GLAUBER EVOLUTION WITH KAC POTENTIALS

I. MESOSCOPIC AND MACROSCOPIC LIMITS, INTERFACE DYNAMICS.

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Abstract. This is the first of three papers on the Glauber evolution of Ising spin systems with Kac potentials. We begin with the analysis of the mesoscopic limit, where space scales like the diverging range, γ^{-1}, of the interaction while time is kept finite: we prove that in this limit the magnetization density converges to the solution of a deterministic, non linear, non local evolution equation. We also show that the long time behavior of this equation describes correctly the evolution of the spin system till times which diverge as $\gamma \to 0$ but are small in units $\log \gamma^{-1}$. In this time regime we can give a very precise description of the evolution and a sharp characterization of the spin trajectories. As an application of the general theory, we then prove that for ferromagnetic interactions, in the absence of external magnetic fields and below the critical temperature, on a suitable macroscopic limit, an interface between two stable phases moves by mean curvature. All the proofs are consequence of sharp estimates on special correlation functions, the v-functions, whose analysis is reminiscent of the cluster expansion in equilibrium Statistical Mechanics.

1. Introduction

The Van der Waals theory of phase transitions describes systems with forces which are repulsive at short distances and have long attractive tails. By scaling the attractive part of the interaction, it is possible to construct a family of models where the above condition is satisfied arbitrarily well. This idea was proposed by M. Kac in the context of equilibrium Statistical Mechanics, where the limit case reproduces exactly the Van der Waals phase diagram, as proven in a wide variety of systems, [17], [20]. Also the metastable effects predicted by the Van der Waals theory are to some extent recovered by this approach, [22].

Non equilibrium properties for systems with Kac potentials have been studied in [5], [6], [19] and, more recently, [24]. Inspired by these works, we begin in this paper a systematic analysis of the Glauber dynamics in Ising spin systems with Kac potentials, which will be further developed in two successive papers, [II] and [III], we will refer to this one as [I]. (An unpublished version of [I] and [II] has appeared a year ago, [IV]: it contains the analysis of the system with different rates of the Glauber dynamics. It also gives a short survey of the physical backgrounds and a list of open problems). As we shall see, the model has a

The research has been partially supported by CNR, GNFM and by grant SC1-CT91-0695 of the Commission of European Communities

surprisingly rich structure and it exhibits a great variety of physically interesting effects. Phase separation, development of interfaces and interface dynamics are the main issues of our analysis. In particular in [III] we study the phase separation after quenching down, below the critical value, a high temperature state: we observe the development of the interfaces and characterize their structure. The successive interface dynamics is investigated here, in the simpler case of a single interface. Fluctuations theory and critical phenomena in relation to stochastic quantization will be discussed in [II].

We denote by γ the scaling parameter of the Kac interaction, that will eventually go to 0. The characteristic feature of a Kac potential is that its range diverges like γ^{-1}, while the total interaction energy of any single spin with all the others is kept finite. We first study the limit when $\gamma \to 0$, scaling the space by the same γ: we will prove that the limiting magnetization density $m(r,t)$ solves the deterministic, non local evolution equation

$$\frac{\partial m}{\partial t} = -m + \tanh\{\beta(J \star m + h)\}, \quad (J \star m)(r) = \int dr' J([r-r'])m(r') \tag{1.1}$$

Times in this limit are not rescaled, so that each individual spin in a time unit undergoes only a finite, random, number of flips. The deterministic behavior described by (1.1) is a mean field effect due to the scaling of the interaction: many spins (infinitely many in the limit $\gamma \to 0$) feel essentially the same potential and while each of them, individually, has a random behavior, the collectivity evolves deterministically, due to a law of large numbers which dampens the fluctuations. The full effect of the interaction only arises at longer times, when each spin, after many flips, reaches a (local) equilibrium distribution.

We call "mesoscopic" the above limit, with space scaled by γ and time kept finite, to distinguish it from other, macroscopic, limits, where also the time is scaled with γ. The equation (1.1) obtained in the mesoscopic limit will be called the mesoscopic equation. As explained before, we expect the more interesting effects to appear at longer times. The real question then is whether they are correctly predicted by the mesoscopic equation. There is no answer valid for all the cases: we have indeed situations where the limits can be interchanged, thus finding the true behavior of the Glauber dynamics at times which diverge as $\gamma \to 0$ by investigating the long time behavior of (1.1). This happens for instance in the development of the interfaces (at temperatures below the critical one and when the magnetization profile is away from its critical value, except on regular surfaces where the interfaces develop) and also in the successive stage of interface dynamics, at least when the curvature of the interface is not too small. In Section 5 we will study the latter case.

It may also happen that the long time predictions of the mesoscopic equation are wrong: the spinodal decomposition after quenching a state from high temperature down below the critical value and the macroscopic fluctuations at the critical temperature are intrinsically random. The stochastic forces responsible for these effects are absent in (1.1), having disappeared in the mesoscopic limit. These aspects will be studied in [II] and [III] while this paper covers cases where the fluctuations are not relevant, as in the mesoscopic limit that we study first. This is easy to investigate, as it is essentially a mean field limit, but we are interested in the behavior of the system past the mesoscopic times and for that we need sharper estimates. As an outcome of our analysis, we obtain a characterization of most of the individual trajectories of the spin process, both in space and time, proving that with large probability they only slightly deviate from what predicted by the mesoscopic equation, at least when times are small in units $\log \gamma^{-1}$.

The small deviations mentioned above are not only consequence of errors and approximations in our estimates, but they have an intrinsic origin, related to the buildup of correlations as time grows and γ is kept finite. This issue is treated in [II], in the context of the Fluctuation Theory for Glauber dynamics with Kac potentials. Here we ignore these effects considering the deviations of the spins distribution from the product measure as errors. We prove, with the help of cluster expansion techniques, that these deviations are small and that, if the evolution described by the mesoscopic equation is sufficiently stable, then they do not have significant effects. As an application, we study in Section 5 the case of ferromagnetic interactions without any external magnetic field and below the critical temperature. We fix an initial magnetization profile which decribes an interface between the two stable phases and prove that, on a suitable macroscopic limit, the interface moves by mean curvature.

We would like to draw the attention of the reader not only on the results outlined so far, but also on the techniques used in their proofs, as they are based on powerful methods which may be useful also in other cases. Kinetic theory and cluster expansion play here an important role. The small parameter of the latter, which is the inverse temperature in Statistical Mechanics, is here played simultaneously by γ and by the time, which, at first, is supposed vanishingly small, as $\gamma \to 0$. To obtain estimates at finite and longer times we then work out an iterative procedure based on special (truncated) correlation functions, the v-functions. This part of the analysis is common to several other models, see [13] for a survey on the method, which, in particular, has been applied to stochastic, discrete-velocity models of the Boltzmann equation, hence the relation with kinetic theory.

The paper is organized as follows. Section 2 contains the main definitions, and the results concerning the derivation of the mesoscopic equation, the interface dynamics and the bounds on the v-functions. The "short time" bounds are proven in Section 3 and extended, in Section 4, to longer times. In Section 5 we apply the previous considerations to prove that a cluster of one phase moves by mean curvature in a macroscopic limit where also time is scaled with γ. In an Appendix we prove Proposition 4.8 of Section 4.

2. Main definitions and results

This Section is divided in three subsections. In the first one, §2.1, we define the model and state the theorem on the convergence to the mesoscopic equation. In §2.2 we consider a class of initial states which describe, in a macroscopic limit, magnetic profiles with an interface. We then have a theorem which states that in the limiting macroscopic evolution the interfaces move by mean curvature. Finally in §2.3 we present results on the propagation of chaos, that is proven in a very strong form and till times which diverge as $\gamma \to 0$ but that are small in units $\log \gamma^{-1}$. In this subsection we define the v-functions and state bounds on these functions which provide the main technical tools for studying the transition from discrete to continuum.

§2.1. The mesoscopic limit.

We consider an Ising spin system and start by recalling the main notation and definitions. A spin configuration is a specification of the values of the spins at all the lattice sites, it is therefore a function σ: $\mathbb{Z}^d \to \{-1,1\}$, that is an element of $\{-1,1\}^{\mathbb{Z}^d}$. The value $\sigma(x)$ of the spin at x is thus a function of the configuration σ, thus a random variable on $\{-1,1\}^{\mathbb{Z}^d}$, the space of all the spin configurations. The restriction to $\Delta \subset \mathbb{Z}^d$ of a configuration σ, is denoted by σ_Δ, which is therefore a function on Δ with values $\{-1,1\}$.

We next recall the definition of Kac potentials.

2.1.1. Definition. *Kac potentials.*

A Kac potential (in this paper) is a function $J_\gamma : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}$, which depends on a (scaling) parameter γ and has the form

$$J_\gamma(x,y) = \gamma^d J(\gamma[x-y]) \tag{2.1}$$

We suppose that γ varies in the set $\{2^{-n}, n \in \mathbb{Z}_+\}$. We assume that $J(r)$ depends on $|r|$, $J(r) = 0$ for all $|r| > 1$ and that $J(r) \in C^3(\mathbb{R}^d)$.

