MOTION BY CURVATURE BY SCALING NON LOCAL EVOLUTION EQUATIONS

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Abstract. We prove convergence to a motion by mean curvature by scaling diffusively a non linear, non local evolution equation. This equation has been introduced in [9] to describe the macroscopic behavior of a ferromagnetic spin system with Kac interaction which evolves with Glauber dynamics. The convergence is proven in any time interval in which the limiting motion is regular.

§1. *General motivations.*

Under quite general conditions and in many instances the interface between two species in competition moves by mean curvature. Namely each point of the interface has a velocity

$$v = \frac{\theta}{R}\nu$$

θ is a phenomenological constant and $1/R$ is proportional to the local mean curvature by a dimensional factor; ν is the unit vector normal to the interface. The sign of the mean curvature is defined in such a way that the velocity is directed toward the local concavity of the interface, so that, in particular, a closed curve in $\mathbb{R}^2$ shortens under this evolution and eventually disappears. In physics the motion by curvature describes the interface

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dynamics after phase separation in isotropic systems with non conserved order parameter and order-disorder phase transition.

The derivation of the motion by curvature from "microscopic evolutions" is a fascinating problem of basic theoretical importance and great applicative interest, which has therefore attracted physicists and mathematicians, [1,2,3,5,14,15,19,20,23,25], just to mention a few papers appeared on this subject that are more closely related to what we do here. There are several aspects of the problem of pure mathematical relevance, which have been largely debated and studied. In the more recent developments, the motion by curvature is viewed as a singular limit of regularized equations, [6,16,24]. The analysis of this limit involves sophisticated mathematical notions, see for instance [7, 22] and, in this context, it is possible to define the motion by curvature in a generalized sense, even past the appearence of singularities, [1,15,19].

Here we are mainly concerned with the derivation problem: we can divide the known results on these specific topics into two groups. In the first one, [1,3,5,14,19], the "basic evolution" is described by a PDE, see (7) below, while, in the second one, [2, 20, 25], it is given by some stochastic dynamics on a Ising spin system. With the noticeable exception of the work of Spohn, [25], the explicit or implicit presence of a diffusive term in the evolution plays a common and dominant role. In this paper we present a derivation of the motion by curvature (limited to times when the motion is regular) where such a feature is absent. We start in fact from the non local evolution equation, see (1) below, introduced in [9] to describe the macroscopic behavior of a spin system with Kac potential, [18], and Glauber dynamics. We present here the basic ingredients of the proof in the simplest version of the problem, details and extensions to more general cases (in particular to the spin system itself) will be given elsewhere.

By avoiding the burden of a formal proof, we omit the proofs of a few technical lemmas, we hope to present more clearly the basic ingredients responsible for the motion by curvature. The absence of a differential structure forces us to proofs somewhat less computational, that may give a better insight on the true mechanisms that rule the phenomenon.

§2. *The evolution equation.*

We consider the evolution equation

$$\frac{\partial m}{\partial t} = -m + \tanh\{\beta J \star m\} \tag{1}$$

where $m = m(r,t)$, $r \in \mathbb{R}^d$, $t \geq 0$, $\beta > 0$, $J = J(|r|)$. We assume $0 \leq J \in C^2$, and $J(|r|) = 0$ for $|r| \geq 1$. We study the case when $\beta > 1$, with the normalization condition $\int dr J(|r|) = 1$. We have used the notation $J \star m$ to denote the convolution of J and m.

As already mentioned, (1) describes the evolution of an Ising spin system with Kac potential J, in the limit when the inverse range of the potential, γ, goes to 0, [9].

The constants $\pm m_\beta$ are stationary solutions of (1), with m_β the strictly positive solution of

$$m_\beta = \tanh\{\beta m_\beta\} \tag{2}$$

whose existence follows from the assumption that $\beta > 1$. In the spin system $\pm m_\beta$ are the equilibrium magnetizations at the inverse temperature β, [21], thus they represent the magnetizations of the pure phases of the system. The existence of two phases corresponds in the spin system to a phase transition, which, therefore, is present whenever $\beta > 1$. $\beta = 1$ is then the critical temperature. Our purpose is to characterize the evolution of an initial datum which has two coexisting phases, precisely a datum which is close to m_β inside a region Λ_0 and to $-m_\beta$ outside Λ_0. We will prove that this situation persists at later times, in the sense that when $t > 0$ the solution is close to m_β inside a region Λ_t and to $-m_\beta$ outside Λ_t, where Λ_t is defined by letting its boundary move by mean curvature. This result holds for "very large regions" and observing the evolution for accordingly "long times", namely in a scaling limit that we explain in the next Section.

In [10] the analysis of the first stage of phase separation, in the spin system which gives rise to (1), shows that the pure phases appear in "large clusters" whose boundaries are regular fulfilling all the requirements that we need for proving the convergence to a motion by mean curvature, as stated in the next Section. The analysis in the present paper is hopefully a guideline for studying the interface dynamics after phase separation, in what should be the sequel of [10].

§3. *Scalings.*

Let Γ_0 be a C^2 surface which is the boundary of a connected, bounded, open set Λ_0. Then the motion by mean curvature starting from Γ_0 is defined as

$$\frac{d\xi}{d\tau} = \frac{\theta}{R}\nu \tag{3}$$

with $\xi = \xi(\tau)$ the generic point of the surface Γ_τ, $R = R(\xi)$ with R^{-1} equal to $(d-1)$ times the mean curvature of Γ_τ at ξ; θ a constant and ν the unit vector normal to Γ_τ at ξ and pointing toward the interior of Γ_τ.

It is then known that there is a positive time interval when Γ_τ is well defined and C^2.

It is easy to verify that (3) is invariant under the scaling:

$$\xi \to r = \lambda^{-1}\xi; \quad \tau \to t = \lambda^{-2}\tau \tag{4}$$

While (3) is a fixed point for the transformation (4), (1) is not left invariant, but, as we shall see, it is attracted by (3) under (4) in the limit $\lambda \to 0$. More precisely we define

$$m^{(\lambda)}(\xi, \tau) = m(\lambda^{-1}\xi, \lambda^{-2}\tau) \tag{5}$$

With a terminology borrowed from the spin systems, we will refer to the variables ξ and τ in the argument of $m^{(\lambda)}$ as the macroscopic variables and to r and t as the mesoscopic variables, to distinguish them from the microscopic ones which appear in the original spin system from which (1) is derived.

We impose convergence in (5) at time 0 by setting

$$m(r, 0) = m_0(\lambda r; \lambda) \tag{6}$$

with $m_0(\xi; \lambda)$ converging as $\lambda \to 0$ to m_β, $[-m_\beta]$, inside and, respectively, outside Λ_0. More assumptions on m_0 will be specified in §8. We shall prove that also $m^{(\lambda)}(\xi, \tau)$ converges as $\lambda \to 0$ to m_β, $[-m_\beta]$, strictly inside and, respectively, outside Λ_τ, where Λ_τ is defined by letting the boundary evolve by mean curvature. θ, in (3), is given by (27a) below. The precise statements are given in §8.

The same procedure applies to the Allen-Cahn equation:

$$\frac{\partial u}{\partial t} = \Delta u - V'(u) \tag{7}$$

with $V(u)$ a symmetric, double well potential with minima at $\pm m_\beta$. Then the function

$$u^{(\lambda)}(\xi,\tau) = u(\lambda^{-1}\xi, \lambda^{-2}\tau) \tag{8}$$

solves

$$\frac{\partial u^{(\lambda)}}{\partial \tau} = \Delta u^{(\lambda)} - \lambda^{-2} V'(u^{(\lambda)}), \qquad u^{(\lambda)}(\xi,0) = u_0(\xi;\lambda) \tag{9}$$

with u_0 equal to m_0.

In [1,5,14,15,19] it is proven that $u^{(\lambda)}(\xi,\tau)$ converges, as $\lambda \to 0$, to $\pm m_\beta$ if ξ is inside, respectively outside Λ_τ, if the boundary of Λ_τ moves by mean curvature with $\theta = 1$. We will give later more details on this result, here we only want to comment on the value of θ. As explained in [25], there is an Einstein relation which involves θ, the mobility of the surface and the surface tension. By this relation we can determine θ in terms of the mobility and of the surface tension. We thus have two ways to compute θ, one based on the Einstein relation, the other one by looking directly at the limiting motion by curvature, which also specifies the value of θ. Both ways lead to the same value of θ, as proven in [25] for the Allen-Cahn equation and in [4] for the case considered here.

