Phase Transition in the 1d Random Field Ising Model with long range interaction. *

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Abstract We study one–dimensional Ising spin systems with ferromagnetic, long–range interaction decaying as $n^{-2+\alpha}$, $\alpha \in (\frac{1}{2}, \frac{\ln 3}{\ln 2} - 1)$, in the presence of external random fields. We assume that the random fields are given by a collection of symmetric, independent, identically distributed real random variables, gaussian or subgaussian. We show, for temperature and strength of the randomness (variance) small enough, with $I\!P = 1$ with respect to the random fields, that there are at least two distinct extremal Gibbs measures.

1 Introduction

It is well known that the one dimensional ferromagnetic Ising model exhibits a phase transition when the forces are sufficiently long range. A fundamental work on the subject is due to Dyson [9]. He proved, by comparison to a hierarchical model, that for a two body interaction $J(n) = \frac{\ln \ln(n+3)}{n^2+1}$, where n denotes the distance, there is spontaneous magnetization at low enough temperature.

On the other hand Rogers &Thompson [14] proved that the spontaneous magnetization vanishes for all temperatures when

$$\lim_{N\to\infty} \frac{1}{[\ln N]^{\frac{1}{2}}} \sum_{n=1}^{N} nJ(n) = 0.$$

Later, Fröhlich & Spencer [11] proved the existence of spontaneous magnetization when $J(n) = n^{-2}$. For the same model Aizenman, Chayes, Chayes, & Newman [1] proved the discontinuity of the magnetization at the critical temperature, the so-called Thouless effect. When $J(n) = n^{-2+\alpha}$, $\alpha < 0$ there is only one Gibbs state [15,6,7] and the free energy is analytic in the thermodynamic parameters, see [8]. More recently the notion of contours introduced in [11] was implemented in [5], by giving a graphical description of the spin configurations better suited for further generalizations. The case studied in [5] covers the regime $0 \le \alpha \le (\ln 3/ \ln 2) - 1$. By applying Griffiths inequalities the existence of a phase transition in the full interval $0 \le \alpha < 1$ can be deduced either by [5], or by [11].

A natural extension of this analysis is its application to disordered systems. One of the simplest prototype models for disordered spin systems is obtained by adding random magnetic fields, say gaussian independent identically distributed with mean zero and finite variance. The problem of (lower) critical dimension for the d–dimensional Random Field Ising Model was very challenging at the end of the eighties since the physical literature predicted conflicting results. For finite range interaction the problem was rigorously solved by two complementary articles, Bricmont & Kupiainen [4] and Aizenman & Wehr [2]. In [4] a renormalization group argument was used to show that if $d \ge 3$ and the variance of the random magnetic field is small enough then almost surely there are at least two distinct Gibbs states (the plus and the minus Gibbs states). In [2] it was proved that for $d \le 2$, almost surely there is an unique Gibbs state. The guide lines of these proofs are suggested by a heuristic argument due to Imry & Ma [13].

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In the long–range one–dimensional setting the Imry & Ma argument is the following: the deterministic cost to create a run of -1 in an interval of length L with respect to the state made of $+1$ at each site, is of order L^α, while the cumulative effect of the random field inside this interval is just $L^{1/2}$. So when $0 \le \alpha \le 1/2$ the randomness is dominant and there is no phase transition. This has been proved by Aizenman & Wehr [2]. They show that the Gibbs state is unique for almost all realizations of the randomness.

When $1/2 < \alpha < 1$, the above Imry & Ma argument suggests the existence of a phase transition since the deterministic part is dominant with respect to the random part as in the case of the three–dimensional random field Ising model. However a rigorous result is this direction was missing.

In this paper we study the random field one–dimensional Ising model with long range interaction $n^{-2+\alpha}$, $\alpha \in (\frac{1}{2}, \frac{\ln 3}{\ln 2} - 1) \simeq (\frac{1}{2}, \frac{58}{100})$. We assume that the random field $h[\omega] := \{h_i[\omega], i \in \mathbb{Z}\}$ is given by a collection of independent random variables, with mean zero and symmetrically distributed. We take $h_i[\omega] = \pm 1$ with $p = \frac{1}{2}$ and we introduce the strength parameter θ. However one could take different distributions, for example gaussian distribution with mean zero and variance θ^2, or subgaussian. In fact all that is needed is $E(e^{th_1}) \le e^{c\theta^2 t^2}$ for some positive constant c and for all $t \in \mathbb{R}$. We prove that for $\frac{1}{2} < \alpha < \frac{\ln 3}{\ln 2} - 1$ the situation is analogous to the three-dimensional short range random field Ising model: for temperature and variance of the randomness small enough, with $\mathbb{P} = 1$ with respect to the randomness, there exist at least two distinct infinite volume Gibbs states, namely the $\mu^+[\omega]$ and the $\mu^-[\omega]$ Gibbs states. The proof is based on the representation of the system in term of the contours as defined in [5]. A Peierls argument is obtained by using the lower bound of the deterministic part of the cost to erase a contour and controlling the contribution of the stochastic part. This control is done applying an exponential Markov inequality and the so-called Yurinski's martingale difference sequences method. We do not need to use any coarse-grained contours as in [4], a fact that simplifies the proof. In the one dimensional case the contours can be described in terms of intervals and the Imry& Ma argument can be implemented. Namely in our case bad configurations of the random magnetic field, the ones for which the naive Imry & Ma argument fails, are treated probabilistically. A kind of energy entropy argument is successfully used, see (3.14), to prove that they can be neglected. In 3 dimensions this specific energy entropy argument fails. The coarse grained contours in [4] allow to control these bad contours on various length scales by using a renormalization group argument. As a by-product an estimate on the decay of the truncated two point correlation functions is given in [4]. Our method does not give any information on this decay. Therefore we do not think that it can be directly applied to give an alternative proof of Bricmont & Kupiainen results [4]. For $\alpha \in [(\ln 3/\ln 2) - 1, 1)$ we still expect the same result to hold but we are not able to prove it. In this case the lower bound for the deterministic contribution to the cost of erasing a contour does not hold, see Lemma 2.3. Known correlations inequalities are not relevant to treat this range of values of α as in the case where the random field is absent.