Given (a magnetic field) $h \in \mathbb{R}$, we define the energy of the spin configuration σ_Δ as

$$H_\gamma(\sigma_\Delta) = -h \sum_{x\in\Delta} \sigma(x) - \frac{1}{2} \sum_{x\neq y\in\Delta} J_\gamma(x,y)\sigma(x)\sigma(y) \tag{2.2}$$

while its energy inclusive of the interaction with the spins in the complement, Δ^c, of Δ, is

$$H_\gamma(\sigma_\Delta|\sigma_{\Delta^c}) = H_\gamma(\sigma_\Delta) - \sum_{x\in\Delta, y\notin\Delta} J_\gamma(x,y)\sigma(x)\sigma(y) \tag{2.3}$$

The class of the Kac potentials is more general than in §2.1.1, the only requirement on J being that it is in $L^1(dr, \mathbb{R}^d)$, see [17], [20]. The restriction on the values of γ is only made with the purpose of simplifying notation when discussing the block spin variables and could be easily lifted.

2.1.2. Definition. *The Glauber dynamics.*

Given (the "inverse temperature") $\beta > 0$ and $\gamma > 0$, we denote by Glauber dynamics the unique Markov process on $\{-1,1\}^{\mathbb{Z}^d}$ whose pregenerator is the operator L_γ with domain the set of all the cylinder functions f on which it acts as

$$L_\gamma f(\sigma) = \sum_{x\in\mathbb{Z}^d} c_\gamma(x,\sigma)[f(\sigma^x) - f(\sigma)] \tag{2.4}$$

In (2.4) σ^x is the configuration obtained from σ by flipping the spin at x, i.e.

$$\sigma^x(y) = \begin{cases} \sigma(y) & \text{if } y \neq x \\ -\sigma(x) & \text{if } y = x \end{cases} \tag{2.5}$$

The "flip rate" $c_\gamma(x,\sigma)$ of the spin at x in the configuration σ is

$$c_\gamma(x,\sigma) = \frac{e^{-\beta h_\gamma(x)\sigma(x)}}{e^{-\beta h_\gamma(x)} + e^{\beta h_\gamma(x)}} \tag{2.6a}$$

$$h_\gamma(x) = h + (J_\gamma \circ \sigma)(x), \qquad (J_\gamma \circ \sigma)(x) = \sum_{y\neq x} J_\gamma(x,y)\sigma(y) \tag{2.6b}$$

The proof of the existence and uniqueness of the Markov process, used in the above definition, may be found in [21]. The space of realizations of the Glauber dynamics is $D(\mathbb{R}_+, \{-1,1\}^{\mathbb{Z}^d})$, the Skorohod space of cadlag trajectories, (continuous from the right and with limits from the left). The value of the process at time t, that is the spin configuration at time t, is denoted by σ_t, the value of the spin in x at time t is $\sigma(x,t)$ which is thus a random variable on $D(\mathbb{R}_+, \{-1,1\}^{\mathbb{Z}^d})$.

Notice that

$$c_\gamma(x,\sigma) = Z_\gamma(\sigma_{x^c})^{-1} e^{-\beta/2\Delta_x H_\gamma(\sigma)}$$

where $\Delta_x H_\gamma(\sigma)$ is the change of energy due to the spin flip at x, namely

$$\Delta_x H_\gamma(\sigma) = H_\gamma\big((\sigma^x)_\Lambda\big) - H_\gamma(\sigma_\Lambda)$$

where Λ is any set which contains x and such that the spin at x does not interact with those in Λ^c. $Z_\gamma(\sigma_{x^c})^{-1}$ is the denominator in (2.6a), but, for what we say below, it may be any other function, provided it is independent of $\sigma(x)$, as implied by the notation. In fact the important point about the rates is that they verify the "detailed balance" condition

$$\frac{c_\gamma(x,\sigma^x)}{c_\gamma(x,\sigma)} = e^{-\beta\Delta_x H_\gamma(\sigma)} \tag{2.6c}$$

The Glauber dynamics is thus intimately related to the notion of:

2.1.3. Definition. *The Gibbs measures.*

The Gibbs measure $\mu_{\beta,h,\gamma}$ is any probability on $\{-1,1\}^{\mathbb{Z}^d}$ which satisfies the DLR equations: namely, such that for any $x \in \mathbb{Z}^d$ and any σ,

$$\mu_{\beta,h,\gamma}\Big(\sigma(x) = \pm 1 \big| \{\sigma(y), y \neq x\}\Big) = \frac{e^{\pm\beta h_\gamma(x)}}{e^{-\beta h_\gamma(x)} + e^{\beta h_\gamma(x)}}, \qquad \mu_{\beta,h,\gamma} \text{ almost surely} \tag{2.7}$$

where the left hand side is the probability that $\sigma(x) = \pm 1$ conditioned on the σ-algebra generated by all the spins $\sigma(y)$, $y \neq x$.

β in §2.1.3 has the physical meaning of an inverse temperature and h of an external magnetic field, J_γ of the spin-spin interaction strength. Notice that the left hand side of (2.7) is a function of $\sigma(x)$ and all $\sigma(y)$, $y \neq x$, it is thus a function of the whole spin configuration σ. Then, from (2.6c) and (2.7), it follows that

$$\mu_{\beta,h,\gamma}\big(\sigma(x)|\{\sigma(y),\ y \neq x\}\big)c_\gamma(x,\sigma) = \mu_{\beta,h,\gamma}\big(\sigma^x(x)|\{\sigma(y),\ y \neq x\}\big)c_\gamma(x,\sigma^x)$$

so that the operator $L_\gamma^{(x)}$ defined by (2.4) after setting $c_\gamma(y,\sigma) = 0$ for all $y \neq x$, is selfadjoint in $L_2\big(\{-1,1\}^{\mathbb{Z}^d}, \mu_{\beta,h,\gamma}\big)$. It then follows that also the full generator of the Glauber dynamics is selfadjoint and that $\mu_{\beta,h,\gamma}$ is stationary, the Glauber dynamics then being a reversible process. We will not exploit this feature of the dynamics and the associated theory of Dirichlet forms, which in many instances has been proven to be a very useful and powerful method. We will use though the existence of the Gibbs measures for a physical interpretation of the results. Many aspects of our analysis are common to other models, as for instance the Glauber + Kawasaki dynamics, which is a very well studied model for reaction-diffusion equations, [8], and phase separation, [13], [12], [15], [16]. The advantage in our case is the explicit connection with equilibrium Statistical Mechanics, so that we can properly talk of thermodynamic phases, surface tension, critical temperature and so forth. We finally mention that (2.6c) does not depend on the choice of Z_γ, which appears in the definition of c_γ, thus different choices of Z_γ define other, equally acceptable, reversible evolutions. The choice (2.6a) gives rise to a simpler limiting mesoscopic equation.

We have so far discussed the Glauber dynamics at the microscopic level, we next turn to the mesoscopic one. The "scale separation" between the two levels is specified by γ: in the transition micro-mesoscopic

$$(x,t) \to (r,t) = (\gamma x, t)$$

time is thus unchanged while space is shrunk by γ. The microscopic points $x \in \mathbb{Z}^d$ are represented in the mesoscopic space $\mathbb{R}^d$ by the lattice $\gamma\mathbb{Z}^d$. It is thus convenient to partition $\mathbb{R}^d$ into the "elementary squares" $\{r : [r]_\gamma = \gamma x\}$, with $x \in \mathbb{Z}^d$ and, denoting by $r = (r_1,..,r_d)$, $x = (x_1,..,x_d)$,

$$[r]_\gamma = \gamma x \ \text{ if } x \in \mathbb{Z}^d \text{ and } \quad \gamma x_i \le r_i < \gamma(x_i+1) \text{ for all } i = 1,..,d \tag{2.8a}$$

2.1.4. Definition. *The mesoscopic representation.*

We denote by $\mathcal{M}(X)$, X a measurable space, the space of all the real valued, measurable functions on X. We then define $\Gamma_\gamma : \mathcal{M}(\mathbb{Z}^d) \to \mathcal{M}(\mathbb{R}^d)$ as

$$\big(\Gamma_\gamma(f)\big)(r) = f(x), \qquad x = \gamma^{-1}[r]_\gamma \text{ and } f \in \mathcal{M}(\mathbb{Z}^d) \tag{2.8b}$$

where $[r]_\gamma$ is defined in (2.8a).

In particular we denote by

$$\sigma_\gamma = \Gamma_\gamma(\sigma),\ \sigma_{\gamma,t} = \Gamma_\gamma(\sigma_t) \tag{2.8c}$$

$\sigma_\gamma(r)$ is thus the image of the spin configuration σ in the mesoscopic representation. Our first theorem proves that for small γ's, the Glauber dynamics in the mesoscopic representation is almost deterministic. The statement refers to a smoothened version of σ_γ, defined in terms of the block spin transformation: this amounts to replace the value $f(r)$ of a function in $\mathcal{M}(\mathbb{R}^d)$ by its average in a region containing r and made by elementary squares, whose size vanishes as $\gamma \to 0$.