§4. *Instantons.*

The equation for $m^{(\lambda)}$ is

$$\frac{\partial m^{(\lambda)}}{\partial \tau} = \lambda^{-2}\Big\{ - m^{(\lambda)} + \tanh\{\beta J^{(\lambda)} \star m^{(\lambda)}\}\Big\} \tag{10a}$$

where

$$J^{(\lambda)}(|\xi|) = \lambda^{-d} J(\lambda^{-1}|\xi|) \tag{10b}$$

(10a) is similar in many ways to the Boltzmann equation with small free path, namely when there is a divergent factor in front of the collision kernel. In our case the divergent factor λ^{-2} forces the curly bracket term in (10a) to be small, hence $m^{(\lambda)}$ should be close to a stationary solution of (1). With respect to the Boltzmann case, the situation here is simpler because of the dissipative nature of (1), which limits drastically the number of stationary states: only two of them will be needed, which correspond to the stable phases $\pm m_\beta$. However, the problem of matching them together will bring in a third stationary and spatially non homogeneous solution of (1). In the Boltzmann case, this is the analogue of a Milne problem, but with moving boundary.

The stationary equation in the macroscopic coordinates is

$$0 = -m^{(\lambda)} + \tanh\{\beta J^{(\lambda)} \star m^{(\lambda)}\} \equiv R + D \tag{11a}$$

$$R = -m^{(\lambda)}(\xi) + \tanh\{\beta m^{(\lambda)}(\xi)\} \tag{11b}$$

$$D = \tanh\{\beta (J^{(\lambda)} \star m^{(\lambda)})(\xi)\} - \tanh\{\beta m^{(\lambda)}(\xi)\} \tag{11c}$$

The first term, R, is strictly local and it plays the role of the reactive term in (9), while the second one, D, is non local and it is in a sense analogous to the term with the laplacian in (9). Like this one it has order λ^2, when $m^{(\lambda)}$ is smooth.

In conclusion, we look for a solution close to m_β when ξ is in Λ_τ and to $-m_\beta$ when ξ is outside, namely wherever the reactive term in (11) is dominant. The transition region is where the D-term in (11) cannot be neglected. As $J^{(\lambda)}$ is an approximate δ-function, the transition region turns out to be infinitesimal in λ, hence, for its analysis, it is convenient to go back to the original mesoscopic coordinates. Then Γ_τ becomes $\lambda^{-1}\Gamma_t$, $t = \lambda^{-2}\tau$, and the interface looks then extremely flat, assuming, as we do, that Γ_τ is regular. It is therefore natural to look for stationary solutions

$$\bar{m} = \tanh\{\beta J \star \bar{m}\} \tag{12}$$

which have a planar symmetry, namely, modulo translations, $\bar{m}(r) = m^{(dd)}(r \cdot \nu)$, ν a unit vector. Then, calling $x = r \cdot \nu$, we have that $m^{(dd)}(x)$ solves the stationary, d=1, problem:

$$m^{(dd)} = \tanh\{\beta \tilde{J} \star m^{(dd)}\} \tag{13}$$

with interaction

$$\tilde{J}(x) = \int_{\mathbb{R}^{d-1}} dy J(|x^2 + y^2|^{1/2}) \tag{14}$$

Dal Passo and de Mottoni, [8], have proven that if J is non increasing, (13) has a unique solution in the class of the non constant, odd, non decreasing functions. They have also shown that

$$\lim_{x\to\pm\infty} m^{(dd)}(x) = \pm m_\beta \tag{15}$$

The result has been generalized in [12] to the interactions considered here, with a stronger uniqueness theorem.

As the convergence in (15) is exponentially fast, [11], $\bar{m}$ is the right candidate for matching the solutions $\pm m_\beta$ at both sides of the interface. Thus, avoiding the initial layer problem, considered in [10], we suppose that at time 0, in a neighborhood of Γ_0, see §8 for a precise statement,

$$m_0(\xi;\lambda) = m^{(dd)}\big(\frac{d(\xi,\Gamma_0)}{\lambda}\big) \tag{16}$$

with $d(\xi,\Gamma_0)$ the signed distance from Γ_0, positive for ξ inside Λ_0.

An instanton solution $\bar{m}$, with properties similar to those stated above, exists also in the Allen-Cahn equation, [17]. de Mottoni and Schatzman, [14], have proven that the instanton-like structure (corresponding to (16)) is persistent, at least till times when the surface is regular. It is a remarkable result, as it shows that in the mesoscopic variables the form of the solution remains the same, to leading orders in λ, for times proportional to λ^{-2}, even though the interface, during this time, moves by distances proportional to λ^{-1}. There are corrections to the shape and the location of the interface, but they vanish as $\lambda \to 0$.

Chen's result, [5], is weaker, as he proves only that super and sub-solutions have the same instanton structure they have initially. They are however shifted from each other by distances which grow as $\log \lambda^{-1}$, in mesoscopic coordinates, so that, in principle, the interface might have flattened by that amount. In the macroscopic variables, however, where spaces are shrunk by λ, this fine structure disappears and the interface becomes sharp. The advantages of this approach are twofold. From one side the analysis is much simpler, from the other one it evidentiates the use of inequalities which play an important role in the proof of the convergence past the appearence of singularities (to a "generalized motion by curvature"), see for instance [15].

One of the basic ingredients in [5] for constructing super and sub-solutions is that the signed distance function from a surface moving by mean curvature is closely related to the heat equation. In (1) this is not directly of help, as a Laplacian is neither present nor likely to be "hidden" somewhere (opposite to what happens in Allen-Cahn where it appears explicitly). We overcome this problem by exploiting an invariance principle, applied to the linearization of (1) around the planar instanton defined earlier. We shall prove that, on a suitable space-time scale, this behaves as the Allen-Cahn equation (in the corresponding approximation) so that in this scaling, inequalities à la Chen will work. We are thus half way between [14] and [5], being close to the former when we exploit the good mixing behavior of (1), and to the latter when we use inequalities to estimate the non linear terms of the expansion, (loosing in this way the accurate control of the shape and the location of the instanton, as in [14]). At variance with [14], we localize our analysis by studying separately the evolution for short times and in small neighbourhoods of the interface. We then use a "patching and iterating" procedure to obtain a global solution. The advantage of working locally is that we only need to consider perturbations of the planar instanton and this greatly simplifies the analysis. In particular we can use a L_∞ setting, as we prove that the linearization around the planar instanton is strictly related to a Markov generator. We explain all that in the next paragraph.

§5. *A probabilistic interpretation.*

To have lighter notation we write $\bar m$ for $m^{(dd)}$. We linearize (1) around $\bar m$ and work in d=1 with J replaced by $\tilde J$. We then obtain

$$\frac{\partial \phi}{\partial t} = L\phi, \qquad L\phi = -\phi + (1-\bar m^2)\beta \tilde J \star \phi \tag{17}$$

$\bar m$ and the equation (17) are studied in [11] and [13], for ease of reference we recall the main results.

Properties of $\bar m$:

$$\bar m'(x) > 0, \qquad \text{for all } \ x \in \mathbb{R} \tag{18a}$$

There are α and M, both positive, so that

$$\lim_{x\to\pm\infty} e^{\alpha|x|}\left|\bar{m}(x) \mp [m_\beta - \frac{M}{\alpha}e^{-\alpha|x|}]\right| = 0 \tag{18b}$$

$$\lim_{|x|\to\infty} e^{\alpha|x|}\left|\bar{m}'(x) - Me^{-\alpha|x|}\right| = 0 \tag{18c}$$

$$\lim_{x\to\pm\infty} e^{\alpha|x|}\left|\bar{m}''(x) \pm \alpha Me^{-\alpha|x|}\right| = 0 \tag{18d}$$

Next we state the results on the asymptotic behavior of the solutions of (17): there is $a > 0$ and, for any $|\delta| < \alpha$, there is c so that

$$\|\phi(\cdot,t) - C_\phi \bar{m}'(\cdot)\|_\delta \le ce^{-at}\|\phi(\cdot) - C_\phi \bar{m}'(\cdot)\|_\delta \tag{19a}$$

where

$$\|u\|_\delta = \sup_x e^{-\delta|x|}|u(x)| \tag{19b}$$

$$C_\phi = \int \mu(dx)\phi(x)\bar{m}'(x)^{-1} \tag{19c}$$

$$\mu(dx) = N\frac{\bar{m}'(x)^2}{1-\bar{m}(x)^2}dx, \qquad N^{-1} = \int_{\mathbb{R}} dx\frac{\bar{m}'(x)^2}{1-\bar{m}(x)^2} \tag{19d}$$

