Approximate solution of the Cahn-Hilliard equation via corrections to the Mullins-Sekerka motion.

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Abstract We develop an alternative method to matched asymptotic expansions for the construction of approximate solutions of the Cahn-Hilliard equation suitable for the study of its sharp interface limit. The method is based on the Hilbert expansion used in kinetic theory. Besides its relative simplicity, it leads to calculable higher order corrections to the interface motion.

1. Introduction

1.1 The Cahn–Hilliard equation and phase segregation

The purpose of this paper is to present a method for constructing approximate solutions to a class of evolution equations typified by the Cahn–Hilliard equation. The method produces solutions suitable for studying the sharp interface limit, and for studying higher order corrections to the sharp interface limit. The method itself is based on the Hilbert expansion used in kinetic theory [5]. The work of Caflisch [4] on constructing solutions of the Boltzmann equation from solutions of the Euler equations can be considered as a paradigm for this sort of investigation. The sharp interface limit of the Cahn–Hilliard equation itself has been rigorously investigated by Alikakos, Bates and Chen [1], following the original heuristic analysis of Pego [13]. Both [13] and [1] are based on matched asymptotic expansions. We aim to show that the Hilbert expansion approach has advantages in the presence of the non–locality inherent in this class of problems, and that in any case, it provides a means to calculate higher order corrections to the sharp interface limit. We begin by recalling some background.

* Work partially supported by U.S. National Science Foundation grant DMS 92–07703

** Work partially supported by E.U. grant HPRN–CT–2002–00282 and PRAXIS XXI/BPD/16400/98

*** Work partially supported by the CNR-GNFM,MURST Cofin 2003/2004

Let Ω be a compact domain in $I\!R^2$. The restriction to two dimensions is for simplicity only; we seek to explain the main ideas in the simplest interesting setting. Let m be an integrable function on Ω. We think of $m(x,t)$ as representing the value of a *conserved "order parameter"* at x in Ω at time t. The order parameter is conserved in the sense that $\int_\Omega m(x,t)\mathrm{d}x$ is independent of t. Therefore, the evolution equation for m can be written in the form

$$\frac{\partial}{\partial t}m(x,t) = \nabla \cdot \vec{J}(x,t)$$

where the *current* $\vec{J}$ is orthogonal to the normal at the boundary of Ω. In the class of equations to be considered, the current will have the form

$$\vec{J}(x,t) = \sigma(m(x,t))\nabla\mu(x,t)$$

where $\sigma(m)$ is the *mobility* and $\mu(x,t)$ is the *chemical potential* of x at time t. The mobility is positive, so that the conserved order parameter m "flows" in the direction of increasing chemical potential.

Finally, the chemical potential is the $L^2(\Omega)$ Frechet derivative of a *free energy functional* $\mathcal{F}$:

$$\mu(x) = \frac{\delta\mathcal{F}}{\delta m}(x)\ .$$

The simplest and most familiar example is known as the Cahn–Hilliard equation. It results from the choices $\sigma(m) = 1$; i.e., constant mobility, and *

$$\mathcal{F}(m) = \frac{1}{2}\int_\Omega |\nabla m(x)|^2\mathrm{d}x + \frac{1}{4}\int_\Omega (m^2(x)-1)^2\mathrm{d}x\ .$$

This leads to

$$\frac{\partial}{\partial t}m(x,t) = \Delta\left(-\Delta m(x,t) + f(m(x,t))\right)\ ,$$

where

$$f(m) = m^3 - m\ . \tag{1.1}$$

If $m(x,t)$ is a solution of this equation, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(m(\cdot,t)) = -\int_\Omega \frac{1}{\sigma}|\vec{J}(x,t)|^2\mathrm{d}x\ ,$$

so that the evolution decreases the free energy. Also clearly, the minimizers of the free energy are the constant functions $m(x) = \pm 1$. These minimizers represent

* The choice of the nonlinearity $\frac{1}{4}(m^2-1)^2$ is used only to carry out the explicit computations in Section 3 and Section 4. Different choices can be made, provided they have the form of a double well potential with equal absolute minima and are smooth enough.

the "pure phases" of the system. However, unless the initial data m_0 happens to satisfy $\int_\Omega m_0(x)\mathrm{d}x = \pm|\Omega|$, these "pure phases" cannot be reached because of the conservation law. Instead, what will eventually be produced is a region in which $m(x) \approx +1$, with $m(x) \approx -1$ in its complement, and with a smooth transition across its boundary. This is referred to as *phase segregation*, and the boundary is the *interface* between the two phases. If we "stand far enough back" from Ω, all we see is the interface, and we do not see any structure across the interface – the structure now being on an invisibly small scale. The evolution of m under the Cahn Hilliard equation, or another such evolution equation of this type, drives an evolution of the interface, and we wish to determine how it evolves. To see any evolution of the interface, one must wait a long time. More specifically, let λ be a small parameter, and introduce new variables τ and ξ through

$$\tau = \lambda^3 t \qquad \text{and} \qquad \xi = \lambda x\ .$$

Then of course

$$\frac{\partial}{\partial t} = \lambda^3 \frac{\partial}{\partial \tau} \qquad \text{and} \qquad \frac{\partial}{\partial x} = \lambda \frac{\partial}{\partial \xi}\ .$$

Hence if $m(x,t)$ is a solution of the Cahn–Hilliard equation, and we define $m^\lambda(\xi,\tau)$ by $m^\lambda(\xi,\tau) = m(x(\xi),t(\tau))$, we obtain

$$\frac{\partial}{\partial \tau} m^\lambda(\xi,\tau) = \Delta_\xi \left(-\lambda \Delta_\xi m^\lambda(\xi,\tau) + \frac{1}{\lambda} f(m^\lambda(\xi,\tau)) \right)\ . \tag{1.2}$$

Following Pego [13], we will be studying solutions of the equation (1.2) in the limit as λ tends to zero. If we think of λ as representing the inverse of a large length scale, the variable ξ will be dimensionless, and indeed, one often refers to the components of ξ as being "dimensionless variables". The dimensionless variables are "slow" and the original variables "fast" for small λ. In what follows we keep the notation ξ for the slow spatial variables, but drop the use of τ and replace it by t for convenience. One should just bear in mind that now we are looking at the evolution over a *very* long time scale when λ is small. For the reasons indicated above, we shall consider initial data $m_0(\xi)$ that is -1 in the region bounded by a smooth closed curve Γ_0 in Ω, and $+1$ outside this region. We refer to such initial data as "sharp interface initial data". At later times t there will still be a fairly sharp interface between a region where $m(\xi,t) \approx +1$ and where $m(\xi,t) \approx -1$, centered on a smooth curve Γ_t. One might hope that for small values of λ, *all information about the evolution of* $m(\xi,t)$ *is contained in the evolution of the interface* Γ_t. This is indeed the case. To explain the situation more clearly, let $\mathcal{M}$ denote the set of all smooth simple closed curves in Ω. As we will explain in Section 2, $\mathcal{M}$ can be viewed as a differentiable manifold. A vector field V on $\mathcal{M}$ is a functional associating to each Γ in $\mathcal{M}$ a function in $C^\infty(\Gamma)$. This function gives the normal velocity of a point on Γ, and thus describes a "flow" on $\mathcal{M}$. We may formally write

$$\frac{\mathrm{d}}{\mathrm{d}t}\Gamma_t = V(\Gamma_t)\ . \tag{1.3}$$

Now, given a flow on $\mathcal{M}$, we can produce from it an evolution in $C^\infty(\Omega)$ through the following device: Let m be any function from $\mathcal{M}$ to $C^\infty(\Omega)$. We write $m(\xi,\Gamma)$ to denote $m(\Gamma)$ evaluated at $\xi \in \Omega$. We can then define a time dependent function on Ω, $m(\xi,t)$, through

$$m(\xi,t) = m(\xi,\Gamma_t)\ . \tag{1.4}$$

Notice that time dependence in $m(\xi,t)$ enters *only* through the evolution of Γ_t. A simple example of such a function is the following: Let $d(\xi,\Gamma)$ denote the signed distance from ξ to Γ, where the sign is negative in case ξ is in the interior of Γ, and positive in case ξ is in the exterior of Γ. The signed distance function, unlike the distance function itself, is smooth near Γ. Let g be any smooth function on $I\!R$ and define $m(\xi,\Gamma) = g(d(\xi,\Gamma))$. All the functions appearing in this paper are essentially of this type, or only slightly more elaborate.

Now if, for small λ and sharp interface initial data, all of the information about the evolution of solutions of the Cahn–Hilliard equation were contained in the motion of the interface, then one might hope to find a vector field V on $\mathcal{M}$ governing the evolution of the interface, and a function m from $\mathcal{M}$ to $C^\infty(\Omega)$ so that (1.4) defines the corresponding solution of the Cahn–Hilliard equation.

In this paper, we prove a result of this type. We construct a sequence of vector fields $V_0, V_1, V_2, \ldots$ on $\mathcal{M}$ such that with $V = \sum_{j=0}^\infty \lambda^j V_j$, the interface for the solution of (1.2) satisfies (1.3). For any $N \geq 1$,

$$V^{(N)} = \sum_{j=0}^{N-1} \lambda^j V_j\ , \tag{1.5}$$

and let let $\Gamma_t^{(N)}$ satisfy $\dfrac{\mathrm{d}}{\mathrm{d}t}\Gamma_t^{(N)} = V^{(N)}(\Gamma_t^{(N)})$, where $\Gamma_0^{(N)} = \Gamma$, the initial interface. Then $\Gamma_t^{(N)}$ is *the Nth order approximate interface.* We also construct a sequence of functions $m_0, m_1, m_2, \ldots$ from $\mathcal{M}$ to $C^\infty(\Omega)$ so that

$$m^{(N)}(\xi,t) = \sum_{j=0}^{N} \lambda^j m_j(\xi,\Gamma_t^{(N)}) \tag{1.6}$$

is an approximate solution of (1.2), with arbitrarily high accuracy for large enough N.

The physical picture described above provides a natural guess for the form of the leading term $m_0(\xi,\Gamma)$. We would expect this to be of the form

$$m_0(\xi,\Gamma) = g\left(\frac{d(\xi,\Gamma)}{\lambda}\right)\ ,$$

where g is "transition profile" across the boundary of the interface, so that as y strictly increases from $-\infty$ to $+\infty$, $g(y)$ increases from -1 to $+1$, with $g(0) = 0$.

Thus, for small enough λ, $m_0(\xi,\Gamma) \approx -1$ if ξ is in the interior of the region bounded by Γ, and $m_0(\xi,\Gamma) \approx +1$ if ξ is in the exterior. This will turn out to be correct, with g chosen to be the transition profile that minimizes the free energy cost of the transition.

The remaining terms will be more complicated. For $j \geq 1$, m_j will have the form

$$m_j(\xi,\Gamma) = h_j(\xi,\Gamma) + \phi_j(\xi,\Gamma) \ . \tag{1.7}$$

The second term in the right, ϕ_j, will satisfy a global Lipschitz condition independent of λ, and will account for the smooth, long range part of the correction at the jth order. It will be specified in terms of the solution of a Poisson type equation in Ω. The first term, h_j, accounts for further corrections that are needed very close to Γ. It will be specified by solving an ordinary differential equation in a variable measuring the signed distance from the boundary. These equations are derived by insisting that $m^{(N)}(\xi,t)$ satisfy (1.2) order by order in λ, up to order λ^N.

This requirement relates the two expansions (1.5) and (1.6). V_0 will be determined by the Fredholm conditions for solvability arising when we seek to solve for h_1 and ϕ_1. Likewise, for $j > 1$, V_{j-1} will be determined by Fredholm conditions for solvability arising when we seek to solve for h_j and ϕ_j. In this way, the two expansions are produced in alternation: We start from a natural guess for the leading order approximation m_0. We then see that we can find a next order correction $m_0 + \lambda m_1$ if and only if we make a specific choice for the leading term V_0 in the description of the interface motion. Doing so, we next seek to find a second order approximation $m_0 + \lambda m_1 + \lambda^2 m_2$, and this determines the next term λV_1 in the description of the interface motion. Continuing in this way, we produce both (1.5) and (1.6) order by order.

This procedure is very reminiscent of the Hilbert expansion in kinetic theory, and we explain this analogy in more detail in Section 3.2 below. From the Hilbert expansion point of view, the two component prescription (1.7) for m_j is somewhat novel. From the matched asymptotic expansion point of view, one might expect h_j to represent an "inner layer" and ϕ_j to represent an "outer layer". However, these two components do not have disjoint support, and we do not match them up at any boundary. They simply account for different parts of the solution: The functions ϕ_j are smooth functions accounting for the very regular nature of the jth order correction far from the interface. Closer to the interface, the Lipschitz function ϕ_j cannot provide all of the correction necessary, and so the function h_j is required to provide the necessary corrections to $m^{(j-1)} + \lambda^j \phi_j$ that are needed close to the interface Γ. Away from Γ, $m^{(j-1)} + \lambda^j \phi_j$ does well enough, and so $h_j(\xi,\Gamma)$ will decay to zero exponentially fast, on the length scale λ even, as ξ moves away from Γ.

The fact that the short range correction h_j and the long range correction ϕ_j do not have disjoint support, and do not have to be matched across any boundary between "inner and outer layers" is particularly advantageous when one is investigating non local analog of the Cahn–Hilliard equation. We shall return to this point later.

Ansatz (Brief Version): *Let $V_0, V_1, V_2, \ldots$ be a sequence of vector fields on $\mathcal{M}$ and $m_0, m_1, m_2, \ldots$ be functions from $\mathcal{M}$ to $C^\infty(\Omega)$. For any given initial interface*

Γ_0 in $\mathcal{M}$, and all $N > 0$, let $\Gamma_t^{(N)}$ be the solution of

$$\frac{\mathrm{d}\Gamma_t^{(N)}}{\mathrm{d}t} = \left[\sum_{j=0}^{N-1} \lambda^j V_j\right] \left(\Gamma_t^{(N)}\right) \qquad \text{with} \qquad \Gamma_0^{(N)} = \Gamma_0. \tag{1.8}$$

Then define the function $m^{(N)}(\xi, t)$ by

$$m^{(N)}(\xi, t) = m_0\left(\frac{d(\xi, \Gamma_t^{(N)})}{\lambda}\right) + \sum_{j=1}^{N} \lambda^j m_j(\xi, \Gamma_t^{(N)}) \ .$$

Our main result is that if the vector fields V_j and the functions m_j are chosen according to a scheme to be specified below, the function $m^{(N)}(\xi, t)$, just defined in the ansatz, is a good approximate solution of the Cahn–Hilliard equation. (Indeed, as we shall soon explain, if N is large enough and λ is small enough, there is an *exact* solution very close by).

Theorem 1.1 *For any $N > 1$ there are vector fields V_j, $j = 0, ..., (N-1)$ and functions m_j, $j = 0, ..., N$ as prescribed in the ansatz having the following properties: Let T denote the lifetime of the solution of (1.8) in $\mathcal{M}$. Then there is a constant C_N so that for all $t < T$,*

$$\frac{\partial}{\partial t} m^{(N)}(\xi, t) = \Delta \left(-\lambda \Delta m^{(N)}(\xi, t) + \frac{1}{\lambda} f(m^{(N)}(\xi, t)) \right) + \Delta R^{(N)}(\xi, t) \tag{1.9}$$

where

$$\sup_{\xi \in \Omega, t \in [0,T]} \left| R^{(N)}(\xi, t) \right| \le C_N \lambda^{N-1} \ . \tag{1.10}$$

Finally the sequences of vector fields and functions are essentially uniquely determined: Given V_j for $j < k$, then V_k is determined up to $\mathcal{O}(\lambda^{k+1})$, and similarly given m_j for $j < k$, then m_k is determined up to $\mathcal{O}(\lambda^{k+1})$.

As we have mentioned, the construction behind Theorem 1.1 is patterned on the Hilbert expansion of kinetic theory. In particular, the work of Caflisch [4] showed how to construct actual solutions of the Boltzmann equation starting from smooth solutions of the Euler equations for small values of a scaling parameter known as the Knudsen number. His first step was to construct close *approximate solutions* of the Boltzmann equation using a Hilbert expansion. His second step was to show that there is an *actual solution* of the Boltzmann equation very close by, for small enough values of the Knudsen number and high enough order approximate solutions.

In our case, the second step has already been taken care of in the work of Alikakos, Bates and Chen [1]. They proved and applied the spectral estimates for the Cahn–Hilliard equation necessary to prove that an actual solution exists nearby a sufficiently nice approximate solution. In their approach, the approximate solutions were constructed using a matched asymptotic expansion, which gives no information on higher order corrections to the flow, while our approach, based on the Hilbert expansion, does, as we explain below.

Later, in Section 3.2, we will return to discuss the analogy with the work of Caflish, as well as with the recent and very significant work of Shih-Hsien Yu, [16], who introduced a *generalized Hilbert expansion with shock layer corrections* to construct approximations to the solutions of the Boltzmann equations with small Knudsen number. This enabled him to extend Caflisch's correspondence between solutions of the Euler equation and solutions of the Boltzmann equation beyond the appearance of shocks, whereas the analysis of Caflisch is valid only up until the appearance of the first shock.

In the meantime however, it is necessary to explain more fully what we actually do here. In Theorem 1.1, and in the following, $\mathcal{O}(\lambda^m)$ denotes terms which are of order λ^m uniformly in all of their variables.

The qualified nature of the uniqueness in the theorem is an indication that there will be choices to be made at every stage of the approximation, and that the manageability of the approximation will depend on how those choices are made. The full result, which amplifies Theorem 1.1, will be given in Theorem 5.3. In the next few sections, we explain what the right choices for the V_j and m_j turn out to be.

The leading term V_0 in the vector field $\sum_{j=0} \lambda^j V_j$ governing the interfacial flow turns out to be something quite well known: It is the vector field generating the *Mullins–Sekerka* flow, as one would expect from Pego's pioneering work [13] on the connection between flows of curves and the Cahn–Hilliard equation. The Mullins–Sekerka vector field is defined as follows:

Fix a number $S > 0$ that will later be interpreted as a "surface tension" and denote by $K(\xi) \equiv K(\xi, \Gamma)$ the curvature at $\xi \in \Gamma$. Then for each Γ in $\mathcal{M}$, let $\mu_{0,0}$ be the solution of

$$\Delta \mu_{0,0}(\xi) = 0 \qquad \text{for} \qquad \xi \in \Omega \setminus \Gamma \tag{1.11}$$

subject to the boundary conditions

$$\mu_{0,0}(\xi) = S\left(K(\xi) - \frac{2\pi}{|\Gamma|}\right) \text{ on } \Gamma \qquad \text{and} \qquad \frac{\partial}{\partial \nu}\mu_{0,0} = 0 \quad \text{on} \quad \partial\Omega\ , \tag{1.12}$$

where $|\Gamma|$ denotes the arc length of Γ, and $\partial/\partial\nu$ denotes the normal derivative. (The role of the double subscript will become clear later in the context of our expansion). Now define $V_0(\Gamma)$ to be the real valued function on Γ given by

$$V_0(\xi, \Gamma) = \left[\frac{\partial}{\partial \nu}\mu_{0,0}\right]_\Gamma (\xi) \qquad \xi \in \Gamma \tag{1.13}$$

where the brackets on the right denote the jump in the normal derivative across Γ. *This defines a vector field on $\mathcal{M}$, and the flow it generates is known as the Mullins–Sekerka flow.*

Concerning the existence of the generated flow; i.e., the solution of the free boundary problem (1.11), (1.12) and (1.13), Chen [6] established the local (in time) existence of a solution in the two dimensional case and, when the initial curve is nearly circular, the global existence and long time behavior. The local existence of a unique smooth solution in any space dimension has been established in [8]. Introducing an alternative approach, Escher and Simonett [10] established the local existence and uniqueness of classical solution to the Mullins–Sekerka problem in any dimension with and without superficial tension, granted enough regularity for the initial hypersurface.

As it is well known, the Mullins–Sekerka flow conserves the area enclosed by Γ_t, and decreases the arc length of Γ_t. To see this, let Ω_Γ^- denote the interior of Γ, and let Ω_Γ^+ denote its exterior. It is an easy consequence of Green's identity that

$$\int_{\Gamma_t} V_0(\eta,\Gamma_t)\mathrm{d}S_\eta = \int_{\Gamma_t}\left[\frac{\partial}{\partial n}\mu_{0,0}\right]_{\Gamma_t}(\eta)\mathrm{d}S_\eta = \int_{\Omega\backslash\Gamma_t}\Delta\mu_{0,0}(\eta)\mathrm{d}\eta = 0\ , \tag{1.14}$$

where, using a standard potential theoretic notation, $\mathrm{d}S_\eta$ denotes the element of arclength along Γ. Therefore, since $\frac{\mathrm{d}}{\mathrm{d}t}|\Omega_{\Gamma_t}^+| = \int_{\Gamma_t} V_0(\eta,t)\mathrm{d}S_\eta$, the Mullins–Sekerka flow conserves the area of $\Omega_{\Gamma_t}^+$, and hence $\Omega_{\Gamma_t}^-$ as well. Furthermore,

$$\frac{\mathrm{d}}{\mathrm{d}t}|\Gamma_t| = -\int_{\Gamma_t} K(\eta)V_0(\eta,\Gamma_t)\mathrm{d}S_\eta = -\frac{1}{S}\int_{\Gamma_t}\mu_{0,0}\left[\frac{\partial}{\partial n}\mu_{0,0}\right]_{\Gamma_t}(\eta)\mathrm{d}S_\eta = -\frac{1}{S}\int_\Omega[\nabla\mu_{0,0}]^2\mathrm{d}\xi$$

so that the Mullins–Sekerka flow diminishes the length of the boundary. Clearly a single sphere or multiple spheres of the same radius are equilibria for this evolution.

The higher order terms in $\sum_{j=0}\lambda^j V_j$ are somewhat more complicated. In this paper, we explicitly compute V_1, the next correction to V_0, and show how all higher terms could be computed. The description of V_1, like that of V_0, is potential theoretic.