2.1.5 Definition. *The block spin transformation.*

We define for any $0 < \alpha < 1$ and γ as in 2.1.1 the block spin transformation $f \to f^{(\alpha,\gamma)}$, f and $f^{(\alpha,\gamma)}$ both in $\mathcal{M}(\mathbb{R}^d)$, as

$$f^{(\alpha,\gamma)}(r) = N_\gamma^{-1} \int dr' \mathbf{1}\big(\{\big|[r]_\gamma - [r']_\gamma\big| \le \gamma^{1-\alpha}\}\big) f(r') \tag{2.9a}$$

$$N_\gamma = \int dr' \mathbf{1}\big(\{\big|[r]_\gamma - [r']_\gamma\big| \le \gamma^{1-\alpha}\}\big) \tag{2.9b}$$

We use the shorthand notation

$$\sigma_\gamma^{(\alpha)} := \big(\sigma_\gamma\big)^{(\alpha,\gamma)}; \qquad \sigma_{\gamma,t}^{(\alpha)} := \big(\sigma_{\gamma,t}\big)^{(\alpha,\gamma)} \tag{2.9c}$$

to avoid redundancy in the formulas, and in general we may omit the superscript γ in (α,γ), when γ already appears as a subscript.

The more familiar form of the block spin transformation is recovered when we apply the transformation to a function $g = \Gamma_\gamma(f)$, $f \in \mathcal{M}(\mathbb{Z}^d)$. In that case $g^{(\alpha,\gamma)}(r)$, $[r]_\gamma =: \gamma x$, is given by

$$g^{(\alpha,\gamma)}(r) = \mathcal{A}_{\gamma^{-\alpha},x}(f) := \frac{1}{|B_{\gamma^{-\alpha}}|} \sum_{y\in B_{\gamma^{-\alpha},x}} f(y) \tag{2.9d}$$

where

$$B_{\gamma^{-\alpha},x} = \{|y-x| \le \gamma^{-\alpha}\}, \quad |B_{\gamma^{-\alpha}}| = \text{cardinality of } B_{\gamma^{-\alpha},x} \tag{2.9e}$$

We will use extensively the above notation in Section 4, where we work in the microscopic rather than in the mesoscopic representation. We will prove there the following result.

2.1.6 Theorem.

For any $\alpha \in (0,1)$ there are ζ, a and b all positive and for any n and any $k^\star \ge 2$, there is c so that the following holds. For all γ small enough and, given γ, for all $\sigma \in \{-1,1\}^{\mathbb{Z}^d}$ and $m \in [-1,1]^{\mathbb{Z}^d}$, $\|m\|_\infty \le 1$, for which (see (2.9a)-(2.9c) for notation)

$$\sup_{|r|\le k^\star\gamma^{-1}} \left|\sigma_\gamma^{(\alpha)}(r) - m^{(\alpha,\gamma)}(r)\right| \le \gamma^\zeta \tag{2.10a}$$

we have that (see again (2.9c) for notation)

$$\mathbb{P}_\sigma^\gamma\Big(\sup_{t\le a\log\gamma^{-1}} \sup_{|r|\le(k^\star-1)\gamma^{-1}} \left|\sigma_{\gamma,t}^{(\alpha)}(r) - m_\gamma^{(\alpha,\gamma)}(r,t)\right| > \gamma^b\Big) \le c\gamma^n \tag{2.10b}$$

where $\mathbb{P}_\sigma^\gamma$ is the law of the Glauber dynamics when the process starts at time 0 from σ;

$$m_\gamma^{(\alpha,\gamma)}(\cdot,t) = \Big(m_\gamma(\cdot,t)\Big)^{(\alpha,\gamma)} \tag{2.10c}$$

and $m_\gamma(r,t)$ is the unique solution of the Cauchy problem (1.1) with intial datum

$$m_\gamma(r,0) = \big(\Gamma_\gamma m\big)(r) \tag{2.11}$$

Observe that, given σ, the choice $m(r) = \sigma_\gamma(r)$ automatically satisfies (2.10a). The proof of Theorem 2.1.6 is essentially the same if we replace, in the sup over r, γ^{-1} by γ^{-k}, no matter how large is k. γ^{-1} is however sufficient for our needs here, as it corresponds to regions unbounded both in mesoscopic and macroscopic units, when the latter are defined as in §2.2 below.

Theorem 2.1.6 expresses in a very strong form the deterministic nature of the Glauber dynamics in the mesoscopic limit. Notice however that Theorem 2.1.6 does not say that $m_\gamma(r,t)$ has a limit when $\gamma \to 0$, in fact we are not imposing that the limit exists at time 0. There are many interesting situations, on the other hand, where there is convergence at time 0, and this will be the main issue in the sequel of the subsection.

We start with a definition:

2.1.7. Definition. *Initial states and mesoscopic profiles.*

A mesoscopic profile is a function $m_0 \in \mathcal{M}(\mathbb{R}^d)$, such that its sup norm $\|m_0\|_\infty \le 1$. We call initial mesoscopic state a family (m_0, μ^γ), where m_0 is a mesoscopic profile and, for each γ, μ^γ is any product probability measure on $\{-1,1\}^{\mathbb{Z}^d}$ such that, setting

$$m_{\gamma,0} := \Gamma_\gamma\Big(\mathbb{E}_{\mu^\gamma}\left(\sigma(\cdot)\right)\Big) \tag{2.12a}$$

$$\lim_{\gamma\to 0} m_{\gamma,0} = m_0, \qquad \textit{Lebesgue almost everywhere} \tag{2.12b}$$

Remarks.

There are two typical examples of families (m_0, μ^γ). In the first one, that we call the "standard initial state", $m_0 \in C(\mathbb{R}^d)$ and

$$\mathbb{E}^\gamma_{\mu^\gamma}\left(\sigma(x)\right) = m_0(\gamma x) \tag{2.12c}$$

In the second example, m_0 is an arbitrary mesoscopic profile and the function $m_{\gamma,0}$ that defines μ^γ via (2.12a) is $m_{\gamma,0} = m_0^{(0,\gamma)}$, which is obtained by averaging m_0 over each elementary square, see (2.8a). Then, by the martingale convergence theorem, $m_{\gamma,0}$ converges Lebesgue almost everywhere to m_0. Observe that this latter example proves that there is an initial state, in the sense of 2.1.7, for any given mesoscopic state m_0, so that the Definition 2.1.7 is well posed.

2.1.8 Theorem.

Let (m_0, μ^γ) be an initial mesoscopic state in the sense of Definition 2.1.7. Let α, ζ, a, b, n, $k^\star$ and c as in Theorem 2.1.6. Then

$$\mu^\gamma\Big(\sup_{|r|\le k^\star\gamma^{-1}} \left|\sigma^{(\alpha)}_\gamma(r) - m^{(\alpha)}_{\gamma,0}(r)\right| \le \gamma^\zeta\Big) \ge 1 - c\gamma^n \tag{2.13a}$$

with $m^{(\alpha)}_{\gamma,0}$ the block spin transform of $m_{\gamma,0}$, defined in (2.12a).

Observe then that as a consequence of (2.13a) and (2.10b),

$$\mu^\gamma\Big(\Big\{\sigma_0 : \mathbb{P}^\gamma_{\sigma_0}\Big(\sup_{t\le a\log\gamma^{-1}}\ \sup_{|r|\le(k^*-1)\gamma^{-1}}\Big|\sigma^{(\alpha)}_{\gamma,t}(r) - m^{(\alpha,\gamma)}_\gamma(r,t)\Big| > \gamma^b\Big) \le c\gamma^n\Big\}\Big) \ge 1 - c\gamma^n \tag{2.13b}$$

$m^{(\alpha,\gamma)}_\gamma(\cdot,t)$ being the block spin transformation of $m_\gamma(\cdot,t)$ this latter solving (1.1) with initial condition $m_\gamma(r,0) = m_{\gamma,0}(r)$.

Furthermore, for any t, $m_\gamma(r,t) \to m(r,t)$, Lebesgue almost everywhere, where $m(r,t)$ is the solution of (1.1) with initial condition m_0. Finally, let (m_0,μ^γ) be the standard initial state defined in the Remark following 2.1.7. Suppose that $m_0 \in C^1(\mathbb{R}^d)$, with bounded derivative, then $m^{(\alpha)}_\gamma(r,t)$ in (2.13b) may be replaced by $m(r,t)$.

As a straight corollary of (2.13b):

$$\mathbb{P}^\gamma_{\mu^\gamma}\Big(\sup_{t\le a\log\gamma^{-1}}\ \sup_{|r|\le(k^*-1)\gamma^{-1}}\Big|\sigma^{(\alpha)}_{\gamma,t}(r) - m^{(\alpha)}_\gamma(r,t)\Big| > \gamma^b\Big) \le c\gamma^n \tag{2.13c}$$

where $\mathbb{P}^\gamma_{\mu^\gamma}$ is the law of the Glauber dynamics starting from μ^γ. Moreover, if (m_0,μ^γ) is a standard initial state with $m_0 \in C^1(\mathbb{R}^d)$ having a bounded derivative, then

$$\mathbb{P}^\gamma_{\mu^\gamma}\Big(\sup_{t\le a\log\gamma^{-1}}\ \sup_{|r|\le(k^*-1)\gamma^{-1}}\Big|\sigma^{(\alpha)}_{\gamma,t}(r) - m(r,t)\Big| > \gamma^b\Big) \le c\gamma^n \tag{2.13d}$$

Further results are stated in Theorem 4.9. The proofs of all the statements so far are given in Sections 3 and 4. Proposition 4.8 and the Appendix are not needed in this part of the analysis.

§2.2. Interface dynamics, motion by mean curvature.