This is a Perron-Frobenius theorem for the operator e^{Lt}. The proof is based on probabilistic methods typical of Equilibrium Statistical Mechanics. The starting point is the observation that $L\bar{m}' = 0$, which is readily obtained by differentiating (13). Therefore L has eigenvalue 0 and eigenvector $\bar{m}'$, which, by (18a), is strictly positive, just like in the Perron-Frobenius theorem. It is therefore natural to make the following mapping:

$$\phi = \bar{m}'\psi, \quad \mathcal{L}\psi = \frac{1}{\bar{m}'}L(\bar{m}'\psi) \tag{20a}$$

$$\mathcal{L}\psi(x) = (1-\bar{m}(x)^2)\beta\int dx'\tilde{J}(|x-x'|)\frac{\bar{m}'(x')}{\bar{m}'(x)}[\psi(x')-\psi(x)] \tag{20b}$$

As evident from (20b), $\mathcal{L}$ is the generator of a jump Markov process with jump intensity

$$K(x,x') = (1-\bar{m}(x)^2)\beta\tilde{J}(|x-x'|)\frac{\bar{m}'(x')}{\bar{m}'(x)} \tag{20c}$$

The validity of the Perron-Frobenius theorem in the present context is related to the fact that by (18c) the transition rate $K(x,y)$ has asymptotically a drift toward the origin and that K is a smooth integral kernel.

Recall now that if m is the solution of (1) and it is close to $\bar{m}$, then $m - \bar{m}$ will be well approximated by (17). In this approximation, $\psi = c$, a constant, means that

$$m - \bar{m} \approx c\bar{m}', \qquad m(x) \approx \bar{m}(x + c) \tag{21}$$

Thus ψ is a displacement; if ψ depends on x, then $\psi(x)$ is the displacement necessary to make $\bar{m}$ equal to m at x, in the approximation (21). It is proven in [11] that $e^{\mathcal{L}t}\psi$ converges to a constant as $t \to \infty$, hence, in the linear approximation, m converges to a translate of the instanton.

In $d > 1$ the conclusions are similar. Given a unit vector ν, we linearize around the instanton $\bar{m}(r) = m^{(dd)}(r \cdot \nu)$, call $x = r \cdot \nu$ and simply write $\bar{m}(x)$. We then define

$$L\phi(r) = -\phi(r) + \int dr' \big(1 - \bar{m}(x)^2\big)\beta J(|r - r'|)\phi(r') \tag{22a}$$

$$\mathcal{L}\psi(r) = \int dr' K(r, r')[\psi(r') - \psi(r)] \tag{22b}$$

$$K(r, r') = \big(1 - \bar{m}(x)^2\big)\beta J(|r - r'|)\frac{\bar{m}'(x')}{\bar{m}'(x)} \tag{22c}$$

where $\bar{m}'$ shorthands the derivative of $m^{(dd)}$ with respect to its argument. Observe that the integral of K is equal to 1.

Thus $\mathcal{L}$ is still the generator of a jump Markov process, with a drift toward the plane $\{r \cdot \nu = 0\}$ and with invariant measure $\mu(dx)dy$, $x = r \cdot \nu$ and y the coordinates in the plane $\{r \cdot \nu = 0\}$. In the scaling limit (4) with $\lambda \to 0$, the process converges to a degenerate brownian motion on the plane $\{r \cdot \nu = 0\}$.

§6. *The linear approximation.*

In this Section we study the evolution in the time interval $[0, T]$, where $T = \lambda^{-\delta}$, $\delta > 0$. For δ sufficiently small the evolution, in this time interval, produces very small changes, so that the linear approximation is rather accurate. In §8 we make this precise, here we just stick to the linear approximation with no further justification. As already mentioned we localize our analysis. Let then $\xi_0 \in \Gamma_0$ and $r_0 = \lambda^{-1}\xi_0$. We study the evolution in a small neighborhood of r_0 and, to this end, we choose a coordinate frame (in mesoscopic variables) with origin at r_0. The x axis is directed along the normal to $\lambda^{-1}\Gamma_0$ at r_0. We choose the other axes of the reference frame along the principal axes of curvature and choose the x axis pointing toward $\lambda^{-1}\Lambda_0$, so that, to first order, the equation for the surface is $x = x^\star(y)$, where

$$x^\star(y) = \frac{1}{2}\lambda \sum_{i=1}^{d-1} \frac{y_i^2}{R_i}, \qquad \sum_{i=1}^{d-1} \frac{1}{R_i} = \frac{1}{R} \tag{23a}$$

$1/R$ is therefore $(d-1)$ times the mean curvature of Γ_0 at ξ_0. The factor λ appears when writing the equation in mesoscopic variables, as done in (23a).

As an example, suppose that Γ_0 is a circle of radius R in $\mathbb{R}^2$. Then the x axis is along the radius and connects $r_0 \in \lambda^{-1}\Gamma_0$ to the center of the circle, directed toward the latter. The y axis passes through r_0 and it is perpendicular to the x axis.

According to (16), in this frame of reference and for r in a neighborhood of the origin,

$$m(r,0) = \bar{m}(x) - \frac{1}{2}\lambda(\sum_{i=1}^{d-1} \frac{y_i^2}{R_i})\bar{m}'(x) + \mathcal{R} \tag{23b}$$

The remainder term, $\mathcal{R}$, will be disregarded in this Section. We next linearize (1) around the planar instanton $\bar{m}(x)$, we have only made explicit the coordinate x in the argument of $\bar{m}(r)$, as this is the only one on which it depends. Calling $\phi(r,t)$ the solution of

$$\frac{\partial \phi}{\partial t} = L\phi, \qquad \phi(r,0) = -\frac{1}{2}\lambda \sum_{i=1}^{d-1} \frac{y_i^2}{R_i}\, \bar{m}'(x) \tag{23c}$$

we have, by (20a),

$$\phi(r,t) = \bar{m}'(x) e^{\mathcal{L}t}\psi_0(r), \qquad \psi_0(r) = -\frac{1}{2}\lambda \sum_{i=1}^{d-1} \frac{y_i^2}{R_i} \tag{24}$$

We are interested in $r = (x,0)$, namely the values along the normal to $\lambda^{-1}\Gamma_0$ at r_0. We then have

$$(e^{\mathcal{L}T}\psi_0)(x,0) = \int_0^T dt \big(e^{\mathcal{L}t}\mathcal{L}\psi_0\big)(x,0) \tag{25a}$$

From (22) it follows that for any $r = (x, y)$,

$$\mathcal{L}\psi_0(r) = \frac{1}{2}\lambda \sum_{i=1}^{d-1} \frac{1}{R_i} \int dx'dy' K\big((x,y),(x',y')\big)[y_i'^2 - y_i^2]$$
$$= \frac{1}{2}\lambda \sum_{i=1}^{d-1} \frac{1}{R_i} (1 - \bar{m}(x)^2)\beta \int dx'dz J\big(|(x'-x)^2 + z^2|^{1/2}\big)$$
$$\times \frac{\bar{m}'(x')}{\bar{m}'(x)} \big[(z_i + y_i)^2 - y_i^2\big] \tag{25b}$$

hence

$$(e^{\mathcal{L}T}\psi_0)(x, 0) = \int_0^T dt (e^{\mathcal{L}^{(1)}t} f)(x) \tag{26a}$$

with $\mathcal{L}^{(1)}$ the operator in d=1 defined in (20b) and

$$f(x) = -\big(1 - \bar{m}(x)^2\big)\beta \int dx'dz J\big(|(x'-x)^2 + z^2|^{1/2}\big) \frac{\bar{m}'(x')}{\bar{m}'(x)} \frac{\lambda}{2} \sum_i z_i^2 / R_i \tag{26b}$$

We define

$$\theta = \int \mu(dx)\big(1 - \bar{m}(x)^2\big)\beta \int dx'dy J\big(|(x'-x)^2 + y^2|^{1/2}\big) \frac{\bar{m}'(x')}{\bar{m}'(x)} y_1^2/2 \tag{27a}$$

then

$$\int \mu(dx) f(x) = -\frac{\lambda}{R}\theta, \qquad \frac{1}{R} = \sum_i \frac{1}{R_i} \tag{27b}$$

Using the exponential convergence rate to equilibrium of the d=1 process, whose proof will be omitted, it follows that there is C so that for all x and t

$$\Big|(e^{\mathcal{L}t}\psi_0)(x, 0) + \frac{1}{R} t\lambda\theta\Big| \le \lambda C(|x| + 1) \tag{28}$$

The rate of convergence of $e^{\mathcal{L}t}$ depends on the starting point x: as the process has a drift toward the origin, it takes a time proportional to $|x|$ to reach a neighbourhood of the origin and then the process approaches equilibrium exponentially fast. Hence, by (26a), the term $|x|$ on the right hand side of (28), which bounds the contribution to the integral in (26a) of $t \le c|x|$, with c a suitable constant necessary to forget the initial condition.