In the following, $G(\xi,\eta)$ is the Neumann Green's function for Ω, and $\mathcal{T}_\Gamma$ is the Dirichlet–Neumann operator for Γ; i.e., the linear operator that transforms the Dirichlet data on Γ for the solution ϕ of (1.11) into the corresponding Neumann boundary data $[\partial\phi/\partial\nu]_\Gamma$. Some relevant potential theoretic background is recalled in an appendix. Finally, for any real valued function $f(\Gamma)$ on $\mathcal{M}$, and any vector field V, $D_V f(\Gamma)$ denotes the derivative of f along the flow through Γ generated by V. If $g(\xi,\Gamma)$ is a function from $\mathcal{M}$ to $C^\infty(\Omega)$, then for fixed ξ, $\Gamma\mapsto g(\xi,\Gamma)$ is a real valued function on $\mathcal{M}$, and we denote the derivative of this function by $D_V g(\xi,\Gamma)$.

Theorem 1.2 *The vector field V_1 on $\mathcal{M}$ giving the next corrections to V_0, the Mullins-Sekerka vector field, is given by $V_1 = V_1^{(0)} + \langle V_1\rangle$ where*

$$\langle V_1\rangle_\Gamma = \frac{1}{4|\Gamma|}\int_\Omega D_{V_0}\,\mu_{0,0}(\xi,\Gamma)\mathrm{d}\xi\quad , \tag{1.15}$$

$$
\begin{aligned}
V_1^{(0)}(\xi,\Gamma) =& \frac{1}{4}(2S+C)\mathcal{T}_\Gamma V_0(\xi,\Gamma) - \frac{1}{2}\mathcal{T}_\Gamma\left[p(\cdot) - \frac{1}{|\Gamma|}\int_\Gamma p(\eta,\Gamma)\mathrm{d}S_\eta\right](\xi) \\
& - \langle V_1\rangle_\Gamma \mathcal{T}_\Gamma\left[\int_\Gamma G(\cdot,\eta)\mathrm{d}S_\eta - \frac{1}{|\Gamma|}\int_\Gamma\int_\Gamma G(\zeta,\eta)\mathrm{d}S_\zeta\mathrm{d}S_\eta\right](\xi).
\end{aligned}
\tag{1.16}
$$

Here, $\mu_{0,0}(\cdot,\Gamma)$ is the harmonic function in $\Omega\backslash\Gamma$ defining V_0 in (1.11) and (1.12). Also, S and C are explicit constants computed below in (3.31) and (3.49),

$$
p(\xi,\Gamma) = \frac{1}{2}\int_\Omega G(\xi,\eta)\left[D_{V_0}\,\mu_{0,0}(\xi,\Gamma)\right]\mathrm{d}\eta\ , \tag{1.17}
$$

and $D_{V_0}\,\mu_{0,0}(\xi,\Gamma)$ denotes the rate of change of $\mu_{0,0}(\xi,\Gamma)$ under the flow induced by V_0.

A formula for computing $D_{V_0}\,\mu_{0,0}(\xi,\Gamma)$, see (4.12), is derived in Section 4. Though complicated, it reduces the computation to standard potential theoretic integrals over Γ.

Pego's work relating the Cahn–Hilliard equation and the Mullins–Sekerka flow was made rigorous by Alikakos, Bates and Chen [1]. Their construction also yields high order approximate solutions, but does not yield higher order corrections to the sharp interface flow. Their work, like Pego's, was based on matched asymptotic expansions. Our approach is modeled on the Hilbert expansion of kinetic theory [4],[5]. Another alternative to matched asymptotic expansions, for the spherically symmetric case, has been developed by Stoth [15].

The plan of the paper is as follows: In Section 2 we describe $\mathcal{M}$ and vector fields on $\mathcal{M}$ in a more precise fashion. In Section 3 we explain the Hilbert expansion, and carry out the computations for the first two terms in explicit detail, proving Theorem 1.2. This gives us, in Section 4, the formula for V_1 mentioned above. Finally, we show that the computations can be carried out to any order, and that they yield approximate solutions of the Cahn–Hilliard equation, as claimed in Theorem 1.1. This is accomplished in the remaining sections. We recall the potential theory and some technical lemmas in an Appendix.

The strategy employed here was devised to treat a non–local variant of the Cahn–Hilliard equation that has been rigorously derived from a scaling limit of a spin system with exchange dynamics and local mean field Kac potentials [11]. The sharp interface limit has been investigated by [12] on a formal level as in Pego's original work. The present approach was developed to facilitate a rigorous treatment, which shall appear in a forthcoming paper.

Finally, E. Orlandi would like to acknowledge discussions with Giorgio Fusco and Nicholas Alikakos. She further thanks the department of Mathematics of Georgia Tech, where part of the work has been completed, for warm hospitality.

2. Vector fields and flows on $\mathcal{M}$

2.1 A local coordinate system near $\Gamma \in \mathcal{M}$

In order to discuss motion in $\mathcal{M}$, it is convenient to first introduce local coordinates in the neighborhood of any given $\Gamma \in \mathcal{M}$.

Let Γ be a smooth closed simple curve in Ω. Let $s \mapsto \eta(s)$ be any arc length parametrization of Γ. (The position of $\eta(0)$ on Γ does not matter). In the following, we will often denote by s the corresponding point on the curve. This slight abuse of notation will prove very convenient. Let $\kappa(\Gamma)$ be given by

$$\kappa(\Gamma) = \max_{\xi \in \Gamma} |K(\xi)| \ . \tag{2.1}$$

Recall that the *signed distance* from ξ to Γ, $d(\xi, \Gamma)$, is defined so that $d < 0$ inside Γ and $d > 0$ outside Γ. As long as $|d(\xi, \Gamma)| < 1/\kappa(\Gamma)$, there is a uniquely determined point $\eta \in \Gamma$ such that $|\xi - \eta| = |d(\xi, \Gamma)|$; this is the point in Γ that is closest to ξ. Therefore, define for all $0 < \lambda_0 < \frac{1}{\kappa(\Gamma)}$,

$$\mathcal{N}(\lambda_0) \equiv \mathcal{N}(\lambda_0, \Gamma) = \{\xi \in I\!R^d \ : \ |d(\xi, \Gamma)| < \lambda_0\} \ .$$

In what follows, for $\xi \in \mathcal{N}(\lambda_0)$, we define

$$s(\xi) = \text{the unique } \eta \text{ with } |\xi - \eta| = |d(\xi, \Gamma)| \ . \tag{2.2}$$

There is a natural set of coordinates in $\mathcal{N}(\Gamma)$: First, fix some arc length parameterization of Γ. With this parameterization fixed, let s denote arc length coordinate of $s(\xi)$. This definition extends the domain of definition of the arc length coordinate from Γ itself to all of $\mathcal{N}(\lambda_0, \Gamma)$. Denoting the corresponding coordinate function $s(\xi)$ introduces a useful ambiguity in notation: Either $s(\xi)$ denotes a number, when it is to be interpreted as a coordinate function, or it denotes a point $\eta \in \Gamma$, when it is used as in (2.2). The ambiguity is harmless since the s coordinate of the point $s(\xi)$ is the number $s(\xi)$, and whenever $s(\xi)$ appears as the argument of a function, the type of variable on which that function depends determine the interpretation. For the second coordinate, define

$$z(\xi) = \frac{d(\xi, \Gamma)}{\lambda} \ .$$

The coordinate transformation $\xi \mapsto (z, s)$ has a simple inverse:

$$\xi = s(\xi) + z\lambda n(s(\xi)) \ ,$$

where $n(s)$ denote the unit outward normal to Γ at $s(\xi)$. Notice that a small variation in ξ produces a small variation in s, but can produce a large variation in z. For this reason, we speak of s as the *slow variable*, and z as the *fast variable*.

2.2 Motion in $\mathcal{M}$

The coordinates that we have just introduced in $\mathcal{N}(\lambda_0, \Gamma)$ provide the means to give $\mathcal{M}$ the structure of a differentiable manifold, and to study motions in this manifold.

Fix any $\Gamma \in \mathcal{M}$, and let $s \mapsto \eta(s)$ denote the arc length parameterization of Γ, as above. Let U_Γ be the subset of $\mathcal{M}$ consisting of curves $\tilde{\Gamma}$ such that for each $\xi \in \tilde{\Gamma}$, $|d(\xi,\Gamma)| < \kappa(\Gamma)$. Each $\tilde{\Gamma} \in U_\Gamma$ has a parameterization

$$s \mapsto \eta(s) + r_{\tilde{\Gamma}}(s)n(s) \tag{2.3}$$

for a uniquely determined smooth function $r_{\tilde{\Gamma}}$. The map $\tilde{\Gamma} \mapsto r_{\tilde{\Gamma}}$ maps U_Γ onto an open subset in $C^\infty(\Gamma)$, which can of course be identified with $C^\infty(S^1)$, where S^1 is the unit circle. Clearly this map is invertible, and we may regard it as a local coordinate map. For each Γ in $\mathcal{M}$, and each ϵ with $0 < \epsilon < \kappa(\Gamma)$, let $U_{\Gamma,\epsilon}$ consist of all $\tilde{\Gamma}$ in $\mathcal{M}$ so that for each ξ in $\tilde{\Gamma}$,

$$|d(\xi,\Gamma)| < \epsilon\ .$$

We take these sets as a basis for the topology on $\mathcal{M}$. The local coordinates just introduced are very useful for studying the motion of curves in $\mathcal{M}$. Let $t \mapsto \Gamma_t$ be a continuous map from some open interval about $t = 0$ into $\mathcal{M}$ such that $\Gamma_0 = \Gamma$. It follows from the continuity that for some $a > 0$, and each t with $|t| < a$, Γ_t has a parameterization

$$s \mapsto \eta(s) + r(s,t)n(s)\ .$$

In this case, knowledge of the function $r(s,t)$ and its evolution provides complete knowledge about Γ_t and its evolution. In particular, the function*

$$V(s) = \frac{\partial}{\partial t} r(s,t)\Big|_{t=0} \tag{2.4}$$

can be viewed as the tangent vector to the curve $t \mapsto \Gamma_t$ in $\mathcal{M}$ at $t = 0$. Hence we call V the *velocity field* of $t \mapsto \Gamma_t$ at $t = 0$. In this sense we write

$$V = \frac{\partial}{\partial t}\Gamma_t\Big|_{t=t_0}\ , \tag{2.5}$$

and identify the tangent space to $\mathcal{M}$ at Γ as the set of all smooth real valued functions $V(s)$ on Γ. Thus, a vector field on $\mathcal{M}$ is a map V assigning to each Γ in $\mathcal{M}$ a smooth real valued function $s \mapsto V(s,\Gamma)$ on Γ. A sufficiently nice vector field on $\mathcal{M}$ defines a flow on $\mathcal{M}$. Given a vector field V on $\mathcal{M}$, and a path $t \mapsto \Gamma_t$ in $\mathcal{M}$, we say that

$$\frac{\partial}{\partial t}\Gamma_t = V(\Gamma_t) \tag{2.6}$$

in case computing the left hand side in the sense of (2.4) and (2.5) gives the same result as evaluating $V(\Gamma_t)$ according to whatever rule defines it. Then $t \mapsto \Gamma_t$ is an

* This function will be differentiable by the implicit function theorem.

integral curve of the flow given by $\Gamma \mapsto V(\cdot, \Gamma)$. Further, we denote the lifetime T of the flow (2.6), starting at $\Gamma \in \mathcal{M}$ as

$$T = \inf\{t > 0 : \kappa(\Gamma_t) \leq \kappa_0\} \tag{2.7}$$

where κ_0 is any arbitrarily chosen positive number so that $\kappa(\Gamma) \leq \kappa_0 < \infty$. If $V(\cdot, \Gamma) = K(\cdot, \Gamma)$, the curvature at $s \in \Gamma$, one obtains the curve shortening *flow by curvature*. The nature of this flow in $d = 2$ has been completely clarified by Grayson. A more pertinent example of a vector field on $\mathcal{M}$ is the Mullins–Sekerka vector field that we have described in the previous section.

There is an obvious but useful decomposition of vector fields on $\mathcal{M}$. For any vector field V on $\mathcal{M}$, define

$$\langle V \rangle_\Gamma = \frac{1}{|\Gamma|} \int_\Gamma V(\xi, \Gamma) \mathrm{d}S_\xi \tag{2.8}$$

and

$$V^{(0)}(\cdot, \Gamma) = V(\cdot, \Gamma) - \langle V \rangle_\Gamma \ .$$

This gives us the decomposition

$$V(\cdot, \Gamma) = V^{(0)}(\cdot, \Gamma) + \langle V \rangle_\Gamma \ . \tag{2.9}$$

In this decomposition, $\langle V \rangle_\Gamma$ is constant, while $V^{(0)}$ is orthogonal to the constants, and thus generates a flow that does not alter the enclosed area. In what follows, we shall derive separate equations for the components $V^{(0)}(\cdot, \Gamma)$ and $\langle V \rangle_\Gamma$ for each of the vector fields V_j in the ansatz.

We close the section giving another example of a class of functions from $\mathcal{M}$ to $C^\infty(\Omega)$: Let some number $\lambda_0 > 0$ be given. Let h be a smooth function from $I\!R^2$ to $I\!R$ such that $h(x, y) = 0$ for all $|x| \geq \lambda_0/\lambda$. Define a function – also denoted by h – on $\Omega \times \mathcal{M}$ by

$$h(\xi, \Gamma) = h\left(\frac{d(\xi, \Gamma)}{\lambda}, s(\xi, \Gamma)\right) \ . \tag{2.10}$$

Then $h(\xi, \Gamma)$ vanishes identically in the region $|d(\xi, \Gamma)| \geq \lambda_0$, and it will be smooth in the region $|d(\xi, \Gamma)| < \lambda_0$ – and hence everywhere – in case $\lambda_0 < 1/\kappa(\Gamma)$. The functions from $\mathcal{M}$ to $C^\infty(\Omega)$ that we use in the ansatz are all functions of this type, or else potentials of them.

3. The Hilbert Expansion

3.1 The Hilbert expansion to first order and the Mullins–Sekerka flow

We begin with a derivation of Pego's result relating the Mullins–Sekerka problem and the Cahn-Hilliard equation from the point of view that will be the basis of our rigorous construction. Our approach is based on a Hilbert expansion, adapted from kinetic theory, instead of matched asymptotic expansions. Nonetheless, the first step is the same; we write the Cahn–Hilliard equation as a system: For each $\xi \in \Omega$ and each $t > 0$,

$$\frac{\partial}{\partial t} m^\lambda(\xi, t) = \Delta \mu^\lambda(\xi, t) \tag{3.1}$$

$$\mu^\lambda(\xi, t) = -\lambda \Delta m^\lambda(\xi, t) + \frac{1}{\lambda} f(m^\lambda(\xi, t)) \tag{3.2}$$

where Δ denotes the Neumann Laplacian on Ω. Let Γ_0 be a smooth closed simple curve in Ω, and consider initial data $m^\lambda(\xi, 0)$ such that $m^\lambda(\xi, 0) \simeq -1$ in the region enclosed by Γ_0 and $m^\lambda(\xi, 0) \simeq +1$, outside Γ_0. The precise profile of $m^\lambda(\xi, 0)$ across Γ_0 will be specified later.

Because the free energy decreases under the evolution described by the Cahn–Hilliard equation, we expect that for initial data that is very close to $+1$ outside Γ_0 and to -1 inside, the solution $m(\xi, t)$ will remain very close to $+1$ outside some new curve Γ_t and to -1 inside. We seek an approximate solution m_1 of the form (to be explained below)

$$m_1(\xi, t) = m_0\left(\frac{d(\xi, \Gamma_t)}{\lambda}\right) + \lambda h_1\left(\frac{d(\xi, \Gamma_t)}{\lambda}, s(\xi, \Gamma_t)\right) + \lambda \phi_1(\xi, \Gamma_t) \ , \tag{3.3}$$

together with an approximate chemical potential μ_0 so that (3.1) and (3.2) are satisfied to leading order in λ:

$$\frac{\partial}{\partial t} m_1(\xi, t) = \Delta \mu_0(\xi, t) + \mathcal{O}(\lambda) \tag{3.4}$$

and

$$\mu_0(\xi, t) = -\lambda \Delta m_1(\xi, t) + \frac{1}{\lambda} f(m_1(\xi, t)) + \mathcal{O}(\lambda). \tag{3.5}$$

We have in mind expansions for m^λ and μ^λ of the form

$$m^\lambda = m_0\left(\frac{d(\cdot, \Gamma_t)}{\lambda}\right) + \sum_{k \geq 1} \lambda^k (h_k + \phi_k) \qquad \text{and} \qquad \mu^\lambda = \sum_{k \geq 0} \lambda^k \mu_k \ , \tag{3.6}$$

of which m_1 and μ_0 are simply the leading order.

The two term prescription for the form of m_j, $j \geq 1$, requires some explanation. The first term on the right side of (3.3) is the easiest to explain.

The free energy minimizing transition profile across a planar interface, which we denote by $\bar{m}$, is a natural candidate for the function m_0. The function $\bar{m}$ is the unique solution of the Euler–Lagrange equation

$$\begin{aligned} & -m''(z) + f(m(z)) = 0 \quad \text{for} \qquad z \in I\!R \\ & \lim_{z \to \pm\infty} m(z) = \pm 1 \qquad m(0) = 0 \ . \end{aligned} \tag{3.7}$$

It is easy to see that for $f(m) = m^3 - m$, $\bar{m}(z) = \tanh(z/\sqrt{2})$ and so

$$\bar{m}'(z) > 0, \qquad |\bar{m}(z) \pm 1| \le C_0 e^{-\alpha|z|}, \qquad |\frac{d^\ell}{dz^\ell}\bar{m}(z)| \le C_\ell e^{-\alpha|z|}\ ,\ \ell = 1,2\ldots \tag{3.8}$$

for all $z \in I\!R$, where $\alpha = \sqrt{2}$ and C_l, $\ell = 0, 1\ldots$ are positive real constants.*

The problem with this choice is that $\bar{m}\left(\frac{d(\xi,\Gamma_t)}{\lambda}\right)$ does not define a C^∞ function on Ω. We can remedy this as follows. Fix a number λ_0. Let $r(u)$ be a smooth, even, unimodal cut–off function so that $r(u) = 1$ for $|u| < 1/2$, and $r(u) = 0$ for $u > 1$. For $\lambda < \lambda_0$, define

$$m_0(z) = r\left(\frac{\lambda}{\lambda_0}z\right)\bar{m}(z) + \left(1 - r\left(\frac{\lambda}{\lambda_0}z\right)\right)\ \mathrm{sgn(z)}\ . \tag{3.9}$$

Notice that for $|z| < \lambda_0/(2\lambda)$, $m_0(z) = \bar{m}(z)$, and for other values of z, the difference is exponentially small in $\frac{\lambda}{\lambda_0}$ because of the bounds (3.8). As long as $\kappa(\Gamma) < 1/\lambda_0$, perpendicular lines through Γ meet only at points that are at a distance from Γ that is greater than λ_0, and no singularities arise. In what follows, whenever m_0 is used to denote a function on $I\!R$, it will be this function defined in (3.9). As a function from $\mathcal{M}$ to $C^\infty(\Omega)$, it will always denote $m_0\left(\frac{d(\xi,\Gamma_t)}{\lambda}\right)$, the first term on the right in (3.3).

The next two terms on the right side of (3.3) require more explanation. The functions h_1 and ϕ_1 give important corrections to the leading term $m_0\left(\frac{d(\xi,\Gamma_t)}{\lambda}\right)$. Long range corrections are given by ϕ_1. It will be a smooth function with derivatives of all orders, and will satisfy a Lipschitz bound that is *independent of* λ. Close to the surface, more rapidly varying corrections may be required, and if so, these are to be encoded in h_1. We shall derive an equation for h_1, and as a consequence of this equation, we shall see that h_1 is a rapidly decaying function of z, like m_0' above.

As we shall see, there is essentially only one way to choose the motion of Γ_t, m_0, h_1 and ϕ_1 so that a solution of (3.4) and (3.5) is possible. As we shall see, (3.4) and (3.5) force the motion of the interface Γ_t to be given, in leading order, by the Mullins–Sekerka flow. This necessity will arise through the Fredholm alternative when we try to solve an equation for the function h_1. As in the Hilbert expansion of kinetic theory, we shall need to satisfy a Fredholm condition at each order, and this shall determine V at each order.

We explain how (3.4) and (3.5) lead to the Mullins–Sekerka flow. We shall see that our prescription (3.9) for m_0 is correct, and derive equations for ϕ_1 and h_1. We shall

* For more general double well potentials, the same sorts of bounds would hold with different α; see [14]. It is these sorts of bounds that we will use, and not really the explicit formula for $\bar{m}(z)$.

extract these from (3.5), so we first need to determine μ_0. In what follows, we shall successively determine μ_0, then ϕ_1 and then h_1. We shall see that V_0 is determined by the Fredholm alternative condition on the solvability of the equation for h_1.

Determination of μ_0: To leading order,

$$\frac{\partial}{\partial t} m(\xi,t) \approx \frac{1}{\lambda} m_0'\left(\frac{d(\xi,\Gamma_t)}{\lambda}\right) V_0(s(\xi)) \ , \tag{3.10}$$

and we wish to extract the leading order of μ^λ from (3.1) and this approximation. First recall that the Cahn–Hilliard equation is conservative in that $\int_\Omega m(\xi,t)\mathrm{d}\xi$, does not depend on t. If this conservation law is to hold at every order, we would require

$$\int_\Omega \left(\frac{1}{\lambda} m_0'\left(\frac{d(\xi,\Gamma_t)}{\lambda}\right) V_0(s(\xi))\right) \mathrm{d}\xi = 0 \ . \tag{3.11}$$

Since

$$\mathrm{d}\xi = \lambda(1-\lambda z K(s))\mathrm{d}s\mathrm{d}z \ , \tag{3.12}$$

and since m_0' is even, this holds if and only if

$$\int_{\Gamma_t} V_0(\eta,\Gamma_t)\mathrm{d}S_\eta = 0 \qquad \text{for all} \quad t \geq 0 \ . \tag{3.13}$$

This of course corresponds to the fact that the flow will not change the area enclosed by Γ_t. We therefore suppose that V_0 satisfies (3.13), and we can now find the chemical potential μ to leading order: The condition (3.11) is the solvability condition of

$$\frac{1}{\lambda} m_0'\left(\frac{d(\xi,\Gamma_t)}{\lambda}\right) V_0(s(\xi),\Gamma_t) = \Delta\mu_0(\xi,t) \ , \tag{3.14}$$

the equation for the leading term in the chemical potential μ^λ that we have obtained from (3.1) using (3.10). Therefore, by (3.14), we have that to leading order μ^λ is given by

$$\mu_0(\xi,t) = \int_\Omega G(\xi,\eta)\left(\frac{1}{\lambda} m_0'\left(\frac{d(\eta,\Gamma_t)}{\lambda}\right) V_0(s(\eta),\Gamma_t)\right)\mathrm{d}\eta + c_0(t) \tag{3.15}$$

where $G(\xi,\eta)$ is the Neumann Greens function for Ω, and $c_0(t)$ is a constant (in ξ) to be determined.