2 Model, notations and main results

2.1. The model and the main results

Let $(\Omega, \mathcal{B}, \mathbb{P})$ be a probability space on which we define $h \equiv \{h_i\}_{i \in \mathbb{Z}}$, a family of independent, identically distributed Bernoulli random variables with $\mathbb{P}[h_i = +1] = \mathbb{P}[h_i = -1] = 1/2$. The spin configuration space is $\mathcal{S} \equiv \{-1, +1\}^{\mathbb{Z}}$. If $\sigma \in \mathcal{S}$ and $i \in \mathbb{Z}$, σ_i represents the value of the spin at site i. The pair interaction among spins is given by $J(|i-j|)$ defined as following[1]:

$$J(n) = \begin{cases} J(1) >> 1 \\ \dfrac{1}{n^{2-\alpha}} \quad \text{if} \quad n > 1, \quad \text{with} \quad \alpha \in [0,1). \end{cases}$$

For $\Lambda \subseteq \mathbb{Z}$ we set $\mathcal{S}_\Lambda = \{-1,+1\}^\Lambda$; its elements are denoted by σ_Λ; also, if $\sigma \in \mathcal{S}$, σ_Λ denotes its restriction to Λ. Given $\Lambda \subset \mathbb{Z}$ finite and a realization of the magnetic fields, the Hamiltonian in the volume Λ, with $\tau = \pm 1$ boundary conditions, is the random variable on $(\Omega, \mathcal{A}, \mathbb{P})$ given by

$$H^\tau(\sigma_\Lambda)[\omega] = H_0^\tau(\sigma_\Lambda) + \theta G(\sigma_\Lambda)[\omega] \tag{2.1}$$

where

$$H_0^\tau(\sigma_\Lambda) := \frac{1}{2} \sum_{(i,j)\in\Lambda\times\Lambda} J(|i-j|)(1-\sigma_i\sigma_j) + \sum_{i\in\Lambda}\sum_{j\in\Lambda^c} J(|i-j|)(1-\tau\sigma_i), \tag{2.2}$$

and

$$G(\sigma_\Lambda)[\omega] := -\sum_{i\in\Lambda} h_i[\omega]\sigma_i. \tag{2.3}$$

In the following we drop the ω from the notation. The corresponding *Gibbs measure* on the finite volume Λ, at inverse temperature $\beta > 0$ and $+$ boundary condition is then a random variable with value on the space of probability measures on $\mathcal{S}_\Lambda$ defined by

$$\mu_\Lambda^+(\sigma_\Lambda) = \frac{1}{Z_\Lambda^+} \exp\{-\beta H^+(\sigma_\Lambda)\} \qquad \sigma_\Lambda \in \mathcal{S}_\Lambda, \tag{2.4}$$

where Z_Λ^+ is the normalization factor. Using FKG inequalities, one can construct with $\mathbb{P} = 1$ the infinite volume Gibbs measure $\mu^+[\omega]$ as limits of local specifications with homogeneous plus boundary conditions along any deterministic sequence of increasing and absorbing finite volumes Λ_n. Of course the same holds with minus boundary conditions, see for example Theorem 7.2.2 in [3] or Theorem IV.6.5 in [10]. The main results are the following.

Theorem 2.1 *Let* $\alpha \in (\frac{1}{2}, \frac{\ln 3}{\ln 2} - 1)$ *and*

$$\zeta = \zeta(\alpha) = 1 - 2(2^\alpha - 1) > 0. \tag{2.5}$$

There exist positive $\theta_0 := \theta_0(\alpha) > 0$ *and* $\beta_0 := \beta_0(\alpha) > 0$ *so that for* $0 < \theta \le \theta_0$ *and* $\beta \ge \beta_0$ *there exists* $\Omega_1 \subset \Omega$ *such that*

$$\mathbb{P}[\Omega_1] \ge 1 - e^{-\frac{\bar b}{200}}, \tag{2.6}$$

and for any $\omega \in \Omega_1$,

$$\mu^+\Big(\{\sigma_0 = -1\}\Big)[\omega] < e^{-\frac{\bar b}{200}} \tag{2.7}$$

where

$$\bar b = \min\left(\frac{\beta\zeta}{4}, \frac{\zeta^2}{2^{10}\theta^2}\right). \tag{2.8}$$

[1] The condition $J(1) >> 1$ is essential to apply the results of [5], reported in Subsection 2.2.

Remark: Since the translation invariant, $\mathcal{B}$ measurable event $A \equiv \{\exists i \in \mathbb{Z} : \mu^+[\omega](\sigma_i = +1) > 1 - e^{-\frac{\bar{b}}{200}}\}$ has strictly positive probability, see (2.6) and (2.7), by ergodicity $\mathbb{P}[A] = 1$. Therefore almost surely the two extremal Gibbs states $\mu^{\pm}[\omega]$ are distinct.

The proof of Theorem 2.1 is given in Section 3. In the next subsection we recall the definition of contours and in Section 4 we prove the main probabilistic estimate.

2.2. Geometrical description of the spin configurations

We will follow the geometrical description of the spin configuration presented in [5] and use the same notations. We will consider homogeneous boundary conditions, i.e the spins in the boundary conditions are either all +1 or all −1. Actually we will restrict ourself to + boundary conditions and consider spin configurations $\sigma = \{\sigma_i, i \in \mathbb{Z}\} \in \mathcal{X}_+$ so that $\sigma_i = +1$ for all $|i|$ large enough.

