GLAUBER EVOLUTION WITH KAC POTENTIALS

II. FLUCTUATIONS.

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Abstract. We continue in this paper the analysis of the Glauber evolution in Ising systems with Kac interactions. In the previous paper, [5], it is proven that in a continuum limit, called the mesoscopic limit, the magnetization density converges to the solution of a non local deterministic equation. Here we study the fluctuations around the limit proving convergence to a generalized Ornstein-Uhlenbeck process. We also prove asymptotic formulas for the correlation functions that improve those established in [5] and that will be used in a successive paper to study phase separation.

1. Introduction.

This is the second paper in a series devoted to the analysis of the Glauber evolution of ± 1 valued spins on the lattice $\mathbb{Z}^d$, interacting via a Kac potential. The purpose of the series is to extend to non equilibrium the equilibrium theory of Kac interactions developed in the late 60's, thus giving a full justification to the Van der Waals theory in the context of Statistical Mechanics.

We refer to [5] and reference therein for more comments on the model, we only remind here the two main features of a Kac potential. The first one is that its range diverges as γ^{-1}, where γ is a positive scaling parameter that goes eventually to 0 and the second one is that the total interaction energy of any single spin with all the others is bounded uniformly in γ. In [5] it is proven that in the meso-limit $\gamma \to 0$, and space scaled by the same γ, the limit magnetization density $m(r,t)$ solves the non local equation

$$\frac{\partial m}{\partial t} = -m + \tanh\{\beta(J \star m + h)\} \tag{1.1}$$

$$(J \star m)(r) = \int dr' J([r - r'])m(r') \tag{1.2}$$

The deterministic behavior described by (1.1) is a mean field effect due to the scaling of the interaction and therefore to the validity of a law of large numbers for the density field. We study here the fluctuation theory proving (see Theorem 2.8 below) that the small deviations from the deterministic limit converge to a generalized Ornstein-Uhlenbeck process.

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They have a relevant role in non equilibrium Statistical Mechanics as explained by the Fluctuating Hydrodynamic Theory, see [11] where the theory is applied also to stochastic systems. Fluctuations are particularly important when the system is in a critical state, in which case they may produce large macroscopic effects. This will be discussed at length in the case of phase separation, in the next paper [6]. We also refer to [2] and [8] for the study of the fluctuations of the interface in the Ginzburg-Landau equation with noise and to [1] and [7] for the analysis of the one dimensional fluctuations at the critical point for the same model considered here. The critical fluctuations are proven to solve in a continuum limit the stochastic quantization equation of the Euclidean field theory. A comprehensive discussion of all these aspects may be found in the old version of this paper, [4], that for the sake of brevity is not reported here.

All the proofs in this and in the successive paper, [6], are consequence of sharp estimates on some special functions, the v and the ω functions, whose analysis is reminiscent of the cluster expansion in equilibrium Statistical Mechanics. The v and the ω functions are linear combination of the correlation functions. The v functions define "a distance" between the actual measure at time t and the product measure with averages $m(\gamma x, t)$, see (1.1). In [5] it is proven a bound of the form $\gamma^{dn/2}$, for the v functions of " order" n, i.e. n is the largest order of the correlations which enter in the definition of v. The ω are more similar to the truncated correlations functions, they infact give a measure of "the distance" between the v functions and the moments of the limit Gaussian process. The main result in this paper is Theorem 2.6 where we show that the ω functions converge to 0 as $\gamma^{2dn/3}$, namely faster than the corresponding v functions. This will imply the convergence of all the moments of the fluctuation fields. The weak convergence of the whole process is then a simple consequence of the Holley and Stroock theory, [9].

The paper is organized as follows: in Section 2 we give the main results, in Section 3 we study the ω functions proving Theorem 2.6 and finally in Section 4 we prove the convergence of the fluctuation field to the generalized Ornstein-Uhlenbeck process.

2. Main definitions and results.

We start by recalling, for ease of reference, some basic definitions from [5]. The Glauber dynamics is a Markov process with state space $\{-1,1\}^{\mathbb{Z}^d}$, whose elements, the spin configurations, are denoted by σ. The generator of the process acts on the cylindrical functions f as

$$L_\gamma f(\sigma) = \sum_{x\in\mathbb{Z}^d} c_\gamma(x,\sigma)[f(\sigma^x) - f(\sigma)] \tag{2.1}$$

where

$$\sigma^x(y) = \begin{cases} \sigma(y) & \text{if } y \neq x \\ -\sigma(x) & \text{if } y = x \end{cases}$$

$$c_\gamma(x,\sigma) = \frac{e^{-\beta h_\gamma(x)\sigma(x)}}{e^{-\beta h_\gamma(x)} + e^{\beta h_\gamma(x)}} \tag{2.2}$$

$$h_\gamma(x) = h + (J_\gamma \circ \sigma)(x), \qquad (J_\gamma \circ \sigma)(x) = \sum_{y \neq x} J_\gamma(x,y)\sigma(y) \tag{2.3}$$

and, finally,

$$J_\gamma(x,y) = \gamma^d J(\gamma[x-y]) \tag{2.4}$$

with $J \in C^3(\mathbb{R}^d)$, spherically symmetric and vanishing when $|r| \geq 1$, see [5].

$\beta > 0$ is the inverse temperature, $h \in \mathbb{R}$ an external magnetic field and $\gamma > 0$ the scaling parameter of the Kac potential, that goes eventually to 0.

2.1. Definition. *(The initial distribution)*

We will always consider initial distribution μ that are product measure on $\{-1,1\}^{\mathbb{Z}^d}$. Sometimes we restrict to a particular class μ^γ of product measures determined by a function $m_0 \in C^0(\mathbb{R}^d)$ such that $\|m_0\|_\infty \leq 1$; μ^γ is then the product measure such that

$$\mathbb{E}_{\mu^\gamma}(\sigma(x)) = m_0(\gamma x) \qquad x \in \mathbb{Z}^d \tag{2.5}$$

Finally for any probability μ, we denote by $\mathbb{P}^\gamma_\mu$ the law of the Markov process generated by L_γ with inital measure μ; $\mathbb{E}^\gamma_\mu$ denotes the corresponding expectation.

In many applications μ will be supported by a single configuration σ_0 so that $\mathbb{E}_\mu(\sigma(x)) = \sigma_0(x)$.

For a while we will work with $\gamma > 0$ fixed and for notational simplicity we will drop the dependence on γ in quantities which do depend on γ. The only exclusion will be J_γ where the subfix γ will be retained.

The basic quantities that we have studied in [5] are the moments of the process, i.e. the expectation of product of spins. Using throughout the sequel the notation $\underline{x}$ for a finite subset of $\mathbb{Z}^d$ and S for the collection of all $\underline{x}$, we write

$$\sigma_t(\underline{x}) = \prod_{x \in \underline{x}} \sigma_t(x) \tag{2.6}$$

We will study the deviation of the expectation of $\sigma_t(\underline{x})$ from the product of the expectations of its factors $\sigma_t(x)$'s and determine the leading term as $\gamma \to 0$.

Rather than centering the spins it is convenient to consider the variables

$$\tilde{\sigma}_t(\underline{x}) = \prod_{x \in \underline{x}} [\sigma_t(x) - m_t(x)] \tag{2.7}$$

where m_t solves

$$\frac{dm_t(x)}{dt} = -m_t(x) + \tanh\{\beta[(J_\gamma \circ m_t)(x) + h]\} \tag{2.8}$$

with initial condition

$$m_0(x) = \mathbb{E}_\mu\left(\sigma(x)\right) \tag{2.9}$$

In [5] we have studied the functions:

$$v_t(\underline{x}) = \mathbb{E}_\mu\left(\tilde{\sigma}_t(\underline{x})\right) \tag{2.10}$$

called v functions. The initial measure in (2.10) is any product measure, which thus specifies m_0 by (2.9), m_t by (2.8) and then $\tilde{\sigma}_t$ by (2.7). Everything in (2.10) depends on γ, the law of the process, the variable σ_t and hence v_t. For ease of reference we report below Theorem 2.3.4 of [5].

2.2. Theorem.

There are $a > 0$, c_n, $n \geq 1$, and C so that for any initial product measure, for any $t \leq a \log \gamma^{-1}$ and $\underline{x} \in S$,

$$|v_t(\underline{x})| \leq c_{|\underline{x}|} e^{C|\underline{x}|t} \gamma^{d|\underline{x}|/2} \tag{2.11}$$

where $|\underline{x}|$ denotes the cardinality of $\underline{x}$.

Observe that if $aC < d/2$, the right hand side of (2.11) vanishes as a power of $\gamma^{|\underline{x}|}$.

We will study in this paper the asymptotic behavior of the $v$functions proving that the leading term is given by "a Gaussian approximation". We need several definitions before stating this result, which is reported in Theorem 2.6.

We call V_t and $V_t^{(n)}$ respectively the linear space of all the cylindrical functions of the process at time t and its subspace generated by all $\sigma_t(\underline{x})$, $\underline{x} \in S^{(n)}$, i.e. such that $|\underline{x}| = n$.

We also call $M(S)$ the linear space of functions on S so that

$$\underline{\sigma}_t := \{\sigma_t(\underline{x}), \underline{x} \in S\}; \qquad \underline{\tilde{\sigma}}_t := \{\tilde{\sigma}_t(\underline{x}), \underline{x} \in S\} \tag{2.12}$$

are random variables with values in $M(S)$. We have set in (2.12)

$$\sigma_t(\emptyset) = \tilde{\sigma}_t(\emptyset) = 1 \tag{2.13}$$

As well known in equilibrium Statistical Mechanics, it is useful to introduce an algebraic structure in $M(S)$ defined by the $*$ product

$$(f * g)(\underline{x}) = \sum_{\underline{y} \subset \underline{x}} f(\underline{y}) g(\underline{x} - \underline{y}) \tag{2.14}$$

It is then readily seen that, setting

$$\underline{m}_t = \left\{ m_t(\underline{x}) = \prod_{x \in \underline{x}} m_t(x) \right\}, \quad m_t(\emptyset) = 1 \tag{2.15}$$

$$\underline{\sigma}_t = \underline{\tilde{\sigma}}_t * \underline{m}_t, \qquad \underline{\tilde{\sigma}}_t = \underline{\sigma}_t * \underline{m}_t^{-1} \tag{2.16}$$

where, $\underline{m}_t^{-1}$, the $*$ inverse of $\underline{m}_t$, is

$$\underline{m}_t^{-1}(\underline{x}) = (-1)^{|\underline{x}|} m_t(\underline{x}) \tag{2.17}$$

For future reference we observe that the identity for the $*$ product is I, where $I(\underline{x}) = 0$ if $|\underline{x}| > 0$ and $I(\emptyset) = 1$. Moreover $f \in M(S)$ has a $*$ inverse if and only if $f(\emptyset) \neq 0$: in fact we can construct f^{-1} iteratively, setting $f^{-1}(\emptyset) = f(\emptyset)^{-1}$, solving then

$$0 = (f^{-1} * f)(x) \ \text{ that gives } \ f^{-1}(x) = -f(\emptyset)^{-1}[f^{-1}(\emptyset) f(x)]$$

and so forth.