Since $\frac{1}{\lambda} m_0'\left(\frac{x}{\lambda}\right) \approx 2\delta(x)$, (3.14) says that μ_0 itself is approximately equal to a single layer potential plus a time dependent constant $c_0(t)$:

$$\mu_{0,0}(\xi,\Gamma_t) = 2\int_{\Gamma_t} G(\xi,\eta) V_0(\eta,\Gamma_t)\mathrm{d}S_\eta + c_0(t) \ . \tag{3.16}$$

Because μ_0 is a "smeared" version of $\mu_{0,0}$, it will be C^∞, unlike $\mu_{0,0}$ which will only be Lipschitz, with a jump in the normal derivative across Γ_t. However, the only quantitative smoothness bound we have on μ_0 that is independent of λ is that it is globally Lipschitz: There is a constant C depending only on Γ_t so that

$$\|\mu_0\|_{\mathrm{Lip}(\Omega)} \leq C \tag{3.17}$$

We can now explain why we have written the first order corrections to m_0 as a sum of two terms, $\lambda h_1 + \lambda\phi_1$. The point is that μ_0, being an approximate single layer potential, cannot decay rapidly to a constant: Single layer potentials decay quite slowly, especially in $I\!R^2$. Therefore, we split the correction into two pieces: One, given by ϕ_1 will be long range, and will have a purely potential theoretic origin and specification. It will however be defined in the whole domain, not just in an "outer layer". The function h_1 will provide corrections to $m_0 + \lambda\phi_1$ that are required near Γ. As we shall see, such corrections are *only* required very near Γ; the equation determining h_1 shall force its support to be exponentially localized near Γ.

Determination of ϕ_1: We now use use (3.5) to determine ϕ_1. Assuming that ϕ_1 encodes all first order long range corrections to m_0, so that m_0' and h_1 decay rapidly for ξ far away from Γ_t,

$$m_1(\xi, \Gamma_t) \approx \pm 1 + \lambda\phi_1(\xi, \Gamma_t) \qquad \text{and} \qquad \Delta m_1(\xi, \Gamma_t) \approx \lambda\Delta\phi_1(\xi, \Gamma_t)\ . \tag{3.18}$$

Now consider (3.5). Because of (3.18), and because of the rapid decay of h_1, for ξ far from Γ_t,

$$\mu(\xi, \Gamma_t) \approx \frac{1}{\lambda} f(\pm 1 + \lambda\phi_1(\xi, \Gamma_t))\ .$$

To leading order in λ, $f(\pm 1 + \lambda\phi_1(\xi, \Gamma_t)) = \lambda f'(1)\phi_1(\xi, \Gamma_t)$. Hence we must have

$$\phi_1(\xi, \Gamma_t) = \frac{1}{f'(1)}\mu_0(\xi, \Gamma_t) \tag{3.19}$$

for ξ such that $|d(\xi, \Gamma_t)| > 1/\kappa(\Gamma_0)$. This specifies ϕ_1 away from Γ_t. It will prove very convenient to take this as the *global* definition of ϕ_1, which we do.

Determination of an equation for h_1: With ϕ_1 determined, we now determine h_1. The point is that closer to Γ_t, further short range corrections may be needed, and it is the job of h_1 to provide these, if they are needed, so that (3.2) is satisfied at $\mathcal{O}(\lambda)$.

Because ϕ_1 is defined globally in Ω by (3.19), it follows immediately from (3.17) that

$$\|\phi_1\|_{\mathrm{Lip}(\Omega)} \leq \frac{C}{f'(1)}\ . \tag{3.20}$$

Moreover, with this definition

$$\lambda\Delta(\lambda\phi_1(\xi, t)) = \frac{\lambda}{f'(1)} m_0'\left(\frac{d(\xi, \Gamma_t)}{\lambda}\right) V_0(s(\xi), \Gamma_t)\ . \tag{3.21}$$

Now we examine (3.2) close to Γ_t to deduce an equation for h_1 and the motion of Γ_t. Since in (3.2) time enters simply as a parameter we avoid writing it in the following, when no confusion arises. We need to express the Laplacian in the (z, s) coordinate system. This is easily worked out to be

$$\lambda^2 \Delta f = (f_{zz} + \lambda^2 f_{ss}) - \lambda K(s) f_z - \lambda^2 K^2(s) z f_z + \mathcal{O}(\lambda^3) \ . \tag{3.22}$$

Using (3.22), (Here and in what follows, subscripted variables denote derivatives). we easily compute $\lambda \Delta m_1$ to $\mathcal{O}(\lambda)$. Note that because of (3.21), the term in (3.2) involving $\Delta \phi_1$ makes no contribution at order λ^0. As for m_0 and h_1, we have from (3.22) that

$$\lambda^2 \Delta h_1 \left(\frac{d(\xi, \Gamma)}{\lambda}, s(\xi, \Gamma) \right) = \frac{\partial^2}{\partial z^2} h_1 \left(z, s(\xi)\right) \Big|_{z = d(\xi,\Gamma)/\lambda} + \mathcal{O}(\lambda) \ .$$

and likewise

$$\lambda \Delta m_0 \left(\frac{d(\xi, \Gamma)}{\lambda} \right) = \left[\frac{1}{\lambda} \frac{\partial^2}{\partial z^2} - K(s(\xi)) \frac{\partial}{\partial z} \right] m_0 \left(z\right) \Big|_{z = d(\xi,\Gamma)/\lambda} + \mathcal{O}(\lambda) \ ,$$

In what follows, we shall use primes to denote derivatives with respect to z. We obtain from the calculations above and (3.2) that

$$\mu_0(\xi) = \frac{1}{\lambda} \left[-m_0''(z) + f(m_0(z)) \right] + \left[h_1''(z, s) + K(s) m_0'(z) + f'(m_0)(\phi_1 + h_1) \right] + \mathcal{O}(\lambda) \tag{3.23}$$

where on the right hand side $z = z(\xi)$ and $s = s(\xi)$ as in Section 2.

Observe first that the term proportional to λ^{-1} in (3.23) must vanish, and so m_0 must satisfy

$$-m_0''(z) + f(m_0(z)) = 0 \ .$$

This equation is satisfied by the free energy minimizing profile $\bar{m}$, and this forces m_0 to be equal to $\bar{m}$ – up to corrections that are exponentially small in λ. This is the case with m_0 as we have defined it.

Introducing the operator $\mathcal{L}$ defined by

$$\mathcal{L} g(z) = -g''(z) + f'(\bar{m}(z)) g(z) \ , \tag{3.24}$$

we write (3.23), replacing $f'(m_0)$ with $f'(\bar{m})$, as

$$\begin{aligned} \mathcal{L} h_1(z, s) &= \mu_0(z, s) - K(s) m_0'(z) - f'(m_0) \phi_1 + \mathcal{O}(\lambda) \\ &= \left(1 - \frac{f'(m_0(z))}{f'(1)} \right) \mu_0(z, s) - K(s) m_0'(z) + \mathcal{O}(\lambda) \ . \end{aligned} \tag{3.25}$$

We are finally in a position to determine an equation h_1. Notice that $f'(\bar{m})$ is even – indeed, $f'(\bar{m}) = 3(\bar{m})^2 - 1$. This means that $\mathcal{L}$ is a parity preserving operator; a fact we shall use later on. Now, with $F(m) = \frac{1}{4}(m^2-1)^2$, $f = F'$ gives

$$f(m) = (m^3 - m), \qquad f'(m) = (3m^2 - 1) \qquad \text{and} \qquad f''(m) = 6m \ . \tag{3.26}$$

For this potential, we have that $\bar{m}(x) = \tanh(x/\sqrt{2})$, and so

$$\bar{m}' = \frac{1}{\sqrt{2}}(1 - \bar{m}^2)$$

Hence it follows that

$$\left(1 - \frac{f'(\bar{m}(z))}{f'(1)}\right) = \frac{3}{2}(1 - \bar{m}^2(z)) = \frac{3}{\sqrt{2}}\bar{m}'(z) \ .$$

Therefore, (3.25) reduces to

$$\mathcal{L}h_1(z,s) = \left(\frac{3}{\sqrt{2}}\mu_0(z,s) - K(s)\right)\bar{m}'(z) + \mathcal{O}(\lambda) \ . \tag{3.27}$$

Since $\bar{m}'(z)$ tends to zero exponentially as $|z|$ increases, and since $\mu_0 = \mu_{0,0} + \mathcal{O}(\lambda)$, and since both are Lipschitz, we finally have

$$\mathcal{L}h_1(z,s) = \left(\frac{3}{\sqrt{2}}\mu_{0,0}(s,0) - K(s)\right)\bar{m}'(z) + \mathcal{O}(\lambda) \ . \tag{3.28}$$

Thus, we take as our equation for h_1,

$$\mathcal{L}h_1(z,s) = \left(\frac{3}{\sqrt{2}}\mu_{0,0}(s,0) - K(s)\right)\bar{m}'(z) \ . \tag{3.29}$$

Determination of V_0 via the Fredholm alternative: The operator $\mathcal{L}$ is self adjoint on $L^2(I\!R)$, and has a null space spanned by $\bar{m}'$. Therefore, the condition for solvability of $\mathcal{L}h_1 = g$ is

$$\int_{I\!R} g(z)\bar{m}'(z)\mathrm{d}z = 0 \ . \tag{3.30}$$

Evidently this is possible in the case at hand if and only if

$$\mu_{0,0}(s,0) = SK(s), \qquad \text{where} \qquad S = \frac{1}{4}\int_{I\!R}(\bar{m}'(z))^2\,dz = \frac{\sqrt{2}}{3} \tag{3.31}$$

in which case the right hand side of (3.28) vanishes up to $\mathcal{O}(\lambda)$, and so we take $h_1 \equiv 0$. In other words, the compatibility condition (3.30) forces (3.31) and allows us to take $h_1 = 0$, so that there is no short range correction at the first order.* (Short range corrections will be required at higher orders).

Next, we identify V_0: It is clear from (3.31) that $\mu_{0,0}$ is the Dirichlet extension of SK on Γ, with $S = \sqrt{2}/3$. By standard elements of the theory of single layer potentials (see the the appendix) and their Dirichlet data,

$$\mu_{0,0}(\xi,\Gamma) - \frac{1}{|\Gamma|}\int_\Gamma \mu_{0,0}(\eta)\mathrm{d}\eta = \int_\Gamma G(\xi,\eta)V_0(\eta)\mathrm{d}S_\eta - \frac{1}{|\Gamma|}\int_\Gamma\int_\Gamma G(\xi,\eta)V_0(\eta)\mathrm{d}S_\eta\mathrm{d}S_\xi \quad \xi\in\Omega \tag{3.32}$$

where, with $\mathcal{T}_\Gamma$ denoting the Dirichlet–Neumann operator for Γ,

$$V_0(\xi) = S\mathcal{T}_\Gamma\left(K(\cdot) - \frac{1}{|\Gamma|}\int_\Gamma K(s)\mathrm{d}s\right)(\xi) \ . \tag{3.33}$$

We see that (3.33) specifies the velocity field V_0, to be the one corresponding Mullins–Sekerka flow.

Determination of $c_0(t)$: So far, we have determined μ_0, and hence ϕ_1, only up to the additive constant $c_0(t)$. This can now be determined, completing the specification of m_1.

For any simple closed curve Γ, $\int_\Gamma K(s)\mathrm{d}s = 2\pi$, so that $\displaystyle\int_\Gamma \mu_{0,0}(\xi,\Gamma)\mathrm{d}S_\xi = 2\pi S$. Therefore, (3.32) becomes

$$\mu_{0,0}(\xi,\Gamma) = \int_\Gamma G(\xi,\eta)V_0(\eta)\mathrm{d}S_\eta - \frac{1}{|\Gamma|}\int_\Gamma\int_\Gamma G(\xi,\eta)V_0(\eta)\mathrm{d}S_\eta\mathrm{d}S_\xi + \frac{2\pi S}{|\Gamma|} \quad \xi\in\Omega \ .$$

Since we require that μ_0 simply be a "smeared" version of $\mu_{0,0}$, we must use this same constant as the constant $c_0(t)$ in (3.15). We finally have that

$$\begin{aligned}\mu_0(\xi,\Gamma) = &\int_\Omega G(\xi,\eta)\left(\frac{1}{\lambda}m_0'\left(\frac{d(\eta,\Gamma_t)}{\lambda}\right)V_0(s(\eta),t)\right)\mathrm{d}\eta \\ &- \frac{1}{|\Gamma|}\int_\Gamma\int_\Gamma G(\xi,\eta)V_0(\eta)\mathrm{d}S_\eta\mathrm{d}S_\xi + \frac{2\pi S}{|\Gamma|} \qquad \xi\in\Omega \ .\end{aligned} \tag{3.34}$$

and of course

$$\phi_1(\xi,\Gamma) = \frac{1}{f'(1)}\mu_0(\xi,\Gamma) = \frac{1}{2}\mu_0(\xi,\Gamma) \ . \tag{3.35}$$

* This is a consequence of the choice $f(m) = (m^3 - m)$.

Now that ϕ_1 is determined, the approximate solution $m_1(\xi,t)$ in (3.3) is completely specified.

3.2 Relation with the Hilbert expansion of kinetic theory

At this point, the analogy with the Hilbert expansion in kinetic theory [5] can be made clear. In this analogy, the Cahn–Hilliard equation corresponds to the Boltzmann equation

$$\frac{\partial f}{\partial t} + \nabla_x \cdot (vf) = \frac{1}{\lambda}\mathcal{Q}(f,f)$$

with a small parameter λ known as the Knudsen number. When λ is small, one must have $\mathcal{Q}(f,f) \approx 0$ and so $f \approx M$, a "local Maxwellian" density on phase space. This has the form

$$M(x,v,t) = \rho(x,t)\left(\frac{1}{2\pi\theta(x,t)}\right)^{3/2} e^{-|v-u(x,t)|^2/2\theta(x,t)}$$

In our problem, the function $\bar{m}\left(\frac{d(\xi,\Gamma_t)}{\lambda}\right)$ plays the role of a local Maxwellian. In the kinetic theory problem, to determine the evolution of the local Maxwellian, one just needs to determine the evolution of the "hydrodynamic moments" $\rho(x,t)$, $u(x,t)$ and $\theta(x,t)$. The functions "center" the local Maxwellian in exactly the same way that Γ_t centers the front $m_0\left(\frac{d(\xi,\Gamma_t)}{\lambda}\right)$ in our problem. In the Hilbert expansion, to leading order, one writes

$$f = M(1+\lambda h)$$

and seeks a solution of the equation in powers of λ just as we did here. The Fredholm criterion provides a compatibility condition for solving an equation involving the linearized Boltzmann operator, and this provides the equations of motion for $\rho(x,t)$, $u(x,t)$ and $\theta(x,t)$ just as the compatibility condition for solving an equation involving our operator $\mathcal{L}$ led to the conclusion that Γ_t evolves under the Mullins-Sekerka flow.

If one continues the Hilbert expansion to higher order, one obtains further refinements to the evolution equations for the hydrodynamical moments: Next come the Navier–Stokes equations, and then the Burnett equations. Continuing it still further, one can construct high order approximate solutions of the Boltzmann equation. These in turn, as was shown by Caflisch [4], can be used to produce solutions of the Boltzmann equation.

This has recently been extended to go beyond the appearance of the first shocks by Yu [16]. This very significant advance is based on the construction of a *generalized* Hilbert expansion which includes shock layer corrections. There is some analogy between these and the short range components h_j in our Hilbert expansion, and in any case, his work is a clear demonstration of the utility of developing generalized Hilbert expansions, such as the one considered here.

For recent work on a model with phase segregation, and hence quite directly relevant to the present paper, see [2].

Our goal in the next section is to push this analogy further, and to obtain higher order corrections to the evolution of Γ_t, and higher order approximate solutions of the Cahn–Hilliard equation, sufficient for showing that the unique solution of the Cahn–Hilliard equation with initial data representing phase segregation with a smooth interface has, for later times that are $\mathcal{O}(1)$, a smooth interface that has evolved according to the Mullins–Sekerka flow.

We next explicitly carry out the second order expansion, and then prove that the expansion can be continued to arbitrary order.

3.3 The prescription at second order

In this section, we prove Theorem 1.2. For this purpose, we seek an approximate solution m of the Cahn–Hilliard equation of the form

$$m_2(\xi,t) = m_0\left(\frac{d(\xi,\Gamma_t)}{\lambda}\right) + \lambda\phi_1(\xi,\Gamma_t) + \lambda^2\left[h_2(\xi,\Gamma_t) + \phi_2(\xi,\Gamma_t)\right] \tag{3.36}$$

where ϕ_1 is the function determined in the previous section, and h_2 and ϕ_2 are to be determined here, so that (3.1) and (3.2) is satisfied to $\mathcal{O}(\lambda^2)$. This time, we will require a short range correction, and h_2 will not vanish.

It is worth doing the expansion to second order explicitly. One reason is that new features concerning the compatibility conditions enter at second order, but after that, the pattern is essentially the same. The second reason is that this provides the form of the leading corrections to the Mullins–Sekerka flow.

Indeed, to carry out the expansion to second order, we let $\Gamma_t^{(1)}$ denote the solution to

$$\frac{\partial}{\partial t}\Gamma_t^{(1)} = V_0(\Gamma_t^{(1)}) + \lambda V_1(\Gamma_t^{(1)}) \qquad , \quad \Gamma_0^{(1)} = \Gamma_0\ , \tag{3.37}$$

where V_0 is the Mullins–Sekerka vector field on $\mathcal{M}$, and V_1, as it will be soon explained, is to be determined by two different types of compatibility conditions. Our first step will be to determine a higher order approximate chemical potential using (3.1). Keeping terms out to first order in λ in both m and the chemical potential μ, we have the equation:

$$\frac{\partial}{\partial t}\left(m_0\left(\frac{d(\xi,\Gamma_t^{(1)})}{\lambda}\right) + \lambda\phi_1(\xi,\Gamma_t^{(1)})\right) = \Delta(\mu_0 + \lambda\mu_1)\ . \tag{3.38}$$

The quantity $\phi_1(\xi,\Gamma_t^{(1)})$ is not so easy to differentiate, even apart from the fact that $\Gamma_t^{(1)}$ is evolving under $V_0 + \lambda V_1$. We must first obtain an equation specifying V_1, and for this purpose, we must isolate the leading contribution from the evolution under V_0: For each t, let $\tilde\Gamma_{t+s}$ be given by

$$\frac{\mathrm{d}}{\mathrm{d}s}\tilde\Gamma_{t+s} = V_0(\tilde\Gamma_{t+s}) \qquad \text{with} \qquad \tilde\Gamma_t = \Gamma_t^{(1)}\ .$$

We then define $D_{V_0}\phi_1$ by

$$D_{V_0}\phi_1(\xi, \Gamma_t^{(1)}) = \lim_{s\to 0} \frac{1}{s}\left(\phi_1(\xi, \tilde{\Gamma}_{t+s}) - \phi_1(\xi, \tilde{\Gamma}_t)\right) . \tag{3.39}$$

This way,

$$\frac{\partial}{\partial t}\phi_1(\xi, \Gamma_t^{(1)}) = D_{V_0}\phi(\xi, \Gamma_t^{(1)}) + \mathcal{O}(\lambda)$$

since in computing $D_{V_0}\phi(\xi, \Gamma_t^{(1)})$ we have only suppressed λV_1. We therefore replace (3.38) by

$$\frac{\partial}{\partial t}\left(m_0\left(\frac{d(\xi, \Gamma_t^{(1)})}{\lambda}\right)\right) + \lambda D_{V_0}\phi_1(\xi, \Gamma_t^{(1)}) = \Delta(\mu_0 + \lambda\mu_1) . \tag{3.40}$$

The compatibility condition for the solvability of (3.40) is that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(m_0\left(\frac{d(\xi, \Gamma_t^{(1)})}{\lambda}\right)\right)\mathrm{d}\xi + \lambda\int_\Omega D_{V_0}\phi_1(\xi, \Gamma_t^{(1)})\mathrm{d}\xi = 0 . \tag{3.41}$$

As one sees from the formula (3.35) for ϕ_1, there is no reason that $\int_\Omega D_{V_0}\phi_1(\xi, \Gamma_t^{(1)})\mathrm{d}\xi$ will vanish in general. We shall deduce a formula for this quantity in the next section, but what is relevant now is that it must be cancelled by the term in (3.41) involving m_0. When $\Gamma_t^{(1)}$ evolves under (3.37), we have that

$$\frac{\partial}{\partial t}m_0\left(\frac{d(\xi, \Gamma_t^{(1)})}{\lambda}\right) = \frac{1}{\lambda}m_0'\left(\frac{d(\xi, \Gamma_t^{(1)})}{\lambda}\right)[V_0(s(\xi)) + \lambda V_1(s(\xi))] .$$

Integrating the right hand side over Ω, we find

$$\int m_0'(z)\mathrm{d}z \int_{\Gamma_t^{(1)}} [V_0(s) + \lambda V_1(s)]\,\mathrm{d}s = 2\lambda\int_{\Gamma_t^{(1)}} V_1(s)\mathrm{d}s .$$

Hence, the compatibility condition for solvability of (3.38) will hold if and only if

$$2\int_{\Gamma_t^{(1)}} V_1(s)\mathrm{d}s = -\int_\Omega D_{V_0}\phi_1(\xi, \Gamma_t^{(1)})\mathrm{d}\xi . \tag{3.42}$$

We therefore decompose V_1 into two pieces $V_1 = V_1^{(0)} + \langle V_1\rangle_\Gamma$ as in (2.9). The part $\langle V_1\rangle_\Gamma$ denotes the average of V_1 over Γ. It has a constant value is determined by (3.42):

$$\langle V_1\rangle_\Gamma = -\frac{1}{2|\Gamma_t^{(1)}|}\int_\Omega D_{V_0}\phi_1(\xi, \Gamma_t^{(1)})\mathrm{d}\xi . \tag{3.43}$$

where the ϕ_1 is given in (3.19). (This is somewhat different from the expression for $\langle V_1 \rangle$ given in Theorem 1.2, but as we shall see, the difference is $\mathcal{O}(\lambda)$).