In one dimension an interface at $(x, x+1)$ means $\sigma_x \sigma_{x+1} = -1$. Due to the above choice of the boundary conditions, any $\sigma \in \mathcal{X}_+$ has a finite, even number of interfaces. The precise location of the interface is immaterial and this fact has been used to choose the interface points as follows: For all $x \in \mathbb{Z}$ so that $(x, x+1)$ is an interface take the location of the interface to be a point inside the interval $[x + \frac{1}{2} - \frac{1}{100}, x + \frac{1}{2} + \frac{1}{100}]$, with the property that for any four distinct points r_i, $i = 1, \ldots, 4$ $|r_1 - r_2| \neq |r_3 - r_4|$. This choice is done once for all so that the interface between x and $x+1$ is uniquely fixed. Draw from each one of these interfaces points two lines forming respectively an angle of $\frac{\pi}{4}$ and of $\frac{3}{4}\pi$ with the $\mathbb{Z}$ line. We have thus a bunch of growing $\vee-$ lines each one emanating from an interface point. Once two $\vee-$ lines meet, they are frozen and stop their growth. The other two lines emanating from the the same interface points are erased. The $\vee-$ lines emanating from others points keep growing. The collision of the two lines is represented graphically by a triangle whose basis is the line joining the two interfaces points and whose sides are the two segment of the $\vee-$ lines which meet. The choice done of the location of the interface points ensure that collisions occur one at a time so that the above definition is unambiguous. In general there might be triangles inside triangles. The endpoints of the triangles are suitable coupled pairs of interfaces points. The graphical representation just described maps each spin configuration in $\mathcal{X}_+$ to a set of triangles.

Notation *Triangles will be usually denoted by T, the collection of triangles constructed as above by $\{\underline{T}\}$ and we will write*

$$|T| = \textit{cardinality of} \quad T \cap \mathbb{Z} = \textit{mass of} \quad T,$$

and by $supp(T) \subset \mathbb{R}$ the basis of the triangle.

We have thus represented a configuration $\sigma \in \mathcal{X}_+$ as a collection of $\underline{T} = (T_1, \ldots, T_n)$. The above construction defines a one to one map from $\mathcal{X}_+$ onto $\{\underline{T}\}$. It is easy to see that a triangle configuration $\underline{T}$ belongs to $\{\underline{T}\}$ iff for any pair T and T' in $\underline{T}$

$$\mathrm{dist}(T, T') \geq \min\{|T|, |T'|\}. \tag{2.9}$$

We say that two collections of triangles $\underline{S}'$ and $\underline{S}$ are compatible and we denote it by $\underline{S}' \simeq \underline{S}$ iff $\underline{S}' \cup \underline{S} \in \{\underline{T}\}$ (*i.e.* there exists a configuration in $\mathcal{X}_+$ such that its corresponding collection of triangles is the collection made of all triangles that are in $\underline{S}'$ or in $\underline{S}$.) By an abuse of notation, we write

$$H_0^+(\underline{T}) = H_0^+(\sigma), \quad G(\sigma(\underline{T}))[\omega] = G(\sigma)[\omega], \quad \sigma \in \mathcal{X}_+ \iff \underline{T} \in \{\underline{T}\}.$$

Definition 2.2 The energy difference *Given two compatible collections of triangles $\underline{S} \simeq \underline{T}$, we denote*

$$H^+(\underline{S}|\underline{T}) := H^+(\underline{S} \cup \underline{T}) - H^+(\underline{T}). \tag{2.10}$$

Let $\underline{T} = (T_1, \dots, T_n)$ with $|T_i| \le |T_{i+1}|$ then using (2.10) one has

$$H^+(\underline{T}) = H^+(T_1|\underline{T} \setminus T_1) + H^+(\underline{T} \setminus T_1). \tag{2.11}$$

The following Lemma proved in [5], see Lemma 2.1 there, gives a lower bound on the cost to "erase" triangles sequentially starting from the smallest ones.

Lemma 2.3 [5] *For $\alpha \in (0, \frac{\ln 3}{\ln 2} - 1)$ and $\zeta := \zeta(\alpha)$ as defined in (2.5) one has*

$$H_0^+(T_1|\underline{T} \setminus T_1) \ge \zeta |T_1|^\alpha, \tag{2.12}$$

and by iteration, for any $1 \le i \le n$

$$H_0^+(\cup_{\ell=1}^i T_\ell | \underline{T} \setminus [\cup_{\ell=1}^i T_\ell]) \ge \zeta \sum_{\ell=1}^i |T_\ell|^\alpha. \tag{2.13}$$

The estimate (2.13) involves contributions coming from the full set of triangles associated to a given spin configuration, starting from the triangle having the smallest mass. To implement a Peierls bound in our set up we need to "localize" the estimates to compute the weight of a triangle or of a finite set of triangles in a generic configuration. In order to do this [5] introduced the notion of contours as clusters of nearby triangles sufficiently far away from all other triangles.

Contours A contour Γ is a collection $\underline{T}$ of triangles related by a hierarchical network of connections controlled by a positive number C, see (2.14), under which all the triangles of a contour become mutually connected. We denote by $T(\Gamma)$ the triangle whose basis is the smallest interval which contains all the triangles of the contour. The right and left endpoints of $T(\Gamma) \cap \mathbb{Z}$ are denoted by $x_\pm(\Gamma)$. We denote $|\Gamma|$ the mass of the contour Γ

$$|\Gamma| = \sum_{T \in \Gamma} |T|$$

i.e. $|\Gamma|$ is the sum of the masses of all the triangles belonging to Γ. We denote by $\mathcal{R}(\cdot)$ the algorithm which associates to any configuration $\underline{T}$ a configuration $\{\Gamma_j\}$ of contours with the following properties.