We next introduce the linear operator K_t on $M(S)$ as follows:

2.3. Definition.

For any x and y in $\mathbb{Z}^d$ let

$$k_t(x, y, m) = -\mathbf{1}_{x=y} + f_x^{(1)}(m_t)\beta J_\gamma(x, y) \tag{2.18}$$

where $\mathbf{1}_{x=y}$ is the Kronecker delta, and

$$f_x^{(n)}(m) = \tanh^{(n)} \Big\{ \beta[J_\gamma \circ m(x) + h] \Big\} \tag{2.19}$$

$\tanh^{(n)}(\cdot)$ is the n-th derivative of hyperbolic tangent. We then define the linear operator K_t on $M(S)$ by setting

$$(K_t g)(\underline{x}) = \sum_{x \in \underline{x}} \sum_{y \notin \underline{x} - x} k_t(x, y) g(\underline{x} - x + y) \tag{2.20}$$

We denote by $K_t(\underline{x}, \underline{y})$ its matrix elements:

$$(K_t g)(\underline{x}) = \sum_{\underline{y}} K_t(\underline{x}, \underline{y}) g(\underline{y}) \tag{2.21}$$

Observe that $k_t(x, y, m)$ is the kernel of the operator obtained by linearizing (2.8). By the help of K_t we next define the W functions:

2.4. Definition.

Let $\kappa_t : S^{(2)} \to \mathbb{R}$

$$\kappa_t(x, y) = \beta J_\gamma(x, y) \Big\{ f_y^{(1)}(m_t)[1 - m_t(x)^2] + f_x^{(1)}(m_t)[1 - m_t(y)^2] \Big\} \tag{2.22}$$

and let $w_t : S^{(2)} \to \mathbb{R}$ *be the solution of*

$$\frac{dw_t}{dt} = K_t^{(2)} w_t + \kappa_t, \qquad w_0 = 0 \tag{2.23}$$

where, for any n, $K_t^{(n)}$ *is the restriction of* K_t *to* $M(S^{(n)})$.

We then define W_t *on* $S^{(2n)}$ *as*

$$W_t(\underline{x}) = \sum_{\{(i_1,j_1),..,(i_n,j_n)\}} \prod_{\ell=1}^{n} w_t(x_{i_\ell}, x_{j_\ell}), \qquad \underline{x} = (x_1, ..., x_{2n}) \tag{2.24}$$

where the sum is over all the partitions $\{(i_1, j_1), .., (i_n, j_n)\}$ *of* $\{1, .., 2n\}$ *into* n *disjoint pairs. We complete the definition of* W_t *by setting it equal to 0 when* $|\underline{x}|$ *is odd, so that* W_t *is an element of* $M(S)$.

Observe that if ξ_x, $x \in \underline{x}$, are centered Gaussian variables with covariance $w_t(x, y)$, then $W_t(\underline{x})$ is the expectation of their product. We will prove that $W_t(\underline{x})$ is the leading term in the expectation of $\tilde{\sigma}(\underline{x})$. It is easy to see that there is a constant C such that

$$\|K_t^{(2)}\| \le C \tag{2.25}$$

as on operator on the space $M(S^{(2)})$, equipped with sup norm. Moreover there is a constant c such that $\|k_t\|_\infty \le c\gamma^d$. We then have by (2.23)

$$\|w_t\|_\infty \le ce^{Ct}\gamma^d \tag{2.26}$$

Then

$$|W_t(\underline{x})| \le \left(ce^{Ct}\gamma^d\right)^{|\underline{x}|/2} \tag{2.27}$$

The relation between $\underline{\tilde{\sigma}}_t$ and W_t is conveniently expressed in terms of the $\hat{\omega}$ functions as

$$\underline{\tilde{\sigma}}_t = \hat{\omega}_t * W_t; \qquad \hat{\omega}_t = \underline{\tilde{\sigma}}_t * W_t^{-1} \tag{2.28}$$

where the $*$ inverse

$$W_t^{-1}(\underline{x}) = (-1)^{|\underline{x}|/2} W_t(\underline{x}) \tag{2.29}$$

Observe that $\hat{\omega}(x) = \tilde{\sigma}(x)$ for $x \in \mathbb{Z}^d$. The proof of (2.29) is obtained by checking that for $|\underline{x}| > 0$

$$0 = \sum_{\underline{y} \subset \underline{x}} (-1)^{|\underline{y}|/2} W_t(\underline{y}) W_t(\underline{x} - \underline{y}) \tag{2.30}$$

We define

$$\omega_t(\underline{x}) = \mathbb{E}_\mu\left(\hat{\omega}_t(\underline{x})\right) \tag{2.31}$$

observing that

$$v_t = \omega_t * W_t \qquad \omega_t = v_t * W_t^{-1} \tag{2.32}$$

As we shall see in Theorem 2.6 the ω functions are much smaller than the v functions so that the leading term in the first equality in (2.32) is the one with only W_t, the other terms are corrections and are smaller the larger is the order (i.e. the number of sites) of ω_t. We next specify the assumptions needed in Theorem 2.6.

2.5. Definition. *(The assumptions).*

Given $t \ge s \ge 0$, we denote by $U_{t;s}$ the flow on $M(S)$ defined as it follows. Let $f \in M(S)$, $f_{s'}$, $s \le s' \le t$ the solution of

$$\frac{df_{s'}}{ds'} = K_{s'} f_{s'}, \qquad f_s = f \tag{2.33}$$

Then $U_{t;s} f = f_t$. We also denote by $U_{t;s}(\underline{x}, \underline{y})$ the matrix elements of $U_{t;s}$, in agreement with Definition 2.3, and by $U_{t;s}^{(n)}$ the restriction of $U_{t;s}$ to $M(S^{(n)})$.

We suppose that there are increasing functions a_t, b_t, ($b_t \ge ct$ for some c), both larger than 1, and coefficients c'_n, so that, for all $t \ge s$

$$\|U_{t;s}^{(n)}\|_\infty \le c'_n a_{t-s}^n, \qquad \|w_t\|_\infty \le c'_2 b_t \gamma^d \tag{2.34}$$

and that

$$b_t^{-1} a_{t-s} b_s \le c'_0 \tag{2.35}$$

It is readily seen that the above assumptions are always satisfied by the choice

$$a_t = b_t = e^{Ct} \tag{2.36}$$

for a suitable C. In some cases however, other choices for a_t and b_t are available and more convenient.

2.6. Theorem.

Let a_t, b_t and c'_n as in the Assumptions 2.5; then there are $a > 0$, ℓ_n and c_n so that for all γ, all $t \le a \log \gamma^{-1}$, all $\underline{x}$ and all initial measures μ (that are product measures)

$$|\omega_t(\underline{x})| \le c_{|\underline{x}|} t^{\ell_{|\underline{x}|}} b_t^{|\underline{x}|} \gamma^{2d|\underline{x}|/3}, \quad \text{for all } |\underline{x}| \ge 3 \tag{2.37}$$

$$|\omega_t(\underline{x})| \le c_2 t^{\ell_2} b_t^5 \gamma^{2d}, \quad \text{for all } |\underline{x}| = 2 \tag{2.38}$$

$$|\omega_t(x)| \le c_1 t b_t^2 \gamma^d, \quad \text{for all } x \in \mathbb{Z}^d \tag{2.39}$$

By recalling (2.32) and (2.34) we then see that within the accuracy $(b_t\gamma^{d/2})^{|\underline{x}|}$, the leading contribution to $v_t(\underline{x})$ is given by $W_t(\underline{x})$.

This allows to go beyond [5] where a law of large numbers is proven, and to study the fluctuation fields that we prove in Theorem 2.8 to converge to a Ornstein-Uhlenbeck process. We prove convergence of all the moments and also the weak convergence of the process. The latter follows easily from Theorem 2.6 using the Holley-Stroock theory, [9].

For $\phi \in \mathcal{S}(\mathbb{R}^d)$, the Schwartz space of smooth functions with fast decay to infinity, we define the magnetic density field

$$X_t^\gamma(\phi) = \gamma^d \sum_{x\in\mathbb{Z}^d} \phi(\gamma x)\sigma_t(x) \tag{2.40}$$

By using the Chebishev inequality and (2.11), (we do a similar computation in Section 4), we can easily see that for any $\delta > 0$

$$\lim_{\gamma\to 0} \sup_{0\le t\le a\log\gamma^{-1}} \mathbb{P}^\gamma_{\mu^\gamma}\left(\left| X_t^\gamma(\phi) - \int dr\phi(r)m_t(r)\right| > \delta\right) = 0 \tag{2.41}$$

where $m_t(r)$ is the solution of (1.1) with initial condition $m_0(r)$, m_0 and μ^γ as in Definition 2.1. By using martingale techniques, see for instance [11], it is also possible to show that if $\mathbb{P}^\gamma$ is the law of the fields $\{X_t^\gamma(\phi)\}$ thought of as the canonical variables $\{X_t(\phi)\}$ in $D(\mathbb{R}_+, \mathcal{S}'(\mathbb{R}^d))$, then $\mathbb{P}^\gamma$ converges weakly on the compacts of $\mathbb{R}_+$ to $\mathbb{P}$, the law supported by the trajectory $t \to m_t(r)dr$.