The part $V_1^{(0)}$ will be orthogonal to the constants. It will be determined by a compatibility condition that arises when we solve for h_2. The pattern is the same in all higher orders: The constant part of the kth order velocity field, $\langle V_k \rangle_\Gamma$ will be determined by the compatibility condition needed to solve Laplace's equation for μ_k. The non-constant part $V_k^{(0)}$ will be determined by the compatibility condition needed to solve for h_k.

Since $\langle V_1 \rangle_\Gamma$ is not zero, the area enclosed by $\Gamma_t^{(1)}$ as it evolves under (3.37) will not be constant. This should not be surprising. Only at the sharp interface limit does the conservation of $\int_\Omega m(\xi, t)\mathrm{d}\xi$ coincide with the conservation of the area enclosed by Γ_t. At higher order, the interactions between the shape of the curve and shape of the interface matter. With this choice of $\langle V_1 \rangle_\Gamma$, the compatibility condition (3.41) is satisfied and we can now solve (3.40) for $\mu_0 + \lambda\mu_1$. First, we use the full evolution, under (3.37), to differentiate the first term in (3.40). Then we apply the Green's function to each of the pieces. Taking in account (3.14) we obtain that

$$\mu_1(\xi, t) = \frac{1}{\lambda} \int_\Omega G(\xi, \eta) m_0' \left(\frac{d(\xi, \Gamma_t^{(1)})}{\lambda} \right) V_1(s(\eta))\mathrm{d}\eta + p(\xi, t) + c_1(t) \tag{3.44}$$

where we set

$$p(\xi, t) = \int_\Omega G(\xi, \eta) \left[D_{V_0} \phi_1(\xi, \Gamma_t^{(1)}) \right] \mathrm{d}\eta \ , \tag{3.45}$$

and $c_1(t)$ is a constant (in ξ) to be determined. (Again, this is somewhat different from the expression for $p(\xi, t)$ given in Theorem 1.2, but as we shall see, the difference is $\mathcal{O}(\lambda)$). In the next section, we shall derive a more explicit formula, at least in terms of potential theory, for $p(\xi, t)$. For the time being, it is convenient to work with this compact form. The first term on the right is an approximate single layer potential and it is still to be fully determined, since we do not know $V_1^{(0)}$ yet.

Toward this end, we first determine ϕ_2. As before, consider ξ far from $\Gamma_t^{(1)}$ where $m_0^2 - 1$ and h_2 are negligible. We then have

$$\mu_0 + \lambda\mu_1 \approx \frac{1}{\lambda} f\left(1 \pm \lambda\phi_1 + \lambda^2\phi_2\right) \approx f'(1)\left(\phi_1 + \lambda\phi_2\right) + \frac{f''(1)}{2}\lambda\phi_1^2 \ .$$

Since $\mu_0 = f'(1)\phi_1$, we define ϕ_2 by $\phi_2 = \dfrac{1}{f'(1)}\left[\mu_1 - \dfrac{f''(1)}{2(f'(1))^2}\mu_0^2\right] = \dfrac{1}{2}\mu_1 + \dfrac{3}{8}\mu_0^2$. As before, we use this definition globally in Ω, and ϕ_2 is Lipschitz with a norm bounded independently of λ. We now return to (3.2), and replace μ^λ by $\mu_0 + \lambda\mu_1$

and m^λ by (3.36). From (3.22), and the potential theoretic definition of ϕ_1,

$$
\begin{aligned}
&-\lambda\Delta(m_0+\lambda\phi_1+\lambda^2 h_2+\lambda^2\phi_2) = \\
&-\frac{1}{\lambda}[m_0''] + Km_0' + \lambda\left[-h_2'' + \frac{1}{f'(1)}m_0' V_0 + K^2 z m_0'\right] + \mathcal{O}(\lambda^2)\ .
\end{aligned}
$$

Clearly,

$$
\begin{aligned}
&\frac{1}{\lambda}f(m_0+\lambda\phi_1+\lambda^2 h_2+\lambda^2\phi_2) = \\
&\frac{1}{\lambda}f(m_0) + f'(m_0)\phi_1 + \lambda f'(m_0)[h_2+\phi_2] + \lambda\frac{f''(m_0)}{2}\phi_1^2\ .
\end{aligned}
$$

Hence from (3.2), since $\bar{m}$ solves (3.7), and since $m_0=\bar{m}$ up to exponentially small corrections,

$$
\begin{aligned}
\mu_0+\lambda\mu_1 = {}& [Km_0' + f'(m_0)\phi_1] \\
&+\lambda\left[\mathcal{L}h_2 + K^2 z m_0' + \frac{1}{f'(1)}m_0' V_0 + f'(m_0)\phi_2 + \frac{1}{2}f''(m_0)\phi_1^2\right] + \mathcal{O}(\lambda^2)\ .
\end{aligned}
$$

Therefore, since $\phi_1 = (1/f'(1))\mu_0 = (1/2)\mu_0$, we can use the identity $(1 - f'(m_0)/f'(1)) = (3/\sqrt{2})m_0'$, and have

$$
\begin{aligned}
\mu_1 = {}& \frac{1}{\lambda}\left(K - \frac{3}{\sqrt{2}}\mu_0\right)m_0' \\
&+\left[\mathcal{L}h_2 + K^2 z m_0' + \frac{1}{f'(1)}m_0' V_0 + f'(m_0)\phi_2 + \frac{1}{2}f''(m_0)\phi_1^2\right] + \mathcal{O}(\lambda)\ .
\end{aligned}
\tag{3.46}
$$

Notice that the first term on the right is bounded uniformly in λ because μ_0 is Lipschitz, $\mu_0 = \mu_{0,0} + \mathcal{O}(\lambda)$, and m_0' decays rapidly. From (3.46) we obtain

$$
\begin{aligned}
\mathcal{L}h_2 = {}& \mu_1 - \frac{1}{\lambda}\left(K - \frac{3}{\sqrt{2}}\mu_0\right)m_0' \\
&-\left[K^2 z m_0' + \frac{1}{f'(1)}m_0' V_0 + f'(m_0)\phi_2 + \frac{1}{2}f''(m_0)\phi_1^2\right] + \mathcal{O}(\lambda).
\end{aligned}
\tag{3.47}
$$

To solve (3.47), the compatibility condition (3.30) need to be satisfied. This condition will determine $V_1^{(0)}$ and therefore μ_1 will be fully determined. Denote

$$
g_1(s) = \int_{I\!R}\left[K^2 z m_0' + \frac{1}{f'(1)}m_0' V_0 + f'(m_0)\phi_2 + \frac{1}{2}f''(m_0)\phi_1^2\right]m_0'\mathrm{d}z.
$$

Since $m_0'(z)$ is even, $f''(m_0) = 6m_0$, ϕ_1 and ϕ_2 have a Lipschitz bound independently on λ we obtain

$$
\begin{aligned}
g_1(s) &= \int_{I\!R} \left[\frac{1}{f'(1)} m_0' V_0 + f'(m_0)\phi_2(0,s) + \frac{1}{2} f''(m_0)\phi_1^2(0,s) \right] m_0' \mathrm{d}z + \mathcal{O}(\lambda) \\
&= 2SV_0(s) + \mathcal{O}(\lambda).
\end{aligned}
\tag{3.48}
$$

where S is the surface tension defined in (3.31). We next investigate

$$
g_2(s) = \frac{1}{\lambda} \int_{I\!R} \left(K - \frac{3}{\sqrt{2}} \mu_0 \right) (m_0')^2(z) \mathrm{d}z.
$$

Then

$$
g_2(s) = \frac{1}{\lambda} \int_{I\!R} \left(K - \frac{3}{\sqrt{2}} \mu_{0,0} \right) (m_0')^2(z) \mathrm{d}z - \frac{3}{\sqrt{2}\lambda} \int_{I\!R} (\mu_0 - \mu_{0,0}) \left(m_0'(z) \right)^2 \mathrm{d}z \ .
$$

The second term, involving the difference between the "smeared" and exact single layer potentials is easily seen to be $\mathcal{O}(\lambda^2)$. As for the first one, note that we have $\mu_{0,0}(z,s) = SK + a\lambda z + \mathcal{O}(\lambda^2)$ for $z > 0$ and $\mu_{0,0}(z,s) = SK + b\lambda z + \mathcal{O}(\lambda^2)$ for $z > 0$. The quantity $b - a$ is just the jump in the normal derivative of $\mu_{0,0}$ across the interface at s, which is $V_0(s)$. Hence with C defined by

$$
C = \int_{I\!R} |z| (m_0')^2 \mathrm{d}z = \frac{4\ln(2) - 1}{6} \ , \tag{3.49}
$$

we have

$$
g_2(s) = CV_0(s) + \mathcal{O}(\lambda) \ . \tag{3.50}
$$

The compatibility condition that we must have in order to solve for h_2 is that

$$
\int \mu_1(\lambda z, s) m_0'(z) \mathrm{d}z = g_1(s) + g_2(s) \ , \tag{3.51}
$$

where μ_1 is given in (3.44). It is convenient to single out from μ_1 the part still unknown which will be determined so that (3.51) holds. We denote

$$
\mu_{1,0}(\xi, \Gamma) = \frac{1}{\lambda} \int_\Omega G(\xi,\eta) m_0' \left(\frac{d(\xi,\Gamma)}{\lambda} \right) V_1^{(0)}(s(\eta)) \mathrm{d}\eta + c_1(t) \tag{3.52}
$$

and

$$
\tilde{\mu}_1(\xi, \Gamma) = < V_1 > \frac{1}{\lambda} \int_\Omega G(\xi,\eta) m_0' \left(\frac{d(\eta,\Gamma}{\lambda} \right) \mathrm{d}\eta + p(\xi,\Gamma) \ . \tag{3.53}
$$

Taking in account (3.48) and (3.50) we then write (3.51) as

$$\int \mu_{1,0}(\lambda z, s) m_0'(z) \mathrm{d}z = (2S+C)V_0 - \int \tilde{\mu}_1(\lambda z, s) m_0'(z) \mathrm{d}z = (2S+C)V_0 - 2\tilde{\mu}_1(0,s) + \mathcal{O}(\lambda)$$

having $\tilde{\mu}_1$ a Lipschitz bound independent on λ. As in the previous section, we may substitute $\mu_{1,0}$ by the corresponding single layer potential

$$\mu_{1,0,0}(\xi, \Gamma) = 2 \int_\Gamma G(\xi, \eta) V_1^{(0)}(\eta) \mathrm{d}S_\eta + c_1(t) \tag{3.54}$$

and further restrict to $z = 0$, obtaining

$$\begin{aligned} &\int \mu_{1,0,0}(0,s) m_0'(z) \mathrm{d}z = 2\mu_{1,0,0}(0,s) = (2S+C)V_0 - 2\tilde{\mu}_1(0,s) \\ &= (2S+C)V_0 + 4\langle V_1 \rangle_\Gamma \int_\Gamma G(s,\eta) \mathrm{d}S_\eta + 2p(s) + \mathcal{O}(\lambda). \end{aligned} \tag{3.55}$$

Inserting (3.54) into (3.55) and integrating over Γ we obtain that

$$\begin{aligned} 2c_1(t) = &-\frac{4}{|\Gamma|} \left\{ \int_\Gamma \int_\Gamma G(\xi,\eta) V_1^{(0)}(\eta) \mathrm{d}S_\eta \mathrm{d}S_\xi + \int_\Gamma \mathrm{d}S_\xi \int_\Gamma G(\xi,\eta) \mathrm{d}S_\eta \right\} \\ &- \frac{2}{|\Gamma|} \int_\Gamma p(\eta) \mathrm{d}S_\eta \ . \end{aligned} \tag{3.56}$$

In this way $c_1(t)$ is written in term of $V_1^{(0)}$, still to be determined. Taking in account (3.56) we obtain from (3.55) an equation for $V_1^{(0)}$

$$\begin{aligned} \mathcal{S}_\Gamma V_1^{(0)}(\xi) = &\frac{1}{4}(2S+C)V_0(\xi) - \langle V_1 \rangle_\Gamma \left[\int_\Gamma G(\xi,\eta) \mathrm{d}S_\eta - \frac{1}{|\Gamma|} \int_\Gamma \mathrm{d}S_\xi \int_\Gamma G(\xi,\eta) \mathrm{d}S_\eta \right] \\ &- \frac{1}{2} \left[p(\xi) - \frac{1}{|\Gamma|} \int_\Gamma p(\eta) \mathrm{d}S_\eta \right] \ , \end{aligned}$$

where $\mathcal{S}_\Gamma$ is the operator defined in (10.9). Finally applying the Dirichlet-Neumann operator, see (10.8) we obtain

$$\begin{aligned} V_1^{(0)} = &\frac{1}{4}(2S+C)\mathcal{T}_\Gamma V_0 - \langle V_1 \rangle_\Gamma \mathcal{T}_\Gamma \left[\int_\Gamma G(\cdot,\eta) \mathrm{d}S_\eta - \frac{1}{|\Gamma|} \int_\Gamma \mathrm{d}S_\xi \int_\Gamma G(\xi,\eta) \mathrm{d}S_\eta \right] \\ &- \frac{1}{2} \mathcal{T}_\Gamma \left[p(\cdot) - \frac{1}{|\Gamma|} \int_\Gamma p(\eta) \mathrm{d}S_\eta \right] \ . \end{aligned} \tag{3.57}$$

With $V_1 = V_1^{(0)} + \langle V_1 \rangle_\Gamma$ determined, we can solve (3.46) for h_2 and we have our approximation.

At this stage it is a simple matter to prove Theorem 1.2:

Proof of Theorem 1.2 We have seen that up to an adjustment of size $\mathcal{O}(\lambda)$, we must have that $\langle V_1 \rangle_\Gamma$ is given by (3.43). To see that this agrees, up to an adjustment of size $\mathcal{O}(\lambda)$, with the expression (1.15) given in Theorem 1.2, note first that by (3.19), and the fact that $f'(1) = 2$, $\phi_1 = \mu_0/2$. It then remains to show that

$$D_{V_0}\mu_0(\xi, \Gamma) = D_{V_0}\mu_{0,0}(\xi, \Gamma) + \mathcal{O}(\lambda) \ . \tag{3.58}$$

For this purpose, let $s \mapsto \eta(s)$ be any arclength parameterization of Γ, and let Γ_z denote the curve parametrized by $s \mapsto \eta(s) + \lambda z n(s)$, using the notation of Section 2. Then, by the definition of μ_0, (3.14), and from (3.12), we have

$$2\mu_0(\xi, \Gamma) = \int_{I\!R} \mu_{0,0}(\xi, \Gamma_z) m_0'(z) \mathrm{d}z + \mathcal{O}(\lambda) \ ,$$

and

$$2D_{V_0}\mu_0(\xi, \Gamma) = \int_{I\!R} D_{V_0}\mu_{0,0}(\xi, \Gamma_z) m_0'(z) \mathrm{d}z + \mathcal{O}(\lambda) \ .$$

To draw the desired conclusion, we must know that $\Gamma \mapsto D_{V_0}\mu_0(\xi, \Gamma)$ is continuous from $\mathcal{M}$ to, say, $L^2(\Omega)$. This can be seen from formula (4.12) in the next section, and hence

$$D_{V_0}\mu_{0,0}(\xi, \Gamma_z) = D_{V_0}\mu_{0,0}(\xi, \Gamma) + \mathcal{O}(\lambda) \ ,$$

in $L^2(\Omega)$, which gives us (3.58).

Next, we have seen that $V_1^{(0)}$ must be given by (3.57), up to an adjustment of size $\mathcal{O}(\lambda)$. This coincides with the formula (1.16) in Theorem 1.2, except that the formulae for p differ: The formula (1.17) in Theorem 1.2 involves $D_{V_0}\mu_{0,0}(\xi, \Gamma)$, while the formula (3.45) for the p in (3.57) involves $D_{V_0}\mu_0(\xi, \Gamma)$. However, by (3.58) once again, these differ by $\mathcal{O}(\lambda)$. $\square$

4. Some differentiation formulas

4.1 The general problem

In this section, we produce a potential theoretic formula for the rate of change of V_0 under its own time evolution. This permits us to give a potential theoretic formula for the function $D_{V_0}\phi_1(\xi, \Gamma_t^{(1)})$, and hence $p(\xi, t)$, so that (1.16) becomes more explicit.

From the formula (3.34) for μ_0 and hence ϕ_1, we see that the main problem to be dealt with here is of the following type: Suppose that we are given two vector fields V and W on $\mathcal{M}$. Suppose further that the first vector field does not affect the enclosed area; i.e., for all Γ, $\int_\Gamma V(s)\mathrm{d}s = 0$. Using V, form the Neumann harmonic extension (see the Appendix)

$$\psi_V(\xi,\Gamma) = \int_\Gamma G(\xi,\eta)V(\eta,\Gamma)\mathrm{d}S_\eta - \frac{1}{|\Gamma|}\int_\Gamma\int_\Gamma G(\xi,\eta)V(\eta,\Gamma)\mathrm{d}S_\eta\mathrm{d}S_\xi \qquad \xi\in\Omega \tag{4.1}$$

and take

$$\frac{\mathrm{d}\Gamma_t}{\mathrm{d}t} = W(\Gamma_t)\ , \qquad \Gamma_0 = \Gamma\ . \tag{4.2}$$

We wish to compute $D_W\psi_V = \dfrac{\mathrm{d}}{\mathrm{d}t}\psi_V(\xi,\Gamma_t)$. In the particular case that V is V_0, the Mullins–Sekerka vector field, $\psi_{V_0}(\xi,\Gamma) = \mu_{0,0}$, and hence if we take $W = V_0$ as well, the quantity we are computing is $D_{V_0}\mu_{0,0}$, which figures in Theorem 1.2. We first derive a general result. We parameterize Γ_t by the arc length of Γ as follows: Let $s\mapsto\xi(s)$ be an arc length parametrization of Γ. For t sufficiently small, every point on Γ_t belongs to $\mathcal{N}(\lambda_0,\Gamma)$, for some strictly positive $\lambda_0\le\frac{1}{\kappa(\Gamma)}$. Hence we can use the coordinates introduced in Section 2 to write a parametrization

$$s\mapsto\xi(s)+r_{\Gamma_t}(s)n(s) \qquad 0\le s\le|\Gamma|$$

of Γ_t. Clearly, $r_{\Gamma_t}(s) = W(s)t+\mathcal{O}(t^2)$. Let $V(s,t)$ be the coordinate representation of a vector field $V(\cdot,\Gamma_t)$ on $\mathcal{M}$ at Γ_t. That is, if $n(s,t)$ is the outward normal to Γ_t at $\xi(s)+r_{\Gamma_t}(s)n(s)$, then $V(\cdot,\Gamma_t)$ is given by the vector field

$$V(s,t)n(s,t)$$

on Γ_t. We seek a formula for $\partial V(s,t)/\partial t$. In the case of the Mullins–Sekerka vector field, and others that we shall encounter here, V is explicitly defined through the Dirichlet–Neumann operator, or more precisely, its inverse: We are given a function $f(\cdot,\Gamma)$ such that $\int_\Gamma f(s,\Gamma)\mathrm{d}s = 0$, and then $V(\cdot,\Gamma) = \mathcal{T}_\Gamma f(\cdot,\Gamma)$. For the Mullins–Sekerka vector field, as we have seen,

$$f(s,\Gamma) = SK(s,\Gamma) - \frac{S}{|\Gamma|}\int_\Gamma K(s,\Gamma)\mathrm{d}s = S\left(K(s,\Gamma)-\frac{2\pi}{|\Gamma|}\right)\ . \tag{4.3}$$

It is relatively easy to compute the evolution of $f(\cdot,\Gamma)$ as Γ evolves according to (4.2). We will use this to compute $\dfrac{\partial}{\partial t}V(\cdot,t)$ in terms of $\dfrac{\partial}{\partial t}f(\cdot,\Gamma_t)$. In the following we will denote simply by $K(s)\equiv K(s,\Gamma)$ being Γ the initial curve for the evolution (4.2). The result is the following:

Theorem 4.1 *Let V and W be two smooth vector fields on $\mathcal{M}$, and suppose that V is defined through $V = \mathcal{T}_{\Gamma_t}(f(\cdot,\Gamma_t))$. Suppose that Γ_t evolves according to (4.2). Let $\mathcal{Q}$ be the operator on $L^2(\Gamma)$ defined by*

$$\mathcal{Q}_{\Gamma,W}h(s) = \int_0^{|\Gamma|}\left[\nabla_\xi G(\xi(s),\eta(r))\cdot W(s)n(s) + \nabla_\eta G(\xi(s),\eta(r))\cdot W(r)n(r)\right]h(r)\mathrm{d}r\ . \tag{4.4}$$

Then we have

$$\begin{aligned}
&\frac{\partial}{\partial t}V(s,0) + W(s)K(s)V(s,0) \\
&\qquad = \mathcal{T}_\Gamma\left(\frac{\partial}{\partial t}f(s,0) + \frac{1}{|\Gamma|}\int_0^{|\Gamma|} f(s,0)W(s)K(s)\mathrm{d}s\right) \\
&\qquad - \mathcal{T}_\Gamma\left(\mathcal{Q}_{\Gamma,W}V(s,0) - \frac{1}{|\Gamma|}\int_\Gamma \mathcal{Q}_{\Gamma,W}V(s,0)\mathrm{d}s\right) \ .
\end{aligned} \tag{4.5}$$

Notice that the operator $\mathcal{Q}_{\Gamma,W}$ is a bounded smoothing operator, the singularities in the two derivatives of the Green's function cancel.