In this subsection we restrict to $J \ge 0$, $h = 0$ and $\beta > 1$ after having imposed the normalization condition $\int dr J(r) = 1$. These assumptions are essential as we expect a completely different behavior in the other cases.

$\beta = 1$ is the inverse critical temperature in the Lebowitz-Penrose limit, [20]: we are thus considering temperatures below the critical one. We denote by m_β the strictly positive solution of

$$m_\beta = \tanh\{\beta m_\beta\}, \qquad \text{(recall that } \int dr J(r) = 1) \tag{2.14}$$

The values $\pm m_\beta$ are the magnetizations of the two extremal Gibbs states in the limit $\gamma \to 0$ and are the thermodynamic values of the magnetization, [20]. Thus the function identically equally to m_β (or to $-m_\beta$), which is a stationary solution of (1.1), is interpreted as the m_β (respectively the $-m_\beta$) pure phase.

The interfaces are then the regions which separate the two thermodinamically pure phases. We are interested here in the dynamical problem, namely the evolution of an initial state where the two phases coexist and are separated by an interface. We study the case when the phase m_β occupies a "large, but bounded region", while the outside is filled up by the other phase $-m_\beta$. If the region is large and the interface sufficiently flat, then we will show that the evolution is simply described by the motion of the interface, namely with the phase m_β in the region enclosed by the moving interface and the phase $-m_\beta$ outside. The dynamics of the interface, as we shall see, obeys the law of motion by mean curvature.

2.2.1 Definition. *Initial states with an interface.*

Let Λ_0 be a compact domain whose boundary, Σ_0, is a C^∞ connected surface in $\mathbb{R}^d$. Let

$$\epsilon = \lambda\gamma, \qquad \lambda = \frac{1}{\sqrt{\log \gamma^{-1}}} \tag{2.15}$$

We denote hereafter by μ^ϵ the product measure on $\{-1,1\}^{\mathbb{Z}^d}$ with $\mathbb{E}^\gamma_{\mu^\epsilon}\big(\sigma(x)\big) = m_\beta$ for all x in $\epsilon^{-1}\Lambda_0$ and $= -m_\beta$ in the complement.

The above state does not describe the interface Σ_0 in the sense of Definition 2.1.7. In fact the function

$$m_{\gamma,0} := \Gamma_\gamma\big(\mathbb{E}_{\mu^\epsilon}(\sigma(\cdot))\big)$$

converges pointwise to the function constantly equal to m_β which is thus the corresponding mesoscopic profile. The size of the cluster of the phase m_β becomes infinite in mesoscopic variables, which are thus inadequate for describing the interface.

2.2.2 Definition. *Macroscopic coordinates.*

We denote by ξ and τ the macroscopic space and time coordinates which are related to the microscopic ones by

$$\xi = \epsilon x, \qquad \tau = \lambda^2 t \tag{2.16}$$

and to the mesoscopic ones (r,t) by

$$\xi = \lambda r, \qquad \tau = \lambda^2 t \tag{2.17}$$

We are denoting by λ^{-1} the parameter which separates the mesoscopic and the macroscopic scales. Times are then separated by the factor λ^{-2}: this is the right scale to observe the motion of the interface, in other phenomena the relevant macroscopic scaling may be different.

The choice (2.15), $\lambda = (\log\gamma^{-1})^{-1/2}$, is motivated by [III] where it is proven that the clusters size after phase separation scales as $(\log\gamma^{-1})^{-1/2}$. Our analysis therefore describes the interface dynamics right after the phase separation, with the further simplifying assumption of considering a single interface. The extension to many interfaces, as they appear in the analysis of [III], should not bring in serious difficulties.

The initial state μ^ϵ in Definition 2.2.1 describes a profile with the pure phases m_β and $-m_\beta$ respectively inside and outside of $\epsilon^{-1}\Lambda_0$. Σ_0 is thus the sharp macroscopic interface between the two pure phases. It is sharp in macroscopic coordinates, as it should, but also in the mesoscopic ones, which is less realistic. The interface, after the phase separation, in fact has a different shape, with a smooth profile, in mesoscopic variables, connecting the values $\pm m_\beta$, [III]. As clear from the proofs in Section 5, our analysis applies as well to this and to any other choice of the initial state, provided its interface becomes sharp in macroscopic coordinates. The choice 2.2.1 is dictated by the simplicity of its presentation.

2.2.3 Definition. *The classical motion by mean curvature.*

We say that the surface Σ_τ evolves according to the classical motion by mean curvature with parameter $\theta > 0$ in the time interval $0 \le \tau \le \tau^\star$, if Σ_τ, for any such τ, is the connected boundary of a compact region $\Lambda_\tau \subset \mathbb{R}^d$. We also require that there is a C^∞, $d-1$ dimensional, compact manifold S_0 and a $C^\infty\big([0,\tau^\star]\times S_0\big)$ function $\xi = \xi(\tau,\xi_0)$, with values in $\mathbb{R}^d$, such that Σ_τ is equal to the set $\{\xi = \xi(\tau,\xi_0),\ \xi_0 \in S_0\}$ and

$$\frac{d\xi}{d\tau} = \theta\kappa\nu \tag{2.18}$$

where ν is the unit vector normal to Σ_τ at ξ and pointing toward the interior of Σ_τ; κ is $d-1$ times the (signed) mean curvature of Σ_τ at ξ.

If Σ_0 is a sphere of radius R_0, then Σ_τ is the sphere of radius R_τ, with

$$\frac{dR_\tau}{d\tau} = -\theta(d-1)\frac{1}{R_\tau}$$

There is a local existence and uniqueness theorem regarding the motion by mean curvature, in the context of Definition 2.2.3, see [2] and references therein, which follows from general results on parabolic equations. It is known that in $d > 2$ singularities may develop after a finite time, while in $d = 2$ the only singularity which may arise is the disappearence of a cluster. More recent results describe what happens after the appearence of singularities yielding global existence theorems for the evolution, see [14] and references therein. Our results only cover the classical case.

2.2.4 Theorem. *Macroscopic limit and motion by mean curvature.*

Let Σ_τ be as in Definition 2.2.3, with θ as in (5.2a) and μ^ϵ as in Definition 2.2.1. Then there is $\zeta > 0$ and for any $\tau < \tau^\star$ and any $n \geq 1$ there is c, so that, for all γ small enough,

$$\mathbb{P}^\gamma_{\mu^\epsilon}\Big(\sup_{t\leq \lambda^{-2}\tau} \sup_{\substack{|r|\leq\gamma^{-1}\\ |d(r,\lambda^{-1}\Sigma_{\lambda^2 t})|\geq \lambda^{-1+\zeta}}} |\sigma^{(\alpha)}_{\gamma,t}(r) \mp m_\beta| \leq \lambda^{3/2}\Big) \geq 1 - c\gamma^n \tag{2.19a}$$

where $|d(r,\lambda^{-1}\Sigma_\tau)|$ denotes the distance of r from the surface $\lambda^{-1}\Sigma_\tau$; the $-$ sign in (2.19a) is for r inside $\lambda^{-1}\Sigma_\tau$ and the $+$ sign for r outside.

More expressively, in macroscopic coordinates the set in (2.19a) is

$$\Big\{ \sup_{\tau'\leq\tau} \sup_{\substack{|\xi|\leq\gamma^{-1}\lambda\\ |d(\xi,\Sigma_{\tau'})|\geq\lambda^\zeta}} |\sigma^{(\alpha)}_{\lambda,t'}(r) \mp m_\beta| \leq \lambda^{3/2}\Big\} \tag{2.19b}$$

with $t' \equiv \lambda^{-2}\tau'$ and $r \equiv \lambda^{-1}\xi$. Therefore "inside" and "outside" Σ_τ we see respectively the phases m_β and $-m_\beta$. The quotation marks recall that the statement does not cover the neighborhood of Σ_τ of the points which are at distance less than λ^ζ from Σ_τ. As the distance vanishes as γ (and λ) go to 0, Theorem 2.2.4 gives a rather complete description of the evolution in the macroscopic representation. The result is however completely unsatisfactory in the mesoscopic description where space is magnified by a factor λ^{-1}. Then the size of the strip around the interface that is not covered by 2.2.4 diverges as $\gamma \to 0$ and since this is where the interface has developed, our result misses entirely the structure of the interface, which, in macroscopic coordinates, was simply the sharp discontinuity that separates the two phases.

The proof of Theorem 2.2.4 is given in Section 5, it does not use the results stated in Proposition 4.8 and in Appendix A, so that it avoids the more refined and certainly more complex arguments involved in the proof of the statements in the next §2.3.