We next consider the fluctuation fields

$$Y_t^\gamma(\phi) = \gamma^{d/2} \sum_{x\in\mathbb{Z}^d} \phi(\gamma x)[\sigma_t(x) - m_t^\gamma(x)] \tag{2.42}$$

The distribution of $\{\sigma_t\}$ in (2.42) is $\mathbb{P}^\gamma_{\mu^\gamma}$ with μ^γ as in Definition 2.1 and $m_t^\gamma(x)$ is the solution of (2.8) with initial condition $m_0(\gamma x)$.

We then consider the space $D(\mathbb{R}_+, \mathcal{S}'(\mathbb{R}^d))$ and its canonical variables

$$Y_t(\phi) = \int dr\phi(r)\xi(r,t)$$

where $\{\xi(r,t),\ r \in \mathbb{R}^d,\ t \ge 0\}$ are the elements of $D(\mathbb{R}_+, \mathcal{S}'(\mathbb{R}^d))$. We denote by $\hat{\mathbb{P}}^\gamma$ the law on $D(\mathbb{R}_+, \mathcal{S}'(\mathbb{R}^d))$ whose marginal on the variables $Y_t(\phi)$ is the same as the distribution of the variables $Y_t^\gamma(\phi)$ induced by $\mathbb{P}^\gamma_{\mu^\gamma}$. We finally let $\hat{\mathbb{E}}^\gamma$ denote the expectation with respect to $\hat{\mathbb{P}}^\gamma$.

2.7 Definition. *(The Ornstein- Uhlenbeck process)*

Let $\hat{\mathbb{P}}$ be the law of the Gaussian process on $D(\mathbb{R}_+, \mathcal{S}'(\mathbb{R}^d))$ with mean zero and covariance

$$\hat{\mathbb{E}}\big(Y_s(\phi)Y_t(\psi)\big) = C_{s,t}(\phi,\psi) \equiv \int drdr' C_{s,t}(r,r')\phi(r)\psi(r') \tag{2.43}$$

where, for any $0 \le s < t$:

$$C_{s,t}(r_1, r_2) = \int dr_2' e^{\mathcal{L}(t-s)}(r_2, r_2') C_{s,s}(r_1', r_2') \tag{2.44}$$

and for any $t \ge 0$

$$C_{t,t}(r, r') = (1 - m_t(r)^2)\delta(r - r') + C_t^0(r, r') \tag{2.45}$$

where

$$\mathcal{L}(r, r', t) = -\delta(r - r') + \mathcal{L}^0(r, r', t) \qquad \mathcal{L}^0(r, r', t) = \ell(r, t)\beta J(|r - r'|) \tag{2.46}$$

$$\ell(r, t) = \frac{1}{\cosh^2\{\beta(J * m_t)(r) + h\}} \tag{2.47}$$

and $C_t^0(r, r')$ *solves*

$$\begin{aligned} \frac{dC_t^0(r, r')}{dt} &= \int dr''[\mathcal{L}(r, r'', t)C_t^0(r'', r') + \mathcal{L}(r', r'', t)C_t^0(r, r'')] \\ &\quad + \mathcal{L}^0(r, r', t)[1 - m_t(r')^2] + \mathcal{L}^0(r, r', t)[1 - m_t(r)^2] \end{aligned} \tag{2.48}$$

$$C_0^0(r, r') \equiv 0$$

Observe that $\mathcal{L}(r, r', t)$ in (2.46) and $\ell(r, t)$ in (2.47) are the continuous version of $k_t(x, y, m)$ and $f_x^{(1)}(m)$ defined in (2.18) and (2.19).

It can be seen that $\hat{\mathbb{P}}$ is supported by the solutions $\xi(r, t)$ of the stochastic equation

$$\frac{d\xi(r, t)}{dt} = \int dr' \mathcal{L}(r, r', t)\xi(r', t) + \frac{dB(r, t)}{dt} \tag{2.49}$$

The noise $B(r, t)$ is Gaussian with mean zero and covariance

$$\hat{\mathbb{E}}\left(\frac{dB(r, t)}{dt}\frac{dB(r', t')}{dt'}\right) = \delta(t - t')\delta(r - r')2b(r, t) \tag{2.50}$$

where

$$b(r, t) = 1 - m_t(r)\tanh\{\beta[(J \star m_t)(r) + h]\} \tag{2.51}$$

The initial condition $\xi(r, 0)$ of (2.49) is a Gaussian process with mean 0 and covariance

$$C_{0,0}(r, r') = \delta(r, r')\big(1 - m_0(r)^2\big)$$

The distribution $\hat{\mathbb{P}}$ of the solution (2.49) is the Gaussian process with mean zero and covariances (2.44)-(2.45).

Observe that $\hat{\mathbb{P}}$ is a "generalized Ornstein-Uhlenbeck process" since only the variables $Y_t(\phi)$ have a meaning as real functions, the variables $\xi(r, t)$ being only defined as distributions.

2.8 Theorem. *With the above notation, $\hat{\mathbb{P}}^\gamma$ converges weakly to $\hat{\mathbb{P}}$ on $D([0,T],\mathcal{S}'(\mathbb{R}^d))$, for any given $T>0$. Also the $\hat{\mathbb{P}}^\gamma$-expectation of the product of any finite number of $Y_t(\phi)$'s converges to the corresponding $\hat{\mathbb{P}}$-expectation, for any fixed t.*

3. Proof of Theorem 2.6.

The proof starts from the computation of the derivative of $\omega_t(\underline{x})$:

$$\frac{d\omega_t(\underline{x})}{dt} = \mathbb{E}_{\mu^\gamma}\left(D_t^\star \hat{\omega}_t(\underline{x})\right) \tag{3.1}$$

where

$$D_t^\star = L_\gamma + \frac{\partial}{\partial t} \tag{3.2}$$

L_γ is the generator of the process, see (2.1), and $\partial/\partial t$ the derivative with repect to the time t in m_t and W_t.

$D_t^\star$ is a linear operator on V_t and the whole proof follows from a good characterization of its action. We first need some more notation:

3.1 Definition.

Given a linear operator $A^\star$ on V_t we define $A(\underline{x},\underline{y})$ so that

$$A^\star\hat{\omega}_t(\underline{x}) = \sum_{\underline{y}} A(\underline{x},\underline{y})\hat{\omega}_t(\underline{y}) \tag{3.3}$$

and the operator A on $M(S)$ as

$$(Af)(\underline{x}) = \sum_{\underline{y}} A(\underline{x},\underline{y})f(\underline{y}) \tag{3.4}$$

We call χ_n the projection onto $M(S^{(n)})$:

$$(\chi_n f)(\underline{x}) = \begin{cases} f(\underline{x}), & \text{if } |\underline{x}| = n \\ 0, & \text{otherwise} \end{cases} \tag{3.5}$$

and

$$\chi_{\le n} = \sum_{k\le n}\chi_k \tag{3.6}$$

We also define the operators $p^\pm_{t,\alpha}$ on $M(S)$ as

$$(p^\pm_{t,\alpha} f)(\underline{x}) = p^\pm_{t,\alpha}(|\underline{x}|) f(\underline{x}); \qquad p^\pm_{t,\alpha}(|\underline{x}|) = [\gamma^{2d/3-\alpha} b_t]^{\pm|\underline{x}|} \tag{3.7}$$

with either $\alpha = d/2$ or $\alpha = 2d/9$; b_t is the function introduced in (2.34).

We finally denote by $\|A\|_\infty$ the norm of the operator A on the space $M(S)$ equipped with sup norm and set

$$|A|_{t,\alpha} = \|p^-_{t,\alpha} A p^+_{t,\alpha}\|_\infty; \qquad |A|_{t,\alpha,n} = \|\chi_n p^-_{t,\alpha} A p^+_{t,\alpha}\|_\infty \tag{3.8}$$

3.2 Proposition.

Given any $N > 2$ there are two linear operators on V_t, $A^\star_t$ and $R^\star_t$, and coefficients $c(n)$ so that

$$D^\star_t = K^\star_t + A^\star_t + R^\star_t \tag{3.9}$$

with

$$K^\star_t \hat\omega_t(\underline{x}) = \sum_{\underline{y}} K_t(\underline{x}, \underline{y}) \hat\omega_t(\underline{y})$$

see (2.21);

$$|R^\star_t \hat\omega_t(\underline{x})| \le c(|\underline{x}|) \sum_{x \in \underline{x}} \left| J_\gamma \circ \tilde\sigma_t(x) \right|^N \tag{3.10}$$

Moreover $A^\star_t : V_t^{(n)} \to V_t^{(n+N)}$ and, recalling (3.8),

$$|A_t|_{t,\alpha,n} \le c(n)\gamma^\alpha, \qquad \|\chi_n A_t\|_\infty \le c(n) \tag{3.11}$$

Proof.

To have lighter notation we drop t from $\hat\omega_t$, $D^\star_t$, $K^\star_t$, $R^\star_t$,..... By the definition of $\hat\omega$, if $x \notin \underline{x}$:

$$\hat\omega(\underline{x} + x) = \tilde\sigma(x)\hat\omega(\underline{x}) - \sum_{y \in \underline{x}} \mathrm{w}(x, y)\hat\omega(\underline{x} - y) \tag{3.12}$$

Since

$$D^\star \tilde\sigma(x) = -\tilde\sigma(x) + f_x(\sigma) - f_x(m) \tag{3.13}$$

we get

$$D^\star \hat\omega(\underline{x}) = \sum_{x \in \underline{x}} \left\{ \hat\omega(\underline{x} - x)[f_x(\sigma) - f_x(m) - \tilde\sigma(x)] - \frac{1}{2} \sum_{y \in \underline{x} - x} \hat\omega(\underline{x} - x - y)\dot{\mathrm{w}}(x, y) \right\} \tag{3.14}$$

where $\dot{\mathrm{w}}(x, y)$ is the right hand side of (2.23). By taking Taylor's expansions, we get

$$f_x(\sigma) - f_x(m) = \sum_{k=1}^{N-1} \frac{1}{k!} f_x^{(k)}(m)[\beta J_\gamma \circ \tilde\sigma(x)]^k + r(x, \sigma, m, N)[\beta J_\gamma \circ \tilde\sigma(x)]^N \tag{3.15}$$

and identify

$$R^\star \hat\omega(\underline{x}) := \sum_{x\in\underline{x}} \hat\omega(\underline{x}-x) r(x,\sigma,m,N)[\beta J_\gamma \circ \tilde\sigma(x)]^N \tag{3.16}$$

Since $|\tilde\sigma(x)|\le 2$, by (2.34)

$$|\hat\omega(\underline{x})| \le c(|\underline{x}|), \qquad c(|\underline{x}|) \text{ a combinatorial factor}$$

hence (3.10).