P.0 *Let $\mathcal{R}(\underline{T}) = (\Gamma_1, \dots, \Gamma_n)$, $\Gamma_i = \{T_{j,i}, 1 \le j \le k_i\}$, then $\underline{T} = \{T_{j,i}, 1 \le i \le n, 1 \le j \le k_i\}$*

P.1 *Contours are well separated from each other.* Any pair $\Gamma \ne \Gamma'$ verifies one of the following alternatives.

$$T(\Gamma) \cap T(\Gamma') = \emptyset$$

i.e. $[x_-(\Gamma), x_+(\Gamma)] \cap [x_-(\Gamma'), x_+(\Gamma')] = \emptyset$, in which case

$$dist(\Gamma, \Gamma') := \min_{T \in \Gamma, T' \in \Gamma'} dist(T, T') > C \left\{|\Gamma|^3, |\Gamma'|^3\right\} \tag{2.14}$$

where C is a positive number.If

$$T(\Gamma) \cap T(\Gamma') \ne \emptyset,$$

then either $T(\Gamma) \subset T(\Gamma')$ or $T(\Gamma') \subset T(\Gamma)$; moreover, supposing for instance that the former case is verified, (in which case we call Γ an inner contour) then for any triangle $T'_i \in \Gamma'$, either $T(\Gamma) \subset T'_i$ or $T(\Gamma) \cap T'_i = \emptyset$ and

$$dist(\Gamma, \Gamma') > C|\Gamma|^3, \quad \text{if} \quad T(\Gamma) \subset T(\Gamma'). \tag{2.15}$$

P.2 *Independence.* Let $\{\underline{T}^{(1)}, \ldots, \underline{T}^{(k)}\}$, be $k > 1$ configurations of triangles; $\mathcal{R}(\underline{T}^{(i)}) = \{\Gamma_j^{(i)}, j = 1, \ldots, n_i\}$ the contours of the configurations $\underline{T}^{(i)}$. Then if any distinct $\Gamma_j^{(i)}$ and $\Gamma_{j'}^{(i')}$ satisfies **P.1**,

$$\mathcal{R}(\underline{T}^{(1)}, \ldots, \underline{T}^{(k)}) = \{\Gamma_j^{(i)}, j = 1, \ldots, n_i; i = 1, \ldots, k\}.$$

As proven in [5], the algorithm $\mathcal{R}(\cdot)$ having properties **P.0**, **P.1** and **P.2** is unique and therefore there is a bijection between families of triangles and contours. Next we report the estimates proven in [5] which are essential for this paper.

Theorem 2.4 [5] *Let $\alpha \in (0, \frac{\ln 3}{\ln 2} - 1)$ and the constant C in the definition of the contours, see (2.14), be so large that*

$$\sum_{m \geq 1} \frac{4m}{[Cm]^3} \leq \frac{1}{2}, \tag{2.16}$$

where $[x]$ denotes the integer part of x. For any $\underline{T} \in \{\underline{T}\}$, let $\Gamma_0 \in \mathcal{R}(\underline{T})$ be a contour, $\underline{S}^{(0)}$ the triangles in Γ_0 and $\zeta(\alpha)$ as in (2.5) Then

$$H_0^+(\underline{S}^{(0)} | \underline{T} \setminus \underline{S}^{(0)}) \geq \frac{\zeta}{2} |\Gamma_0|^\alpha, \tag{2.17}$$

where

$$|\Gamma_0|^\alpha := \sum_{T \in \underline{\Gamma}_0} |T|^\alpha. \tag{2.18}$$

Theorem 2.5 [5] *For any $\gamma > 0$ there exists $C_0(\gamma)$ so that for $b \geq C_0(\gamma)$ and for all $m > 0$*

$$\sum_{0 \in \Gamma |\Gamma| = m} w_b^\gamma(\Gamma) \leq 2m e^{-bm^\gamma}, \tag{2.19}$$

where

$$w_b^\gamma(\Gamma) := \prod_{T \in \Gamma} e^{-b|T|^\gamma}. \tag{2.20}$$

In the sequel, it is convenient to identify in each contour Γ the families of triangles having the same mass.

Definition 2.6

$$\Gamma = \{\underline{T}^{(0)}, \underline{T}^{(1)}, \ldots \underline{T}^{(k_\Gamma)}\}$$

where for $\ell = 0, \ldots k_\Gamma$, $\underline{T}^{(\ell)} := \{T_1^{(\ell)}, T_2^{(\ell)}, \ldots T_{n_\ell}^{(\ell)}\}$, and each triangle of the family $\underline{T}^{(\ell)}$ has the same mass, i.e. for all $i \in \{1, \ldots n_\ell\}$, $|T_i^{(\ell)}| = \Delta_\ell$ for $\Delta_\ell \in I\!N$. According to (2.18)

$$|\Gamma|^\rho = \sum_{\ell=0}^{k_\Gamma} |\underline{T}^{(\ell)}|^\rho, \quad |\underline{T}^{(\ell)}|^\rho = \sum_{T \in \underline{T}^{(\ell)}} |T|^\rho = n_\ell \Delta_\ell^\rho, \quad \rho \in I\!R^+. \tag{2.21}$$

3 Proof of Theorem 2.1

The proof of Theorem 2.1 is an immediate consequence of the following proposition and the Markov inequality.

Proposition 3.1 *Let $\alpha \in (\frac{1}{2}, \frac{\ln 3}{\ln 2} - 1)$. There exist positive $\theta_0 := \theta_0(\alpha) > 0$ and $\beta_0 := \beta_0(\alpha) > 0$ so that for $0 < \theta \le \theta_0$ and $\beta \ge \beta_0$*

$$\mathbb{E}\left[\mu^+\Big(\frac{1-\sigma_0}{2}\Big)\right] \le e^{-\frac{\bar b}{100}}, \tag{3.1}$$

where $\bar b$ is the quantity defined in (2.8).