In conclusion, in the linear approximation,

$$m\big((x, 0), T\big) \approx \bar{m}(x) - \bar{m}'(x)\frac{1}{R}T\lambda\theta \approx \bar{m}\big(\bar{x} - \frac{1}{R}T\lambda\theta\big) \tag{29}$$

The interface has thus moved in the time T by $T\lambda\theta/R$. Recall that we are using the mesoscopic units, therefore, in the macroscopic ones, where spaces are shrunk by λ, the displacement becomes $T\lambda^2\theta/R$. Since the macroscopic times are λ^2 times the mesoscopic ones, the term $\lambda^2 T$ above is just the macroscopic time corresponding to the mesoscopic time T. In this approximation, therefore, we obtain (3) with θ equal to the expression in (27a). More details are given in §8.

§7. *The level set equation and the generalized motion by curvature.*

The same arguments presented in §6 apply as well to the Allen-Cahn equation. The analysis in that case is simpler, because the x and y motions are independent in the process generated by the analogue of $\mathcal{L}$. As a consequence, the estimates on the approach to equilibrium, used in (28) are not necessary. In both cases the main point is the convergence of the processes with generator $\mathcal{L}$ and its analogue in Allen-Cahn, under the diffusive limit (4), to a degenerate Brownian motion on the plane $\{r\cdot\nu\}$. As the initial deviations from the planar instanton grow quadratically along the plane (see the factor $\sum y_i^2/R_i$) the square displacement of the Brownian motion in the tangent plane is proportional to the velocity of the interface. In our case the analysis is complicated by the correlation between the x and y motions, which reflects in the more complex formula for θ.

A degenerate Brownian motion is also present in the level set equation:

$$\frac{\partial u}{\partial t} = \sum_{i,j} \Big(\delta_{i,j} - \frac{\nabla_i u \nabla_j u}{|\nabla u|^2}\Big) \nabla_i \nabla_j u \tag{30}$$

The evolution given by (30) is such that all the level sets of u move by mean curvature, provided they are regular surfaces. The regularized motion by curvature can then be defined by first adding to (30) the extra diffusive term $\epsilon\Delta u$ and then letting $\epsilon \to 0$. Notice that both (30) and its regularized version are invariant under the diffusive scaling (4).

The right hand side of (30) gives rise to a degenerate Brownian motion on the planes tangent to the level sets, similarly to the diffusion considered earlier and associated to (1) and (7). It escapes to us, though, whether the analogy carries over to a more direct procedure for proving convergence to the motion by curvature, maybe based on relating the corresponding Brownian motions.

§8. *Statements and proofs.*

The initial datum: Let Λ_0 be an open, connected, bounded set whose boundary Γ_0 is a C^2 surface. We fix $0 < \zeta < 1$ and assume that for all λ sufficiently small, (16) holds for all ξ such that $|d(\xi, \Gamma_0)| \le \lambda^{1-\zeta}$. For the other values of ξ, inside and, respectively, outside Λ_0, we set $m_0(\xi; \lambda) = \pm m^{(dd)}(\lambda^{-\zeta})$.

The result: Let Γ_τ, $\tau \ge 0$, be the motion by curvature defined by (3) with θ as in (27a) and let Λ_τ be the bounded region whose boundary is Γ_τ. Let $\tau^\star > 0$ be such that Γ_τ is regular for a time interval strictly longer than $\tau^\star$. Then there are $a > 0$ and $b > 0$ so that for all $\tau \le \tau^\star$ and all λ small enough

$$|m^{(\lambda)}(\xi, \tau) \mp m_\beta| \le \lambda^b \tag{31}$$

for all ξ inside and, respectively, outside Λ_τ, and such that $|d(\xi, \Gamma_\tau)| \ge \lambda^a$.

We prove (31) by constructing super and sub-solutions of (1), which squeeze the solution in a way that yields the desired result. Actually this is an abuse of language, since we construct functions which are above and below the solution, but only at the times kT, k integer. However, since $\lambda^2 T \to 0$, this time grid in macroscopic time units has an infinitesimal spacing, so that our bounds are indeed close to true super and sub-solutions. By symmetry we can limit ourselves to the case of the super-solutions.

The biased motion by curvature. Given an open, bounded, connected region Λ whose boundary Γ is C^2, and given $\xi \in \Gamma$, we denote by ν the unit vector normal to Γ at ξ and pointing toward the interior of Λ. The mean curvature (times the factor (d-1)) of Γ at ξ

is denoted by R^{-1}. Then, for any real h, we define the h-biased motion $\Gamma_\tau^{(h)}$ of Γ_0 so that the points of $\Gamma_\tau^{(h)}$ satisfy the equation

$$\frac{d\xi^{(h)}}{d\tau} = \Big(\frac{\theta}{R} - h\Big)\nu \tag{32}$$

with θ as in (27a).

We omit the proof that:

Lemma. Let $\tau^\star > 0$ be such that the unbiased motion by curvature is regular in a time interval longer than $\tau^\star$. Then there are h_0 and c so that $\Gamma_\tau^{(h)}$ exists and it is regular for all $|h| \le h_0$ and all $\tau \le \tau^\star$. Furthermore if $\xi^{(h)}(\tau)$ and $\xi(\tau)$ verify (32) and, respectively, (3) with $\xi^{(h)}(0) = \xi(0)$, then

$$\Big|\xi^{(h)}(\tau) - \xi(\tau)\Big| \le ch \tag{33}$$

The constant c is independent of the starting point in Γ_0.

The supersolution $m^\star(\xi,\tau)$. We fix δ and R_0 as follows:

$$1/40 < \delta < 1/20; \quad 2 - 10\delta > \alpha R_0 > 3/2 \tag{34a}$$

with α as in (18). We then define, for all λ sufficiently small, $m^\star(\xi,\tau)$ as

$$m^\star(\xi,\tau) = m^{(dd)}\Big(\frac{d(\xi,\Gamma_\tau^{(h)})}{\lambda}\Big), \qquad h = \lambda^{\delta/2} \tag{34b}$$

whenever $\lambda^{-1}|d(\xi,\Gamma_\tau^{(h)})| \le R_0 \log\lambda^{-1}$. For the other values of ξ, $m^\star(\xi,\tau) = \pm m_\beta + \lambda^{3/2}$, inside and respectively outside $\Lambda_\tau^{(h)}$. By recalling (18b) and that $\alpha R_0 \ge 3/2$, it is readily seen that for all λ small enough, $m^\star(\xi,0) \ge m_0(\xi;\lambda)$. Later we will introduce the functions $\tilde m$ and $\hat m$, which are the same as $m^\star$, but expressed in other coordinates.

The iterative procedure. For $k \in \mathbb{Z}_+$, let $t_k = kT \equiv k\lambda^{-\delta}$, with δ as in (34a). We then call $m_{(k)}(r,t)$ the solution of (1) for $t \ge t_k$ such that $m_{(k)}(r,t_k) = m^\star(\lambda r, \lambda^2 t_k)$, for all $r \in \mathbb{R}^d$. We are going to prove that for all λ small enough and all $t_k \le \lambda^{-2}\tau^\star$,

$$m_{(k)}(r,t_{k+1}) \le m^\star(\lambda r, \lambda^2 t_{k+1}), \qquad \text{for all } r \in \mathbb{R}^d \tag{35a}$$

We have already seen that

$$m_0(\lambda r;\lambda) \le m^\star(\lambda r, 0)$$

Then, by the Comparison Theorem below, it follows that the solution $m(r,t)$ of (1) with initial datum $m_0(\lambda r;\lambda)$ is such that

$$m(r,t_1) \le m_{(0)}(r,t_1) \le m^\star(\lambda r, \lambda^2 t_1)$$

by (35a) with $k=0$. We are now in the same situation as at $t=0$, hence, by using again the Comparison Theorem and the inequality (35a),

$$m(r,t_k) \le m^\star(\lambda r, \lambda^2 t_k), \qquad \text{for all } \ t_k \le \lambda^{-2}\tau^\star \tag{35b}$$

Since the same proof works as well for time intervals of size χT, $1 \le \chi \le 2$, $m^\star(\xi,\tau)$ is an upper bound for $m^{(\lambda)}(\xi,\tau)$, for all ξ and all τ such that $\lambda^2 T \le \tau \le \tau^\star$, for all λ small enough. An analogous property holds for the lower bound and, using these bounds, we derive (31) with $a = 1/80$ and $b = 3/2$, as we are going to show.