We thus have

$$D^\star\hat\omega(\underline{x}) = \Omega_1+\Omega_2+R^\star\hat\omega(\underline{x}) \tag{3.17}$$

$$\Omega_1 = \sum_{x\in\underline{x}} \Big\{ \hat\omega(\underline{x}-x)\Big(-\tilde\sigma(x) + f_x^{(1)}(m)\beta J_\gamma\circ\tilde\sigma(x)\Big) - \frac12 \sum_{y\in\underline{x}-x} \hat\omega(\underline{x}-x-y)\dot{\mathrm{w}}(x,y)\Big\} \tag{3.18}$$

$$\Omega_2 = \sum_{x\in\underline{x}} \hat\omega(\underline{x}-x)\sum_{k=2}^{N-1}\frac{1}{k!} f_x^{(k)}(m)[\beta J_\gamma\circ\tilde\sigma(x)]^k =: \sum_{\underline{y}} \Omega_2(\underline{x},\underline{y})\hat\omega(\underline{y}) \tag{3.19}$$

We write

$$\Omega_1 = \Omega_{1,1}+\Omega_{1,2}+\Omega_{1,3} \tag{3.20}$$

with, see (2.18)

$$\begin{aligned}\Omega_{1,1} = \sum_{x\in\underline{x}} \Big\{ & f_x^{(1)}(m)\sum_{z\notin\underline{x}} \beta J_\gamma(x,z)\tilde\sigma(z)\hat\omega(\underline{x}-x) \\ & -\frac12 \sum_{y\in\underline{x}-x} \hat\omega(\underline{x}-x-y)\sum_{z\notin\underline{x}}\Big(k(x,z,m)\mathrm{w}(z,y)+k(y,z,m)\mathrm{w}(x,z)\Big)\Big\}\end{aligned} \tag{3.21}$$

$$\Omega_{1,2} = \sum_{x\in\underline{x}}\Big\{\sum_{y\in\underline{x}-x} \mathrm{w}(x,y)\hat\omega(\underline{x}-x-y) - \tilde\sigma(x)\hat\omega(\underline{x}-x)\Big\} \tag{3.22}$$

$$\begin{aligned}\Omega_{1,3} = \sum_{x\in\underline{x}}\sum_{y\in\underline{x}-x} \Big\{ & f_x^{(1)}(m)\beta J_\gamma(x,y)\tilde\sigma(y)\hat\omega(\underline{x}-x) - \frac12 \hat\omega(\underline{x}-x-y)\kappa(x,y) \\ & -\frac12 \sum_{z\in\underline{x}-x-y} \hat\omega(\underline{x}-x-y)[k(x,z,m)\mathrm{w}(z,y)+k(y,z,m)\mathrm{w}(x,z)]\Big\}\end{aligned} \tag{3.23}$$

Thus the first term, $\Omega_{1,1}$, takes into account the contribution from $J_\gamma\circ\tilde\sigma(x)$ and $\dot{\mathrm{w}}(x,y)$ due to the sites not in $\underline{x}$. The contribution to $\dot{\mathrm{w}}(x,y)$ from $k(x,x)$ and $k(y,y)$ is in $\Omega_{1,2}$, that also accounts for the first term on the right hand side of (3.18). The remaining terms are collected in $\Omega_{1,3}$.

We observe that the last term on the right hand side of (3.21) is equal to

$$-\sum_{x\in\underline{x}}\sum_{y\in\underline{x}-x}\sum_{z\notin\underline{x}} \hat\omega(\underline{x}-x-y)k(x,z,m)\mathrm{w}(z,y)$$

Thus, using (3.12) and (2.18) we have

$$\Omega_{1,1} = \sum_{x\in \underline{x}} \sum_{z\in \underline{x}-x} k(x,z,m)\hat\omega(\underline{x}+z-x)$$

By (3.12)

$$\Omega_{1,2} = -\sum_{x\in\underline{x}} \hat\omega(\underline{x})$$

hence

$$\Omega_{1,1}+\Omega_{1,2} = K^\star\hat\omega(\underline{x}) \tag{3.24}$$

We next consider $\Omega_{1,3}$. Let $x\in\underline{x}$ and $y\in\underline{x}-x$ be fixed, then, using (3.12),

$$\hat\omega(\underline{x}-x) = \tilde\sigma(y)\hat\omega(\underline{x}-x-y) - \sum_{z\in\underline{x}-x} \mathrm{w}(y,z)\hat\omega(\underline{x}-x-y-z) \tag{3.25}$$

Since

$$\tilde\sigma(y)^2 = 1-m(y)^2-2m(y)\tilde\sigma(y) \tag{3.26}$$

and recalling the definition of $\kappa(x,y)$, see (2.22), the second term on the right hand side of (3.23) is cancelled by a term contributing to the first one, precisely the one that comes from $1-m(y)^2$ in (3.25). Thus

$$\begin{aligned}\Omega_{1,3} = &-\sum_{x\in\underline{x}}\sum_{y\in\underline{x}-x}\Big\{k(x,y,m)2m(y)\tilde\sigma(y)\hat\omega(\underline{x}-x-y)\\ &+\sum_{z\in\underline{x}-x-y}\mathrm{w}(y,z)\Big(k(x,y,m)\tilde\sigma(y)\hat\omega(\underline{x}-x-y-z)+k(x,z,m)\hat\omega(\underline{x}-x-y)\Big)\Big\}\\ &=:\sum_{\underline{y}}\Omega_{1,3}(\underline{x},\underline{y})\hat\omega(\underline{y})\end{aligned} \tag{3.27}$$

We have proven so far that

$$D^\star\hat\omega = K^\star\hat\omega+\Omega_2+\Omega_{1,3}+R^\star\hat\omega \tag{3.28}$$

with $\Omega_{1,3}$ as in (3.27) and Ω_2 as in (3.19). We have also proved the bound (3.10) for $R^\star\hat\omega(\underline{x})$.

All the terms in Ω_2 and $\Omega_{1,3}$ are of the form $\tilde\sigma(x_1)\cdots\tilde\sigma(x_n)\hat\omega(\underline{z})$, for some $n\ge 1$, $x_1\dots x_n$ and $\underline{z}$. Multiplication by $\tilde\sigma(x)$ is a linear operator on V_t, identified by the matrix elements $S_x(\underline{x},\underline{y})$ via the identity

$$\tilde\sigma(x)\hat\omega(\underline{x}) = \sum_{\underline{y}} S_x(\underline{x},\underline{y})\hat\omega(\underline{y}) \tag{3.29}$$

Using (3.12) we obtain that for $x\notin\underline{x}$

$$\sum_{\underline{y}} S_x(\underline{x},\underline{y})\hat\omega(\underline{y}) = \hat\omega(\underline{x}+x)+\sum_{y\in\underline{x}} w(x,y)\hat\omega(\underline{x}-y) \tag{3.30}$$

The same equation can be used to find also the other ones. Let $x \in \underline{x}$ and denote by $\underline{y} = \underline{x} - x$, then

$$
\begin{aligned}
\tilde\sigma(x)\hat\omega(\underline{y}+x) &= \tilde\sigma(x)^2\hat\omega(\underline{y}) - \tilde\sigma(x)\sum_{y\in\underline{y}} \mathrm{w}(x,y)\hat\omega(\underline{y}-y) \\
&= (1-m(x)^2)\hat\omega(\underline{y}) - 2m(x)\tilde\sigma(x)\hat\omega(\underline{y}) - \sum_{y\in\underline{y}} \mathrm{w}(x,y)\tilde\sigma(x)\hat\omega(\underline{y}-y)
\end{aligned}
\tag{3.31}
$$

Thus for $x \notin \underline{x}$, $|\underline{x}| = n$ from (3.30) and (2.34) we get,

$$
\begin{aligned}
p^-_{t,\alpha}(|\underline{x}|)\sum_{\underline{y}} |S_x(\underline{x},\underline{y})| p^+_{t,\alpha}(|\underline{y}|) &\le b_t\gamma^{2d/3-\alpha} + n\gamma^{-(2d/3-\alpha)}b_t^{-1}c_2' b_t\gamma^d \\
&\le A_{\gamma,n} := c_1 n\gamma^{d/3+\alpha/2}b_t
\end{aligned}
\tag{3.32}
$$

because $\alpha = 0$ or $\alpha = 2d/9$ and

$$
\gamma^{2d/3-\alpha} \le \gamma^{d/3+\alpha/2}
$$

for all $t \le a\log\gamma^{-1}$ and all γ small enough.