Proof: A necessary condition to have $\sigma_0 = -1$ is that the site zero is contained in the support of some contour Γ so that

$$\mu_\Lambda^+(\sigma_0 = -1) \le \mu_\Lambda^+(\{\exists \Gamma : 0 \in \Gamma\}) \le \sum_{\Gamma \ni 0} \mu_\Lambda^+(\Gamma). \tag{3.2}$$

By definition, see (2.4),

$$\mu_\Lambda^+(\Gamma)[\omega] := \frac{1}{Z_\Lambda^+[\omega]} \sum_{\underline{T}:\underline{T}\simeq\Gamma} e^{-\beta H^+(\underline{T}\cup\Gamma)[\omega]}, \tag{3.3}$$

where $\sum_{\underline{T}:\underline{T}\simeq\Gamma}$ means that the sum is over all families of triangles compatible with the contour Γ. Recalling (2.2) and (2.10), for any j such that $0 \le j \le k_\Gamma$, we write for the deterministic part of the Hamiltonian

$$H_0^+(\underline{T}\cup\Gamma) = H_0^+(\underline{T}\cup\Gamma \setminus (\cup_{\ell=0}^j \underline{T}^{(\ell)})) + H_0^+(\underline{T}\cup\Gamma \setminus (\cup_{\ell=0}^j \underline{T}^{(\ell)})|(\cup_{\ell=0}^j \underline{T}^{(\ell)})). \tag{3.4}$$

Using estimate (2.17) and recalling notation (2.21)

$$H_0^+(\underline{T}\cup\Gamma \setminus (\cup_{\ell=0}^j \underline{T}^{(\ell)})|(\cup_{\ell=0}^j \underline{T}^{(\ell)})) \ge \frac{\zeta}{2}\sum_{\ell=0}^j n_\ell |\Delta_\ell|^\alpha. \tag{3.5}$$

Therefore

$$\mu_\Lambda^+(\Gamma) \le e^{-\beta\frac{\zeta}{2}\sum_{\ell=0}^j n_\ell|\Delta_\ell|^\alpha} \frac{1}{Z_\Lambda^+} \sum_{\underline{T}:\underline{T}\simeq\Gamma} e^{-\beta H_0^+(\underline{T}\cup\Gamma\setminus(\cup_{\ell=0}^j\underline{T}^{(\ell)}))+\beta\theta G(\sigma(\underline{T}\cup\Gamma))[\omega]}. \tag{3.6}$$

We multiply and divide (3.6) by

$$\sum_{\underline{T}:\underline{T}\simeq\Gamma} e^{-\beta H_0^+(\underline{T}\cup\Gamma\setminus(\cup_{\ell=0}^j\underline{T}^{(\ell)}))+\beta\theta G(\sigma(\underline{T}\cup\Gamma\setminus\cup_{\ell=0}^j\underline{T}^{(\ell)}))[\omega]}, \tag{3.7}$$

and reconstruct $\mu_\Lambda^+(\Gamma\setminus[\cup_{\ell=0}^j\underline{T}^{(\ell)}])$, observing that $\sum_{\underline{T}:\underline{T}\simeq\Gamma} 1 \le \sum_{\underline{T}:\underline{T}\simeq\Gamma\setminus\cup_{\ell=0}^j\underline{T}^{(\ell)}} 1$. We get

$$\mu_\Lambda^+(\Gamma) \le e^{-\frac{\beta\zeta}{2}\sum_{\ell=0}^j |\underline{T}^{(\ell)}|^\alpha} \mu_\Lambda^+(\Gamma\setminus[\cup_{\ell=0}^j\underline{T}^{(\ell)}]) e^{\beta F_j[\omega]} \tag{3.8}$$

where

$$F_j[\omega] := \frac{1}{\beta}\ln\left\{\frac{\sum_{\underline{T}:\underline{T}\simeq\Gamma} e^{-\beta H_0^+(\underline{T}\cup\Gamma\setminus\cup_{\ell=0}^j\underline{T}^{(\ell)})+\beta\theta G(\sigma(\underline{T}\cup\Gamma))[\omega]}}{\sum_{\underline{T}:\underline{T}\simeq\Gamma} e^{-\beta H_0^+(\underline{T}\cup\Gamma\setminus(\cup_{\ell=0}^j\underline{T}^{(\ell)}))+\beta\theta G(\sigma(\underline{T}\cup\Gamma\setminus(\cup_{\ell=0}^j\underline{T}^{(\ell)}))[\omega]}}\right\}. \tag{3.9}$$

In (3.8) we explicitly quantify the deterministic cost of the first smaller families of triangles $\{\underline{T}^{(0)},\dots,\underline{T}^{(j)}\}$ and express the main random contribution $F_j[\omega]$ so that it is antisymmetric with respect to the sign exchange of the random field inside $\cup_{\ell=0}^j\underline{T}^{(\ell)}$, see (4.6). This observation allows to estimate this random contribution in a very convenient way, see Lemma 3.2. To this aim we define for each Γ the partition: $\Omega = \cup_{j=-1}^{k_\Gamma} B_j$ where for $j \in \{0, \dots k_\Gamma - 1\}$

$$B_j = B_j(\Gamma) := \{\omega : F_j[\omega] \le \frac{\zeta}{4} \sum_{\ell=0}^{j} |\underline{T}^{(\ell)}|^\alpha, \text{ and } \forall i > j, F_i[\omega] > \frac{\zeta}{4} \sum_{\ell=0}^{i} |\underline{T}^{(\ell)}|^\alpha\}, \tag{3.10}$$

$$B_{k_\Gamma} = B_{k_\Gamma}(\Gamma) := \{\omega : F_{k_\Gamma}[\omega] \le \frac{\zeta}{4} \sum_{\ell=0}^{k_\Gamma} |\underline{T}^{(\ell)}|^\alpha\}, \tag{3.11}$$

and

$$B_{-1} = B_{-1}(\Gamma) := \{\omega : \forall i > -1; F_i[\omega] > \frac{\zeta}{4} \sum_{\ell=0}^{i} |\underline{T}^{(\ell)}|^\alpha\}. \tag{3.12}$$

The relevant properties of the partition are given in the following lemma, whose proof is given in Section 4.

Lemma 3.2 *For $-1 \le j \le k_\Gamma$,*

$$I\!E\left[1\!\!1_{B_j}\right] \le e^{-\frac{\zeta^2}{2^{10}\theta^2} \sum_{\ell=j+1}^{k_\Gamma} |\underline{T}^{(\ell)}|^{2\alpha-1}}. \tag{3.13}$$

with the convention that an empty sum is zero.