As already mentioned, Theorem 2.2.4 describes the evolution of the system right after the phases separate, (under the assumption that the different interfaces move independently). The structure of the system much after phase separation should look essentially similar, if distances are measured in units $t^{1/2}$, which is thus the typical size of the clusters of the two phases. These later stages may be studied, to a first approximation, in the same context of Definition 2.1.1, but with a different choice of λ. The regime $\lambda = \gamma^\zeta$, with $\zeta > 0$ small, can be treated, we believe, with techniques similar to those emploied here. When the condition that ζ is small is relaxed, the analysis becomes more and more complex. As discussed in [II], the fluctuations of the interface in a time t and in the simpler case d=2 are of the order of $\gamma t^{1/4}$. They should therefore produce finite displacements of the surface (in mesoscopic units!) on the time scale $t \approx \gamma^{-4}$. The space correlations of the displacements have order $t^{1/2} = \gamma^{-2}$, which thus produce local changes of the curvature of the order of γ^4. By this argument, at $\zeta = 4$ there is a competiotion between fluctuations and the motion by curvature. When $\zeta > 4$ the leading contribution on a first time regime is purely due to stochastic effects, only later the effect of the initial curvature will influence the evolution, but the way this happens is still not clear to us. The real challenge is when λ is independent of γ and we take first $\lambda \to 0$ and then $\gamma \to 0$: this is the "true hydrodynamic limit". The interface is still expected to move by mean curvature and, we believe, with the same parameter. But the mechanism of convergence will be drastically different, because the stochastic effects to take into account will include events that in our analysis here are large deviations and have negligibly small probability.

We conclude this subsection with a few bibliographical remarks. As clear from its proof, given in Section 5, Theorem 2.2.4 is essentially a corollary of Theorem 2.1.6 and of a result in [9], where the motion by mean curvature is derived by scaling (1.1) according to (2.17). Results on convergence to the motion by mean curvature have been obtained for other models: for the Glauber dynamics in Ising spin systems in $d = 2$, with nearest neighbor, ferromagnetic, interactions at 0 temperature, [23]; for the Glauber+Kawasaki dynamics in [1] and [18]. In particular the analysis in [1] allows to characterize the magnetization pattern also at the interface. In [18], the convergence is that of Theorem 2.2.4 (i.e. not as sharp as required for determining the interface), but the result is proven even past the appearence of singularities. The limiting evolution, in this case, is the generalized motion by mean curvature.

The value of the coefficient θ in the motion by mean curvature should be related, according to [23], to the mobility of the interface and to the surface tension by an Einstein relation. The validity of such a relation in the present context has been proven in [3].

§2.3. Propagation of chaos and bounds on the v-functions.

In this subsection we state theorems on the factorization properties of the spin distribution at any given time t showing that the distribution is close (and converges as $\gamma \to 0$) to a product measure with means given by the solution of the mesoscopic equation (1.1). We consider the process starting from an arbitrary product measure, in particular from single configurations. We remain in the lattice, without going to the mesoscopic representation, and introduce the lattice analogue of (1.1) as follows.

2.3.1. Definition. *The discretized equation.*

For any $\gamma > 0$ the discretized evolution equation is

$$\frac{dm_\gamma(x,t)}{dt} = -m_\gamma(x,t) + \tanh\{\beta[(J_\gamma \circ m_\gamma)(x,t) + h]\} \tag{2.20a}$$

where $(f \circ g)$ denotes the discrete convolution of f and g, as in (2.6b). We then call $m_\gamma(x,t|\sigma)$ the solution of (2.20a) with initial condition

$$m_\gamma(x,0|\sigma) = \sigma(x), \qquad \text{for all } x \in \mathbb{Z}^d \text{ and } \sigma \in \{-1,1\}^{\mathbb{Z}^d} \tag{2.20b}$$

If μ is a measure, $m_\gamma(x,t|\mu)$ denotes the solution of (2.20a) with initial condition $m_\gamma(x,0|\mu) = \mathbb{E}_\mu(\sigma(x))$, for all $x \in \mathbb{Z}^d$.

In Section 4, see Proposition 4.7.2, we discuss the relation between (2.20a) and (1.1). (2.20a) and the actual spin flip dynamics are related because there is c such that, for all x,

$$\left| \frac{dm_\gamma(x,t)}{dt} - \mathbb{E}^\gamma_{\nu_t}\left(L_\gamma \sigma(x)\right) \right| \le c\gamma^d \tag{2.21a}$$

if ν_t is the product measure on $\{-1,1\}^{\mathbb{Z}^d}$ with means

$$\mathbb{E}^\gamma_{\nu_t}\left(\sigma(x)\right) = m_\gamma(x,t), \qquad \text{for all } x \text{ in } \mathbb{Z}^d \tag{2.21b}$$

In an ideal case where at all times the measure is a product measure, (2.21a) allows to compute the evolution of the expectations of the spins, hence to determine completely the distribution of the process at any single time. This property is called propagation of chaos, because in product measures there are no correlations between the spins, hence no "order" is present. We say "propagation" because factorization is supposed to hold initially. One can easily check, though, that in our case, and in general, propagation of chaos does not hold (the condition is verified for the independent particles, see for instance Chapter II of [13], but this is "the typical exception").

If some weak form of propagation of chaos holds, with the measure at any time t suitably close to a product measure, then, conceivably, the average values of the spins are also close to the solution of (2.20). We thus introduce "a distance" between the actual measure at time t and the product measure ν_t^γ, defined so that its averages are $m_\gamma(x,t|\sigma)$, and we prove that this distance vanishes when $\gamma \to 0$. The distance is defined in terms of the v-functions, i.e. special linear combinations of the correlation functions and the function m_γ defined in 2.3.1. We will see that the v-functions satisfy an integral equation, whose solution can be characterized quite explicitly yielding the desired results. A similar strategy has already been used in several other models, see [13] for a survey on the method. A refinement of the v-functions, which leads to the introduction of the ω functions, is developed in [II] to characterize the behavior of the fluctuation fields and to evidentiate the effects of the random forces which, though infinitesimal as $\gamma \to 0$, are nonetheless present. When there are instabilities, their effect is amplified and becomes macroscopic, as we shall see in [III] in the case of the spinodal decomposition after quenching from a high temperature pure phase.

Behind the definition of the v and of the ω-functions there are algebraic considerations classical in Statistical Mechanics, that will be made explicit in [II]. Before the definition of the v-functions we introduce some notation:

2.3.2 Notation.

For any positive integer n we denote by $\mathbb{Z}^{dn}_{\neq}$ the collection of all the sets $\underline{x} = (x_1, \dots, x_n)$ in $\mathbb{Z}^d$ with n distinct elements and write $|\underline{x}| = n$. S is the union of $\mathbb{Z}^{dn}_{\neq}$ over all $n \geq 0$, i.e. it is the collection of all the finite subsets of $\mathbb{Z}^d$, while S_∞ includes also the subsets of $\mathbb{Z}^d$ with infinite cardinality.

2.3.3. Definition. *The v-functions.*

When the Glauber dynamics starts from a single configuration σ, the v functions are

$$v^\gamma(\underline{x}, t|\sigma) = \mathbb{E}^\gamma_\sigma\Big(\prod_{x\in\underline{x}}[\sigma(x,t) - m_\gamma(x,t|\sigma)]\Big), \quad \underline{x} \in S, \;\; t \geq 0 \tag{2.22}$$

Analogous expression defines $v^\gamma(\cdot, t|\mu^\gamma)$, when the process starts from a general product measure μ^γ.

2.3.4. Theorem. *Bounds on the v functions.*

(i) There are $a > 0$ and C' and for any n c so that, for all $t \leq a \log \gamma^{-1}$,

$$\sup_\sigma \sup_{|\underline{x}|=n} |v^\gamma_n(\underline{x}, t|\sigma)| \leq c e^{C'nt} \gamma^{dn/2} \tag{2.23}$$

(ii) The same bound holds when the process starts from a product measure μ^γ and with $m_\gamma(x,t|\sigma)$ (in the espression defining v^γ) replaced either by $m_\gamma(x,t|\mu^\gamma)$ or by $m_\gamma(r,t)$, $r = \gamma x$), with m_γ solution of (1.1) with initial datum $m_{\gamma,0}$, as defined in (2.12a).

If (m_0, μ^γ) is a "standard initial state" (see the Remarks after Definition 2.1.7) and $m_0 \in C^1(\mathbb{R}^d)$ with bounded derivative, then

$$\lim_{\gamma\to 0} \sup_{0\leq t\leq a\log\gamma^{-1}} \sup_{|\underline{x}|=n} \left| \mathbb{E}^\gamma_{\mu^\gamma}\Big(\prod_{i=1}^n \sigma(x_i,t)\Big) - \prod_{i=1}^n m(\gamma x_i, t) \right| = 0 \tag{2.24}$$

where $m(r,t)$ solves (1.1) with initial datum $m_0(r)$.

Observe that from (2.23) it follows that given any $\zeta > 0$ there is $a > 0$ and, for any n, c so that

$$\sup_\sigma \sup_{|\underline{x}|=n} \sup_{t \le a \log \gamma^{-1}} |v_n^\gamma(\underline{x}, t|\sigma)| \le c\gamma^{n(d/2-\zeta)} \tag{2.25}$$

More precise estimates when $t = \tau \log \gamma^{-1}$, $0 < \tau \le a$ are reported in Theorem 4.9.

The mesoscopic limit is defined by letting $\gamma \to 0$ with the mesoscopic position r and the time t kept fixed: by (2.24) it then follows that, for the standard initial state and for any distinct $r_1, .., r_n$ and for any t

$$\lim_{\gamma \to 0} \mathbb{E}^\gamma_{\mu^\gamma} \Big(\prod_{i=1}^n \sigma(\gamma^{-1}[r_i]_\gamma, t) \Big) = \prod_{i=1}^n m(r_i, t) \tag{2.26}$$

We have thus derived the mesoscopic equation (1.1) also in the sense of (2.26), showing that in the limit the spin distribution factorizes.

