$

Theorems 1.1 and 1.3 can be easily derived from Proposition 2.1. Indeed, taking a disorder average we have

$$
\frac{\mathbb{E}(\eta_0^2)}{2}\sum_{i\in\Lambda}(-\Delta_\Lambda)^{-1}_{i,i} - |\Lambda|\log\frac{C_{\mathrm{nG},d}+\varepsilon}{c_{\mathrm{G},d}} \leq \sum_{i\in\Lambda}\mathbb{E}\Big(\eta_i\mu_{\varepsilon,\Lambda}[\eta](\varphi_i)\Big).
\tag{16}
$$

Proof of Theorem 1.1 It is known, see [14], page 40, that in $d = 2$, the asymptotics of the Green's-function in a square, at fixed i is $(-\Delta_{\Lambda_L})^{-1}_{i,i} \sim \log L$. This and (16) imply immediately (2). □

Finally let us note in passing that a constant magnetic field is always winning against an arbitrarily strong pinning, and even more strongly than a random field. Indeed, let $d \geq 2$, let $\eta_i = h \geq 0$ for all sites i and let $\varepsilon \geq 0$ be arbitrary. Then, there is a constant $c_d > 0$, independent of h and ε, such that

$$\liminf_{L\uparrow\infty} \frac{1}{L^{d+2}} \sum_{i\in\Lambda_L} \mu_{\varepsilon,\Lambda_L}[h](\varphi_i) \geq c_d h. \tag{17}$$

This again follows from Proposition 2.1, using $\sum_{i,j\in\Lambda}(-\Delta_{\Lambda_L})^{-1}_{i,j} \sim L^{d+2}$.

3 Proof of Pinning-type results

To prove the lower bound on the fraction of pinning sites in dimension $d \geq 3$ given in Theorem 1.4 we will in fact prove the following fixed-disorder lower bound:

Proposition 3.1 *Under the assumptions given in Theorem 1.4, for all finite volumes Λ and for all realizations η we have, for any ε_0, $\varepsilon > \varepsilon_0 > 0$,*

$$\begin{aligned} &\frac{1}{|\Lambda|}\sum_{i\in\Lambda}\mu_{\varepsilon,\Lambda}[\eta](\varphi_i = 0) \\ &\geq \frac{1}{\log\frac{\varepsilon}{\varepsilon_0}}\left(\log\frac{\varepsilon c_-^{\frac{d}{2}}}{\left(1+\frac{\varepsilon_0 c_-^{\frac{d}{2}}}{(2\pi)^{\frac{d}{2}}}\right)C_{G,d}} - \frac{1}{2c_-|\Lambda|}\sum_{i,j\in\Lambda}(-\Delta_\Lambda)^{-1}_{i,j}\eta_i\eta_j\right) \end{aligned} \tag{18}$$

with a constant $C_{G,d}$ defined in (26).

Proof of Theorem 1.4: The proof follows from the existence of the infinite-volume Green's-function in $d \geq 3$. Taking a disorder-expectation of (18) and setting $\varepsilon_0 = 1$ we get the thesis. □

Proof of Proposition 3.1: The proof is based on the trick to differentiate and integrate back the log of the partition function, now w.r.t. ε: Differentiation gives

$$\varepsilon\frac{d}{d\varepsilon}\log Z_{\varepsilon,\Lambda}[\eta] = \sum_{i\in\Lambda}\mu_{\varepsilon,\Lambda}[\eta](\varphi_i = 0). \tag{19}$$

We integrate this relation back, and it will be important for us to do it starting from a positive $\varepsilon_0 > 0$. So we get

$$\log\frac{Z_{\varepsilon,\Lambda}[\eta]}{Z_{\varepsilon_0,\Lambda}[\eta]} = \int_{\varepsilon_0}^{\varepsilon}\frac{d\tilde\varepsilon}{\tilde\varepsilon}\sum_{i\in\Lambda}\mu_{\tilde\varepsilon,\Lambda}[\eta](\varphi_i = 0) \leq \log\frac{\varepsilon}{\varepsilon_0}\cdot\sum_{i\in\Lambda}\mu_{\varepsilon,\Lambda}[\eta](\varphi_i = 0), \tag{20}$$