We then write

$$\mu_\Lambda^+(\Gamma) = \sum_{j=-1}^{k_\Gamma} \mu_\Lambda^+(\Gamma) 1\!\!1_{\{B_j\}}$$

and apply to each $\mu_\Lambda^+(\Gamma) 1\!\!1_{\{B_j\}}$ estimate (3.8). We obtain

$$\begin{aligned} I\!E\left[\mu_\Lambda^+(\Gamma)\right] &= \sum_{j=-1}^{k_\Gamma} I\!E\left[\mu_\Lambda^+(\Gamma)) 1\!\!1_{\{B_j\}}\right] \\ &\le \sum_{j=-1}^{k_\Gamma} e^{-\frac{\beta\zeta}{4}\sum_{\ell=0}^{j} |\underline{T}^{(\ell)}|^\alpha} e^{-c\frac{\zeta^2}{\theta^2}\sum_{\ell=j+1}^{k_\Gamma} |\underline{T}^{(\ell)}|^{2\alpha-1}} \\ &\le (k_\Gamma + 1) e^{-\bar{b}\sum_{\ell=0}^{k_\Gamma} |\underline{T}^{(\ell)}|^{2\alpha-1}}, \end{aligned} \tag{3.14}$$

where $\bar{b} := \min(\frac{\beta\zeta}{4}, \frac{\zeta^2}{2^{10}\theta^2})$. Recalling (2.20), one has

$$I\!E\left[\mu_\Lambda^+(\{0 \in \Gamma\})\right] \le \sum_{\Gamma \ni 0} (k_\Gamma + 1) w_{\bar{b}}^{2\alpha-1}(\Gamma) = \sum_{m \ge 3} (m+1) \sum_{0 \in \Gamma : |\Gamma| = m} w_{\bar{b}}^{2\alpha-1}(\Gamma). \tag{3.15}$$

Using (2.19), after a few lines computation one gets (3.1) ■

Remark: The upper bound $\alpha < \frac{\ln 3}{\ln 2} - 1$ in Theorem 2.1 follows from Theorem 2.4, the lower bound $\alpha > \frac{1}{2}$ from Theorem 2.5 and (3.15).

4 Probabilistic estimates

Let $h = h[\omega]$ be a realization of the random magnetic fields and $A \subset \mathbb{Z}$. Define

$$(S_A h)_i = \begin{cases} -h_i, & \text{if } i \in A; \\ h_i, & \text{otherwise} \end{cases}, \tag{4.1}$$

and denote $h[S_A\omega] \equiv S_A h[\omega]$. In the following to simplify notation we wet $S_{\underline{T}} h = S_{\text{supp}(T)} h$. Recalling (2.3), it is easy to see that

$$G(\sigma(\underline{T} \cup \Gamma \setminus \underline{T}^{(0)}))[\omega] = G(\sigma(\underline{T} \cup \Gamma))[S_{\underline{T}^{(0)}}\omega]. \tag{4.2}$$

In general

$$G(\sigma(\underline{T} \cup \Gamma \setminus \cup_{\ell=0}^{i} \underline{T}^{(\ell)}))[\omega] = G(\sigma(\underline{T} \cup \Gamma))[S_{D_i}\omega] \tag{4.3}$$

where

$$D_i \subset \cup_{\ell=0}^{i} \left(\text{supp}(\underline{T}^{(\ell)}) \right) \tag{4.4}$$

is the non–empty set so that

$$S_{D_i} = S_{\underline{T}^{(i)}} S_{\underline{T}^{(i-1)}} \dots S_{\underline{T}^{(1)}} S_{\underline{T}^{(0)}}. \tag{4.5}$$

When all the triangles in $(\underline{T}^{(\ell)}, \ell = 0, \dots, j)$ have disjoint supports (4.4) becomes an equality. In general there are triangles inside triangles and in this case the inclusion in (4.4) is strict. By construction the $F_j[\omega]$ defined in (3.9) are such that

$$F_j(h(D_j^c), h(D_j)) = -F_j(h(D_j^c), -h(D_j)), \qquad j \in 0, \dots, k_\Gamma, \tag{4.6}$$

where for a set $A \subset \mathbb{Z}$, we denote by $h(A) = \{h_i : i \in A\}$. Therefore one gets that $\mathbb{E}[F_j] = 0$.

Proof of Lemma 3.2 Set

$$A_i := \frac{\zeta}{4} \sum_{\ell=0}^{i} |\underline{T}^{(\ell)}|^\alpha. \tag{4.7}$$

We have $\mathbb{P}[B_j] \leq \mathbb{P}[\forall i > j; F_i[\omega] > A_i]$. Let λ_i for $i = j+1, \dots, k_\Gamma$ be positive parameters, by exponential Markov inequality we have

$$\mathbb{P}[\forall i > j : F_i[\omega] \geq A_i] \leq e^{-\sum_{\ell=j+1}^{k_\Gamma} \lambda_\ell A_\ell} \mathbb{E}\left[e^{\sum_{\ell=j+1}^{k_\Gamma} \lambda_\ell F_\ell} \right]. \tag{4.8}$$