For $x \in \underline{x}$ we use (3.31) and we observe that we can apply the same estimates as before for the second and third term on the right hand side of (3.31). Then from (2.34) we get,

$$
\begin{aligned}
p^-_{t,\alpha}(|\underline{x}|)\sum_{\underline{y}} |S_x(\underline{x},\underline{y})| p^+_{t,\alpha}(|\underline{y}|) &\le \gamma^{-(2d/3-\alpha)}b_t^{-1} + 2\gamma^{-(2d/3-\alpha)}b_t^{-1}A_{\gamma,n} \\
&\quad + n[\gamma^{-(2d/3-\alpha)}b_t^{-1}]^2 c_2' b_t\gamma^d A_{\gamma,n} \ \le c_2 n^2\gamma^{-(2d/3-\alpha)}
\end{aligned}
\tag{3.33}
$$

Recalling (3.27) and using (3.32) we have

$$
\begin{aligned}
p^-_{t,\alpha}(|\underline{x}|)\sum_{\underline{y}} |\Omega_{1,3}(\underline{x},\underline{y})| p^+_{t,\alpha}(|\underline{y}|) &\le c\Big([\gamma^{-(2d/3-\alpha)}b_t^{-1}]^2\gamma^{d/3+\alpha/2}\gamma^d b_t \\
&\quad + [\gamma^{-(2d/3-\alpha)}b_t^{-1}]^3\gamma^{d/3+\alpha/2}\gamma^{2d}b_t + [\gamma^{-(2d/3-\alpha)}b_t^{-1}]^2\gamma^{2d}b_t \Big)
\end{aligned}
\tag{3.34}
$$

where c is proportional to $|\underline{x}|^3$. We thus have

$$
p^-_{t,\alpha}(|\underline{x}|)\sum_{\underline{y}} |\Omega_{1,3}(\underline{x},\underline{y})| p^+_{t,\alpha}(|\underline{y}|) \le c\Big\{\gamma^{5\alpha/2}b_t^{-1} + \gamma^{d/3+7\alpha/2}b_t^{-2} + \gamma^{2d/3+2\alpha}b_t^{-1}\Big\}
\tag{3.35}
$$

We next observe that from (3.32) and (3.33), given $x \in \underline{x}$,

$$
\begin{aligned}
\sum_y |J_\gamma(x,y)| p^-_{t,\alpha}(|\underline{x}|)\sum_{\underline{y}} |S_y(\underline{x},\underline{y})| p^+_{t,\alpha}(|\underline{y}|) &\le c_1' A_{\gamma,n} + \sum_{y\in\underline{x}} |J_\gamma(x,y)| p^-_{t,\alpha}(|\underline{x}|)\sum_{\underline{y}} |S_y(\underline{x},\underline{y})| p^+_{t,\alpha}(|\underline{y}|) \\
&\le c_1' A_{\gamma,n} + c_2' n^3\gamma^d\gamma^{-(2d/3-\alpha)} \le c\gamma^{d/3+\alpha/2}
\end{aligned}
\tag{3.36}
$$

Hence, recalling (3.19):

$$p^-_{t,\alpha}(|\underline{x}|)\sum_{\underline{y}}|\Omega_2(\underline{x},\underline{y})|p^+_{t,\alpha}(|\underline{y}|)\le c\max_{2\le k\le N}[\gamma^{-(2d/3-\alpha)}b_t^{-1}][\gamma^{d/3+\alpha/2}]^k\le c\gamma^{2\alpha}b_t^{-1} \tag{3.37}$$

We have thus concluded the proof of Proposition 3.2. $\square$

Proof of (2.37)**.**

Using Proposition 3.2 we rewrite (3.1) as follows. Let $\omega_t=\{\omega_t(\underline{x}),\underline{x}\in S\}$ in $M(S)$, $\omega_t(\emptyset)\equiv 1$. Then

$$\frac{d\omega_t}{dt}=K_t\omega_t+A_t\omega_t+R_t \tag{3.38}$$

$$R_t(\underline{x})=\mathbb{E}_{\mu^\gamma}\left(R_t^\star\hat{\omega}_t(\underline{x})\right) \tag{3.39}$$

K_t as in Definition 2.3 with $K^{(0)}\equiv 0$; A_t is related to $A_t^\star$ by Definition 3.1, with $A_t(\emptyset,\underline{y})\equiv 0$.

Both A_t and R_t depend on N, N will be specified later in terms of the value $n=|\underline{x}|$ for which we wish to prove (2.37). Recalling that $I(\emptyset)=1$ and $I(\underline{x})=0$, $|\underline{x}|>0$, we have

$$\omega_t=I+\int_0^t dsU_{t,s}\big(A_s\omega_s+R_s\big) \tag{3.40}$$

because $\omega_0(\underline{x})\equiv 0$, $|\underline{x}|>0$; $U_{t,s}$ is defined in Definition 2.5.

Given $h^\star>n$ we call

$$\chi^\star=\chi_{\le h^\star};\quad \omega_t^-=\chi^\star\omega_t,\quad \omega_t^+=(1-\chi^\star)\omega_t \tag{3.41}$$

We then have

$$\omega_t^-=I+\int_0^t dsU_{t,s}\chi^\star\big(A_s\omega_s^-+A_s\omega_s^++R_s\big) \tag{3.42}$$

After H iterations we get

$$\begin{aligned}\omega_t^-&=I+\int_0^t dsU_{t,s}\chi^\star[A_s\omega_s^++R_s]+\sum_{k=1}^{H-1}\{\Gamma_k(t)+\Lambda_k(t)+\Delta_k(t)\}+\Delta(t)\\&=\sum_{k=0}^{H-1}\{\Gamma_k(t)+\Lambda_k(t)+\Delta_k(t)\}+\Delta(t)\end{aligned} \tag{3.43}$$

where

$$\Gamma_k(t)=\int_0^t ds_1..\int_0^{s_{k-1}}ds_kU_{t,s_1}A^\star_{s_1}..U_{s_{k-1},s_k}A^\star_{s_k}I,\quad A_t^\star=\chi^\star A_t \tag{3.44}$$

$$\Lambda_k(t)=\int_0^t ds_1..\int_0^{s_{k-1}}ds_kU_{t,s_1}A^\star_{s_1}..U_{s_{k-1},s_k}A^\star_{s_k}[\int_0^{s_k}dsU_{s_k,s}\chi^\star R_s] \tag{3.45}$$

$$\Delta_k(t)=\int_0^t ds_1..\int_0^{s_{k-1}}ds_kU_{t,s_1}A^\star_{s_1}..U_{s_{k-1},s_k}A^\star_{s_k}[\int_0^{s_k}dsU_{s_k,s}A^\star_s\omega_s^+] \tag{3.46}$$

$$\Delta(t) = \int_0^t ds_1 .. \int_0^{s_{H-1}} ds_H U_{t,s_1} A^\star_{s_1} .. U_{s_{H-1},s_H} A^\star_{s_H} \omega^-_{s_H} \tag{3.47}$$

We fix n and we need to bound $|\omega_t(\underline{x})|$ with $|\underline{x}| = n$. Observe that

$$\omega_t(\underline{x}) = \omega^-_t(\underline{x}) \qquad \text{for all } \underline{x} \text{ such that } |\underline{x}| = n \tag{3.48}$$

We start by bounding from $\Gamma_k(t)$. We write

$$\chi_n \Gamma_k(t) = \int_0^t ds_1 .. \int_0^{s_{k-1}} ds_k \chi_n U_{t,s_1} A^\star_{s_1} \chi_{\le(n+N)} .. \chi_{\le(n+(k-1)N)} U_{s_{k-1},s_k} A^\star_{s_k} I \tag{3.49}$$

because $A^\star_s : V^{(j)}_s \to V^{(j+N)}_s$, by Proposition 3.2. Recalling (3.7), we write for $\alpha = 0$ and $\alpha = 2d/9$:

$$\tilde U_{t,s} = p^-_{t,\alpha} U_{t,s} p^+_{s,\alpha}; \qquad A^{(\alpha)}_t = p^-_{t,\alpha} A^\star_t p^+_{t,\alpha} \tag{3.50}$$

Observe that $\tilde U_{t,s}$ does not depend on α.

We next choose $\alpha = 0$ and, recalling that $p^\pm_t(\emptyset) = 1$, we get

$$\chi_n \Gamma_k(t) = p^+_{t,0}(n) \int_0^t ds_1 .. \int_0^{s_{k-1}} ds_k \chi_n \tilde U_{t,s_1} A^{(0)}_{s_1} \chi_{\le(n+N)} .. \chi_{\le(n+(k-1)N)} \tilde U_{s_{k-1},s_k} A^{(0)}_{s_k} I \tag{3.51}$$

By (2.34), (2.35) and Proposition 3.2 we have

$$\|\chi_\ell \tilde U_{t,s}\|_\infty \le c'_\ell a^\ell_{t-s} b^{-\ell}_t b^\ell_s \le c'_\ell (c'_0)^\ell, \qquad \|\chi_\ell A^{(0)}_t\|_\infty \le c(\ell) \tag{3.52}$$

Hence there is a constant $c_0(n, h^\star, N, H)$ so that for $k \le H-1$

$$\|\chi_n \Gamma_k(t)\|_\infty \le c_0(n, h^\star, N, H) [b_t \gamma^{2d/3}]^n \frac{t^k}{k!} \tag{3.53}$$

From (3.45) we get

$$\begin{aligned} \|\chi_n \Lambda_k(t)\|_\infty = p^+_{t,\alpha}(n) \int_0^t ds_1 .. \int_0^{s_{k-1}} ds_k \chi_n \tilde U_{t,s_1} A^{(\alpha)}_{s_1} \chi_{\le(n+N)} .. \\ .. \chi_{\le(n+(k-1)N)} \tilde U_{s_{k-1},s_k} A^{(\alpha)}_{s_k} \int_0^{s_k} ds \tilde U_{s_k,s} p^-_{s,\alpha} \chi^\star R_s \end{aligned} \tag{3.54}$$

By Proposition 3.2 we have

$$\|\chi_\ell A^{(\alpha)}_t\|_\infty \le c(\ell) \gamma^\alpha \tag{3.55}$$

Then there is a constant $c_1 \equiv c_1(n, h^\star, N, H)$ so that

$$\begin{aligned} \|\chi_n \Lambda_k(t)\|_\infty &\le c_1 b^n_t \gamma^{(2d/3-\alpha)n} \gamma^{\alpha k} \frac{t^{k+1}}{(k+1)!} \gamma^{-(2d/3-\alpha)h^\star} \sup_{s \le t} \sup_x \mathbb{E}_{\mu^\gamma} \left(|J_\gamma \circ \tilde\sigma(x)|^N \right) \\ &\le c_1 [b_t \gamma^{(2d/3)}]^n \gamma^{\alpha(h^\star - n)} \gamma^{\alpha k} \frac{t^{k+1}}{(k+1)!} \gamma^{-(2d/3)h^\star} \sup_{s \le t} \sup_x \mathbb{E}_{\mu^\gamma} \left(|J_\gamma \circ \tilde\sigma(x)|^N \right) \end{aligned} \tag{3.56}$$

By (2.11), for every even N there is c so that

$$\mathbb{E}_{\mu^\gamma}\left(|J_\gamma \circ \tilde\sigma(x)|^N\right) \le c[e^{Ct}\gamma^{d/2}]^N \tag{3.57}$$

(3.57) is obtained by expanding the product on the left hand side. Terms with all sites distinct give rise to a v function that is bounded using (2.11). This behaves as the right hand side of (3.57). All the other terms are smaller, as there is an extra factor J_γ whenever two sites are equal (and possibly two sites less in the v function).