Proof: In the proof, we shall drop some subscripts. A simple computation shows that the element of arc length along Γ_t in the parameterization that we employ is

$$\rho(s,t)\mathrm{d}s = \left((1+w(s,t)K(s))^2 + (\partial w(s,t)/\partial s)^2\right)^{1/2}\mathrm{d}s \qquad 0 \le s \le |\Gamma| \ .$$

Again, it is easy to see that

$$\rho(s,t) = 1 + tW(s)K(s) + \mathcal{O}(t^2) \ .$$

Since, as explained around (10.10) of the appendix, $f(\cdot,\Gamma_t) = \mathcal{S}_{\Gamma_t}V(\cdot,\Gamma_t)$ we have

$$\begin{aligned}
f(s,t) = &\int_0^{|\Gamma|} G(\xi(s)+w(s,t)n(s), \eta(r)+w(r,t)n(r))V(r,t)\rho(r,t)\mathrm{d}r \\
&- \frac{1}{|\Gamma_t|}\int_0^{|\Gamma|}\int_0^{|\Gamma|} G(\xi(s)+w(s,t)n(s), \eta(r)+w(r,t)n(r))V(r,t)\rho(r,t)\mathrm{d}r\rho(s,t)\mathrm{d}s.
\end{aligned}$$

Recalling the definition (4.4) of $\mathcal{Q}_{\Gamma,W}$, we have that

$$\begin{aligned}
\frac{\partial}{\partial t}f(s,0) = &\ \mathcal{S}_\Gamma\left(\frac{\partial}{\partial t}V(s,0) + W(s)K(s)V(s,0)\right) \\
&+ \mathcal{Q}_{\Gamma,W}V(s,0) - \frac{1}{|\Gamma|}\int_0^{|\Gamma|}\mathcal{Q}_{\Gamma,W}V(s,0)\mathrm{d}s \\
&+ \frac{|\Gamma|'}{|\Gamma|^2}\int_0^{|\Gamma|}\int_0^{|\Gamma|} G(\xi(s),\eta(r))V(r,0)\mathrm{d}r\mathrm{d}s \\
&- \frac{1}{|\Gamma|}\int_0^{|\Gamma|} W(s)K(s)\left(\int_0^{|\Gamma|} G(\xi(s),\eta(r))V(r,0)\mathrm{d}r\right)\mathrm{d}s \ .
\end{aligned}$$

where

$$\frac{d|\Gamma_t|}{\mathrm{dt}}\Big|_{t=0} \equiv |\Gamma|' = \int_0^{|\Gamma|} K(s)W(s)\mathrm{d}s \; . \tag{4.6}$$

From the definition of $f(s,t)$ and (4.6) one easily recognizes the contribution of the last two terms as $-\frac{1}{|\Gamma|}\int_0^{|\Gamma|} f(s,0)W(s)K(s)\mathrm{d}s$. Then since, by assumption, $\int_0^{|\Gamma|} f(s,t)\rho(s,t)\mathrm{d}s = 0$ identically in t, we have that

$$\int_0^{|\Gamma|} \frac{\partial}{\partial t} f(s,0)\mathrm{d}s = -\int_0^{|\Gamma|} f(s,0)W(s)K(s)\mathrm{d}s \; .$$

Combining results, we have the following identity:

$$\begin{aligned}
&\frac{\partial}{\partial t} f(s,0) + \frac{1}{|\Gamma|}\int_0^{|\Gamma|} f(s,0)W(s)K(s)\mathrm{d}s \\
&\qquad = \mathcal{Q}_{\Gamma,W}V(s,0) - \frac{1}{|\Gamma|}\int_0^{|\Gamma|} \mathcal{Q}_{\Gamma,W}V(s,0)\mathrm{d}s \\
&\qquad + \mathcal{S}_\Gamma\left(\frac{\partial}{\partial t}V(s,0) + W(s)K(s)V(s,0)\right) \; .
\end{aligned}$$

Applying the Dirichlet–Neumann operator, see (10.8), we have the result. □

4.2 Application to $D_{V_0}\,\mu_{0,0}$

Let Γ evolve under (4.2), from (4.3), using (4.6) again, we obtain

$$\frac{\partial}{\partial t} f(s,0) = S\frac{\partial}{\partial t}K(s,t)\Big|_{t=0} + \frac{2\pi S}{|\Gamma|^2}\int_\Gamma K(s)W(s)\mathrm{d}s \; .$$

A well known computation yields the result that

$$\frac{\partial}{\partial t}K(s,t)\Big|_{t=0} = -\left(\frac{\mathrm{d}^2}{\mathrm{d}s^2}W(s) + K(s)^2W(s)\right) \tag{4.7}$$

Hence in this case, since from (4.3) $SK(s) - f(s,0) = S\frac{2\pi}{|\Gamma|}$ we obtain

$$\begin{aligned}
&\frac{\partial}{\partial t} f(s,0) + \frac{1}{|\Gamma|}\int_0^{|\Gamma|} f(s,0)W(s)K(s)\mathrm{d}s \\
&\qquad = S\left(-\frac{\mathrm{d}^2}{\mathrm{d}s^2}W(s) - K^2(s)W(s) + \frac{1}{|\Gamma|}\int_0^{|\Gamma|} K(s)^2W(s)\mathrm{d}s\right) \; .
\end{aligned} \tag{4.8}$$

We now have what we need to compute the derivative in t, along the Mullins–Sekerka flow, V_0, of $\psi_{V_0}(\cdot,\Gamma_t) \equiv \mu_{0,0}(\cdot,\Gamma_t)$. We first compute the derivative in t under a flow generated by W of $\psi_V(\cdot,\Gamma_t)$, see (4.1). Then we set $V = V_0$ and $W = V_0$. A computation just as in the proof of Theorem 4.1 yields

$$
\begin{aligned}
&\frac{\partial}{\partial t}\psi_V(\xi,\Gamma_t)\bigg|_{t=0} = \\
&\quad \int_\Gamma \nabla_\eta G(\xi,\eta)\cdot n(\eta)W(\eta)V(\eta,0)\mathrm{d}S_\eta \\
&\quad + \int_\Gamma G(\xi,\eta)\left(\frac{\partial}{\partial t}V(\eta,0) + K(\eta)W(\eta)V(\eta)\right)\mathrm{d}S_\eta \\
&\quad - \frac{1}{|\Gamma|}\int_\Gamma \mathcal{Q}_{\Gamma,W}V(\eta,0)\mathrm{d}S_\eta + \frac{|\Gamma|'}{|\Gamma|^2}\int_\Gamma\int_\Gamma G(\xi,\eta)V(\eta,0)\mathrm{d}S_\eta\mathrm{d}S_\xi \\
&\quad - \frac{1}{|\Gamma|}\int_\Gamma\int_\Gamma G(\xi,\eta)\left(\frac{\partial}{\partial t}V(\eta,0) + K(\eta)W(\eta)V(\eta,0) + K(\xi)W(\xi)V(\eta)\right)\mathrm{d}S_\eta\mathrm{d}S_\xi \ .
\end{aligned}
$$

This simplifies to

$$
\begin{aligned}
\frac{\partial}{\partial t}\psi_V(\xi,\Gamma_t)\bigg|_{t=0} &= \int_\Gamma \nabla_\eta G(\xi,\eta)\cdot n(\eta)W(\eta)V(\eta,0)\mathrm{d}S_\eta \\
&\quad + \mathcal{E}_{\Gamma,N}\left(\frac{\partial}{\partial t}V(\eta,0) + K(\eta)W(\eta)V(\eta,0)\right) - \frac{1}{|\Gamma|}\int_\Gamma \mathcal{Q}_{\Gamma,W}V(\eta,0)\mathrm{d}S_\eta \\
&\quad + \frac{|\Gamma|'}{|\Gamma|^2}\int_\Gamma\int_\Gamma G(\xi,\eta)V(\eta,0)\mathrm{d}S_\eta\mathrm{d}S_\xi \\
&\quad - \frac{1}{|\Gamma|}\int_\Gamma\int_\Gamma G(\xi,\eta)K(\xi)W(\xi)V(\eta,0)\mathrm{d}S_\eta\mathrm{d}S_\xi \ .
\end{aligned}
\tag{4.9}
$$

where $\mathcal{E}_{\Gamma,N}$ is the operator defined in (10.12). From (4.6), arguing as before, the contribution of the the last two terms in (4.9) is given by

$$
-\frac{1}{|\Gamma|}\int_0^{|\Gamma|}\psi_V(\xi,0)W(s)K(s)\mathrm{d}s \ . \tag{4.10}
$$

From (3.31) and (3.32) when $\xi\in\Gamma$ and $V = V_0$, we have that

$$
\begin{aligned}
&SK(\xi) - \frac{S}{|\Gamma|}\int_\Gamma K(s)\mathrm{d}s \\
&= \int_\Gamma G(\xi,\eta)V_0(\eta,0)\mathrm{d}S_\eta - \frac{1}{|\Gamma|}\int_\Gamma\int_\Gamma G(\xi,\eta)V_0(\eta,0)\mathrm{d}S_\eta\mathrm{d}S_\xi = \psi_{V_0}(\xi,\Gamma) \ .
\end{aligned}
\tag{4.11}
$$

Then (4.10), when $V = V_0$, is equal to

$$-S\left(\frac{1}{|\Gamma|}\int_\Gamma K^2(\xi)W(\xi)\mathrm{d}S_\xi - \frac{2\pi}{|\Gamma|^2}\int_\Gamma K(\xi)W(\xi)\mathrm{d}S_\xi\right) .$$

Now using (4.8), Theorem 4.1 and the Dirichlet extension operator $\mathcal{E}_{\Gamma,D}$, (defined in (10.13) of the appendix), we have

$$\begin{aligned}
\frac{\partial}{\partial t}\psi_{V_0}(\xi,\Gamma_t)\Big|_{t=0} &= \int_\Gamma \nabla_\eta G(\xi,\eta)\cdot n(\eta)W(\eta)V_0(\eta,0)\mathrm{d}S_\eta - \mathcal{E}_{\Gamma,D}(\mathcal{Q}_{\Gamma,W}V_0(\eta)) \\
&+ \mathcal{E}_{\Gamma,N}\left(S\mathcal{T}_\Gamma\left(-\frac{\mathrm{d}^2}{\mathrm{d}s^2}W(s) - K(s)^2W(s) + \frac{1}{|\Gamma|}\int_0^{|\Gamma|}K(s)^2W(s)\mathrm{d}s\right)\right) \\
&- S\left(\frac{1}{|\Gamma|}\int_\Gamma K^2(\xi)W(\xi)\mathrm{d}S_\xi - \frac{2\pi}{|\Gamma|^2}\int_\Gamma K(\xi)W(\xi)\mathrm{d}S_\xi\right) \\
&= \int_\Gamma \nabla_\eta G(\xi,\eta)\cdot n(\eta)W(\eta)V_0(\eta,0)\mathrm{d}S_\eta - \mathcal{E}_{\Gamma,D}\left(\mathcal{Q}_{\Gamma,W}V_0(\eta,0)\right) \\
&+ S\mathcal{E}_{\Gamma,D}\left(-\frac{\mathrm{d}^2}{\mathrm{d}s^2}W(s) - K(s)^2W(s)\right) + \frac{2S\pi}{|\Gamma|^2}\int_\Gamma K(\xi)W(\xi)\mathrm{d}S_\xi .
\end{aligned} \tag{4.12}$$

The first term on the right is a double layer potential. Setting $W = V_0$ where V_0 is the Mullins Sekerka flow in (4.12) we obtain $D_{V_0}\,\mu_{0,0}$ which appears in (1.15) and (1.17).

5. Results for general N

We follow the scheme outlined in the previous sections. We start by ammending our ansatz for constructing the approximate solutions by further specifying the nature of the functions m_j.

Ansatz – full version: *Let any number $\lambda_0 > 0$ be given. For any $\Gamma \in \mathcal{M}$ with $\kappa(\Gamma) < 1/(2\lambda_0)$, let $m^{(N)}(\cdot,\Gamma) \in C^\infty(\Omega)$ be*

$$m^{(N)}(\xi,\Gamma) = m_0\left(\frac{d(\xi,\Gamma)}{\lambda}\right) + \sum_{j=1}^N \lambda^j m_j(\xi,\Gamma) . \tag{5.1}$$

Here, the function m_0 is defined in (3.9). For $j \geq 1$, set

$$m_j(\xi,\Gamma) = h_j\left(\frac{d(\xi,\Gamma)}{\lambda}, s(\xi,\Gamma)\right) + \phi_j(\xi,\Gamma) \quad \xi\in\Omega, \quad j = 1,..N. \tag{5.2}$$

Let $h_j(\cdot,\Gamma)$ be a $C^\infty(\Omega)$ function of the type (2.10). The ϕ_j, $j = 1,..N$ satisfy Neuman boundary conditions on $\partial\Omega$ and a global Lipschitz bound $\lambda-$independent, i.e.

$$\|\phi_j\|_{Lip(\Omega)} \leq C \qquad j = 1,..,N, \tag{5.3}$$

where C is a constant independent on λ.

Notational convention: In the following we denote by $m^{(N)}(\xi,t) \equiv m^{(N)}(\xi,\Gamma_t^{(N)})$ the function having the requirements prescribed in the ansatz and evaluated at $\Gamma_t^{(N)}$, $t \in [0,T]$, the solution of (1.8), being T its lifetime, see (2.7). We will fix once for all a small value of $\lambda_0 > 0$, and define $\kappa_0 = \dfrac{1}{2\lambda_0}$. This is the upper bound on the curvature that will be tolerated in our estimates, since they suppose that the local coordinate system introduced in Section 2 is valid for all $|z| < \lambda_0/\lambda$. Hence we use this value of κ_0 in defining the lifelime T of our solution of (2.6); see (2.7). We will write

$$m_j(\xi,t) = h_j(\frac{d(\xi,\Gamma_t^{(N)})}{\lambda},\xi,t) + \phi_j(\xi,t) \qquad j = 1,..,N\ ,$$

whenever we need to stress that h_j depends on $\Gamma_t^{(N)}$ through the fast scale $\frac{d(\xi,\Gamma_t^{(N)})}{\lambda}$. Further we drop in the following the superscript (N) in $\Gamma_t^{(N)}$, writing Γ_t. Through what follows, we write C to designate a generic positive constant independent on λ. Its actual numerical value may change from one occurrence to the next.

Let V_j, $j = 0,..,(N-1)$ be the sequence of vector fields introduced in the ansatz. We split them, according to (2.8) and (2.9), as

$$V_j = V_j^{(0)} + \langle V_j\rangle \qquad j = 0,..,N-1. \tag{5.4}$$

The $V_j^{(0)}$ will be determined applying the Dirichlet-Neuman operator, by potential theory, in Theorem 5.2. The $\langle V_j\rangle$, the part constant on Γ, will be determined in Theorem 5.1, stated next.

Theorem 5.1 *Fix $N > 1$. Let $\Gamma_t^{(N)}$, $t \in [0,T]$, be the solution of (1.8) in $\mathcal{M}$, being T its lifetime. Let $m^{(N)}(\cdot,\Gamma_t^{(N)})$ be as in the ansatz. There is an unique way to determine the $\langle V_j\rangle$, $j = 0,..,(N-1)$, such that there exists an unique (up to a constant in ξ) expansion*

$$\mu^{(N-1)}(\xi,t) = \sum_{i=0}^{N-1} \lambda^i \mu_i(\xi,t) \qquad in\ \Omega\times[0,T], \tag{5.5}$$

with

$$\frac{\partial}{\partial t} m^{(N)}(\xi,t) = \Delta\mu^{(N-1)}(\xi,t) + R_1(\xi,t,\lambda) \qquad in\ \Omega\times(0,T), \tag{5.6}$$

with R_1 given in (6.9). Further $\mu^{(N-1)}(\cdot,t)$, for $t \in [0,T]$, is a $C^\infty(\Omega)$ function satisfying Neumann homogeneous boundary conditions on $\partial\Omega$,

$$\sup_{\xi,t\in\Omega\times[0,T]} |R_1(\xi,t,\lambda)| \le C(T)\lambda^{N-1} \tag{5.7}$$

and

$$\sup_{t\in[0,T]}\int_{\Omega}|R_1(\xi,t,\lambda)|d\xi \leq C(T)\lambda^N \tag{5.8}$$

where $C(T)$ is a constant independent on λ. Moreover, the μ_i in (5.5) are specified by (6.13), (6.17) and (6.27) below.

The proof of Theorems 5.1 is deferred to Section 6. The next theorem assures the existence and (essential) uniqueness of the functions m_j, $j = 0,..N$, having the properties required in the ansatz. Existence and unicity are obtained provided a compatibility condition is satisfied. This determines $V_j^{(0)}$, the orthogonal part of the velocity fields.

Theorem 5.2 *Let T be the lifetime of the solution of (1.8) in $\mathcal{M}$. Let $\mu^{(N-1)}(\cdot,t)$, $t\in[0,T]$, as in Theorem 5.1. Then it is possible to choose the vector fileds $V_j^{(0)}$ so that there exist m_j, $j = 0,..,N$ having the properties prescribeded in the ansatz such that*

$$\mu^{(N-1)}(\xi,t) = -\lambda\Delta m^{(N)}(\xi,t) + \frac{1}{\lambda}f(m^{(N)}(\xi,t)) + R_2(\xi,t,\lambda) \qquad \text{in } \Omega\times(0,T], \tag{5.9}$$

with R_2 given in (8.45). Further $m^{(N)}(\cdot,t)$, for $t\in[0,T]$, is a $C^\infty(\Omega)$ function that satisfies homogeneous Neumann boundary conditions and

$$\sup_{\xi\in\Omega}\sup_{t\in[0,T]}|R_2(\xi,t,\lambda)| \leq C\lambda^N. \tag{5.10}$$

Finally, the chooice of the $V_j^{(0)}$ is specified by the equations (8.18), (8.25) and (8.37) below.

The proof of this result is given in Section 8. Theorem 5.1 and Theorem 5.2 provide the two steps to construct the approximate solution to (1.2). From Theorem 5.1 and 5.2 one obtains easily the following comprehensive result, which amplifies Theorem 1.1. Its proof is given in Section 9.

Theorem 5.3 *For all $N > 1$, there are uniquely defined sequences of vector fields V_j, $j = 0,..,(N-1)$, on $\mathcal{M}$ and functions m_j, $j = 0,..,N$, from $\mathcal{M}$ to $C^\infty(\Omega)$ as in the ansatz such that the following holds. For any $\Gamma_0\in\mathcal{M}$, choose $k_0\geq\kappa(\Gamma_0)$, set $\lambda_0 = \frac{1}{2k_0}$ and let T be the lifetime of the solution of (1.8) in $\mathcal{M}$, according to (2.7). Then for all $t<T$, for all $\lambda\in(0,\lambda_0]$ we can construct $(\tilde{m}^{(N)},\tilde{\mu}^{(N-1)})\in C^\infty(\Omega\times[0,T])$ where $\tilde{m}^{(N)}$ is a λ^N modification of $m^{(N)}$, i.e. $\sup_{(\xi,t)\in\Omega\times[0,T]}|\tilde{m}^{(N)}(\xi,t)-m^{(N)}(\xi,t)|\leq C\lambda^N$ and $\tilde{\mu}^{(N-1)}$ is a λ^{N-1} modification of $\mu^{(N-1)}$, i.e. $\sup_{(\xi,t)\in\Omega\times[0,T]}|\tilde{\mu}^{(N-1)}(\xi,t)-\mu^{(N-1)}(\xi,t)|\leq C\lambda^{N-1}$ satisfying*

$$\frac{\partial}{\partial t}\tilde{m}^{(N)}(\xi,t) = \Delta\tilde{\mu}^{(N-1)}(\xi,t) \qquad \text{in } \Omega\times(0,T)$$
$$\tilde{\mu}^{(N-1)}(\xi,t) = -\lambda\Delta\tilde{m}^{(N)}(\xi,t) + \frac{1}{\lambda}f(\tilde{m}^{(N)}(\xi,t)) + R(\xi,t,\lambda) \qquad \text{in } \Omega\times(0,T). \tag{5.11}$$

where

$$\sup_{\xi\in\Omega}\sup_{t\in[0,T]}|R(\xi,t,\lambda)| \le C\lambda^{N-1} \ .$$

Further, $\tilde{\mu}^{(N-1)}(\cdot,t)$ and $\tilde{m}^{(N)}(\cdot,t)$, for $t\in[0,T]$, satisfy Neumann homogeneous boundary conditions on the boundary of Ω. In addition

$$\sup_{t\in[0,T]}\sup_{\xi\in\Omega}|\tilde{\mu}^{(N-1)}(\xi,t) - \mu_{0,0}(\xi,t)| \le C\lambda, \tag{5.12}$$

where $\mu_{0,0}$ is the solution of (1.11), (1.12),

$$\sup_{t\in[0,T]}\ \sup_{\xi\in\mathcal{N}(\lambda_0,\Gamma_t^{(N)})}\left|\tilde{m}^{(N)}(\xi,t) - \bar{m}\left(\frac{d(\xi,\Gamma_t^{(N)})}{\lambda}\right)\right| \le C\lambda \ , \tag{5.13}$$

$$\sup_{t\in[0,T]}\ \sup_{\xi\in\Omega\setminus\mathcal{N}(\frac{\lambda_0}{2},\Gamma_t^{(N)})}\left|\tilde{m}^{(N)}(\xi,t)\mp 1\right| \le C\lambda \ . \tag{5.14}$$

Once the approximate solution to (1.2) is constructed it remains to show that it is indeed "close" to the solution of (1.2). However, as mentioned in the introduction, the approximate solution, given here satisfies the requirements needed to apply the spectral estimate used in [1] to show that the approximate solution is indeed close, in the Sobolev space H_{-1}, to the solution of Cahn-Hilliard equation.

6. Construction of the approximate chemical potential

In this section we apply classical potential theory to prove Theorem 5.1. We look for a function $\mu^{(N-1)}$ from $\mathcal{M}$ to $C^\infty(\Omega)$ having the form

$$\mu^{(N-1)}(\xi,\Gamma) = \sum_{i=0}^{N-1}\lambda^i\mu_i(\xi,\Gamma) \qquad \xi\in\Omega \ , \tag{6.1}$$

where μ_i, $i=0,..,N-1$, are functions to be determined. We insert $m^{(N)}$, as in the ansatz, and $\mu^{(N-1)}$, as in (6.1), both evaluated at $\Gamma_t^{(N)}$ where $\Gamma_t^{(N)}$ is the solution of (1.8), into (3.1). We obtain $(N-1)$ Laplace equations for $\mu_i(\cdot,\Gamma_t^{(N)})$, $i=1,..(N-1)$. The compatibility condition needed to solve these Laplace equations determines $\langle V_j\rangle(\Gamma_t^{(N)})$ for $j=0,..,(N-1)$.