Set

$$F[\omega] := \sum_{i=j+1}^{k_\Gamma} \lambda_i F_i[\omega]. \tag{4.9}$$

It remains to estimate $\mathbb{E}[e^F]$. Note that $F[\omega]$ depends on all the random fields on Λ. Let N be the number of sites in Λ. To avoid involved notations, we define a bijection Π from Λ to $\{1, \dots, N\}$ as follows: first pick up all the $n_0\Delta_0$ sites in supp$(\underline{T}^{(0)})$ and put them consecutively in $N, \dots, N - n_0\Delta_0 + 1$ (keeping them in the same order as they are for definiteness). Then pick up the sites in supp$(\underline{T}^{(1)})$ that are not in supp$(\underline{T}^{(0)})$ and put them consecutively starting at $N - n_0\Delta_0$ until they are exhausted. Note that if no triangles of size Δ_0 are within triangle of size Δ_1, Π maps $supp(\underline{T}^{(0)}) \cup supp(\underline{T}^{(1)})$ onto $\{N, \dots, N - n_0\Delta_0 - n_1\Delta_1 + 1\}$; otherwise Π maps $supp(\underline{T}^{(0)}) \cup supp(\underline{T}^{(1)})$ onto a proper subset of $\{N, \dots, N - n_0\Delta_0 - n_1\Delta_1 + 1\}$. One can iterate this procedure until all the sites of the support of Γ are exhausted. As above, for all $1 \leq j \leq k_\Gamma - 1$, if all triangles considered are disjoint Π maps $\cup_{\ell=0}^{j+1} supp(\underline{T}^{(\ell)})$ onto $\{N, N-1, \dots, N - M_{j+1} + 1\}$ where $M_{j+1} = \sum_{\ell=0}^{j+1} n_\ell \Delta_\ell$, otherwise on a proper subset of it. Then one can pick up all the remaining sites of Λ

and continue as above. The Π so defined induces a bijection from the random magnetic fields indexed by Λ to a family of random variables $(h_1,\dots,h_N)$ by $(\Pi h)_i := h_{\Pi i}, \forall i \in \Lambda$. Using this bijection, one can work with the random variables $(h_i, 1 \le i \le N)$. Define the family of increasing σ-algebra:

$$(\emptyset,\Omega) = \Sigma_0 \subset \Sigma_1 = \sigma(h_1) \subset \Sigma_2 = \sigma(h_1,h_2) \subset \ldots \subset \Sigma_N = \sigma(h_1,h_2,\dots,h_N)$$

and $\Delta_k(F) = I\!E\left[F|\Sigma_k\right] - I\!E\left[F|\Sigma_{k-1}\right]$ the associated martingale difference sequences. We have

$$I\!E[F|\Sigma_N] = F; \quad I\!E[F|\Sigma_0] = I\!E[F] = 0, \quad F = \sum_{k=1}^{N} \Delta_k(F).$$

Remark that

$$I\!E[F_{j+1}|\Sigma_i] = 0 \quad \forall i \in \{1,\dots N - M_{j+1}\} \tag{4.10}$$

since by (4.6)

$$F_{j+1}(h(D^c_{j+1}), h(D_{j+1})) = -F_{j+1}(h(D^c_{j+1}), -h(D_{j+1}))$$

and

$$\Pi h(\cup_{\ell=0}^{j+1} \underline{T}^{(\ell)}) \subset \{h_{N-M_{j+1}},\dots,h_N\}.$$

Note also that

$$I\!E[F|\Sigma_i] = 0 \quad \forall i \in \{1,\dots N - M_{k_\Gamma}\}, \tag{4.11}$$

and

$$I\!E[e^F] = I\!E[e^{\sum_{k=1}^{N-1}\Delta_k(F)} I\!E[e^{\Delta_N(F)}|\Sigma_{N-1}]].$$

With self explained notations, using Jensen inequality one has

$$I\!E[e^{\Delta_N(F)}|\Sigma_{N-1}] = \int e^{\Delta_N(F)}\, I\!P(dh_N) \le \int e^{\left[F(h_{<N},h_N) - F(h_{<N},\tilde h_N)\right]} I\!P(dh_N) I\!P(d\tilde h_N). \tag{4.12}$$

We then expand the exponential in the right hand side of (4.12). All the odd powers but the constant one vanish. For the even power we recall (4.9) and by the Lipschitz continuity of each term with respect to $(h_1,\dots,h_N)$ we get

$$\begin{aligned} \left|F(h_{<N},h_N) - F(h_{<N},\tilde h_N)\right| &\le \sum_{i=j+1}^{k_\Gamma} \lambda_i |F_i(h_{<N},h_N) - F_i(h_{<N},\tilde h_N)| \\ &\le 2\theta|h_N - \tilde h_N| \sum_{\ell=j+1}^{k_\Gamma} \lambda_\ell. \end{aligned} \tag{4.13}$$

Then estimating $|h_N - \tilde h_N| \le 2$ and $2^{(2n-1)^+}(2n!)^{-1} \le (n!)^{-1}$ to re-sum the series one gets

$$I\!E[e^{\Delta_N}|\Sigma_{N-1}] \le e^{16\theta^2(\sum_{\ell=j+1}^{k_\Gamma}\lambda_\ell)^2}. \tag{4.14}$$

In the case of gaussian or subgaussian variable one just performs all the integration instead of using $|h_N - \tilde h_N| \le 2$. This will modify the result by a constant different from 16. To iterate, one uses again the Jensen inequality obtaining

$$\begin{aligned} I\!E[e^{\Delta_{N-1}}|\Sigma_{N-2}] &= \int e^{\Delta_{N-1}(F)} I\!P(dh_{N-1}) \\ &\le \int e^{\int\left[F(h_{<N-1},h_{N-1},\hat h_N) - F(h_{<N-1},\tilde h_{N-1},\hat h_N)\right] I\!P(d\hat h_N)} I\!P(dh_{N-1}) I\!P(d\tilde h_{N-1}). \end{aligned}$$

It is clear that the random variable

$$\int \left[F(h_{<N-1}, h_{N-1}, \hat{h}_N) - F(h_{<N-1}, \tilde{h}_{N-1}, \hat{h}_N)\right] I\!P(d\hat{h}_N)$$

is a symmetric ones under $I\!P(dh_{N-1})I\!P(d\tilde{h}_{N-1})$ and satisfies an estimate as (4.13) from which one gets

$$I\!E[e^{\Delta_{N-1}(F)}|\Sigma_{N-2}] \le e^{16\theta^2(\sum_{\ell=j+1}^{k_\Gamma}\lambda_\ell)^2}. \tag{4.15}$$