A final remark about (2.23) regarding the fact that the bound improves with n, as $\gamma \to 0$. This is to some extent surprising, as one would not expect that the average of a product of n spins converges to the product of the averages faster than its rate when $n = 1$. This is indeed so, the whole point here is that we are not taking the average of the product of n spins, but rather a special combination of averages involving all the products of $k \le n$ spins. It is just this special combination which makes the convergence faster; notice however that if we keep γ fixed and let $n \to \infty$, then the dependence of the coefficient c on n will spoil (at least in our estimates), the decay rate $\gamma^{dn/2}$.

In [II] we will see that there are other combinations of correlation functions which describe in a more accurate way the factorization properties of the correlation functions.

3. Short time estimates

In this Section we prove a weaker version of (2.23), Theorem 3.7, where t is restricted to $t \le \gamma^\delta$, and δ is any fixed, positive number independent of γ.

We use the following notation: for $\underline{x} \in S$ (the set of all the finite subsets of $\mathbb{Z}^d$) we shorthand

$$\tilde{\sigma}(\underline{x}, t) = \prod_{x \in \underline{x}} [\sigma(x,t) - m_\gamma(x,t|\sigma)] \tag{3.1}$$

Recalling that $\mathbb{E}^\gamma_\sigma$ is the expectation of the process which starts at time 0 from the configuration σ, we have

$$\mathbb{E}^\gamma_\sigma\left(\tilde\sigma(\underline{x},t)\right) = \mathbb{E}^\gamma_\sigma\left(\tilde\sigma(\underline{x},0)\right) + \int_0^t ds\frac{d}{ds}\mathbb{E}_\sigma\left(\tilde\sigma(\underline{x},s)\right) \tag{3.2}$$

Since

$$\tilde\sigma(\underline{x},0)=0 \text{ because, for all } x,\; m_\gamma(x,0|\sigma)=\sigma(x,0) \tag{3.3}$$

(3.2) becomes

$$\mathbb{E}^\gamma_\sigma\left(\tilde\sigma(\underline{x},t)\right) = \int_0^t ds\mathbb{E}^\gamma_\sigma\Big(\sum_x D^{(x)}\tilde\sigma(\underline{x},s)\Big) \tag{3.4a}$$

where, for any function $g(\sigma,m)$, $\sigma\in\{-1,1\}^{\mathbb{Z}^d}$, $m\in[-1,1]^{\mathbb{Z}^d}$,

$$D^{(x)}g(\sigma,m) = \Big\{L^{(x)}_\gamma + \frac{dm_\gamma(x,s|\sigma)}{ds}\frac{\partial}{\partial m_\gamma(x,s|\sigma)}\Big\}g(\sigma,m) \tag{3.4b}$$

$L^{(x)}_\gamma$ is the generator of the process where only the spin at x flips, namely:

$$L^{(x)}_\gamma f(\sigma) = c_\gamma(x,\sigma)[f(\sigma^x)-f(\sigma)] \tag{3.5}$$

We now restrict to $t\le\gamma^\delta$. Since the integrand in (3.4a) is bounded uniformly in γ, we see right away that $|\mathbb{E}^\gamma_\sigma\left(\tilde\sigma(\underline{x},t)\right)|\le c\gamma^\delta$, for a suitable constant c. The bound that we want to prove, however, is much better than that and its proof will require some more work. Our strategy is based on an iterative analysis of (3.4). Since each integral which appears in the iteration is necessarily extended to an interval smaller than γ^δ, the terms with N integrals are then bounded proportionally to $\gamma^{N\delta}$ and, if $\delta N > nd/2$, n the cardinality of $\underline{x}$, we then obtain the desired bound. Unfortunately the argument is not as simple, the trouble is that the expression on the right hand side of (3.4a) is not of the form $\tilde\sigma(\underline{y})$, nor a linear combination of such terms. Therefore when we write the integral equation for such a term, we obtain in general a more complex expression with also terms which do not vanish when computed at time 0, unlike in (3.3)-(3.4a). We need a good characterization of the class of functions obtained after the action of products of the operators $D^{(x_i)}$ on $\tilde\sigma(\underline{x})$. This is accomplished in Proposition 3.6 below, but we preliminarily need extra notation and definitions. They are slightly more complex than what really needed in this Section, but with this extra effort we have the right setup also for the more delicate case of Section 4, where we will extend the proof to finite and longer times.

As we often switch from σ to m, it is convenient to think of them both in the same space $[-1,1]^{\mathbb{Z}^d}$, even though σ belongs to the restricted ensemble $\{-1,1\}^{\mathbb{Z}^d}$. The generic element of $[-1,1]^{\mathbb{Z}^d}$ is denoted by λ, not to be confused with the λ of (2.15), which will appear again only in Section 5.

3.1 Definition.

We first define the operators $\hat{a}_y^{\pm}$, $y \in \mathbb{Z}^d$, *which map* $[-1,1]^{\mathbb{Z}^d}$ *into itself as*

$$\hat{a}_y^{\pm}\lambda(x) = \begin{cases} \lambda(x) & \text{if } x \neq y \\ \pm 1 & \text{if } x = y \end{cases} \tag{3.6}$$

and then the operators $a_y^{\pm}$ *mapping* $\mathcal{M}\big([-1,1]^{\mathbb{Z}^d}\big)$ *into itself, as*

$$a_y^{\pm} f(\lambda) = f(\hat{a}_y^{\pm}\lambda) \tag{3.7a}$$

We also define, for $y \in \mathbb{Z}^d$ *and* $\underline{y} \in S$,

$$\partial_y = \gamma^{-d}\frac{a_y^+ - a_y^-}{2}, \quad a_{\underline{y}}^{\pm} = \prod_{y \in \underline{y}} a_y^{\pm}, \quad \partial_{\underline{y}} = \prod_{y\in\underline{y}} \partial_y \tag{3.7b}$$

Moreover, racalling the notation (2.9d) and setting $y^\star = (y, y') \in \mathbb{Z}^d \times \mathbb{Z}^d$, $y' \in B_{\gamma^{-\alpha},y} - y$, $\alpha \in (0,1)$, *we define*

$$\delta_{y^\star} = \frac{1}{2\gamma^{d+1-\alpha}}[a_y^+ a_{y'}^- - a_{y'}^- a_y^+]; \qquad \delta_{\underline{y}^\star} = \prod_{y^\star \in \underline{y}^\star} \delta_{y^\star} \tag{3.7c}$$

where $\underline{y}^\star = (y_1^\star, .., y_\ell^\star)$ *is a finite collection of elements* $y_i^\star$, *such that the entries* y_i, y_i', *of* $y_i^\star$, $i = 1, .., \ell$, *are all distinct. Their union is denoted by* $\underline{\underline{y}}^\star$. *We call* S^2 *the set of all* $\underline{y}^\star$.

We will also use the convention that when a set is empty the corresponding operator is the identity.

The operators ∂_y and $\delta_{y^\star}$ play the role of derivatives and indeed they are essentially derivatives when acting on functions which depend "weakly" on $\lambda(y)$, as shown in Lemma 3.3 below.

3.2 Definition.

We denote by Ξ *the set of all* $\xi = (x, \underline{y}_1, \underline{y}_2, \underline{y}_3, \underline{y}^\star)$, $x \in \mathbb{Z}^d$, $\underline{y}_i \in S$, $i = 1,2,3$, $\underline{y}^\star \in S^2$, *with the condition that* $\underline{y}_1$, $\underline{y}_2$, $\underline{y}_3$ *and* $\underline{\underline{y}}^\star$ *are all disjoint. Their union is denoted by* V_ξ.

Ξ^0 *is the subset of* Ξ *with* $\underline{y}^\star = \emptyset$, *in that case we simply write* $\xi = (x, \underline{y}_1, \underline{y}_2, \underline{y}_3)$. *We use the convention of dropping the entries of* ξ *which are emptysets, thus writing* x *for* $\xi = (x, \emptyset, \emptyset, \emptyset, \emptyset)$ *as in the definition:*

$$f_x(\lambda) = \tanh\{\beta[\sum_{y\neq x} J_\gamma(x,y)\lambda(y) + h]\} \tag{3.8}$$

We also define for $\xi=(x,\underline{y}_1,\underline{y}_2,\underline{y}_3,\underline{y}^\star)$,

$$f_\xi(\lambda)=a^+_{\underline{y}_1}a^-_{\underline{y}_2}\partial_{\underline{y}_3}\delta_{\underline{y}^\star}f_x(\lambda) \tag{3.9a}$$

In particular if $\xi\in\Xi^0$, *(3.9a) becomes*

$$f_\xi(\lambda)=a^+_{\underline{y}_1}a^-_{\underline{y}_2}\partial_{\underline{y}_3}f_x(\lambda) \tag{3.9b}$$

Finally we call

$$d(\xi)=|\underline{y}_3|+|\underline{y}^\star_4| \tag{3.10a}$$

the "number of derivatives present in ξ*".*

The last notation is justified by the following lemma:

3.3 Lemma.

Let $\xi\in\Xi$, *then*

$$f_\xi(\lambda)=\int d\epsilon\,\lambda_{(\gamma,\xi)}(\epsilon)\tanh^{(d(\xi))}\left(\psi_\xi(\lambda)+\epsilon\right) \tag{3.10b}$$

where $\tanh^{(k)}(\cdot)$ *is the* k*-th derivative of* $\tanh(\cdot)$*;*

$$\psi_\xi(\lambda)=\ \beta\sum_{z\notin V_\xi}\ J_\gamma(x,z)\lambda(z) \tag{3.10c}$$

$\lambda_{(\gamma,\xi)}(\epsilon)$ *is a function supported in an interval* $I_{(\gamma,\xi)}$ *of* $\mathbb{R}$ *containing the point*

$$\beta\{[\sum_{z\in\underline{y}_1}-\sum_{z\in\underline{y}_2}]J_\gamma(x,z)+h]\} \tag{3.10d}$$

and furthermore there is c *so that if* $\underline{y}_3\neq\emptyset$

$$\|\lambda_{(\gamma,\xi)}\|_\infty\le c\gamma^{-d};\quad |I_{(\gamma,\xi)}|\le c\gamma^d \tag{3.11a}$$

while if $\underline{y}_3=\emptyset$ *and* $\underline{y}^\star\neq\emptyset$

$$\|\lambda_{(\gamma,\xi)}\|_\infty\le c\gamma^{-d-1+\alpha};\quad |I_{(\gamma,\xi)}|\le c\gamma^{d+1-\alpha} \tag{3.11b}$$

If both $\underline{y}_3=\underline{y}^\star=\emptyset$ *then the density in (3.10b) is a delta function at* $\epsilon=0$*.*

Proof.

We start by computing $\partial_{\underline{y}_3}f_x(\lambda)$. We write $\underline{y}_3=(y_{3,1},..,y_{3,\ell})$ and

$$\partial_{\underline{y}_3}f_x(\lambda)=\partial_{y_{3,\ell}}\dots\partial_{y_{3,1}}f_x(\lambda)$$

We set, for $j = 1,..,\ell$,

$$F_j(\lambda) = \partial_{y_{3,j}} \dots \partial_{y_{3,1}} f_x(\lambda) \tag{3.12}$$

We claim that

$$F_j(\lambda) = \int d\epsilon \lambda_{\gamma,j}(\epsilon) \tanh^{(j)} \left\{ \beta \sum_{y \in Y^{(j)}} J_\gamma(x,y)\lambda(y) + \beta h + \epsilon \right\} \tag{3.13a}$$

where

$$Y^{(j)} = \text{the complement of } \{y_1, \dots, y_j\} \text{ in } \mathbb{Z}^d$$

$\lambda_{\gamma,j}(\epsilon)$ is supported by an interval $I_{\gamma,j}$ and there is c_j so that

$$\|\lambda_{\gamma,j}\|_\infty \le c_j \gamma^{-d}; \quad |I_{\gamma,j}| \le c_j \gamma^d \tag{3.13b}$$

We will prove the claim (3.13) by induction on j. We thus assume that (3.13) holds for $j < \ell$ and, writing

$$\psi_n := \beta\{ \sum_{y \in Y^{(n)}} J_\gamma(x,y)\lambda(y) + h\}, \qquad z = y_{3,j+1}$$

we have

$$\begin{aligned} \partial_z \tanh^{(j)}(\psi_j + \epsilon) &= \frac{1}{2\gamma^d} \Big\{ \tanh^{(j)} \big(\psi_{j+1} + \epsilon + \beta J_\gamma(x,z)\big) - \tanh^{(j)} \big(\psi_{j+1} + \epsilon - \beta J_\gamma(x,z)\big) \Big\} \\ &= \int_{-\beta J_\gamma(x,z)}^{\beta J_\gamma(x,z)} \frac{d\epsilon'}{2\gamma^d} \tanh^{(j+1)}(\psi_{j+1} + \epsilon + \epsilon') \end{aligned} \tag{3.14}$$

Hence

$$\lambda_{\gamma,j+1}(\epsilon) = \int_{-\beta J_\gamma(x,z)}^{\beta J_\gamma(x,z)} \frac{d\epsilon'}{2\gamma^d} \lambda_{\gamma,j}(\epsilon - \epsilon') \tag{3.15}$$

so that

$$\|\lambda_{\gamma,j+1}\|_\infty \le 2\beta \frac{|J_\gamma(x,z)|}{2\gamma^d} \|\lambda_{\gamma,j}\|_\infty; \quad |I_{\gamma,j+1}| \le |I_{\gamma,j}| + 2\beta |J_\gamma(x,z)| \tag{3.16}$$

hence $c_{j+1} = (1 + 2\beta \|J\|_\infty) c_j$.

The same argument shows that

$$\lambda_{\gamma,1}(\epsilon) = \frac{1}{2\gamma^d} \mathbf{1}\big(|\epsilon| \le \beta |J_\gamma(x, y_{3,1})|\big) \tag{3.17}$$

so that the claim (3.13) is proven.

The same argument applies to $\delta_{\underline{y}^\star}$. The action of $\delta_{y^\star}$, $y^\star = (y,y')$, involves a change in the argument of the function $\tanh^{(k)}$ from $-[J_\gamma(x,y) - J_\gamma(x,y')]$ to its opposite. The difference is bounded proportionally to $\gamma^{d+1-\alpha}$, recalling that, by definition $|y - y'| \le \gamma^{-\alpha}$.

Since $a^+_{\underline{y}_1}$ simply amounts to setting $\lambda(y) = 1$ for all $y \in \underline{y}_1$, and analogously for $a^-_{\underline{y}_2}$, the lemma easily follows, we omit the details. $\square$

Remark

As clear from its proof, Lemma 3.3 extends to the case when f_x is replaced by any function $g \in C^\infty$ of the argument $\sum_y J_\gamma(x,y)\lambda(y) + h$, in particular by any combination or integral of derivatives of $\tanh(\cdot)$.

3.4 Definition.

We call H the collection of all $\eta = (\underline{x}, \underline{y}, \xi_1, .., \xi_p)$, with $\underline{x}$ and $\underline{y}$ in S, ξ_1,..,ξ_p in Ξ and H^0 the subset of all $\eta = (\underline{x}, \emptyset, \xi_1, .., \xi_p)$, with $\xi_i \in \Xi^0$. In this case we simply write $\eta = (\underline{x}, \xi_1, .., \xi_p)$.

We define "the order of η" as

$$|\eta| = |\underline{x}| + |\underline{y}| + p \tag{3.18}$$

We also set

$$g(\eta) = |\underline{y}| + p \tag{3.19}$$

and call it the g-order of η and, finally, recalling (3.10a),

$$d(\eta) = \sum_{i=1}^{p} d(\xi_i) \tag{3.20}$$

We need a last definition:

3.5 Definition.

Given $\eta \in H$, we define the function $\rho_\eta \in \mathcal{M}\big(\{-1,1\}^{\mathbb{Z}^d} \times [-1,1]^{\mathbb{Z}^d}\big)$ as

$$\rho_\eta(\sigma, m) = \tilde{\sigma}(\underline{x}) \sum_{\underline{z}} \chi_{\underline{x},\underline{y}}(\underline{z}) \tilde{\sigma}(\underline{z}) \prod_{i=1}^{p} [f_{\xi_i}(\sigma) - f_{\xi_i}(m)] \tag{3.21}$$

where, if $\ell := |\underline{y}|$, $\underline{z} \in \mathbb{Z}^{d\ell}_{\neq}$,

$$\chi_{\underline{x},\underline{y}}(\underline{z}) = \prod_{i=1}^{\ell} \left\{ \frac{1}{|B_{\gamma^{-\alpha}}|} \sum_{z_i} \mathbf{1}\big(z_i \in B_{\gamma^{-\alpha}, y_i} - \underline{x}\big) \right\} \tag{3.22}$$

In particular, if $\eta \in H^0$, we restrict to this case in the sequel of this Section,

$$\rho_\eta(\sigma, m) = \tilde{\sigma}(\underline{x}) \prod_{i=1}^{p} [f_{\xi_i}(\sigma) - f_{\xi_i}(m)] \tag{3.23}$$

Finally if all the entries of η are the emptyset, we write $\eta = \emptyset$ and define $\rho_\emptyset = 1$.

3.6 Proposition.

There is $M_\gamma(\eta, \eta', m)$, η, η' in H^0, $m \in [-1,1]^{\mathbb{Z}^d}$, so that, for all γ, η, m and $\sigma \in \{-1,1\}^{\mathbb{Z}^d}$,

$$\sum_{x\in\mathbb{Z}^d} D^{(x)}\rho_\eta(\sigma, m) = \sum_{\eta'\in H^0} M_\gamma(\eta,\eta',m)\rho_{\eta'}(\sigma,m) \tag{3.24}$$

with $M_\gamma(\eta,\eta',m) = 0$ if $|\eta'| > |\eta| + 1$ and if $d(\eta') > d(\eta) + |\eta|$. Furthermore there are coefficients $c(n,q)$, $n > 0$, $q \geq 0$, (independent of γ) such that for all γ and all $m \in [-1,1]^{\mathbb{Z}^d}$

$$\sup_{\substack{|\eta|=n\\ d(\eta)=q}} \sum_{|\eta'|=p} |M_\gamma(\eta,\eta',m)| \leq \begin{cases} c(n,q)\gamma^{d(n-p)/2} & \text{if } p \leq n \\ c(n,q) & \text{if } p = n, n+1 \end{cases} \tag{3.25}$$

Proof.