In fact for $\lambda^2 T \le \tau \le \tau^\star$, if $|d(\xi, \Gamma_\tau^{(h)})| \ge R_0 \lambda \log\lambda^{-1}$ and ξ is in the complement of $\Lambda_\tau^{(h)}$,

$$m^\star(\xi,\tau) = -m_\beta + \lambda^{3/2} \ge m(\lambda^{-1}\xi, \lambda^{-2}\tau) \ge -m_\beta$$

The first inequality follows from (35b), the second one by using the Comparison Theorem after recalling that $m_\beta \ge |m_0(\xi;\lambda)|$ for all ξ and that m_β is a stationary solution of (1). An analogous bound is obtained working with the sub-solutions:

$$m_\beta \ge m(\lambda^{-1}\xi, \lambda^{-2}\tau) \ge m_\beta - \lambda^{3/2}$$

if $|d(\xi,\Gamma_\tau^{(-h)})| \ge R_0\lambda\log\lambda^{-1}$ and ξ is inside $\Lambda_\tau^{(h)}$. On the other hand, if

$$|d(\xi,\Gamma_\tau)| \ge R_0\lambda\log\lambda^{-1} + c\lambda^{\delta/2}$$

with $c\lambda^{\delta/2}$ as in (33), recall that, by (34b), $h = \lambda^{d/2}$, then, for any $\xi' \in \Gamma_\tau^{(\pm h)}$,

$$|d(\xi,\xi')| \ge |d(\xi,\xi^\star)| - |d(\xi^\star,\xi')|$$

where $\xi^\star \in \Gamma_\tau$ and with $\xi^\star$ and ξ' related as in (33). Therefore

$$|d(\xi, \Gamma_\tau^{(\pm h)})| = \inf_{\xi' \in \Gamma_\tau^{(\pm h)}} |d(\xi, \xi')| \geq \inf_{\xi^\star \in \Gamma_\tau} |d(\xi, \xi^\star)| - c\lambda^{\delta/2} \geq R_0 \log \lambda^{-1}$$

Hence

$$\left| m^{(\lambda)}(\xi, \tau) \mp m_\beta \right| \leq \lambda^{3/2} \quad \text{if} \quad |d(\xi, \Gamma_\tau)| \geq \lambda^{1/80} \geq R_0 \lambda \log \lambda^{-1} + c\lambda^{\delta/2}$$

for all λ small enough and, respectively, for ξ inside and outside Λ_τ.

The Comparison Theorem, (see [12]). Let $u(r,t)$ and $v(r,t)$ be two solutions of (1) for $t \geq 0$, such that $u(r,0) \geq v(r,0)$ for all r. Then $u(r,t) \geq v(r,t)$ for all r and all $t \geq 0$.

In the remaining of this Section we prove (35a) with $k = 0$. The proof works unchanged for all $t_k \leq \lambda^{-2}\tau^\star$ and, for what said above, this proves (35b), hence (31). Besides the Comparison Theorem we frequently use another basic lemma, also proven in [12]:

The Barrier Lemma. There are V and c_1 positive so that if $u(r,t)$ and $v(r,t)$ solve (1) and, for some $S > 0$, $u(r,0) = v(r,0)$ for all $r \leq VS$, then

$$\left| u(0,S) - v(0,S) \right| \leq c_1 e^{-S} \tag{36}$$

It is convenient to use a special notation for the function $m^\star$ when expressed in mesoscopic coordinates:

$$\tilde{m}(r,t) = m^\star(\lambda r, \lambda^2 t) \tag{37a}$$

Estimates away from the interface. Let r be inside $\lambda^{-1}\Lambda_0$ and let $|d(r, \lambda^{-1}\Gamma_0)| \geq 2VT$, $T = \lambda^{-\delta}$. Then, $\tilde{m}(r,0) = m_\beta + \lambda^{3/2}$, and therefore, by the Barrier Lemma,

$$|m_{(0)}(r,T) - m(T)| \leq c_1 e^{-T} \tag{38}$$

where $m(t)$ solves (1) with initial datum constantly equal to $m_\beta + \lambda^{3/2}$. Since

$$\frac{dm(t)}{dt} = -m(t) + \tanh\{\beta m(t)\}$$

there are a' and b' positive so that

$$|m(t) - m_\beta| \le a' e^{-b' t}$$

Therefore

$$|m_{(0)}(r, T) - m_\beta| \le a' e^{-b'T} + c_1 e^{-T} < \lambda^{3/2}$$

for all λ small enough, hence $m_{(0)}(r, T) \le \tilde{m}(r, T)$.

The same argument works in the complement of $\lambda^{-1}\Lambda_0$.

We next study the solution in the region $\{|d(r, \lambda^{-1}\Gamma_0)| \le 2VT\}$, which is critical for the evolution of the interface.

Notation. Let $\xi_0 \in \Gamma_0$, ν the normal to Γ_0 at ξ_0 pointing toward Λ_0. We introduce a frame with origin at $r_0 = \lambda^{-1}\xi_0$, x-axis along ν and the other ones parallel to the principal axes of curvature of Γ_0 at ξ_0 and we orient the x axis as explained in Section §6.

Calling r' the coordinate in this new frame which corresponds to r in the old one, we set

$$\hat{m}(r', t) := \tilde{m}(r, t) = m^\star(\lambda r, \lambda^2 t) \tag{39a}$$

$$m(r', t) := m_{(0)}(r, t) \tag{39b}$$

What we need to prove is that $m(r', T) \le \hat{m}(r', T)$ for all $r' = (x', 0)$ with $|x'| \le 3VT$. By the arbitrarity of r_0, this completes the proof of (35a) with $k = 0$, having already proven the estimate away from the interface.

Since we always use in the following the new coordinates r', with no risk of confusion we may and will call them again r. We recall and expand the considerations of §6. Writing (16) in mesoscopic variables, the initial condition is a function of the signed distance $d(r, \lambda^{-1}\Gamma_0)$ from the surface $\lambda^{-1}\Gamma_0$. We consider $|r| \le 4VT$ and, for such values of r, there is a constant c_2' so that, denoting by (x, y) the coordinates of r and recalling that $x^\star(y)$ is the expression in (23a), we have

$$\Big| d(r, \lambda^{-1}\Gamma_0) - \big(x - x^\star(y)\big) \Big| \le c_2' \lambda^2 T^3 \tag{40a}$$

Then, for a suitable constant c_2,

$$\Big| \bar m\big(d(r,\lambda^{-1}\Gamma_0)\big) - \bar m(x) + x^\star(y)\bar m'(x) \Big| \le c_2 \lambda^2 T^3 \bar m'(x) \tag{40b}$$

To derive the last expression we have expanded to second order $\bar m(d(r))$ around $\bar m(x)$ and used that, by (18),

$$|\bar m''(x)| \le c\bar m'(x)$$

for a suitable constant c.

We define

$$n(r,0) := \mathbf{1}(|x| \le X_\lambda)[\bar m(x) + \bar m'(x)\omega(y,\lambda)] + \mathbf{1}(|x| > X_\lambda)[m_\beta \mathrm{sign}\, x + \lambda^{3/2}] \tag{40c}$$

$$X_\lambda = R_0 \log \lambda^{-1} - 1 \tag{40d}$$

$$\omega(y,\lambda) = \begin{cases} -\lambda/2 \sum_i y_i^2/R_i + c_2\lambda^2 T^3 & \text{if } \sum_i y_i^2 \le (4VT)^2 \\ c_2\lambda^2 T^3 & \text{otherwise} \end{cases} \tag{40e}$$

By our choice of coordinates, if $|r| \le 4VT$ and, setting $r = (x,y)$, if $|x| \le X_\lambda$, then $|d(r,\lambda^{-1}\Gamma_0)| \le R_0 \log \lambda^{-1}$ for all λ small enough. Then, for such values of r, $\hat m(r,0) = \bar m(d(r,\lambda^{-1}\Gamma_0))$ and, by (40b),

$$\hat m(r,0) \le \bar m(x) - \bar m'(x)x^\star(y) + \bar m'(x)c_2\lambda^2T^3 = n(r,0)$$

We have also when $|x| > X_\lambda$, that

$$\hat m(r,0) \le n(r,0), \qquad \text{for all } \; |r| \le 4VT$$

Using the Barrier Lemma and the Comparison Theorem, we then have that

$$m(r,T) \le n(r,T) + c_1 e^{-T}, \qquad \text{for all } \; |r| \le 3VT \tag{41}$$

We write

$$u(r,t) = n(r,t) - \bar m(x), \qquad u_t \equiv u(\cdot,t) \tag{42}$$

Then

$$u_t = e^{Lt}u_0 + \int_0^t ds e^{L(t-s)}\Psi_s \tag{43a}$$

$$\Psi_t = \tanh\{\beta J \star (\bar m + u_t)\} - \tanh\{\beta J \star \bar m\} - \big(1-\bar m^2\big)\beta J \star u_t \tag{43b}$$

where L is the linearized operator around $\bar m$, see (22a).