When differentiating $m^{(N)}(\cdot,\Gamma_t^{(N)})$ with respect to t we need to take into account that $m^{(N)}$ depends on Γ_t, through a fast and slow scale. The fast scale brings a factor λ^{-1}.

Notation *Let m be a function from $\mathcal{M}$ to $C^\infty(\Omega)$ of the type (2.10). Let V be a vector field on $\mathcal{M}$. We denote by*

$$D_V m(\xi,\Gamma) = \frac{1}{\lambda} h'(\frac{d(\xi,\Gamma)}{\lambda}, s(\xi,\Gamma))V(s(\xi)) \; , \tag{6.2}$$

where we indicate with prime the derivative of h with respect to the first variable $z=\frac{d(\xi,\Gamma)}{\lambda}$. When $W_N=\sum_{j=0}^{N-1}\lambda^j V_j$, with V_0,...V_{N-1} a vector fields on $\mathcal{M}$

$$D_{W_N} m(\xi,\Gamma) = \sum_{j=0}^{N-1} \lambda^j D_{V_j} m(\xi,\Gamma) \; .$$

Note that by the orthogonality of $\nabla_\xi d$ with respect to the surface there is no contribution in (6.2) from $s(\xi,\Gamma)$. We have then

$$\begin{aligned} \frac{\partial m^{(N)}}{\partial t}(\xi,t) &= D_{W_N}(m^{(N)}) = D_{V_0}m_0 + \lambda\left[D_{V_1}m_0 + D_{V_0}m_1\right] \\ &+ \lambda^2\left[D_{V_1}m_1 + D_{V_0}m_2 + D_{V_2}m_0\right] + ... + \lambda^{N-1}\left[\sum_{i=0}^{N-1} D_{V_i} m_{N-1-i}\right] + R_N + E \end{aligned} \tag{6.3}$$

where

$$R_N \equiv \lambda^N \left[\sum_{i=0}^{N-1} D_{V_i} m_{N-i}\right] + \mathcal{O}(\lambda^N). \tag{6.4}$$

The term $E\equiv E(\xi,t,\lambda)$ is obtained by diferentiating $r(\frac{d(\xi,\Gamma_t)}{\lambda_0})$ with respect to the velocity field the function

$$E(\xi,t,\lambda) = \frac{1}{\lambda_0} r'(\frac{d(\xi,\Gamma_t)}{\lambda_0}) \left[\sum_{i=0}^{N-1}\lambda^i V_i(\sigma(\xi),t)\right] \left\{\bar{m} - \left[1\!\!1_{\{d(\xi,\Gamma_t)>0\}} - 1\!\!1_{\{d(\xi,\Gamma_t)<0\}}\right]\right\} \tag{6.5}$$

It is exponentially small, namely r' is different from zero only for $\frac{\lambda_0}{2\lambda} \le |z| \le \frac{\lambda_0}{\lambda}$ and $\bar{m}$ goes exponentially to ± 1. Taking into account (6.3) and (3.1) we obtain a set of N equations for the μ_i, $i = 0, ..N-1$.

Zero order term in λ

$$D_{V_0} m_0 \equiv \frac{1}{\lambda} V_0 m' = \Delta \mu_0 \qquad \text{for } \xi \in \Omega \;, \tag{6.6}$$

First order term in λ

$$[D_{V_1} m_0 + D_{V_0} m_1] = \Delta \mu_1 \quad \text{for} \quad \xi \in \Omega \tag{6.7}$$

n-th order term in λ ($n \le N-1$)

$$\left[\sum_{i=0}^{n} D_{V_i} m_{n-i}\right] = \Delta \mu_n \quad \text{for} \quad \xi \in \Omega \;. \tag{6.8}$$

Remainder term

The remainder term, see (6.4) and (6.5) is given by

$$R_1(\xi, t, \lambda) = R_N(\xi, t) + E(\xi, t, \lambda) \tag{6.9}$$

It can be easily estimated

$$\sup_{(\xi,t) \in \Omega \times [0,T]} |R_1(\xi, t)| \le C(T) \lambda^{N-1} \;. \tag{6.10}$$

Further, one gains an extra power of λ when integrating R_1, since the terms of order λ^{N-1} have support in $\mathcal{N}(\lambda_0)$,

$$\sup_{t \in [0,T]} \int_\Omega |R_1(\xi, t)| \mathrm{d}\xi \le C(T) \lambda^N \;. \tag{6.11}$$

Next we show existence and uniqueness (up to constant) of the solutions of the equations obtained at different order. In Lemma 6.1 and in Lemma 6.2 we consider respectively the first and second order equation, since for $d = 2$ the first order term, does not require the extra device we need for higher order terms. These equations were already discussed in Section 3. We repeat here to make the presentation more systematic. Finally in Lemma 6.3 we outline the proof for solving the equation to a generic order.

Lemma 6.1 *There exists an unique (up to constant in ξ) solution of (6.6) provided*

$$\int_{\Gamma_t} V_0(\eta,\Gamma_t)\mathrm{d}S_\eta = 0 \qquad t\in[0,T]\ . \tag{6.12}$$

It is given by

$$\mu_0(\xi,\Gamma_t) = \int_\Omega G(\xi,\eta)\left(\frac{1}{\lambda}m_0'\left(\frac{d(\eta,\Gamma_t)}{\lambda}\right)V_0(s(\eta),t)\right)\mathrm{d}\eta + c_0(t)\ , \tag{6.13}$$

where $c_0(t)$ is a constant (in ξ) to be determined. It is a $C^\infty(\Omega)$ function for $t\in[0,T]$.

Proof: The solvability of (6.6) requires that for all $t\in[0,T]$

$$\int_\Omega \left(\frac{1}{\lambda}m_0'\left(\frac{d(\eta,\Gamma_t)}{\lambda}\right)V_0(s(\eta),t)\right)\mathrm{d}\eta = 0 \tag{6.14}$$

This forces to take V_0 such that (6.12) holds. Now since

$$\mathrm{d}\eta = \lambda(1-\lambda z K(s))\mathrm{d}s\mathrm{d}z \tag{6.15}$$

and m_0' is even we have that (6.14) is satisfied. □

Remark In dimension $d=2$, the velocity field V_0 coincides with $V_0^{(0)}$, the constant part being zero. If we were working in three or more dimensions, the integral (6.14) would not have vanished identically, but would have been a term of $\mathcal{O}(\lambda^2)$. This would have caused only a slight complication, and we shall explain how to deal with such problems in Lemma 6.2 when we discuss the first order term.

Lemma 6.2 *There exists a unique (up to constant in ξ) solution of (6.7) provided*

$$V_1(\Gamma_t) \equiv V_1^{(0)}(\Gamma_t) + \langle V_1\rangle(\Gamma_t)$$

with

$$\int_{\Gamma_t} V_1^{(0)}(\eta,\Gamma_t)\mathrm{d}S_\eta = 0 \qquad \forall t\in[0,T] \tag{6.16}$$

and $\langle V_1\rangle$ chosen according to (6.23). It is given by

$$\mu_1(\xi,t) = \mu_{1,0}(\xi,t) + \tilde{\mu}_1(\xi,t) \tag{6.17}$$

where $\tilde{\mu}_1$ is given in (6.26),

$$\mu_{1,0}(\xi,t) = \int_\Omega G(\xi,\eta)\left(\frac{1}{\lambda}m_0'\left(\frac{d(\eta,\Gamma_t)}{\lambda}\right)V_1^{(0)}(s(\eta),t)\right)\mathrm{d}\eta + c_1(t), \tag{6.18}$$

and $c_1(t)$ is a constant (in ξ) to be determined. The solution is a $C^\infty(\Omega)$ function for $t \in (0,T]$.

Proof: The solvability of (6.7) requires

$$\int_\Omega \left[D_{V_1}m_0 + D_{V_0}m_1\right] d\xi = 0 \tag{6.19}$$

for any $t \in [0,T]$. Here we are assuming that m_1, m_0 and V_0 are already determined and so we define

$$b_1(t) = \int_\Omega D_{V_0}m_1 \mathrm{d}\xi\ . \tag{6.20}$$

Set

$$V_1(\Gamma_t) \equiv V_1^{(0)}(\Gamma_t) + < V_1 > (\Gamma_t) \tag{6.21}$$

Require (6.16). Then we obtain

$$\int_\Omega D_{V_1}m_0 \mathrm{d}\xi = 2|\Gamma_t| < V_1 > (\Gamma_t)\ . \tag{6.22}$$

Hence, to satisfy (6.19) we must take

$$< V_1(\Gamma_t) >= -\frac{1}{2|\Gamma_t|}b_1(t)\ . \tag{6.23}$$

This determines $< V_1(\Gamma_t) >$, the projection of $V_1(\Gamma_t)$ onto the constants. It still remains to determine the orthogonal part $V_1^{(0)}$. The solution of (6.7) exists and it is given by

$$\mu_1(\xi,t) = \int_\Omega G(\xi,\eta)\left[D_{V_1}m_0 + D_{V_0}m_1\right]\mathrm{d}\eta + c_1(t) \tag{6.24}$$

Because we shall use the decomposition (6.21), it is convenient to write

$$\mu_1(\xi,t) = \mu_{1,0}(\xi,t) + \tilde{\mu}_1(\xi,t) \tag{6.25}$$

where $\mu_{1,0}(\xi,t)$ is given in (6.18) and

$$\tilde{\mu}_1(\xi,t) = \int_\Omega G(\xi,\eta)D_{V_0}m_1\mathrm{d}\eta + < V_1(t) > \int_\Omega G(\xi,\eta)\left(\frac{1}{\lambda}m_0'\left(\frac{d(\eta,\Gamma_t)}{\lambda}\right)\right)\mathrm{d}\eta \tag{6.26}$$

$\square$

Lemma 6.3 *The solution of (6.8), for $2 \leq j \leq N-1$ exists and is unique (up to constant in ξ) provided*

$$V_j(\Gamma_t) \equiv V_j^{(0)}(\Gamma_t) + < V_j(\Gamma_t) >,$$

$$\int_{\Gamma_t} V_j^{(0)}(s, \Gamma_t) \mathrm{d}s = 0 \qquad \forall t \in [0, T]$$

and $< V_j(\Gamma_t) >$ chosen according to (6.34). It is given by

$$\mu_j(\xi, t) = \mu_{j,0}(\xi, t) + \tilde{\mu}_j(\xi, t) \tag{6.27}$$

where

$$\mu_{j,0}(\xi, t) = \int_\Omega G(\xi, \eta) \left(\frac{1}{\lambda} m_0' \left(\frac{d(\eta, \Gamma_t)}{\lambda} \right) V_j^{(0)}(s(\eta), t) \right) \mathrm{d}\eta + c_j(t) \tag{6.28}$$

and $\tilde{\mu}_j$ is

$$\begin{aligned} \tilde{\mu}_j(\xi, t) = & \int_\Omega G(\xi, \eta) \left[\sum_{n=0}^{j-1} D_{V_n} m_{j-n} \right] \mathrm{d}\eta \\ & + < V_j(\Gamma_t) > \int_\Omega G(\xi, \eta) \left(\frac{1}{\lambda} m_0' \left(\frac{d(\eta, \Gamma_t)}{\lambda} \right) \right) \mathrm{d}\eta \end{aligned} \tag{6.29}$$

The solution $\mu_j(\cdot, t)$, for $t \in (0, T]$ is a $C^\infty(\Omega)$ function.

Proof: The proof goes as in Lemma 6.2. The solution exists if

$$\int_\Omega \left[\sum_{n=0}^{j} D_{V_n} m_{j-n} \right] \mathrm{d}\xi = 0 \tag{6.30}$$

for any $t \in [0, T]$. Here, $D_{V_n} m_{j-n}$ for $n = 0, .., j-1$ are determined and so we define

$$b_j(t) = \int_\Omega \left[\sum_{n=0}^{j-1} D_{V_n} m_{j-n} \right] \mathrm{d}\xi \ . \tag{6.31}$$

Requiring

$$V_j(\Gamma_t) \equiv V_j^{(0)}(\Gamma_t) + < V_j(\Gamma_t) >$$

with

$$\int_{\Gamma} V_j^{(0)}(s,\Gamma_t)\mathrm{d}s = 0 \ . \tag{6.32}$$

gives

$$\int_{\Omega} D_{V_j} m_0 \mathrm{d}\xi = 2|\Gamma_t| < V_j(\Gamma_t) > \ . \tag{6.33}$$

Hence to fulfill (6.30) we must take

$$< V_j(\Gamma_t) >= -\frac{1}{2|\Gamma_t|} b_j(t) \ . \tag{6.34}$$

This determines $< V_j(\Gamma_t) >$, the projection of $V_j(\Gamma_t)$ onto the constants. It still remains to determine the orthogonal part $V_j^{(0)}$. The solution of (6.8) exists and, as done before, is represented by (6.27). □

Proof of Theorem 5.1
From Lemma 6.1, Lemma 6.2 and Lemma 6.3 we have that $\mu^{(N-1)}$, satisfies by construction (1.1). The remainder R_1 is defined in (6.9) and estimated in (6.10)) and (6.11). The $\mu^{(N-1)}(\cdot,t)$ for $t \in [0,T]$ satisfies homogeneous Neumann boundary conditions by construction. Theorem 5.1 is then proved. □

7. Proof of Theorem 5.2: Derivation of the equations

In this section we begin the proof of Theorem 5.2. We write (3.2), inserting in the left side the function $\mu^{(N-1)}(\cdot,\Gamma_t^{(N)})$ determined in Theorem 5.1, see (5.5):

$$\lambda\mu^{(N-1)}(\xi,t) = -\lambda^2\Delta m(\xi,t) + f(m(\xi,t)) \qquad \text{in } \Omega\times(0,T) \tag{7.1}$$

The $\mu^{(N-1)}$ are written in terms of the $m^{(N)}$, chosen according to the ansatz. Here we prove that there exists an unique way to find the function $m^{(N)}$, having indeed the property required in the ansatz and satisfying equation (7.1) in the sense of Theorem 5.2. The existence at any order of the m_j, $j = 0,..,N$ is obtained provided a compatibility condition is satisfied. This compatibility condition forces us to take $V_j^{(0)}$, $j = 0,1..,(N-1)$, according to (8.9), (8.25) and (8.37). In the proof of the Theorem 5.2 we distinguish two main steps

• **step 1:** *Determination at any order of the equations. This is carried out in this section.*

• **step 2:** *Analysis of the equations derived in the first step. This will be done in the next section.*

In this and in the next section Γ_t is kept fixed, so to simplify notations we drop t subscript, except where it may add some clarity.

To separate the fast and slow scale of $m^{(N)}$ near the surface Γ, we write the Laplacian in the system of local coordinates introduced in Subsection 2.1. The expansion in λ of the Laplacian written in this coordinate system is reported in the appendix. We match the right and left terms of the equations having the same power of λ, distinguishing the case where $\xi \in \mathcal{N}(\lambda_0)$ from the one with $\xi \in \Omega \setminus \mathcal{N}(\lambda_0)$. We therefore get at any order two sets of equations, one for $\xi \in \mathcal{N}(\lambda_0)$ and the other for $\xi \in \Omega \setminus \mathcal{N}(\lambda_0)$. After simple, however lengthly conputations we obtain the following. Taking into account a formula from the appendix, namely (10.15), and denoting by $'$ the derivative with respect to z, and letting a_n, b_n, c_n denote the quantities defined in (10.16),

$$\begin{aligned}\lambda^2 \Delta m^{(N)}(z,s) = &\left\{ \bar m''(z) + \sum_{n=1}^{N} \lambda^n \left[h_n''(z,s) + a_n(z,s)\bar m' \right] \right\} \\ &+ \left\{ \sum_{n=2}^{N} \lambda^n \sum_{i=1}^{n-1} a_{n-i}(z,s) h_i'(z,s) + \sum_{n=3}^{N} \lambda^n \left[\sum_{i=1}^{n-2} b_{n-i}(z,s) \frac{d^2}{ds^2} h_i(z,s) \right] \right. \\ &\left. + \sum_{n=4}^{N} \lambda^n \sum_{i=1}^{n-3} c_{n-i}(z,s) \frac{d}{ds} h_i(z,s) \right\} + \lambda^2 \Delta \left[\sum_{i=1}^{N} \lambda^i \phi_i(\xi) \right] + E_1(\xi,t,\lambda) + \lambda^{N+1} A(\xi,t,\lambda)\end{aligned} \tag{7.2}$$

with

$$\sup_{(\xi,t)\in\Omega\times[0,T]} |A(\xi,t,\lambda)| \le C(T), \tag{7.3}$$

$$\sup_{t\in[0,T]} \int_\Omega d\xi |A(\xi,t,\lambda)| \le \lambda C(T), \tag{7.4}$$

$$\begin{aligned} E_1(\xi,\lambda) \equiv\ & \lambda^2 \Delta r\left(\frac{d(\xi,\Gamma)}{\lambda_0}\right) \left\{ \bar m\left(\frac{d(\xi,\Gamma)}{\lambda}\right) - \left[\mathbb{I}_{\{d(\xi,\Gamma)>0\}} - \mathbb{I}_{\{d(\xi,\Gamma)<0\}} \right] \right\} \\ & + 2\lambda^2 \nabla r \cdot \nabla \left[\bar m\left(\frac{d(\xi,\Gamma)}{\lambda}\right) \right] \end{aligned},$$

and

$$\lim_{\lambda\to 0} \sup_{(\xi,t)\in\Omega\times[0,T]} |E_1(\xi,t,\lambda)| = 0 \tag{7.5}$$

the convergence being exponentially fast due to the decay of $\bar m$.

Define f_i such that

$$f(m^{(N)}) = f(m_0) + f'(m_0)\left[\sum_{i=1}^{N} \lambda^i m_i\right] + \sum_{i=2}^{N} \lambda^i f_i(m_0, m_1, .., m_{i-1}) \\ + \lambda^{N+1} B_{N+1}(\cdot, \lambda) \tag{7.6}$$

$$\sup_{\xi\in\Omega, t\in[0,T]} |B_{N+1}(\xi, t, \lambda)| \leq C \tag{7.7}$$

One easily obtains the f_i, for $i = 2, .., N$ Taylor expanding up to $N-$ order f around m_0 and collecting terms having the same power of λ. We insert (7.2) and (7.6) into (7.1). We equate terms having the same order (when estimated with the $L^\infty(\Omega)$ norm) in λ obtaining at any order two equations one for $\xi \in \Omega \setminus \mathcal{N}(\lambda_0)$, the other for $\xi \in \mathcal{N}(\lambda_0)$. The one for $\xi \in \Omega \setminus \mathcal{N}(\lambda_0)$, determines the ϕ_i, the slowly varying terms, the other for $\xi \in \mathcal{N}(\lambda_0)$ determines the h_i, the rapidly decaying terms. When deriving the equations for $\xi \in \mathcal{N}(\lambda_0)$ terms of the type $\Delta\phi_i(\lambda z, s)$ appear. As was shown for the first term in the expansion, see Subsection 3.1, the ϕ_i are C^∞ functions, since are proportional to the μ_i, and have the same type of singularity in λ when differentiated in ξ. Therefore, the terms $\lambda^{n+1}\Delta\phi_{n-1}(\lambda z, s)$ are $\mathcal{O}(\lambda^n)$, and we write them in the λ^n order equation.