where we have used that $\sum_{i\in\Lambda}\mu_{\tilde\varepsilon,\Lambda}[\eta](\varphi_i = 0)$ is a monotone function of $\tilde\varepsilon$. Note that the integrand itself is not a monotone function. (Compare [8] for a related non-random pinning scenario, with back-integration from zero.) Expanding as before, see (15), and keeping only the contribution in the expansion where all sites are pinned we have

$$Z_{\varepsilon,\Lambda}[\eta] = \sum_{A\subset\Lambda} \varepsilon^{|A|} Z_{\varepsilon=0,\Lambda\setminus A}[\eta] \geq \varepsilon^{|\Lambda|}. \tag{21}$$

For the upper bound on the partition function of the full model (at ε_0) we first use the lower bound on the potential $V(t) \geq \frac{c_- t^2}{2}$ giving us a comparison with a Gaussian partition function with curvature c_-:

$$Z_{\varepsilon_0,\Lambda}[\eta] \leq Z^{\text{Gauss},c_-}_{\varepsilon_0,\Lambda}[\eta]. \tag{22}$$

It is a simple matter to rescale the Gaussian curvature away

$$Z^{\text{Gauss},c_-}_{\varepsilon_0,\Lambda}[\eta] = c_-{}^{-\frac{d|\Lambda|}{2}} Z^{\text{Gauss}}_{\varepsilon_0 c_-{}^{\frac{d}{2}},\Lambda}[c_-{}^{-\frac{1}{2}}\eta], \tag{23}$$

where the partition function on the r.h.s. is taken with unity curvature potential. For the Gaussian partition function we claim the upper bound (writing again in the original parameters) of the form

$$Z^{\text{Gauss}}_{\varepsilon,\Lambda}[\eta] \leq \Big(1 + \frac{\varepsilon}{(2\pi)^{\frac{d}{2}}}\Big)^{|\Lambda|} Z^{\text{Gauss}}_{\varepsilon=0,\Lambda}[\eta]. \tag{24}$$

Here is an elementary proof: We will replace successively the single-site integrations involving the Dirac measure by integrations only over the Lebesgue measure with the appropriately adjusted prefactor. Indeed, consider one site i and compute the contribution to the partition function while fixing the values of φ_j for j not equal to i. Then use that

$$\begin{aligned}
&\int \Big(d\varphi_i + \varepsilon\delta_0(d\varphi_i)\Big) \exp\Big(-\frac{\varphi_i^2}{2} + (\sum_{j\sim i} \varphi_j + \eta_i)\varphi_i\Big) \\
&= (2\pi)^{\frac{d}{2}} \exp\Big(\frac{(\sum_{j\sim i}\varphi_j + \eta_i)^2}{2}\Big) + \varepsilon \\
&\leq \Big(1 + \frac{\varepsilon}{(2\pi)^{\frac{d}{2}}}\Big)(2\pi)^{\frac{d}{2}} \exp\Big(\frac{(\sum_{j\sim i}\varphi_j + \eta_i)^2}{2}\Big) \\
&= \Big(1 + \frac{\varepsilon}{(2\pi)^{\frac{d}{2}}}\Big) \int d\varphi_i \exp\Big(-\frac{\varphi_i^2}{2} + (\sum_{j\sim i} \varphi_j + \eta_i)\varphi_i\Big)
\end{aligned} \tag{25}$$

and iterate over the sites. For the Gaussian unpinned partition function use, see (12),

$$\begin{aligned}
Z^{\text{Gauss}}_{\varepsilon=0,\Lambda}[\eta] &= \exp\Big(\frac{1}{2}\sum_{i,j\in\Lambda} (-\Delta_\Lambda)^{-1}_{i,j}\eta_i\eta_j\Big) Z^{\text{Gauss}}_{\varepsilon=0,\Lambda}[0] \\
&\leq \exp\Big(\frac{1}{2}\sum_{i,j\in\Lambda} (-\Delta_\Lambda)^{-1}_{i,j}\eta_i\eta_j\Big) C^{|\Lambda|}_{\text{G},d}
\end{aligned} \tag{26}$$

with a suitable constant. From here (5) follows from (20,21,22,23,24,26) □

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