By expanding the right hand side of (43b) around $\beta J \star \bar m$, we get

$$u_t = e^{Lt}u_0 + \int_0^t ds e^{L(t-s)}\Big(-\bar m(1-\bar m^2)(\beta J \star u_s)^2 + A_s(\beta J \star u_s)^3\Big) \tag{44a}$$

$$A_s = \frac{1}{3!} 2\cosh^{-4} a_s [2\sinh^2 a_s - 1] \tag{44b}$$

where a_s is a number between $\beta J \star \bar m$ and $\beta J \star (\bar m + u_s)$. We rewrite (44a) as

$$u_t = e^{Lt}u_0 + \int_0^t ds e^{L(t-s)}\Big(-\bar m(1-\bar m^2)(\beta J \star e^{Ls}u_0)^2 + \mathcal{R}_s^{(1)}\Big) \tag{45}$$

$$\mathcal{R}_s^{(1)} = A_s(\beta J \star u_s)^3 - \bar m(1-\bar m^2)\Big((\beta J \star u_s)^2 - (\beta J \star e^{Ls}u_0)^2\Big) \tag{46}$$

We shall see that the first term in the integral on the right hand side of (45) is of the order $\lambda^2 T^3 \bar m'(x)$ while the one containing $\mathcal{R}_s^{(1)}$ is even smaller, of the order of $\lambda^3 T$. An error of the order $\lambda^2 T^3$ cannot be neglected, as we shall see, for x large, but the term $\lambda^2 T^3 \bar m'(x)$ is all right, due to the decay properties of $\bar m'$, see (18). To recover the term $\bar m'$ we have splitted u as in (45). The error $\lambda^3 T$ will be acceptable also for x large.

We examine separately the terms on the right hand side of (45), but before doing this we need some properties of the semigroup e^{Lt}, that we state below. A short sketch of their proofs is given in §9.

Some properties of the semigroup e^{Lt}. The kernel $e^{Lt}(r,r')$ of the semigroup e^{Lt} is non negative. It has the following expression, see (22a),

$$e^{Lt}(r,r') = \delta(r-r')e^{-t} + e^{-t}\sum_{n\ge 1}\frac{t^n}{n!}H^n(r,r') \tag{47a}$$

$$H(r,r') = \big(1-\bar m(x)^2\big)\beta J(|r-r'|) \tag{47b}$$

Denoting by H the operator whose kernel is $H(r,r')$, we have, see (22b,c),

$$H\bar m' = \bar m', \tag{48a}$$

and also

$$e^{Lt}\phi = \bar m' e^{\mathcal{L}t}\frac{1}{\bar m'}\phi, \qquad \text{for all } \phi \tag{48b}$$

As already noticed, see (24), from (48b) it follows that if $\phi = \bar m'\psi$ then

$$e^{Lt}\phi = \bar m' e^{\mathcal{L}t}\psi \tag{48c}$$

For the semigroup $e^{\mathcal{L}t}$ the following holds. Given $T = \lambda^{-\delta}$ and V as in the Barrier Lemma, we let

$$\psi_\lambda(r) := \frac{\lambda}{2}\sum_i \frac{y_i^2}{R_i}\mathbf{1}\big(\sum_i y_i^2 \ge (4VT)^2\big) \tag{49a}$$

then there is c_1' so that

$$e^{\mathcal{L}T}\psi_\lambda(r) \le c_1' e^{-T} \tag{49b}$$

Furthermore there is $R^\star$ so that

$$\sup_{|r|\ge R^\star}\int dr' H(r,r') < 1 \tag{50}$$

and c_0 so that for all $t \ge 0$

$$\|e^{Lt}\|_\infty \equiv \sup_r \int dr' e^{Lt}(r,r') \le c_0 \tag{51}$$

Finally, there is c_3 so that

$$\|e^{LT}\mathbf{1}(|x| > X_\lambda)\|_\infty \le c_3\lambda^{\alpha R_0} \tag{52}$$

with α as in (18).

For later purposes we need to bound the initial datum u_0 as we are going to explain

Bounds on $u_0(r)$. Recalling the definitions (42) and (40) we have that

$$u_0(r) = \mathbf{1}(|x| \le X_\lambda)\bar m'(x)\omega(y,\lambda) + \mathbf{1}(|x| > X(\lambda))[-\bar m(x) + m_\beta \text{ sign } x + \lambda^{3/2}] \tag{53}$$

Using (40e),

$$
\begin{aligned}
\mathbf{1}(|x| \le X_\lambda)\bar m'(x)\omega(y,\lambda) &= \bar m'(x)\big\{c_2\lambda^2T^3 - \frac{\lambda}{2}\sum_i \frac{y_i^2}{R_i}\mathbf{1}(\sum_i y_i^2 \le (4VT)^2)\big\}\\
&\quad + \mathbf{1}(|x| > X_\lambda)\bar m'(x)\frac{\lambda}{2}\sum_i \frac{y_i^2}{R_i}\mathbf{1}(\sum_i y_i^2 \le (4VT)^2)\\
&\le \bar m'(x)\big\{c_2\lambda^2T^3 - \frac{\lambda}{2}\sum_i \frac{y_i^2}{R_i}\big\} + \bar m'(x)\psi_\lambda(r)\\
&\quad + \frac{\lambda}{2}\frac{d-1}{R}(4VT)^2\mathbf{1}(|x| > X_\lambda)\bar m'(x)
\end{aligned} \tag{54}
$$

where $\psi_\lambda(r)$ is defined in (49a).

By (18) there are c_4' and c_4'' so that

$$
|\bar m(x) - m_\beta \text{sign} x| \le c_4' e^{-\alpha|x|}, \qquad \bar m'(x) \le c_4'' e^{-\alpha|x|} \tag{55}
$$

Therefore, recalling that $X_\lambda = R_0 \log\lambda^{-1} - 1$ and $\alpha R_0 > 3/2$, there is c_4 so that

$$
\begin{aligned}
\mathbf{1}(|x| > X_\lambda)\big[\bar m'(x)\frac{\lambda}{2}\frac{d-1}{R}(4VT)^2 - \bar m(x) + m_\beta \text{sign} x + \lambda^{3/2}\big]&\\
\le c_4\lambda^{3/2}\mathbf{1}(|x| > X_\lambda)&
\end{aligned} \tag{56}
$$

which thus bounds the sum of the last terms on the right hand side of (54) and (53). From (53), (54) and (56) we then have that

$$
u_0(r) \le \bar m'(x)\big\{c_2\lambda^2T^3 - \frac{\lambda}{2}\sum_i \frac{y_i^2}{R_i}\big\} + \bar m'(x)\psi_\lambda(r) + c_4\lambda^{3/2}\mathbf{1}(|x| > X_\lambda) \tag{57}
$$

We will also use another bound on u_0, based on the fact that, by (40e),

$$
\omega(y,\lambda) \le c_2\lambda^2T^3 + \frac{\lambda}{2}\frac{d-1}{R}(4VT)^2 \tag{58}
$$

Then, from (53) and (55), there is c_5 so that

$$
u_0(r) \le \bar m'(x)c_5\lambda T^2 + c_4\lambda^{3/2}\mathbf{1}(|x| > X_\lambda) \tag{59}
$$

We are now ready to bound the three terms on the right hand side of (45). We start with the first one recalling that its leading term has already been studied in §6.

Bounds on the first term on the r.h.s. of (45).