Iterating one gets $I\!E[e^{\Delta_k(F)}|\Sigma_{k-1}] \le e^{4\theta^2(\sum_{\ell=j+1}^{k_\Gamma}\lambda_\ell)^2}$ for $k \in \{N, N-1, \ldots, N-M_{j+1}\}$. When $k = N - M_{j+1} - 1$, a new fact happens. Using (4.10) for $i = N - M_{j+1} \equiv m$ and computing

$$\Delta_m(F) = \sum_{i=j+1}^{k_\Gamma} \lambda_i \left(I\!E[F_i|\Sigma_m] - I\!E[F_i|\Sigma_{m-1}]\right) \tag{4.16}$$

one obtains that the term corresponding to $i = j+1$ in the sum gives zero contribution. Therefore, in this case, one has

$$\left|\int \left[F(h_{<m}, h_m, \hat{h}_{>m}) - F(h_{<m}, \tilde{h}_m, \hat{h}_{>m})\right] I\!P(\hat{h}_{>m})\right| \le 4\theta \sum_{\ell=j+2}^{k_\Gamma} \lambda_\ell. \tag{4.17}$$

Iterating this procedure one gets

$$I\!E\left[e^F\right] \le e^{16\theta^2\left\{\left(\sum_{\ell=0}^{j+1} n_\ell\Delta_\ell\right)\left(\sum_{\ell=j+1}^{k_\Gamma}\lambda_\ell\right)^2 + n_{j+2}\Delta_{j+2}\left(\sum_{\ell=j+2}^{k_\Gamma}\lambda_\ell\right)^2 + \ldots + n_{k_\Gamma}\Delta_{k_\Gamma}\left(\lambda_{k_\Gamma}\right)^2\right\}}. \tag{4.18}$$

The estimate (4.18) suggests to set for $\ell = j+1, \ldots, k_\Gamma$

$$\mu_\ell \equiv \sum_{n=\ell}^{k_\Gamma} \lambda_n \tag{4.19}$$

and the constraints ($\lambda_i \ge 0, j+1 \le i \le k_\Gamma$) become μ_ℓ decreasing with ℓ. We write the first exponent of (4.8) in terms of $\{\mu_\ell\}_{\ell=0}^{j}$ obtaining

$$-\sum_{\ell=j+1}^{k_\Gamma} \lambda_\ell A_\ell = -\frac{\zeta}{4}\mu_{j+1}\left(\sum_{\ell=0}^{j} n_\ell \Delta_\ell^\alpha\right) - \sum_{\ell=j+1}^{k_\Gamma} \frac{\zeta}{4}\mu_\ell n_\ell \Delta_\ell^\alpha, \tag{4.20}$$

and for the exponent in (4.18) we obtain

$$16\theta^2(\mu_{j+1})^2\left(\sum_{\ell=0}^{j} n_\ell\Delta_\ell\right) + \sum_{\ell=j+1}^{k_\Gamma} 16\theta^2(\mu_\ell)^2 n_\ell\Delta_\ell. \tag{4.21}$$

Denote

$$f(\mu_\ell) \equiv -\frac{\zeta}{4}\mu_\ell\Delta_\ell^\alpha + 16\theta^2\mu_\ell^2\Delta_\ell \quad \ell = j+1, \ldots k_\Gamma.$$

Choose $\mu_\ell \equiv \bar{\mu}_\ell$ where

$$\bar{\mu}_\ell = \frac{1}{4\times 32}\frac{\zeta}{\theta^2\Delta_\ell^{1-\alpha}}, \tag{4.22}$$

is the minimizer of $f(\mu_\ell)$. Note that $\bar\mu_\ell$ is a decreasing function of ℓ and

$$f(\bar\mu_\ell) = -\frac{\zeta^2 \Delta_\ell^{2\alpha-1}}{2^{10}\theta^2}. \tag{4.23}$$

Therefore collecting together the last sum in (4.20) and the one in (4.21) we get

$$-\sum_{\ell=j+1}^{k_\Gamma} \frac{\zeta}{4}\bar\mu_\ell n_\ell \Delta_\ell^\alpha + \sum_{\ell=j+1}^{k_\Gamma} 16\theta^2(\bar\mu_\ell)^2 n_\ell \Delta_\ell = -\frac{\zeta^2}{2^{10}\theta^2}\sum_{\ell=j+1}^{k_\Gamma} n_\ell \Delta_\ell^{2\alpha-1} = -\frac{\zeta^2}{2^{10}\theta^2}\sum_{\ell=j+1}^{k_\Gamma} |\underline{T}^{(\ell)}|^{2\alpha-1}. \tag{4.24}$$

Summing up (4.20) and (4.21), taking in account (4.22) and (4.24) we get

$$-\frac{\zeta}{4}\bar\mu_{j+1}\left(\sum_{\ell=0}^{j} n_\ell \Delta_\ell^\alpha\right) + 16\theta^2(\bar\mu_{j+1})^2\left(\sum_{\ell=0}^{j} n_\ell\Delta_\ell\right) = -\sum_{\ell=0}^{j} n_\ell\left(\frac{\zeta}{4}\bar\mu_{j+1}\Delta_\ell^\alpha - 16\theta^2\Delta_\ell(\bar\mu_{j+1})^2\right). \tag{4.25}$$

One can check easily that for all $0 \le \ell \le j$ one has

$$\left(\frac{\zeta}{4}\bar\mu_{j+1}\Delta_\ell^\alpha - 16\theta^2\Delta_\ell(\bar\mu_{j+1})^2\right) = \frac{\zeta^2\Delta_\ell}{2^9\theta^2\Delta_{j+1}^{1-\alpha}}\left(\frac{1}{\Delta_\ell^{1-\alpha}} - \frac{1}{2\Delta_{j+1}^{1-\alpha}}\right) > 0 \tag{4.26}$$

since by construction $\Delta_\ell < \Delta_{j+1}$ for $0 \le \ell \le j$. By (4.8), (4.18), (4.24), and (4.25) one gets (3.13). ■

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