Zero order term in λ

$$0 = -r(\frac{d(\xi, \Gamma)}{\lambda_0})m_0''(z) + f(m_0(z)) \quad \text{for} \quad z \in [-\frac{\lambda_0}{\lambda}, \frac{\lambda_0}{\lambda}] \tag{7.8}$$

$$f(\pm 1) = 0 \quad \text{for } \xi \in \Omega \setminus \mathcal{N}(\lambda_0) \tag{7.9}$$

First order term in λ:

$$\mu_0(\lambda z, s) = -\left[h_1''(z, s) - K(s)m_0'(z)\right] \\ + f'(m_0)\left[h_1(z, s) + \phi_1(\lambda z, s)\right] \quad \text{for} \quad \xi \in \mathcal{N}(\lambda_0) \tag{7.10}$$

and

$$\mu_0(\xi) = f'(1)\phi_1(\xi) \quad \text{for} \quad \xi \in \Omega \setminus \mathcal{N}(\lambda_0) \tag{7.11}$$

Second order term in λ:

$$\mu_1(\lambda z, s) = -\left[h_2''(z, s) - K^2(s)zm_0'(z) - K(s)h_1'(z, s)\right] \\ + f'(m_0(z))\left[h_2(z, s) + \phi_2(\lambda z, s)\right] - \lambda\Delta\phi_1(\lambda z, s) + f_2(m_0, m_1)(\lambda z, s) \tag{7.12}$$

for $\xi \in \mathcal{N}(\lambda_0)$.

$$\mu_1(\xi) = f'(1)\phi_2(\xi) + f_2(1, \phi_1(\xi)) - \lambda\Delta\phi_1(\xi) \qquad \text{for} \qquad \xi \in \Omega \setminus \mathcal{N}(\lambda_0) \tag{7.13}$$

More explicitly the f_2 term is given

$$f_2(m_0, m_1)(\lambda z, s) = \frac{1}{2} f''(m_0(z)) \left[h_1^2(z,s) + \phi_1^2(\lambda z, s) + 2\phi_1(\lambda z, s) h_1(z,s)\right]$$

for $\xi \in \mathcal{N}(\lambda_0)$, and

$$f_2(1, \phi_1(\xi)) = \frac{1}{2} f''(1)\phi_1^2(\xi)$$

for $\xi \in \Omega \setminus \mathcal{N}(\lambda_0)$.

n-th order term in λ ($3 \leq n \leq N$):

$$\begin{aligned}
&\mu_{n-1}(\lambda z, s) \\
&= -\left[h_n''(z,s) + a_n(z,s)m_0'(z) + \sum_{i=1}^{n-1}\left[a_{n-i}(z,s)h_i'(z,s)\right] + \sum_{i=1}^{n-2} b_{n-i}(z,s)\frac{d^2}{ds^2}h_i(z,s)\right. \\
&\left. + \mathbb{I}_{\{n\geq 4\}} \sum_{i=1}^{n-3} c_{n-i}(z,s)\frac{d}{ds}h_i(z,s)\right] - \lambda\Delta\phi_{n-1}(\lambda z, s) + f'(m_0)\left[h_n(z,s) + \phi_n(\lambda z, s)\right] \\
&+ f_n(m_0, m_1, m_2, .., m_{n-1})(\lambda z, s) \qquad \xi \in \mathcal{N}(\lambda_0)
\end{aligned} \tag{7.14}$$

$$\mu_{n-1}(\xi) = -\lambda\Delta\phi_{n-1}(\xi) + f'(1)\phi_n(\xi) + f_n(\pm 1, \phi_1, \phi_2, .., \phi_{n-1})(\xi) \qquad \xi \in \Omega \setminus \mathcal{N}(\lambda_0) \tag{7.15}$$

The Remainder:

The remainder term is given by

$$\lambda\tilde{R}_2(\xi, t, \lambda) = \lambda^{N+1}A(\xi, t, \lambda) + \lambda^{N+2}\Delta\phi_N(\xi, t) + E_1(\xi, t, \lambda) + \lambda^{N+1}B_{N+1}(\xi, t, \lambda). \tag{7.16}$$

¿From (7.3), (7.5), (7.7), we have

$$\sup_{(\xi,t)\in\Omega\times[0,T]} |\tilde{R}_2(\xi, t, \lambda)| \leq C\lambda^N . \tag{7.17}$$

8. Proof of Theorem 5.2: Analysis of compatibility conditions

In this section we analyze the equations obtained in Section 7. As in Section 3, the strategy is to find at each order in λ, first, the slowly varying part, the ϕ_i, solving the equations for $\xi \in \Omega \backslash \mathcal{N}(\lambda_0, \Gamma_t^{(N)})$. Then we extend ϕ_i globally in Ω and determine the rapidly decaying part h_i solving the equations in $\xi \in \mathcal{N}(\frac{\lambda_0}{2}, \Gamma_t^{(N)})$. However here, in order to continue to arbitrary order, it is convenient to modify the way we extract the compatibility condition required to solve the equation for the h_i. The modification is to add and subtract to each order a term of lower order $\lambda^{i+1}\alpha_i(s,\Gamma)\bar{m}'(z)$, with $\alpha_i(\cdot,\Gamma) \in C^\infty(\Gamma)$. Adding and subtracting terms does not change, of course, the total quantity but it modifies the equation we obtain at each single order. We obtain at any $i \geq 1$ order in λ the corresponding m_i split in one part, the function ϕ_i defined globally in Ω, satisfying Neumann condition on the boundary of Ω, the other part, the h_i, is different from zero only in a tubular neighborhood of Γ, $\mathcal{N}(\frac{\lambda_0}{2}, \Gamma_t^{(N)})$ and it is exponentially decaying to 0 far from Γ. The 0 order term is different, in the sense that m_0 far from the interfaces relaxes exponentially fast to ± 1. We first state the following Lemma, taken from[1]. We use this to determine the condition for solvability of equations of the type (8.1), where $\mathcal{L}$ is the operator on $L^2(I\!R)$ defined in (3.24).

Lemma 8.1 **[ABC]** *Let $A(z,s,t)$, $z \in I\!R$, $s \in \Gamma$, $t \in [0,T]$. Assume that there exists $A^\pm(s,t)$ such that for $A(z,s,t) - A^\pm(s,t) = \mathcal{O}(e^{-\alpha|z|})$ as $|z| \to \infty$ for $s \in \Gamma$ and $t \in [0,T]$. Then for each $s \in \Gamma$ and $t \in [0,T]$*

$$
\begin{aligned}
&(\mathcal{L}w)(z,s,t) = A(z,s,t) \quad \text{for} \qquad z \in I\!R \\
&w(0,s,t) = 0, w(\cdot,s,t) \in L^\infty(I\!R)
\end{aligned}
\tag{8.1}
$$

has a solution if and only if

$$
\int_{I\!R} A(z,s,t)\bar{m}'(z)dz = 0 \quad \text{for all} \quad s \in \Gamma, t \in [0,T] \tag{8.2}
$$

In addition if the solution exists, then it is unique and satisfies

$$
D_z^\ell \left[w(z,s,t) + \frac{A^\pm(s,t)}{f'(1)} \right] = \mathcal{O}(e^{-\alpha|z|}) \qquad \text{as} \qquad |z| \to \infty \qquad \text{and } \ell = 0,1,2 \tag{8.3}
$$

Furthermore if $A(z,s,t)$ satisfies

$$
D_s^m D_t^n D_z^\ell \left[A(z,s,t) - A^\pm(s,t) \right] = \mathcal{O}(e^{-\alpha|z|})
$$

then

$$
D_s^m D_t^n D_z^\ell \left[w(z,s,t) + \frac{A^\pm(s,t)}{f'(1)} \right] = \mathcal{O}(e^{-\alpha|z|})
$$

for all $m = 0, 1...M$, $n = 0, 1...N$, and $\ell = 0, 1...L+2$. Further, since $\mathcal{L}$ is a preserving parity operator, the solution $w(z, s, t)$ is odd (even) with respect to z if $A(z, s, t)$ is odd (even) with respect to z for $s \in \Gamma$ and $t \in [0, T]$.

Remark: In case $A(\cdot,\cdot,\cdot) \in C^\infty(I\!R \times \Gamma \times [0,T])$, the solution $w(\cdot,\cdot,\cdot)$ of (8.1) is $C^\infty(I\!R \times \Gamma \times [0,T])$. Whenever we apply Lemma 8.1, the right hand side of (8.7) will be $C^\infty(I\!R \times \Gamma \times [0,T])$, then the solutions will be in $C^\infty(I\!R \times \Gamma \times [0,T])$.

The compatibility conditions must hold for every Γ in $\mathcal{M}$ and so in our derivation we do must refer to Γ_t.

Zero order term in λ:

For $\xi \in \mathcal{N}(\frac{\lambda_0}{2})$ we have from (7.8)

$$0 = -\bar{m}''(z) + f(\bar{m}(z)) \quad \text{for} \qquad z \in [-\frac{\lambda_0}{2\lambda}, \frac{\lambda_0}{2\lambda}] \tag{8.4}$$

and from (7.9)

$$0 = f(\pm 1) \quad \text{for } \xi \in \Omega \setminus \mathcal{N}(\lambda_0) \tag{8.5}$$

The (8.4) and (3.8) are satisfied by $\bar{m}$, see (3.7) and (3.8).

First order term in λ:

As explained at the beginning of this section, it is convenient for solving (7.10) to add a term $\lambda\alpha_1(s,\Gamma)\bar{m}'(z)$, $s \in \Gamma$ and $z \in I\!R$, with $\alpha_1(\cdot,\Gamma)$ to be determined. This term will be subtracted to the second order. In the following we will short notation, writing $\alpha_1(s) \equiv \alpha_1(s,\Gamma)$. Recalling the definition of $\mathcal{L}$, see (3.24), adding $\lambda\alpha_1(s)\bar{m}'(z)$, we write (7.10) as

$$\mu_0(\lambda z, s) - f'(\bar{m}(z))\phi_1(\lambda z, s) - K(s)\bar{m}'(z) + \lambda\alpha_1(s)\bar{m}'(z) = (\mathcal{L}h_1)(z, s) \tag{8.6}$$

for $\xi \in \mathcal{N}(\frac{\lambda_0}{2})$. One has from (7.11) that

$$\phi_1(\xi) = \frac{\mu_0(\xi)}{f'(1)} \quad \text{for} \qquad \xi \in \Omega \setminus \mathcal{N}(\lambda_0)\ . \tag{8.7}$$

We extend this definition of ϕ_1 globally in Ω. We then insert (8.7) into (8.6) obtaining for $s \in \Gamma$,$|z| \le \frac{\lambda_0}{2\lambda}$.

$$\mu_0(\lambda z, s)\left[1 - \frac{f'(\bar{m}(z))}{f'(1)}\right] - K(s)\bar{m}'(z) + \lambda\alpha_1(s)\bar{m}'(z) = (\mathcal{L}h_1)(z, s) \tag{8.8}$$

Since the left hand side of (8.8) tends exponentially to 0 as $z \to \pm\infty$ if the solution of (8.8) exists, see Lemma 8.1, then decays exponentially fast to 0. We can therefore extend (8.8) for z in all $I\!R$. We have the following result.

Lemma 8.2 *Set*

$$V_0^{(0)}(\xi,\Gamma) = S\mathcal{T}_\Gamma\left[K(\cdot) - \frac{1}{|\Gamma|}\int_\Gamma K(\eta)\mathrm{d}S_\eta\right](\xi) \qquad \xi \in \Gamma\ , \tag{8.9}$$

where

$$S = \frac{1}{4}\int_{I\!R}\left(\bar{m}'(z)\right)^2 dz = \frac{\sqrt{2}}{3} \tag{8.10}$$

and $\mathcal{T}_\Gamma$ is the operator defined in (10.8). Then it is uniquely determined $\alpha_1(\cdot,\Gamma) \in C^\infty(\Gamma)$ and it exists an unique solution of (8.8), $h_1(\cdot,s)$, $s \in \Gamma$, such that $h_1(0,s) = 0$ and $h_1(\cdot,s) \in L^\infty(I\!R)$. Moreover $h_1(\cdot,s)$, for $s \in \Gamma$ is even as function of z and its derivatives with respect to z decay exponentially to 0 as z tends to $\pm\infty$.

Proof: For any fixed $s \in \Gamma$, the condition for the existence of h_1, see (8.2), requires

$$\begin{aligned}&\int_{I\!R}\mu_0(\lambda z,s)\left[1 - \frac{f'(\bar{m}(z))}{f'(1)}\right]\bar{m}'(z)dz \\ &= \left[K(s) - \lambda\alpha_1(s)\right]\int_{I\!R}\left(\bar{m}'(z)\right)^2 dz \qquad \text{for } s \in \Gamma\end{aligned} . \tag{8.11}$$

Let

$$\mu_{0,0}(\xi) = 2\int_\Gamma V_0^{(0)}(\eta)G(\xi,\eta)\mathrm{d}S_\eta + c_0(t) \qquad \xi \in \Omega \tag{8.12}$$

since $\mu_{0,0}(\xi) - \mu_0(\xi) \simeq \lambda$, we first choose $V_0^{(0)}$ imposing for $s \in \Gamma$

$$\int_{I\!R}\mu_{0,0}(0,s)\left[1 - \frac{f'(\bar{m}(z))}{f'(1)}\right]\bar{m}'(z)dz = K(s)\int_{I\!R}\left(\bar{m}'(z)\right)^2 dz\ . \tag{8.13}$$

We obtain, since

$$\int_{I\!R} f'(\bar{m}(z))\bar{m}'(z)dz = f(1) - f(-1) = 0 \tag{8.14}$$

$$2\mu_{0,0}(\xi) = K(\xi)\int_{I\!R}\left(\bar{m}'(z)\right)^2 dz \quad \xi \in \Gamma. \tag{8.15}$$

Inserting (8.12) in (8.15) and integrating over Γ we obtain that

$$4\int_\Gamma \mathrm{d}S_\xi\int_\Gamma V_0(\eta)G(\xi,\eta)\mathrm{d}S_\eta + 2c_0(t)|\Gamma| = 2\pi\int_{I\!R}\left(\bar{m}'(z)\right)^2 dz\ . \tag{8.16}$$

Then from (8.16) we obtain

$$c_0(t) = \frac{1}{2|\Gamma|}\left[2\pi \int_{I\!R} (\bar{m}'(z))^2\, dz - 4\int_\Gamma \mathrm{d}S_\xi \int_\Gamma V_0^{(0)}(\eta) G(\xi,\eta)\mathrm{d}S_\eta\right] . \tag{8.17}$$

In this way the constant $c_0(t)$ of the single layer potential, see (8.12), is written in terms of the velocity field $V_0^{(0)}$ still to be determined. We then insert $c_0(t)$ as in (8.17) into (8.15) obtaining the equation determining $V_0^{(0)}$. We have

$$\mathcal{S}_\Gamma V_0^{(0)}(\eta,\Gamma) = S\left[K(\eta) - \frac{2\pi}{|\Gamma|}\right] \qquad \eta \in \Gamma ,$$

where $\mathcal{S}_\Gamma$ is the linear operator defined in (10.9). Applying the Dirichlet-Neumann operator, see (10.10), we obtain

$$V_0^{(0)}(\xi,\Gamma) = S\mathcal{T}_\Gamma\left[K(\cdot) - \frac{2\pi}{|\Gamma|}\right](\xi) \qquad \xi \in \Gamma . \tag{8.18}$$

This determines $V_0^{(0)}$ and then $c_0(t)$, see (8.17). Now that $V_0^{(0)}$ and $c_0(t)$ are chosen, we simply choose $\alpha_1(s)$ so that (8.11) is satisfied. Then for any $s \in \Gamma$, Lemma 8.1 assures the existence of the unique solution of (8.8) with $h_1(0,s) = 0$, exponentially decaying to zero as $|z| \to \infty$. Since the left hand side of (8.8) is even, the solution $h_1(\cdot,s)$ is even as function of z. $\square$

Note that for any $t \in (0,T)$, $\mu_{0,0}(\cdot,\Gamma_t^{(N)})$, defined in (8.12) and (8.17) with $V_0^{(0)}$ given in (8.18), satisifies (1.11) and (1.12). It is the same already derived in Section 3, see (3.16). The function h_1 determined in Lemma 8.2 is λ different from the one of Section 3. In fact the equation determining h_1 here, see (8.6), has terms of order λ not taken in account in Section 3 when solving for h_1.

Second order term in λ:

We proceed as before. Since (7.13) and $\Delta\phi_1 = 0$ in $\Omega \setminus \mathcal{N}(\lambda_0)$ we obtain

$$\mu_1(\xi) = f'(1)\phi_2(\xi) + \frac{1}{2}f''(1)\phi_1^2(\xi) \qquad \xi \in \Omega \setminus \mathcal{N}(\lambda_0) . \tag{8.19}$$

which gives, for $\xi \in \Omega \setminus \mathcal{N}(\lambda_0)$,

$$\phi_2(\xi) = \frac{1}{f'(1)}\left[\mu_1(\xi) - \frac{1}{2}f''(1)\phi_1^2(\xi)\right] \tag{8.20}$$

As done before, we extend the validity of (8.20) globally in Ω. We insert (8.20) into (7.12). We add, subtracting to the next order, $\lambda\alpha_2(s)\bar{m}'(z)$ obtaining

$$\mu_1(\lambda z,s)\left[1-\frac{f'(\bar{m}(z))}{f'(1)}\right]-A_2(z,s)+\lambda\alpha_2(s)\bar{m}'(z)=(\mathcal{L}h_2)(z,s)\qquad \xi\in\mathcal{N}(\frac{\lambda_0}{2})\quad (8.21)$$

where we set

$$\begin{aligned}&A_2(z,s)=\lambda\Delta\phi_1(\lambda z,s)-f_2(\bar{m},m_1)(\lambda z,s)+\frac{f'(\bar{m}(z))}{f'(1)}f_2(1,\phi_1)(\lambda z,s)\\&-K^2(s)z\bar{m}'(z)-K(s)h_1'(z,s)+\alpha_1(s)\bar{m}'(z)\end{aligned}\ . \qquad (8.22)$$

All the quantities in (8.22) have been already determined. Further

$$\lim_{|z|\to\infty}A_2(z,s)=0\qquad s\in\Gamma \qquad (8.23)$$

exponentially fast. Namely, since the exponential convergence of $\bar{m}(\cdot)$ to ± 1 and the one of $\bar{m}'(\cdot)$ and $h_1'(\cdot,s)$, for $s\in\Gamma$, to 0 we need only to verify that

$$\lim_{|z|\to\infty}\left[\lambda\Delta\phi_1(\lambda z,s)-f_2(\bar{m},m_1)(\lambda z,s)+\frac{f'(\bar{m}(z))}{f'(1)}f_2(1,\phi_1)\right]=0\ . \qquad (8.24)$$

From (8.7) and (3.15) we have that $\lambda\Delta\phi_1(\lambda z,s)=\frac{1}{f'(1)}V_0^{(0)}(s)\bar{m}'(z)$. Then as $|z|\to\infty$, it converges exponentially fast to 0. Further since $\lim_{|z|\to\infty}f'(\bar{m}(z))=f'(1)=f'(-1)$ and $\lim_{|z|\to\infty}h_1(z,s)=0$ exponentially fast, we have (8.24) and therefore (8.23). As done before, we extend (8.21) in $I\!R$.

Lemma 8.3 *Set*

$$V_1^{(0)}(\xi,\Gamma)=\mathcal{T}_\Gamma\left[\frac{1}{4}B_1(\cdot)-\frac{1}{4|\Gamma|}\int_\Gamma B_1(s)\mathrm{d}s\right](\xi)\quad \xi\in\Gamma \qquad (8.25)$$

where $B_1(s)$ is defined in (8.28) and $\mathcal{T}_\Gamma$ is the Dirichlet–Neumann operator. Then there are uniquely determined $\alpha_2(\cdot,\Gamma)\in C^\infty(\Gamma)$ and $h_2(\cdot,s)\in\Lambda^\infty(I\!R)$, $h_2(0,s)=0$ with $s\in\Gamma$, solution of (8.21). Moreover $h_2(\cdot,s)$ and its derivatives with respect to z decay exponentially to 0, as z tends to $\pm\infty$.

Proof The solvability condition, see (8.2), is satisfied provided for $s\in\Gamma$ and $t\in[0,T]$

$$\int_{I\!R}\mu_1(\lambda z,s)\left[1-\frac{f'(\bar{m}(z))}{f'(1)}\right]\bar{m}'(z)dz=\int_{I\!R}A_2(z,s)\bar{m}'(z)dz-\lambda\alpha_2(s)\int_{I\!R}\left(\bar{m}'(z)\right)^2dz \qquad (8.26)$$

where μ_1 is defined in (6.25). The term $\tilde{\mu}_1$ of μ_1 has been already completely determined. Still to be determined are, as in the previous case, the constant $c_1(t)$, the velocity $V_1^{(0)}$ and $\alpha_2(s)$. Write (8.26) as

$$\int_{I\!R} \mu_{1,0}(\lambda z, s)\left[1 - \frac{f'(\bar{m}(z))}{f'(1)}\right]\bar{m}'(z)dz = B_1(s) - \lambda\alpha_2(s)\int_{I\!R}(\bar{m}'(z))^2\,dz \tag{8.27}$$

where

$$B_1(s) = \int_{I\!R}\left\{A_2(z,s) - \tilde{\mu}_1(\lambda z, s)\left[1 - \frac{f'(\bar{m}(z))}{f'(1)}\right]\right\}\bar{m}'(z)dz. \tag{8.28}$$

Let

$$\mu_{1,0,0}(\xi) = 2\int_\Gamma V_1^{(0)}(\eta)G(\xi,\eta)\mathrm{d}S_\eta + c_1(t)\xi \in \Omega \tag{8.29}$$

since $\mu_{1,0,0}(\xi) - \mu_{1,0}(\xi) \simeq \lambda$, we first choose V_1 imposing

$$\int_{I\!R}\mu_{1,0,0}(0,s)\left[1 - \frac{f'(\bar{m}(z))}{f'(1)}\right]\bar{m}'(z)dz = B_1(s) \tag{8.30}$$

We obtain

$$\mu_{1,0,0}(0,s) = \frac{1}{2}B_1(s) \qquad s \in \Gamma \tag{8.31}$$

Inserting (8.29) in (8.31) and integrating over Γ we obtain

$$c_1(t) = \frac{1}{|\Gamma_t|}\left[\frac{1}{2}\int_{\Gamma_t} B_1(\eta)\mathrm{d}S_\eta - 2\int_{\Gamma_t}\mathrm{d}S_\xi\int_{\Gamma_t}V_1^{(0)}(\eta)G(\xi,\eta)\mathrm{d}S_\eta\right] \tag{8.32}$$

Since $\int_\Gamma V_1^{(0)}(s)ds = 0$, let $\mathcal{S}_\Gamma$ be the linear operator defined in (10.9). Then (8.31) can be written as

$$\mathcal{S}_\Gamma V_1^{(0)}(\xi) = \frac{1}{4}B_1(\xi) - \frac{1}{4|\Gamma|}\int_\Gamma B_1(s)ds \quad \xi \in \Gamma$$

and applying the Dirichket-Neumann operator, see (10.10) we obtain (8.25). This determines the (constant in ξ) $c_1(t)$. Now that $V_1^{(0)}$ and $c_1(t)$ are determined, we choose $\alpha_2(s)$ so that (8.27) is satisfied. $\square$

Notice that $\mu_{1,0,0}$ solves

$$\begin{aligned} &\Delta\mu_{1,0,0} = 0 \text{ for } \xi \in \Omega \setminus \Gamma \\ &\mu_{1,0,0}(s) = B_1(s) \text{ on } \Gamma. \end{aligned} \tag{8.33}$$

$n-$**th order term in** λ**,** $3 \le n \le N$**.**

As previously, we determine the function ϕ_n for $\xi \in \Omega \setminus \mathcal{N}(\lambda_0)$ from (7.15). Then, we extend the validity in Ω obtaining

$$\phi_n(\xi) = \frac{1}{f'(1)} \left[\mu_{n-1}(\xi) + \lambda\Delta\phi_{n-1}(\xi) - f_n(\pm 1, \phi_1, \phi_2, .., \phi_{n-1})(\xi)\right] \quad \xi \in \Omega \tag{8.34}$$

We then insert (8.34) into (7.14). We add and subtract (at the next order) the quantity $\lambda\alpha_n(s)\bar{m}'(z)$, to the left hand side of (7.14) and we obtain

$$\mu_{n-1}(\lambda z, s)\left[1 - \frac{f'(\bar{m}(z))}{f'(1)}\right] - A_n(z,s) + \lambda\alpha_n(s)\bar{m}'(z) = (\mathcal{L}h_n)(z,s) \tag{8.35}$$

where we set

$$\begin{aligned} A_n(z,s) &= -a_{n-1}(z,s)\bar{m}' - \sum_{i=1}^{n-1}\left[a_{n-i}(z,s)h_i'(z,s)\right] - \sum_{i=1}^{n-2} b_{n-i}(z,s)\frac{d^2}{ds^2}h_i(z,s) \\ &- \mathbb{I}_{n\ge 4}\sum_{i=1}^{n-3}\left[c_{n-i}(z,s)\frac{d}{ds}h_i(z,s)\right] - \lambda\Delta\phi_{n-1}(\lambda z, s)[1 - \frac{f'(\bar{m}(z))}{f'(1)}] \\ &+ \frac{f'(\bar{m}(z))}{f'(1)} f_n(\pm 1, \phi_1, \phi_2, .., \phi_{n-1})(\lambda z, s) - f_n(m_0, m_1, m_2, .., m_{n-1})(\lambda z, s) \end{aligned} \tag{8.36}$$

It is easy to verify that for all $s \in \Gamma$

$$\lim_{|z|\to\infty} A_n(z,s) = 0$$

exponentially fast. Namely there is no problem for those terms involving $\bar{m}'$, $h_i(\cdot, s)$ and their derivatives, because of the exponential convergence to zero of all these terms, for all $s \in \Gamma$. For the remaining terms recall that $\lim_{|z|\to\infty} f'(\bar{m}(z)) = f'(\pm 1)$, $m_i = h_i + \phi_i$ with $h_i(z,s) \to 0$, as $|z| \to \infty$ for all $s \in \Gamma$, all limits being exponentially fast. Then one obtains immediately

$$\lim_{|z|\to\infty}\left[\frac{f'(\bar{m}(z))}{f'(1)} f_n(\pm 1, \phi_1, \phi_2, .., \phi_{n-1})(\lambda z, s) - f_n(m_0, m_1, m_2, .., m_{n-1})(\lambda z, s)\right] = 0$$

exponentially fast. We extend (8.35) to hold on all of $I\!R$, and regard it as an equation for $h_n(\cdot, s)$ for $s \in \Gamma$.