By (57), for $r = (x,0)$

$$(e^{LT}u_0)(x,0) \le e^{LT}\ \bar m'(x)\big\{c_2\lambda^2T^3 - \frac{\lambda}{2}\sum_i \frac{y_i^2}{R_i} + \psi_\lambda(r)\big\}$$
$$+ c_4\lambda^{3/2}e^{LT}\mathbf{1}(|x| > X_\lambda) \tag{60a}$$

From (48b), (48c), (28), (49b) and (52) we then get

$$(e^{LT}u_0)(x,0) \le \bar m'(x)\big[-\frac{\lambda}{R}T\theta + \lambda C(|x|+1) + c_2\lambda^2T^3 + c_1'e^{-T}\big] + c_4c_3\lambda^{3/2+\alpha R_0} \tag{60b}$$

Bounds on the second term on the r.h.s. of (45).

We use the bound (59) for u_0 in the second term in (45), which we compute at $t = T$ and at a point $r = (x,0)$. Then, from (51) and the fact that $\|\bar m\|_\infty < 1$, we have

$$\Big|\int_0^T ds e^{L(T-s)}\Big(\bar m(1-\bar m^2)(\beta J \star e^{Ls}u_0)^2\Big)\Big| \le [\beta\|J\|_\infty c_4]^2c_0^3\lambda^3T$$
$$+ \{\beta\|J\|_\infty c_5\lambda T^2\|\bar m'\|_\infty + 2\beta\|J\|_\infty c_4\lambda^{3/2}c_0\}\int_0^T ds e^{L(T-s)}\big(\beta J\star e^{Ls}\bar m'\big)c_5\lambda T^2 \tag{61}$$

We use (48b) and that $e^{Ls}\bar m' = \bar m'$ for all $s \ge 0$, to conclude that there is a constant c_6 so that

$$e^{L(T-s)}\big(J\star e^{Ls}\bar m'\big) = \bar m' e^{\mathcal{L}(T-s)}\frac{1}{\bar m'}\big(J\star\bar m'\big) \le c_6\bar m' \tag{62}$$

From (61) and (62) we then have that there are c_7 and c_8 so that

$$\Big|\int_0^T ds e^{L(T-s)}\Big(\bar m(1-\bar m^2)(\beta J\star e^{Ls}u_0)^2\Big)\Big| \le c_7\lambda^3T + c_8\lambda^2T^5\bar m' \tag{63}$$

Bounds on the last term on the r.h.s. of (45).

From (59) there is c_9 so that

$$\|u_0\|_\infty \le c_9\lambda T^2 \tag{64}$$

Let

$$t^\star = \inf\{t : \|u_t\|_\infty < 1\} \tag{65}$$

then from (43) and the fact that the kernel of e^{Lt} is positive, we get that there is a constant c_{10} so that

$$|u_t| \le e^{Lt}|u_0| + c_{10}\int_0^t ds e^{L(t-s)}(J\star u_s)^2 \tag{66a}$$

and also

$$|u_t - e^{Lt}u_0| \le c_{10}\int_0^t ds e^{L(t-s)}(J\star u_s)^2 \tag{66b}$$

Then, by (51) and (66a),

$$\|u_t\|_\infty \le c_0\|u_0\|_\infty + c_{10}\int_0^t ds c_0\|J\|_\infty\|u_s\|_\infty^2 \qquad \text{for all } t\le t^\star \tag{67a}$$

so that from (64) we get that there is c_{11} so that

$$\|u_t\|_\infty \le \frac{c_0\|u_0\|_\infty}{1-tc_0c_{10}\|J\|_\infty\|u_0\|_\infty} \le \sum_{n\ge 1}(c_{11}\lambda t T^2)^n \qquad \text{for all } t\le t^\star \tag{67b}$$

Since $3\delta<1$, see (34a), the series on the right hand side of (67b) converges also for $t=T$ if λ is small enough. This implies that $t^\star > T$.

Therefore there is c_{12} so that

$$\|u_t\|_\infty \le c_{12}\lambda T^3 \qquad \text{for all } t\le T \tag{67c}$$

We rewrite the last bracket on the right hand side of (46) as

$$\begin{aligned}(\beta J\star u_s)^2 - &(\beta J\star e^{Ls}u_0)^2 \\ &= \Big(\beta J\star(u_s - e^{Ls}u_0)\Big)^2 + 2\Big(\beta J\star(u_s-e^{Ls}u_0)\Big)\big(\beta J\star e^{Ls}u_0\big)\end{aligned}$$

Using (66b) we then have, for suitable constants c_{13} and c_{14},

$$\int_0^t ds e^{L(t-s)}\mathcal{R}_s^{(1)} \le c_{13}T\big[(\lambda T^3)^3 + \{T(\lambda T^3)^2\}^2 + \lambda T^3\{T(\lambda T^3)^2\}\big] \le c_{14}\lambda^3T^{10} \tag{68}$$

because, by our choice of δ, λ^3T^{10} is the leading term, as $\lambda\to 0$.

Upper bound for $m(r,T)$. Going back to (41) and (42), we thus conclude that for $r=(x,0)$, $|r|\le 3VT$,

$$\begin{aligned}m(r,T) \le \bar m(x) + c_1e^{-T} + \{\bar m'(x)[-\frac{\lambda}{R}T\theta + \lambda C(|x|+1) + c_2\lambda^2T^3] + \|\bar m'\|_\infty c_1' e^{-T} \\ + c_3c_4\lambda^{3/2+\alpha R_0}\} + \{c_7\lambda^3T + c_8\lambda^2T^5\bar m'(x)\} + c_{14}\lambda^3T^{10}\end{aligned} \tag{69a}$$

the first curly bracket coming from (60b) and the second one from (63). The last term comes from (68).

Calling

$$\bar{c}_1 = c_1 + \|\bar{m}'\|_\infty c_1', \quad c_{15} = c_2 + c_8, \quad c_{16} = c_3 c_4 \tag{69b}$$

we have that

$$m(r,T) \le B(x;\lambda), \quad r = (x,0),\ |r| \le 3VT \tag{69c}$$

where

$$\begin{aligned} B(x;\lambda) := \bar{m}(x) + \{\bar{m}'(x)[-\frac{\lambda}{R}T\theta + \lambda C(|x|+1) + c_{15}\lambda^2 T^5] \\ + [\bar{c}_1 e^{-T} + c_{16}\lambda^{3/2+\alpha R_0} + c_7\lambda^3 T + c_{14}\lambda^3 T^{10}] \end{aligned} \tag{69d}$$

Lower bound for $\hat{m}(r,T)$. We first observe the following property of the motion (32), that we state without proof. There is a constant c_{17} so that for all $r = (x,0)$ with $|x| \le R_0 \log \lambda^{-1} + 1$,

$$|d(r, \lambda^{-1}\Gamma^{(h)}_{\lambda^2 T}) - x + [\frac{\lambda}{R}T\theta - h\lambda T]| \le c_{17}\lambda^3 T^2 \tag{70}$$

By (70), setting $r = (x,0)$

$$\begin{aligned} \hat{m}(r,T) \ge & \Big(\bar{m}(x) - \bar{m}'(x)[\frac{\lambda}{R}T\theta - h\lambda T] - c_{17}\bar{m}'(x)\lambda^3 T^2 - c_{18}\bar{m}''(x)\lambda^2 T^2\Big) \\ & \times \mathbf{1}(|x| \le R_0 \log \lambda^{-1} + 1) \\ & + [m_\beta \operatorname{sign} x + \lambda^{3/2}]\mathbf{1}(|x| > R_0 \log \lambda^{-1} + 1) \end{aligned} \tag{71}$$

Conclusion of the proof of (35a) for $k = 0$. We suppose λ so small that $\lambda T\theta/R + h\lambda T < 1$. Then, by (70), the points on the x-axis at distance $R_0 \log \lambda^{-1} + 1$ from the origin have distance larger than $R_0 \log \lambda^{-1}$ from $\lambda^{-1}\Gamma^{(h)}_\tau$, $\tau \equiv \lambda^2 T$. On the other hand, the points on the x-axis at distance $R_0 \log \lambda^{-1} - 1$ have distance not larger than $R_0 \log \lambda^{-1}$ from the interface. We have

$$\lim_{\lambda \to 0} \lambda^{-3/2} \bar{m}'\big(R_0 \log \lambda^{-1} - 1\big) = 0, \qquad \text{since } \alpha R_0 > 3/2 \tag{72}$$