Since $h^\star > n$, $t \le a\log\gamma^{-1}$ and $b_t \le e^{Ct}$ we have for a small enough and for a suitable constant $c_2 \equiv c_2(n, h^\star, N, H)$,

$$\|\chi_n \Lambda_k(t)\|_\infty \le c_2[b_t\gamma^{(2d/3)}]^n \gamma^{-(2d/3)h^\star}\gamma^{(d/2-aC)N} \tag{3.58}$$

Analogously,

$$\begin{aligned}\chi_n\Delta_k(t) = p^+_{t,\alpha}(n)\int_0^t ds_1 .. \int_0^{s_{k-1}} ds_k \chi_n \tilde U_{t,s_1} A^{(\alpha)}_{s_1}\chi_{\le(n+N)} .. \chi_{\le(n+(k-1)N)}\tilde U_{s_{k-1},s_k} A^{(\alpha)}_{s_k}\\ \times\int_0^{s_k} ds \tilde U_{s_k,s} A^{(\alpha)}_s p^-_{s,\alpha}\omega^+_s\end{aligned} \tag{3.59}$$

By Theorem 2.1

$$\sup_{s\le a\log\gamma^{-1}}\ \sup_{|\underline{x}|=h} |\omega_s(\underline{x})| \le c'_h \gamma^{(d/2-aC)h} \tag{3.60}$$

where c'_h is a suitable constant. Hence there is a constant $c_3 \equiv c_3(n, h^\star, N, H)$, such that for $\alpha = 2d/9$ and $t \le a\log\gamma^{-1}$,

$$\begin{aligned}\|\chi_n\Delta_k(t)\|_\infty &\le c_3 b_t^n \gamma^{(2d/3-\alpha)n}\gamma^{\alpha k}\frac{t^{k+1}}{(k+1)!}\sup_{h^\star\le h\le h^\star+N} c'_h\ \gamma^{[(d/2-aC)-(2d/3-\alpha)]h}\\ &\le c_3[b_t\gamma^{2d/3}]^n\gamma^{-2dn/9}\gamma^{[(d/18)-aC)]h^\star}\end{aligned} \tag{3.61}$$

We bound $\Delta(t)$ in an analogous way, getting the existence constants $c_4 \equiv c_4(n, h^\star, N, H)$ and $c'_4 \equiv c'_4(n, h^\star, N, H)$, such that for all $t \le a\log\gamma^{-1}$,

$$\begin{aligned}\|\chi_n\Delta(t)\|_\infty &\le c_4 b_t^n\gamma^{(2d/3-\alpha)n}\gamma^{\alpha H}\frac{t^H}{H!}\gamma^{-(2d/3-\alpha)h^\star}\\ &\le c'_4[b_t\gamma^{2d/3}]^n\gamma^{\alpha(h^\star-n)}\left(\gamma^\alpha\log\gamma^{-1}\right)^H\gamma^{-(2d/3)h^\star}\end{aligned} \tag{3.62}$$

We first chose $h^\star > n$ so that (see (3.61))

$$\gamma^{-2dn/9}\gamma^{[(d/18)-aC]h^\star} \le 1 \tag{3.63}$$

Then we chose H and N so that (see (3.62) and (3.58) respectively)

$$\left(\gamma^\alpha\log\gamma^{-1}\right)^H\gamma^{-(2d/3)h^\star} \le 1, \qquad \gamma^{(d/2)-aC)N}\gamma^{-(2d/3)h^\star} \le 1 \tag{3.64}$$

From (3.43), (3.53), (3.58), (3.61), (3.62) and the choice of $h^\star$, H and N, we finally get that there is a constant $\bar c$ so that

$$\sup_{|\underline{x}|=n} |\omega_t^-(\underline{x})| \le \bar c\big\{[b_t\gamma^{2d/3}]^n t^H + [b_t\gamma^{2d/3}]^n\big\} \tag{3.65}$$

The inequality (2.37) is therefore proven.

Proof of (2.38)-(2.39).

We go back to (3.9) recalling that the action of $A_t^\star$ is given by $\Omega_2 + \Omega_{1,3}$, see (3.28). Let $\underline{x} = (x_1, x_2)$ in (3.19), then, for a suitable constant c,

$$|\Omega_2(\underline{x}, \emptyset)| \le c[b_t\gamma^d]^2 \tag{3.66}$$

as it only arises from values of $k \ge 3$ in (3.19) and the largest contribution comes from the product of two w_t's for which we use (2.34). Analogously we have, for a suitable constant c,

$$\sum_y |\Omega_2(\underline{x}, y)| \le c[b_t\gamma^d], \qquad \sum_{|\underline{y}|=2} |\Omega_2(\underline{x}, \underline{y})| \le c[b_t\gamma^d], \qquad \sum_{|\underline{y}|\ge 3} |\Omega_2(\underline{x}, \underline{y})| \le c \tag{3.67}$$

When $|\underline{x}| = 1$ we get

$$|\Omega_2(x, \emptyset)| \le c[b_t\gamma^d], \qquad \sum_y |\Omega_2(x, y)| \le c[b_t\gamma^d] \tag{3.68}$$

$$\sum_{|\underline{y}|=2} |\Omega_2(x, \underline{y})| \le c, \qquad \sum_{|\underline{y}|\ge 3} |\Omega_2(x, \underline{y})| \le c \tag{3.69}$$

From (3.27) we get $\Omega_{1,3}(\underline{x}) = 0$ if $|\underline{x}| = 1$, while, if $|\underline{x}| = 2$,

$$|\Omega_{1,3}(\underline{x}, \emptyset)| \le c\gamma^d b_t\gamma^d \tag{3.70}$$

(the first factor γ^d comes from $k(x, z, m)$ the other factor from bounding w_t in the second term on the right hand side of (3.27)).

$$\sum_y |\Omega_{1,3}(\underline{x}, y)| \le c\gamma^d \tag{3.71}$$

We call

$$\psi_t = \sup_{|\underline{x}|=2}\sup_{s\le t} |\omega_s(\underline{x})|; \qquad \phi_t = \sup_{|\underline{x}|=1}\sup_{s\le t} |\omega_s(\underline{x})| \tag{3.72}$$

We then get

$$\psi_t \le \int_0^t ds |U_{t,s}\chi_2\big(A_s\omega_s + R_s\big)| \qquad \phi_t \le \int_0^t ds |U_{t,s}\chi_1\big(A_s\omega_s + R_s\big)| \tag{3.73}$$

We use (2.35), (3.16) and (3.57) to conclude that

$$\|U_{t,s}\chi_2\|_\infty \le (b_t c_0')^2, \qquad \|U_{t,s}\chi_1\|_\infty \le b_t c_0', \qquad \sup_{s\le a\log\gamma^{-1}} \|R_s\|_\infty \le \bar c\big(\gamma^{(d/2)-a}\big)^N \tag{3.74}$$

Furthermore using (2.37) that has been already proven, we have from (3.67) that there is $\bar{c}$ so that for all $s \le a \log \gamma^{-1}$

$$\sum_{|\underline{y}|\ge 3} |\Omega_2(\underline{x},\underline{y})||\omega_s(\underline{y})| \le N \sup_{3\le k\le N} s^{\ell_k} b_s^k \gamma^{2dk/3} \le \bar{c}\, s^{\ell_3} b_s^3 \gamma^{2d} \tag{3.75}$$

From all this we then get that, for a suitable constant c,

$$\psi_t \le \int_0^t ds (b_t c_0')^2 c \Big\{ [(b_s\gamma^d)^2 + b_s\gamma^d\phi_s + b_s\gamma^d\psi_s + s^{\ell_3} b_s^3\gamma^{2d}] + [b_s\gamma^{2d} + \gamma^d\phi_s] \Big\} \tag{3.76}$$

Analogously

$$\phi_t \le \int_0^t ds (b_t c_0') c \Big\{ b_s\gamma^d + b_s\gamma^d\phi_s + \psi_s + s^{\ell_3} b_s^3 \gamma^{2d} \Big\} \tag{3.77}$$

Hence, for a suitable c',

$$\psi_t \le c' t b_t^2 \Big\{ (b_t\gamma^d)^2 + b_t\gamma^d\phi_t + t^{\ell_3} b_t^3 \gamma^{2d} \Big\} \tag{3.78}$$

$$\phi_t \le c' t b_t \Big\{ b_t\gamma^d + \psi_t + t^{\ell_3} b_t^3\gamma^{2d} \Big\} \tag{3.79}$$

It then follows that, for a suitable c'', and $\ell_3 \ge 1$

$$\psi_t \le c'' t^{\ell_3+1} b_t^5 \gamma^{2d}; \qquad \phi_t \le c'' t b_t^2 \gamma^d \tag{3.80}$$

Theorem 2.6 is thus proven. □

4. Fluctuations.

In this Section we prove Theorem 2.8. The key ingredients of the proof are the bounds on the ω functions. We start with a Lemma, which follows quite easily from Theorem 2.6. We write explicitly below the dependence on γ of the v functions and ω functions .

4.1. Lemma.

There are $a > 0$ and, for any odd integer n, c_n, $n \ge 1$, and $\bar{C}$ so that for all initial measures μ (that are product measures), for all $t \le a \log \gamma^{-1}$ and for all $\underline{x} \in S$, $|\underline{x}| = n$,

$$|v_t^\gamma(\underline{x})| \le c_{|\underline{x}|} e^{\bar{C}|\underline{x}|t} \gamma^{d(|\underline{x}|+1)/2} \tag{4.1}$$

where $|\underline{x}|$ denotes the cardinality of $\underline{x}$.