Lemma 8.4 *For any positive integer* n*,* $n \le N$*, set*

$$V_{n-1}^{(0)}(\xi, \Gamma) = \mathcal{T}_\Gamma \frac{1}{4}\left[B_{n-1}(\cdot) - \frac{1}{|\Gamma|}\int_\Gamma B_{n-1}(s)\right] \qquad \text{for} \qquad \xi \in \Gamma \tag{8.37}$$

where $B_{n-1}(s)$ is defined in (8.40). Then there are uniquely determined $\alpha_n(\cdot,\Gamma) \in C^\infty(\Gamma)$ and $h_n(\cdot,s) \in L^\infty(I\!R)$ for $s \in \Gamma$, with $h_n(0,s) = 0$ solutions of (8.35). Moreover $h_n(\cdot,s)$, for all $s \in \Gamma$, and its derivatives with respect to z decay exponentially to 0 as $z \to \pm\infty$.

Proof: The solvability condition is satisfied provided

$$\int_{I\!R} \mu_{n-1}(\lambda z, s) \left[1 - \frac{f'(\bar{m}(z))}{f'(1)}\right] \bar{m}'(z) dz = \int_{I\!R} A_n(z,s)\bar{m}'(z)dz - \lambda\alpha_n(s) \int_{I\!R} (\bar{m}'(z))^2 dz. \tag{8.38}$$

Since, see (6.27), $\mu_{n-1} = \mu_{n-1,0} + \tilde{\mu}_{n-1}$ with $\tilde{\mu}_{n-1}$ already determined, to satisfy (8.38) we require that

$$\int_{I\!R} \mu_{n-1,0}(\lambda z, s) \left[1 - \frac{f'(\bar{m}(z))}{f'(1)}\right] \bar{m}'(z) dz = B_{n-1}(s) - \lambda\alpha_n(s) \int_{I\!R} (\bar{m}'(z))^2 dz \tag{8.39}$$

where

$$B_{n-1}(s) = \int_{I\!R} \left\{ A_n(z,s) - \tilde{\mu}_{n-1}(\lambda z, s) \left[1 - \frac{f'(\bar{m}(z))}{f'(1)}\right]\right\} \bar{m}'(z) dz \tag{8.40}$$

We set

$$\mu_{n-1,0,0}(\xi) = 2\int_\Gamma V_{n-1}(\eta) G(\xi,\eta) \mathrm{d}S_\eta + c_{n-1}(t) \tag{8.41}$$

Since $\mu_{n-1,0}(\xi,t) - \mu_{n-1,0,0}(\xi,t) \simeq \lambda$ we determine V_{n-1} imposing

$$\int_{I\!R} \mu_{n-1,0,0}(0,s) \left[1 - \frac{f'(\bar{m}(z))}{f'(1)}\right] \bar{m}'(z) dz = B_{n-1}(s) \tag{8.42}$$

obtaining

$$\mu_{n-1,0,0}(0,s) = \frac{1}{2} B_{n-1}(s) \qquad s \in \Gamma \tag{8.43}$$

Inserting (8.41) in (8.43)and integrating over Γ we obtain

$$c_{n-1}(t) = \frac{1}{|\Gamma|}\left[\frac{1}{2}\int_\Gamma B_{n-1}(\eta)\mathrm{d}S_\eta - 2\int_\Gamma \mathrm{d}S_\xi \int_\Gamma V^{(0)}_{n-1}(\eta) G(\xi,\eta)\mathrm{d}S_\eta\right] \tag{8.44}$$

We insert (8.44) into (8.41) obtaining from (8.43)

$$\mathcal{S}_\Gamma V^{(0)}_{n-1}(\xi) = \frac{1}{4}\left[B_{n-1}(\xi) - \frac{1}{|\Gamma|}\int_\Gamma B_{n-1}(s)\mathrm{d}s\right] \qquad \xi \in \Gamma$$

and then (8.37). This determines $V_{n-1}^{(0)}$ and then $c_{n-1}(t)$. We then chose α_n to satisfy (8.39). $\square$

Proof of Theorem 5.2:

To complete the proof of Theorem 5.2 we need to estimate the remainder term, given, see (7.16), by

$$\lambda R_2(\xi,t,\lambda) = \lambda \tilde{R}_2(\xi,t,\lambda) - \lambda^{N+1}\alpha_N(s(\xi),t)\bar{m}'(\frac{d(\xi,\Gamma)}{\lambda}) \tag{8.45}$$

Since (7.17) we obtain that

$$\sup_{\xi\in\Omega}\sup_{t\in[0,T]} |R_2(\xi,t,\lambda)| \leq C\lambda^N \tag{8.46}$$

Theorem 5.2 is then proved. $\square$

9. Proof of Theorem 5.3

Set

$$\tilde{m}^{(N)}(\xi,t) = m^{(N)}(\xi,t) - \int_0^t \bar{R}_1(\tau,\lambda)d\tau \tag{9.1}$$

where

$$\bar{R}_1(t,\lambda) = \frac{1}{|\Omega|}\int_\Omega R_1(\xi,t,\lambda)\mathrm{d}\xi$$

and $R_1(\xi,t,\lambda)$ is the remainder in Theorem 5.1, defined in (6.9) and estimated in (6.10) and (6.11). Denote

$$\tilde{\mu}^{(N-1)}(\xi,t) = \mu^{(N-1)}(\xi,t) + v(\xi,t) \tag{9.2}$$

where $v(\xi,t)$ solves

$$\begin{aligned} &\Delta v(\xi,t) = R_1(\xi,t,\lambda) - \bar{R}_1(t,\lambda) \qquad \text{for} \qquad \xi\in\Omega \\ &\frac{\partial}{\partial\nu}v = 0 \qquad \text{on} \qquad \partial\Omega \end{aligned} \tag{9.3}$$

with the further requirement

$$\int_\Omega v(\xi,t)\mathrm{d}\xi = 0 \qquad t\in[0,T]\ .$$

Since $|R(\xi, t, \lambda)| \leq C(T)\lambda^{N-1}$ we have that $|v(\xi, t)| \leq C\lambda^{N-1}$. The function $\tilde{m}^{(N)}$ and $\tilde{\mu}^{(N-1)}$ satisfy (5.11). Namely the first equation of (5.11) is satisfied by Theorem 5.1 and by construction, see (9.1) and (9.3). The second equation is obtained from Theorem 5.2 adding and subtracting terms to obtain $\tilde{\mu}^{(N-1)}$ and $\tilde{m}^{(N)}$. We obtain

$$\tilde{\mu}^{(N-1)} = \mu^{(N-1)} + v = -\lambda\Delta\tilde{m}^{(N)} + \frac{1}{\lambda}f(\tilde{m}^{(N)}) + R$$

where

$$R \equiv R(\xi, t, \lambda) = \frac{1}{\lambda}\left[f\left(\tilde{m}^{(N)} + \int_0^t \bar{R}_1(\tau, \lambda)\mathrm{d}\tau\right) - f(\tilde{m}^{(N)})\right] + R_2 + v$$

and R_2 is the remainder in Theorem 5.2, see (8.45). Since $\bar{R}_1 = \mathcal{O}(\lambda^N)$, $R_2 = \mathcal{O}(\lambda^N)$, $v = \mathcal{O}(\lambda^{N-1})$ and

$$\frac{1}{\lambda}\left[f\left(\tilde{m}^{(N)} + \int_0^t \bar{R}_1(\tau, \lambda)\mathrm{d}\tau\right) - f(\tilde{m}^{(N)})\right] \leq \frac{C}{\lambda}\int_0^t \bar{R}_1(\tau, \lambda)\mathrm{d}\tau = \mathcal{O}(\lambda^{N-1}) \tag{9.4}$$

the second equation of (5.11) is satisfied as well. The (5.13) and (5.14) are satisfied by construction of the $m^{(N)}$. Theorem 5.3 is then proved. □

Appendix A

A.1: The Dirichlet–Neumann operator

Let $G(\xi, \eta)$ be the Green function in Ω, with Neumann boundary condition on $\partial\Omega$, satisfying the equation

$$\Delta G(\xi, \eta) = \delta(\xi - \eta) - \frac{1}{|\Omega|}\ , \tag{10.1}$$

so that

$$\int_\Omega G(\xi, \eta)d\eta = \int_\Omega G(\xi, \eta)d\xi = 0\ . \tag{10.2}$$

Under the compatibility condition $\int_\Omega f(\xi)d\xi = 0$, the unique solution of the equation

$$\Delta v(\xi) = f(\xi) \qquad \text{for} \qquad \xi \in \Omega \tag{10.3}$$

with Neumann boundary conditions in $\partial\Omega$, and with $\int_\Omega v(\xi)d\xi = 0$, is given by

$$v(\xi) = \int_\Omega G(\xi, \eta)f(\eta)d\eta\ . \tag{10.4}$$

All other solutions with Neumann boundary conditions differ from this one by a constant. We will be particularly concerned with certain single layer potentials in what follows. Given a smooth function h defined on Γ, consider the *single layer potential*

$$\phi_h(\xi) = \int_\Gamma G(\xi,\eta)h(\eta)\mathrm{d}S_\eta \ ,$$

where $\mathrm{d}S_\eta$ denotes the arclength measure along Γ; this notation is standard in potential theory. The function ϕ_h satisfies Neumann boundary boundary conditions on $\partial\Omega$, and satisfies the equation

$$\Delta\phi_h(\xi) = h(\xi)\mathrm{d}S_\xi - \frac{1}{\Omega}\int_\Gamma h(\eta)\mathrm{d}S_\eta \ .$$

Clearly there is a discontinuity in the the normal derivatives of ϕ_h across Γ, and we have that

$$h(\xi) = \left[\frac{\partial}{\partial n}\phi_h\right]_\Gamma(\xi) \tag{10.5}$$

where the right hand side is the difference in the normal derivatives at $\xi \in \Gamma$:

$$\left[\frac{\partial}{\partial n}\phi_h\right]_\Gamma(\xi) = \left(\frac{\partial\phi_h}{\partial n}\right)_{\Omega_\Gamma^+}(\xi) - \left(\frac{\partial\phi_h}{\partial n}\right)_{\Omega_\Gamma^-}(\xi) \ .$$

This is a well known result from potential theory [9]. For ξ away from Γ,

$$\Delta\phi_h(\xi) = \frac{1}{\Omega}\int_\Gamma h(\eta)\mathrm{d}S_\eta \ .$$

Thus the single layer potential is harmonic away from Γ if and only if $\int_\Gamma h(\xi)\mathrm{d}S_\xi = 0$. Otherwise, it is subharmonic or superharmonic, according to whether $\int_\Gamma h(\xi)\mathrm{d}S_\xi$ is positive or negative. Every continuous function ϕ that satisfies the Neumann boundary condition, and is harmonic away from Γ, and which satisfies

$$\int_\Omega \phi(\xi)\mathrm{d}\xi = 0 \tag{10.6}$$

is the single layer potential of a uniquely determined function h defined on Γ satisfying

$$\int_\Gamma h(\xi)\mathrm{d}S_\xi = 0 \ . \tag{10.7}$$

Indeed, if ϕ_h is such a single layer potential, then from (10.2), $\int_\Omega \phi_h(\xi)\mathrm{d}\xi = 0$.

On the other hand, let ϕ be any continuous function that is harmonic on Ω_Γ^- and Ω_Γ^+, and which satisfies (10.6). Define h in Γ by

$$h(\xi) = \left[\frac{\partial}{\partial n}\phi\right]_\Gamma (\xi) \ ;$$

we refer to this as the Neumann data for ϕ. By the divergence theorem,

$$\int_\Gamma h(\xi)\mathrm{d}S_\xi = \int_\Gamma \nabla\phi \cdot n \mathrm{d}S_\xi + \int_\Gamma \nabla\phi \cdot (-n)\mathrm{d}S_\xi = \int_{\Omega_\Gamma^-} \Delta\phi \mathrm{d}\xi + \int_{\Omega_\Gamma^+} \Delta\phi \mathrm{d}\xi = 0 \ .$$

Hence, h satisfies (10.7).

Notice that $\phi - \phi_h$ satisfies Neumann boundary conditions and

$$\left[\frac{\partial}{\partial n}(\phi - \phi_h)\right]_\Gamma (\xi) = 0 \ .$$

This means that $\phi - \phi_h$ is a constant. Since the integral is zero, it is zero, and so $\phi = \phi_h$.

This proves the one to one correspondence between single layer potentials of functions h satisfying (10.7), and continuous functions ϕ that are harmonic on Ω_Γ^- and Ω_Γ^+, and which satisfy (10.6).

Next, given a continuous function ϕ that is harmonic on on Ω_Γ^- and Ω_Γ^+, whether or not (10.6) is satisfied, define the function g on Γ by $g = \phi|_\Gamma$. We naturally refer to g as the Dirichlet data for ϕ.

The Neumann data is $\left[\frac{\partial}{\partial n}\phi\right]_\Gamma$. The Dirichlet–Neuman operator $\mathcal{T}_\Gamma$ defined by

$$\mathcal{T}_\Gamma g = \left[\frac{\partial}{\partial n}\phi\right]_\Gamma \qquad (10.8).$$

where ϕ is the continuous function that is Harmonic in Ω_Γ^- and Ω_Γ^+, and with $\phi|_\Gamma = g$.

A simple argument shows that $\mathcal{T}_\Gamma$ is a positive Hermitian operator. Indeed, let ψ be continuous on Ω, and harmonic on Ω_Γ^- and Ω_Γ^+, and with $\psi|_\Gamma = h$. Then

$$\begin{aligned}\int_\Gamma h\mathcal{T}_\Gamma g \mathrm{d}s &= \int_\Gamma \psi \left[\frac{\partial}{\partial n}\phi\right]_\Gamma \mathrm{d}s \\ &= \int_{\Omega_\Gamma^+} \nabla\cdot(\psi(\nabla\phi))\mathrm{d}\xi + \int_{\Omega_\Gamma^-} \nabla\cdot(\psi(\nabla\phi))\mathrm{d}\xi \\ &= \int_\Omega \nabla\psi\cdot\nabla\phi \mathrm{d}\xi \ .\end{aligned}$$

Taking $h = 1$, so that $\psi = 1$, we further see that the range of $\mathcal{T}_\Gamma$ is orthogonal to the constants. We let $\mathcal{T}_\Gamma$ denote the Friedrichs extension of $\mathcal{T}_\Gamma$. It is easy to see, and well known, that the form domain of $\mathcal{T}_\Gamma$ is the Sobolev space $H^{1/2}(\Gamma)$, and that the kernel consists exactly of the constants. There is an explicit formula for the inverse of $\mathcal{T}_\Gamma$ restricted to the orthogonal complement of the constants; we denote this by $\mathcal{S}_\Gamma$. Indeed, let v be any function on Γ with $\int_\Gamma v(s)\mathrm{d}s = 0$. Since the single layer potential ϕ_v for v has Neumann data v, all we need to do is to subtract a constant to make this function orthogonal to the constants on Γ, instead of being orthogonal to the constant on Ω. Therefore, the inverse $\mathcal{S}_\Gamma$ is given by

$$\mathcal{S}_\Gamma v(\xi) = \int_\Gamma G(\xi,\eta)v(\eta)\mathrm{d}S_\eta - \frac{1}{|\Gamma|}\int_\Gamma\int_\Gamma G(\xi,\eta)v(\eta)\mathrm{d}S_\eta \mathrm{d}S_\xi \qquad \xi \in \Gamma\ . \tag{10.9}$$

It is easily checked that this is self adjoint on the orthogonal complement of the constants. Now let h be an arbitrary smooth function on Γ satisfying $\int_\Gamma h(s)\mathrm{d}s = 0$. Consider the single layer potential

$$\phi(\xi) = \int_\Gamma G(\xi,\eta)h(\eta)\mathrm{d}S_\eta \qquad \xi \in \Omega\ .$$

In general, the Dirichlet data for ϕ does not integrate to zero on Γ, and hence is not directly related to the Neumann data through the Dirichlet–Neumann operator. However, we can correct for this by subtracting a constant: Form the function

$$\tilde{\phi}(\xi) = \phi(\xi) - \frac{1}{|\Gamma|}\int_\Gamma \phi(\eta)\mathrm{d}S_\eta\ .$$

Then clearly

$$\begin{aligned} \tilde{\phi}|_\Gamma &= \mathcal{S}_\Gamma h \\ h &= \mathcal{T}_\Gamma \tilde{\phi} \end{aligned}\ . \tag{10.10}$$

We can now express the vector field V driving the Mullins–Sekerka flow as

$$V = \mathcal{T}_\Gamma\left(K - \frac{1}{|\Gamma|}\int_\Gamma K(s)\mathrm{d}s\right)\ . \tag{10.11}$$

We close by establishing notation for the two harmonic extension operators that will arise throughout what follows: The *Neumann harmonic extension operator* $\mathcal{E}_{\Gamma,N}$ is defined by

$$\left(\mathcal{E}_{\Gamma,N} v\right)(\xi) = \int_\Gamma G(\xi,\eta)v(\eta)\mathrm{d}S_\eta - \frac{1}{|\Gamma|}\int_\Gamma\int_\Gamma G(\xi,\eta)v(\eta)\mathrm{d}S_\eta \mathrm{d}S_\xi \quad \xi \in \Omega\ , \tag{10.12}$$

where v is a function on Γ satisfying

$$\int_\Gamma v(\xi)\mathrm{d}S_\xi = 0 \ .$$

Notice that $(\mathcal{E}_{\Gamma,N} v)(\xi)$ is the unique function that is continuous on Ω, harmonic on $\Omega\backslash\Gamma$ satisfying Neumann boundary conditions on $\partial\Omega$, with Neuman data v on Λ, and with zero integral over Γ.

The *Dirichlet harmonic extension operator* $\mathcal{E}_{\Gamma,D}$ is defined by setting $\mathcal{E}_{\Gamma,D}g(\xi)$ to be the harmonic function ϕ on $\Omega\backslash\Gamma$ with Neumann boundary conditions on $\partial\Gamma$, and with $\phi|_\Gamma = g$. Here, there is no restriction on the integral of g over Γ.

Naturally, the Dirichlet extension can be expressed in terms of the Neumann extension and the Dirichlet–Neumann operator. We have from (10.9) and (10.12) that

$$\mathcal{E}_{\Gamma,D}g(\xi) = \mathcal{E}_{\Gamma,N}\left(\mathcal{T}_\Gamma\left(g - \frac{1}{|\Gamma|}\int_\Gamma g(\eta)\mathrm{d}S_\eta\right)\right)(\xi) + \frac{1}{|\Gamma|}\int_\Gamma g(\eta)\mathrm{d}S_\eta \qquad \xi\in\Omega\ . \tag{10.13}$$

A.2: The expansion in λ of the Laplacian in local coordinates.

Le $f(z,s)$, $z = \frac{d}{\lambda}$, be a C^2 function from $I\!R\times\Gamma$ to $I\!R$. Then, in dimension $d=2$, we have that

$$\begin{aligned}\lambda^2\Delta f(z,s) &= \frac{1}{1-K(s)\lambda z}\left\{((1-K(s)\lambda z)f_z)_z + \lambda^2\left(\frac{f_s}{1-K(s)\lambda z}\right)_s\right\}\\ &= f_{zz} - \lambda K(s) f_z\frac{1}{1-K(s)\lambda z} + \lambda^2\frac{f_{ss}}{(1-K(s)\lambda z)^2} + \lambda^3 f_s\frac{\frac{d}{ds}K(s)z}{(1-K(s)\lambda z)^3}\end{aligned} \tag{10.14}$$

Recalling that, for $|x|<1$

$$\frac{1}{(1-x)} = \sum_{n=0}^\infty x^n; \qquad \frac{1}{(1-x)^2} = \sum_{n=0}^\infty n x^{n-1}; \qquad \frac{1}{(1-x)^3} = \frac{1}{2}\sum_{n=0}^\infty n(n-1)x^{n-2}$$

we rewrite (10.14) as the following

$$\lambda^2\Delta f = f_{zz} + \sum_{n=0}^\infty \lambda^{n+1}\left\{a_{n+1}(z,s)f_z + b_{n+1}(z,s)f_{ss} + c_{n+1}(z,s)f_s\right\} \tag{10.15}$$

where

$$\begin{aligned} a_{n+1}(z,s) &= -K^{n+1}(s)z^n\ ,\\ b_{n+1}(z,s) &= nK^{n-1}(s)z^{n-1}\ ,\\ c_{n+1}(z,s) &= \frac{1}{2}n(n-1)z^{n-1}K^{n-2}(s)\frac{d}{ds}K(s)\ .\end{aligned} \tag{10.16}$$

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