We finally compare (71) and (69). We begin with $r=(x,0)$ and $|x|\le R_0\log\lambda^{-1}+1$. In order to show that $\hat{m}(r,T)\ge m(r,T)$, it is enough to prove the right hand side of (71) is not smaller than the right hand side of (69d), namely that

$$h\lambda T\ge c_{18}\frac{\|\bar{m}''\|_\infty}{\|\bar{m}'\|_\infty}\|\lambda^2T^2+c_{17}\lambda^3T^2+\lambda C(R_0\log\lambda^{-1}+2)+c_{15}\lambda^2T^5$$
$$+[\bar{c}_1e^{-T}+c_7\lambda^3T+c_{14}\lambda^3T^{10}+c_{16}\lambda^{3/2+\alpha R_0}][\bar{m}'\big(R_0\log\lambda^{-1}+1\big)]^{-1} \qquad (73)$$

which is in fact verified for all λ small enough, recalling (34a). To derive (73), we have also used the inequality

$$\frac{1}{2}h\lambda T\ge c_{14}\lambda^3T^{10}[\bar{m}'\big(R_0\log\lambda^{-1}+1\big)]^{-1},\quad \text{for all } \lambda \text{ small enough}$$

which follows from

$$\bar{m}'(x)\ge c_4'''e^{-\alpha|x|}$$

(see (18b) and from

$$\frac{1}{2}\ge\lambda^{-\delta/2+2}T^9c_4'''\lambda^{\alpha R_0}$$

it remains to show that for $r=(x,0)$, with $|x|>R_0\log\lambda^{-1}+1$,

$$m_\beta\operatorname{sign}x\ge B(x;\lambda)$$

which is verified for all λ small enough because

$$\lambda^{3/2}\ge c_4'\lambda^{\alpha R_0}+c_4''\lambda^{\alpha R_0}\frac{\lambda T\theta}{|R|}+\lambda\sup_{|x|\ge R_0\log\lambda^{-1}}\{\bar{m}'(x)[C|x|+1]\}$$
$$+c_4''\lambda^{\alpha R_0}c_{15}\lambda^2T^5+\bar{c}_1e^{-T}+c_{16}\lambda^{3/2+\alpha R_0}+c_7\lambda^3T^{10} \qquad (74)$$

(74) is verified for all λ small enough, because, by (34a) $\alpha R_0\ge 3/2$ and

$$\lambda\sup_{|x|\ge R_0\log\lambda^{-1}}\{\bar{m}'(x)[C|x|+1]\}\le c_4''\lambda^{1+\alpha R_0}[CR_0\log\lambda^{-1}+1]$$

(for all λ small enough).

§9. *Appendix.*

Here we sketch the proof of the statements in the sub-paragraph *Some properties of the semigroup* e^{Lt}.

The proof of (47) is straightforward and it is omitted, (48a) follows from differentiating (12).

Proof of (49b) From the definition (20) it follows that for $r = (x, 0)$,

$$(e^{\mathcal{L}T}\psi_\lambda)(r) = e^{-T}\sum_{n\geq 1}\frac{T^n}{n!}\int K^n(r,r')\psi_\lambda(r')dr' \tag{75}$$

Since $K(r_1, r_2) = 0$ if $|r_1 - r_2| > 1$, from (75) we have

$$(e^{\mathcal{L}T}\psi_\lambda)(r) \leq e^{-T}\sum_{n\geq 4VT}\frac{T^n}{n!}\frac{\lambda(d-1)}{R}n^2 \tag{76}$$

hence (49b).

The (50) follows from (18), by direct inspection. We next give an outline of the proofs of (51) and (52) which are less obvious.

Proof of (51). This is a Corollary of (19a), but since the proof of (19a) is not yet published, we prove directly (51). We use the representation (47a), so that it is enough to show that

$$\sup_{n,r}\int dr' H^n(r,r') < \infty \tag{77}$$

For notational simplicity we hereafter write R for the parameter $R^\star$ in (50). Then

$$H^n(r,r') = \sum_{p=0}^{n}\int dr'' H^p(r,r'')\mathbf{1}(|x''| \leq R+1)\tilde{H}^{n-p}(r'',r')$$

denoting by x, x' and x'' the x-coordinates of r, r', r'' and so on; furthermore

$$\tilde{H}(r,r') = \mathbf{1}\big(|x'| > R+1\big)H(r,r') \tag{78}$$

Then, denoting by $g < 1$ the left hand side of (50), we have from (50), recalling that $J(|r|) = 0$ when $|r| > 1$,

$$\int dr' H^n(r,r') \leq \sum_{p=0}^{n} g^{n-p}\int dr'' H^p(r,r'')\mathbf{1}(|x''| < R+1)$$

Calling

$$c = \max_{|x|\le R+1} [\bar m'(x)]^{-1} < \infty \tag{79}$$

(see (18a)), we have, by (48a),

$$\int dr' H^n(r,r') \le g^n + c \sum_{p=1}^{n} g^{n-p} \int dr'' H^p(r,r'') \bar m'(x'') \le g^n + c\frac{1}{1-g} \bar m'(x)$$

hence (77), recalling that, by (18), $\bar m'(x)$ is bounded. $\square$

Proof of (52). We proceed as in the proof of (51). We then have

$$H^n(r,r') = \tilde H^n(r,r') + \sum_{p=1}^{n} \int dr'' H^p(r,r'') \mathbf{1}(|x''| < R+1) \tilde H^{n-p}(r'',r') \tag{80}$$

The contribution to the left hand side of (52) coming from $\tilde H^n$ is bounded by

$$e^{-T} \sum_{n=0}^{\infty} \frac{T^n}{n!} g^n \le e^{-T(1-g)} \tag{81}$$

For $|x'| \ge R_0 \log \lambda^{-1}$ and $|x''| < R+1$, we have $n-p > 2$, for all λ small enough. Then

$$\begin{aligned} \tilde H^{n-p}(r'',r') &= \int dr_1 dr_2 \beta J(|r_1 - r'|) \tilde H^{n-p-2}(r_1,r_2) H(r_2,r'') \beta (1 - \bar m(x'')^2) \\ &\le \int dr_1 dr_2 J(|r_1 - r'|) \tilde H^{n-p-2}(r_1,r_2) \mathbf{1}(R+1 \le |x_2| \le R+2) a_1 \end{aligned} \tag{82}$$

with a_1 a suitable constant.

There is $a_2 > 0$ so that

$$\min_{R+1\le |x_2| \le R+2} \int dr dr' \tilde H(r_2,r) \mathbf{1}(|x| > R+1) H(r,r') \mathbf{1}(|x'| \le R+1) \ge {a_2}^{-1} \tag{83}$$

hence, from (82)

$$\tilde H^{n-p}(r'',r') \le a_1 a_2 \int dr_1 dr_2 dr_3 J(|r_1 - r'|) \tilde H^{n-p-1}(r_1,r_2) H(r_2,r_3) \mathbf{1}(|x_3| \le R+1) \tag{84}$$

Similarly to (22), we write

$$H(r,r') = \bar m'(x) K(r,r') \frac{1}{\bar m'(x')}, \qquad \int dr' K(r,r') = 1 \tag{85}$$

so that, recalling (79),

$$\tilde{H}^{n-p}(r'',r') \le a_1 a_2 \int dr_1 J(|r_1 - r'|)\bar{m}'(x_1) c \pi_{n-p}(r_1) \tag{86}$$

where $\pi_q(r_1)$ is the probability of the event: { q is the first "time" when the chain arrives at a point r whose x coordinate is less or equal to $R+1$ (in absolute value) }. The probability refers to the Markov chain starting from r_1 and with transition probability K. Thus, for any r_1,

$$\sum_q \pi_q(r_1) \le 1 \tag{87}$$

(in fact the equality holds).

In conclusion, the second term on the right hand side of (80) becomes

$$\sum_{p=1}^{n} \int dr'' H^p(r,r'') \mathbf{1}(|x''| < R+1) \tilde{H}^{n-p}(r'',r')$$
$$\le \sum_{p=1}^{n} \Big\{ \int dr'' \bar{m}'(x) K^p(r,r'') \mathbf{1}(|x''| < R+1) c \Big\} a_1 a_2 \int dr_1 J(|r - r_1|) \bar{m}'(x_1) c \pi_{n-p}(r_1) \tag{88}$$

The curly bracket term is bounded by $c\bar{m}'(x) \le a_3$, independently of p. Then, using (87), we get

$$\sum_{p=1}^{n} \int dr'' H^p(r,r'') \mathbf{1}(|x''| < R+1) \tilde{H}^{n-p}(r'',r') \le a_3 a_1 a_2 c \|J\|_\infty \bar{m}'(R_0 \log \lambda^{-1} - 1) \tag{89}$$

Thus, recalling (81), we get, for a suitable c', using (18),

$$\|e^{LT} \mathbf{1}(|x| > X_\lambda)\|_\infty \le e^{-T(1-g)} + c' e^{-\alpha R_0 \log \lambda^{-1}}$$

which proves (52). $\square$

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