Proof. Using the first equality in (2.32) we write

$$v_t^\gamma(\underline{x}) = \sum_{\underline{y}} \omega_t^\gamma(\underline{x}-\underline{y}) W_t^\gamma(\underline{y})$$

We recall that $W_t^\gamma(\underline{y}) = 0$ if $|\underline{y}|$ is odd, so that in all terms there is a ω function. (4.1) then follows from (2.27), (2.37) and (2.39). $\square$

Recalling the notation below (2.42) and the Definition 2.7 we first prove in the next lemma the convergence of the moments of $Y_t^\gamma(\phi)$, thus proving the last statement of Theorem 2.8.

4.2 Lemma.

Let $n \geq 2$ and $\phi_1, \dots, \phi_n$ in $\mathcal{S}(\mathbb{R}^d)$. Then, for any $t > 0$

$$\lim_{\gamma \to 0} \hat{\mathbb{E}}^\gamma \Big(\prod_{i=1}^n Y_t^\gamma(\phi_i) \Big) = \hat{\mathbb{E}} \Big(\prod_{i=1}^n Y_t(\phi_i) \Big)$$

Proof.

Let $n \geq 2$ and let $\underline{x} \in \S^{(n)}$, $\underline{x} = (x_1, \dots, x_n)$. We define a partition of the set $\{1, \dots, n\}$ by saying that i and j are in the same atom of the partition if and only if $x_i = x_j$. We call $h(\underline{x})$ the number of atoms of the partition so that

$$\{1, \dots, n\} = \bigcup_{k=1}^{h(\underline{x})} A_k, \qquad A_k \subset \{1, \dots, n\} \qquad A_k \cap A_h = \emptyset \text{ if } k \neq h \tag{4.2}$$

We also denote by $f(\underline{x})$ the number of atoms A_k such that $|A_k| = 1$. We suppose to have ordered the atoms of the partition in such a way that the first A_ℓ, $\ell \leq f(\underline{x})$ are singletons, i.e.

$$A_\ell = \{i_\ell\}, \qquad \ell = 1, \dots, f(\underline{x}) \tag{4.3}$$

We then have

$$n \geq h(\underline{x}) \geq f(\underline{x}) \geq 0, \qquad h(\underline{x}) \leq f(\underline{x}) + \frac{n - f(\underline{x})}{2} \tag{4.4}$$

and also

$$\prod_{i=1}^n \tilde{\sigma}_t(x_i) = \prod_{\ell=1}^{f(\underline{x})} \tilde{\sigma}_t(x_{i_\ell}) \prod_{k=f(\underline{x})+1}^{h(\underline{x})} \tilde{\sigma}_t(x_{i_k})^{|A_k|} \qquad i_k := \min\{i : i \in A_k\} \tag{4.5}$$

We now observe that for any $k \geq 1$

$$\begin{aligned}
\tilde{\sigma}_t(x)^k &= \frac{1 + \sigma(x)}{2}[1 - m_t^\gamma(x)]^k + \frac{1 - \sigma(x)}{2}[-1 - m_t^\gamma(x)]^k \\
&= \tilde{\sigma}_t(x) \left\{ \frac{[1 - m_t^\gamma(x)]^k}{2} - \frac{[-1 - m_t^\gamma(x)]^k}{2} \right\} \\
&+ \frac{1 + m_t^\gamma(x)}{2}[1 - m_t^\gamma(x)]^k + \frac{1 - m_t^\gamma(x)}{2}[-1 - m_t^\gamma(x)]^k \\
&= \tilde{\sigma}_t(x) a(k, m_t^\gamma(x)) + b(k, m_t^\gamma(x))
\end{aligned} \tag{4.6}$$

with a and b defined by the last equality. Thus

$$a(1, m_t^\gamma(x)) = 1, \quad b(1, m_t^\gamma(x)) = 0, \quad a(2, m_t^\gamma(x)) = -2m_t^\gamma(x), \quad b(2, m_t^\gamma(x)) = 1 - m_t^\gamma(x)^2 \tag{4.7}$$

and there are constants c_k so that

$$|a(k, m_t^\gamma(x))| \le c_k, \qquad |b(k, m_t^\gamma(x))| \le c_k \tag{4.8}$$

Since

$$\hat{\mathbb{E}}^\gamma\Big(\prod_{i=1}^n Y_t^\gamma(\phi_i)\Big) = \gamma^{dn/2} \sum_{\underline{x}\in S^{(n)}} \Big(\prod_{i=1}^n \phi_i(\gamma x_i)\Big) \mathbb{E}^\gamma_{\mu^\gamma}\Big(\prod_{i=1}^n \tilde\sigma_t(x_i)\Big) \tag{4.9}$$

Denoting by $[(f+1)/2]$ the integer part of $(f+1)/2$, we then have

$$\begin{aligned}
&\Big|\gamma^{dn/2} \sum_{\underline{x}\in S^{(n)}} \Big(\prod_{i=1}^n \phi_i(\gamma x_i)\Big) \mathbb{E}^\gamma_{\mu^\gamma}\Big(\prod_{i=1}^n \tilde\sigma_t(x_i)\Big)\Big| \\
&\qquad \le c\gamma^{dn/2} \sum_{h,f}^{\star} \sum_{\underline{x}\in S^{(n)}} \mathbf{1}(h(\underline{x}) = h, f(\underline{x}) = f)\Big(\prod_{i=1}^n |\phi_i(\gamma x_i)|\Big)\gamma^{d[(f+1)/2]} \\
&\qquad \le c'\gamma^{dn/2} \sum_{h,f}^{\star} \gamma^{-dh}\gamma^{d[(f+1)/2]}
\end{aligned} \tag{4.10}$$

where the stared sum is restricted to the non negative integers h and f which satisfy (4.4). We have applied (2.11) when f is even, observing that the integer part of $(f+1)/2$, denoted by $[(f+1)/2]$, is equal to $f/2$. We instead use (4.1) when f is odd, in which case $[(f+1)/2] = (f+1)/2$. By (4.4), the right hand side of (4.10) is uniformly bounded in γ and it vanishes in the limit $\gamma \to 0$, unless f is even and $h = (f+n)/2$. This is possible only when also n is even, we shall therefore restrict henceforth to this case.

We thus have

$$\hat{\mathbb{E}}^\gamma\Big(\prod_{i=1}^{2n} Y_t^\gamma(\phi_i)\Big) = G_\gamma + R_1^\gamma \tag{4.11}$$

where

$$\begin{aligned}
G_\gamma = \gamma^{dn} \sum_{k=0}^n \sum_{\substack{I\subset\{1,..2n\}\\ |I|=2k}} \sum_{\underline{z}\in S_{\neq}^{(2(n-k)}} \Big(\prod_{i\notin I} \phi_i(\gamma z_i)\Big) v_t^\gamma(\underline{z}) \\
\times \sum_{\{(i_1,j_1)..(i_k,j_k)\}} \sum_{\substack{\underline{x}\in S_{\neq}^{(k)}\\ \underline{x}\cap\underline{z}=\emptyset}} \Big(\prod_{\ell=1}^k \phi_{i_\ell}(\gamma x_\ell)\phi_{j_\ell}(\gamma x_\ell)[1 - m_t^\gamma(x_\ell)^2]\Big)
\end{aligned} \tag{4.12}$$

where $S_{\neq}^{(i)}$ is the subset of $S^{(i)}$ with configurations with sites different from each other and therefore $\underline{z} = \{z_i,\ i \notin I\}$ is a configuration with $2(n-k)$ sites different from each other. The sum over $(i_1, j_1)..(i_k, j_k)$ is over all the partitions of I into k elements (atoms), each with two indices. The remainder R_1^γ can be estimated as in (4.10) and vanishes when $\gamma \to 0$. By (2.32) we have

$$v_t^\gamma(\underline{y}) = W_t^\gamma(\underline{y}) + \sum_{\emptyset\neq\underline{z}\subset\underline{y}} \omega_t^\gamma(\underline{z})W_t^\gamma(\underline{y} - \underline{z}) \tag{4.13}$$

Then by Theorem 2.6, for any positive p,

$$\lim_{\gamma\to 0}\gamma^{-dp}\sup_{\underline{y}\in S_{\neq}^{(2p)}}\left|v_t^\gamma(\underline{y})-W_t^\gamma(\underline{y})\right|=0 \tag{4.14}$$

We then substitute in (4.12) the following

$$v_t^\gamma(\underline{z})=W_t^\gamma(\underline{z})+[v_t^\gamma(\underline{z})-W_t^\gamma(\underline{z})] \tag{4.15}$$

and we obtain

$$G_\gamma=G'_\gamma+R_2^\gamma$$

where

$$\begin{aligned} G'_\gamma=&\sum_{k=0}^{n}\sum_{\substack{I\subset\{1,..2n\}\\ |I|=2k}}\gamma^{2d(n-k)}\sum_{\underline{z}\in S_{\neq}^{(2(n-k))}}\Big(\prod_{i\notin I}\phi_i(\gamma z_i)\Big)\gamma^{-d(n-k)}W_t(\underline{z})\\ &\times\sum_{\{(i_1,j_1)..(i_k,j_k)\}}\gamma^{dk}\sum_{\substack{\underline{x}\in S_{\neq}^{(k)}\\ \underline{x}\cap\underline{z}=\emptyset}}\Big(\prod_{\ell=1}^{k}\phi_{i_\ell}(\gamma x_\ell)\phi_{j_\ell}(\gamma x_\ell)[1-m_t^\gamma(x_\ell)^2]\Big)\end{aligned} \tag{4.16}$$

and R_2^γ vanishes in the limit $\gamma\to 0$. From (2.8) and the choice of μ^γ in Definition 2.1 we have

$$\lim_{\gamma\to 0}\sup_{x\in\mathbb{Z}^d}|m_t^\gamma(x)-m_t(\gamma x)|=0 \qquad \text{for all } t\geq 0 \tag{4.17}$$

where m_t solves (1.1) with initial condition m_0.

We finally observe that from (4.17), (2.23) and (2.48) it follows that

$$\lim_{\gamma\to 0}\sup_{x,y\in\mathbb{Z}^d}|\gamma^{-d}w_t(x,y)-C_t^0(\gamma x,\gamma y)|=0 \qquad \text{for all } t\geq 0 \tag{4.18}$$

namely that $C_t^0(r,r')$ in (2.48) is the continuous version, suitable normalized, of $w_t(x,y)$.

From (2.45), (4.17) and (4.18) we then have that

$$\lim_{\gamma\to 0}|G'_\gamma-\hat{\mathbb{E}}\Big(\prod_{i=1}^{2n}Y_t(\phi_i)\Big)|=0 \tag{4.19}$$

The Lemma is thus proven $\square$

To complete the proof of Theorem 2.8 we need to prove the convergence of the fluctuation fields as a process on $D([0,T],\mathcal{S}'(\mathbb{R}^d))$. This is done following criteria which are by now standard in this field. Their applicability is granted in our case by the bounds proven on the v and ω functions. We give below some details and references.

Tightness.

We first prove tightness of $\hat{\mathbb{P}}^\gamma$ on $D([0,T],\mathcal{S}'(\mathbb{R}^d))$. By Mitoma's theorem, [10], it is enough to show that the marginal of $\hat{\mathbb{P}}^\gamma$ on $\{Y_t(\phi),\ t\in[0,T]\}$ is tight. For this we use the criterion stated in Theorem 2.6.3 of [3]. We thus need to prove that there is c such that

$$\sup_{0\le t\le T} \hat{\mathbb{E}}_{\mu^\gamma}\Big(F_1^\gamma(t)^2+F_2^\gamma(t)^2+Y_t^\gamma(\phi)^2\Big)\le c \tag{4.20}$$

where

$$F_1^\gamma(t)=(L+\frac{d}{dt})Y_t^\gamma(\phi) \tag{4.21}$$

(the time derivative acts only on m_t^γ)

$$F_2^\gamma(t)=LY_t^\gamma(\phi)^2-2Y_t^\gamma(\phi)LY_t^\gamma(\phi) \tag{4.22}$$

For any function $f\in C^3(\mathbb{R})$ we have

$$\begin{aligned}(L+\frac{d}{dt})f\big(Y_t^\gamma(\phi)\big) &= \sum_x c_\gamma(x,\sigma_t)\Big[f\Big(Y_t^\gamma(\phi)-2\gamma^{d/2}\phi(\gamma x)\sigma_t(x)\Big)-f\big(Y_t^\gamma(\phi)\big)\Big]\\ &\quad -f'\big(Y_t^\gamma(\phi)\big)\gamma^{d/2}\sum_x\phi(\gamma x)\frac{d}{dt}m_t^\gamma x)\\ &= f'\big(Y_t^\gamma(\phi)\big)F_1^\gamma(t)+\frac{1}{2}f''\big(Y_t^\gamma(\phi)\big)\Big(4\gamma^d\sum_x c_\gamma(x,\sigma_t)\phi(\gamma x)^2\Big)+R_1^\gamma(t)\end{aligned} \tag{4.23}$$

$$F_1^\gamma(t)=\gamma^{d/2}\sum_x\phi(\gamma x)(L^x+\frac{d}{dt})\tilde{\sigma}_t(x) \tag{4.24}$$

where

$$L^xg(\sigma)=c_\gamma(x,\sigma)[g(\sigma^x)-g(\sigma)] \tag{4.25}$$

and for any $T>0$ there is c so that

$$|R_1^\gamma(t)|\le c\gamma^{d/2}\qquad t\le T$$

By choosing in (4.23) $f(y)=y^2$, (4.22) can be written as

$$F_2^\gamma(t)=4\gamma^d\sum_x c_\gamma(x,\sigma_t)\phi(\gamma x)^2 \tag{4.26}$$

and

$$|F_2^\gamma(t)|\le c \tag{4.27}$$

We use (2.8) to write

$$(L^x+\frac{d}{dt})\tilde{\sigma}_t(x)=-\tilde{\sigma}_t(x)+[f_x(\sigma_t)-f_x(m_t^\gamma)] \tag{4.28}$$

We then linearize (4.28) around m_t^γ, the solution of (2.8) with initial condition $m_0(\gamma x)$, and we get

$$F_1^\gamma(t) = \gamma^{d/2} \sum_x \phi(\gamma x)(K_t \tilde{\sigma}_t)(x) + \mathcal{R}_2^\gamma(t) \tag{4.29}$$

where

$$\mathcal{R}_2^\gamma(t) = F_1^\gamma(t) - \gamma^{d/2} \sum_x \phi(\gamma x)(K_t \tilde{\sigma}_t)(x) \tag{4.30}$$

By (2.11), for any $T > 0$ there are constants c_1 and c_2 so that

$$\hat{\mathbb{E}}^\gamma \left([\gamma^{d/2} \sum_x \phi(\gamma x)(K_t \tilde{\sigma}_t)(x)]^2 \right) \leq c_1 \qquad t \leq T \tag{4.31}$$

$$\hat{\mathbb{E}}^\gamma \left([\mathcal{R}_2^\gamma(t)]^2 \right) \leq c_2 \gamma^d \qquad t \leq T \tag{4.32}$$

The martingale problem.

To identify the limiting law $\hat{\mathbb{P}}$ along a convergent subsequence $\hat{\mathbb{P}}^\gamma$, we first observe that $\mathbb{P}$ is supported by continuous trajectories. This follows from the fact that the jumps of $Y_t^\gamma(\phi)$ are bounded by the sup norm of ϕ times $\gamma^{d/2}$, see for instance Spohn, [11]. We will then identify the limit as the solution of a martingale problem. We start from the martingale characterization of $\hat{\mathbb{P}}^\gamma$. We have that for any function $f \in C^3(\mathbb{R})$

$$f\big(Y_t^\gamma(\phi)\big) - \int_0^t ds (L + \frac{d}{ds}) f\big(Y_s^\gamma(\phi)\big) \quad \text{is a } \hat{\mathbb{P}}^\gamma\text{-martingale} \tag{4.33}$$

We use (4.23). For the term (4.26) using (2.11) and arguments similar to those used when proving tightness we have that

$$\sup_{t \leq T} \hat{\mathbb{E}}^\gamma \left(\left| F_2^\gamma(t) - 2\gamma^d \sum_x b_\gamma(x,t) \phi(\gamma x)^2 \right|^2 \right) \leq c \gamma^d \tag{4.34}$$

where

$$b_\gamma(x,t) = 1 - m_t^\gamma \tanh\{\beta[(J_\gamma \circ m_t^\gamma)(x) + h]\} \tag{4.35}$$

Note that since $m_t^\gamma \to m_t$ solution of (1.1) b_γ is the discretized version of b of (2.51). From this, by taking the limit along the subsequence which converges to $\hat{\mathbb{P}}$, we get that

$$f\big(Y_t(\phi)\big) - \int_0^t ds f'\big(Y_s(\phi)\big) Y_s(\mathcal{L}^+ \phi) - \frac{1}{2} \int_0^t ds f''\big(Y_s(\phi)\big) \|2b(\cdot, s)\phi^2\|_1 \tag{4.36}$$

is a $\hat{\mathbb{P}}$-martingale, $\mathcal{L}^+$ being the adjoint of $\mathcal{L}$ and $b(r,t)$ is as in (2.51). Since the distribution at time 0 is specified as in Theorem 2.8, this information together with the support properties of $\hat{\mathbb{P}}$ and (4.36) (for arbitrary f, $t \leq T$ and ϕ) identifies $\hat{\mathbb{P}}$ according to the Holley and Stroock's theory, see[9] and [11].

It is easy to compute from (4.36) the covariance at different time of $\hat{\mathbb{P}}$ and verify that they are given by (2.44). It is less straight the identification of the equal time covariance, which we report below for the sake of completeness. Call $C_t(r,r')$ the expression $C_{t,t}(r,r')$ of (2.45) and recalling (2.43) consider

$$\hat{\mathbb{E}}\big(Y_t(\phi)^2\big) = \int dr dr' C_t(r,r')\phi(r)\phi(r')$$

Then, from (4.36):

$$\begin{aligned} C_t(r,r') = \int_0^t ds \int dr'' \big\{\mathcal{L}(r,r'',s)C_s(r'',r') + \mathcal{L}(r',r'',s)C_s(r,r'')\big\} \\ + \int_0^t ds 2b(r,s)\delta(r-r') + \delta(r-r')[1-m_0(r)^2] \end{aligned} \tag{4.37}$$

Defining $C_t^0(r,r')$ so that (2.45) holds and substituting it into (4.37), we get

$$\begin{aligned} &C_t^0(r,r') - \int_0^t ds \int dr'' \big\{\mathcal{L}(r,r'',s)C_s^0(r'',r') + \mathcal{L}(r',r'',s)C_s^0(r,r'')\big\} \\ &\quad = -\delta(r-r')[1-m_t(r)^2] + \delta(r-r')[1-m_0(r)^2] \\ &\quad + \int_0^t ds \int dr'' \big\{\mathcal{L}(r,r'',s)\delta(r'-r'')[1-m_s(r')^2] + \mathcal{L}(r',r'',s)\delta(r-r'')[1-m_s(r)^2]\big\} \\ &\quad + \int_0^t ds 2b(r,s)\delta(r-r') \end{aligned} \tag{4.38}$$

The last two integrals can be rewritten, using (2.46), (1.1) and (2.51) as

$$\begin{aligned} &\int_0^t ds \big\{\mathcal{L}^0(r,r',s)[1-m_s(r')^2] + \mathcal{L}^0(r',r,s)[1-m_s(r)^2]\big\} \\ &\qquad - \int_0^t ds 2\delta(r-r')[1-m_s(r)^2] + 2\int_0^t ds b(r,s)\delta(r-r') \\ &= \int_0^t ds \big\{\mathcal{L}^0(r,r',s)[1-m_s(r')^2] + \mathcal{L}^0(r',r,s)[1-m_s(r)^2]\big\} - \int_0^t ds \frac{d}{ds} m_s(r)^2 \delta(r-r') \end{aligned} \tag{4.39}$$

Using (4.39), (4.38) shows that C_t^0 defined by (2.45) is the same as in (2.48). Theorem 2.8 is thus proven. □

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