We start by computing the action of $D^{(x)}$ on a single spin. By (2.20a),

$$\begin{aligned} D^{(x)}\tilde\sigma(x) &= L_\gamma^{(x)}\sigma(x) - \Big(- m(x) + \tanh\{\beta[J_\gamma \circ m)(x) + h]\}\Big) \\ &= -\tilde\sigma(x) + [f_x(\sigma) - f_x(m)] \end{aligned} \tag{3.26}$$

Notice that if $\underline{x}$ reduces to x with all $\xi_i = \emptyset$, then $\rho_\eta = \tilde\sigma(x)$, hence (3.24) and (3.25) are verified. To study the general case we first write:

$$\begin{aligned} f_\xi(\sigma) &= \{\frac{1+\sigma(x)}{2}a_x^+ + \frac{1-\sigma(x)}{2}a_x^-\}f_\xi(\sigma) \\ &= \frac{a_x^+ + a_x^-}{2}f_\xi(\sigma) + \sigma(x)\gamma^d\partial_x f_\xi(\sigma) \\ &= \{\frac{a_x^+ + a_x^-}{2}[f_\xi(\sigma) - f_\xi(m)] + \frac{a_x^+ + a_x^-}{2}f_\xi(m)\} + \{\tilde\sigma(x)\gamma^d\partial_x[f_\xi(\sigma) - f_\xi(m)] \\ &\quad + \tilde\sigma(x)\gamma^d\partial_x f_\xi(m) + m(x)\gamma^d\partial_x f_\xi(m) + m(x)\gamma^d\partial_x[f_\xi(\sigma) - f_\xi(m)]\} \end{aligned} \tag{3.27}$$

Therefore

$$f_\xi(\sigma) - f_\xi(m) = \Gamma_{1,x} + m(x)\gamma^d\Gamma_{2,x} + \tilde\sigma(x)\gamma^d[\Gamma_{3,x} + \Gamma_{4,x}] + \gamma^{2d}\Gamma_{5,x} \tag{3.28a}$$

$$\begin{aligned} &\Gamma_{1,x} = \frac{a_x^+ + a_x^-}{2}[f_\xi(\sigma) - f_\xi(m)], \qquad \Gamma_{2,x} = \partial_x[f_\xi(\sigma) - f_\xi(m)] \\ &\Gamma_{3,x} = \partial_x[f_\xi(\sigma) - f_\xi(m)], \qquad \Gamma_{4,x} = \partial_x f_\xi(m) \\ &\Gamma_{5,x} = \gamma^{-2d}\{\frac{a_x^+ + a_x^-}{2}f_\xi(m) + m(x)\gamma^d\partial_x f_\xi(m) - f_\xi(m)\} \end{aligned} \tag{3.28b}$$

The equations (3.27)-(3.28) hold for any function $f(\cdot)$, however if $f = f_\xi$ there is a constant c so that

$$|\Gamma_{5,x}| \le c, \quad \text{uniformly in } \gamma,\ m \text{ and } x \tag{3.28c}$$

as we are going to prove.

First observe that if $x \in V_\xi$ then $\Gamma_{5,x} = 0$. We then suppose that $x \notin V_\xi$, then, by Lemma 3.3 there is an interval $I_{(\gamma,\xi)}$ and a function $\lambda_{(\gamma,\xi)}$ supported on $I_{(\gamma,\xi)}$ and such that

$$\rho_\xi(m) = \int_{I_{(\gamma,\xi)}} d\epsilon \lambda_{(\gamma,\xi)}(\epsilon) \tanh^{(d(\xi)}\big(\psi + \epsilon + J_\gamma(\hat{x}, x) m(x)\big)$$

where

$$\xi = (\hat{x}, \underline{\hat{y}}_1, \underline{\hat{y}}_2, \underline{\hat{y}}_3), \qquad \psi = \beta \sum_{y \notin V_\xi} J_\gamma(\hat{x}, y) m(y) + \beta h$$

and, for a suitable constant c

$$\|\lambda_{(\gamma,\xi)}\|_\infty \le c\gamma^{-d}, \qquad |I_{(\gamma,\xi)}| \le c\gamma^d$$

We then have:

$$\Gamma_{5,x} = \int_{I_{(\gamma,\xi)}} d\epsilon \lambda_{(\gamma,\xi)}(\epsilon) \gamma^{-2d} \Phi(\epsilon) \tag{3.29a}$$

$$\begin{aligned}\Phi(\epsilon) =& \frac{1}{2} \tanh^{(d(\xi)}\big(\psi + \epsilon + J_\gamma(\hat{x}, x)\big) + \frac{1}{2} \tanh^{(d(\xi)}\big(\psi + \epsilon - J_\gamma(\hat{x}, x)\big) \\ &+ m(x) \Big\{ \frac{1}{2} \tanh^{(d(\xi)}\big(\psi + \epsilon + J_\gamma(\hat{x}, x)\big) - \frac{1}{2} \tanh^{(d(\xi)}\big(\psi + \epsilon - J_\gamma(\hat{x}, x)\big) \Big\} \\ &- \tanh^{(d(\xi)}\big(\psi + \epsilon + J_\gamma(\hat{x}, x) m(x)\big)\end{aligned} \tag{3.29b}$$

By expanding Φ in powers to second order in the small parameter $J_\gamma(\hat{x}, x)$, we readily see that (3.28c) holds. It follows from this computation that the derivative of $\Gamma_{5,x}$ with respect to $m(x)$ is also bounded.

Notation and remarks.

We denote by $\Gamma^j_{i,x}$, $i = 1,..,5$ *and* $1 \le j \le p$, *the term* $\Gamma_{i,x}$ *in (3.28b) when* $\xi = \xi_j$.

We use the notation $A + B$ *to denote the union of the sets* A *and* B *and* $A - B$ *for the set theoretical difference of* A *and* B.

All the terms $\Gamma^j_{i,x}$ *do not depend on* $\sigma(x)$; $\Gamma^j_{i,x}$, $i \le 4$, *does not depend on* $m(x)$, *and, finally,* $|\partial/\partial m(x) \Gamma^j_{5,x}|$ *is bounded.*

We hereafter set $\eta = (\underline{x}, \xi_1, .., \xi_p)$, $\xi_i = \big(\hat{x}_i, \underline{\hat{y}}_1, \underline{\hat{y}}_2, \underline{\hat{y}}_3\big)$.

We write

$$\rho_\eta = \sum_{x\in\underline{x}} \tilde\sigma(\underline{x}-x)[\prod_{j=1}^{p}\Gamma^j_{1,x}]D^{(x)}\tilde\sigma(x)$$

$$+\gamma^d\sum_{x\notin\underline{x}}\tilde\sigma(\underline{x})\sum_{i=1}^{p}[\prod_{j\neq i}\Gamma^j_{1,x}](\Gamma^i_{3,x}+\Gamma^i_{4,x})D^{(x)}\tilde\sigma(x)+S_1+S_2 \tag{3.30a}$$

where

$$S_1 = \sum_{x\in\underline{x}}\tilde\sigma(\underline{x}-x)\sum_{\substack{\underline{i}_1,..,\underline{i}_5\\ |\underline{i}_1|<p}}\Delta_{\underline{i}_1,..,\underline{i}_5,\underline{x},x} \tag{3.30b}$$

$$S_2 = \sum_{x\notin\underline{x}}\tilde\sigma(\underline{x})\sum_{\substack{\underline{i}_1,..,\underline{i}_5\\ |\underline{i}_1|<p-1}}\Delta_{\underline{i}_1,..,\underline{i}_5,\underline{x},x} \tag{3.30c}$$

$$\Delta_{\underline{i}_1,..,\underline{i}_5,\underline{x},x} = \gamma^{d(|\underline{i}_2|+|\underline{i}_3|+|\underline{i}_4|+2|\underline{i}_5|)}\Gamma^{\underline{i}_1}_{1,x}...\Gamma^{\underline{i}_5}_{5,x}D^{(x)}\{m(x)^{|\underline{i}_2|}\tilde\sigma(x)^k\} \tag{3.30d}$$

the sum is over all the partitions of $1,..,p$ into five atoms, $\underline{i}_1$,.., $\underline{i}_5$; $|\underline{i}_j|$ denotes the number of elements in $\underline{i}_j$ and

$$\Gamma^{\underline{i}_j}_{j,x} = \prod_{i\in\underline{i}_j}\Gamma^i_{j,x}$$

Finally

$$k = k(x,\underline{x},\underline{i}_3+\underline{i}_4) = \begin{cases} 1+|\underline{i}_3|+|\underline{i}_4| & \text{if } x\in\underline{x} \\ |\underline{i}_3|+|\underline{i}_4| & \text{if } x\notin\underline{x}\end{cases} \tag{3.30e}$$

By using (3.26), the first term on the right hand side of (3.30a) can be written as

$$\sum_{\eta'} M_{0,1}(\eta,\eta',m)\rho_{\eta'} \tag{3.31a}$$

where the sum is restricted to $|\eta'|=|\eta|$ and

$$\sum_{\eta'}|M_{0,1}(\eta,\eta',m)|\le 2|\underline{x}| \tag{3.31b}$$

In fact the sum has $|\underline{x}|$ terms and, using (3.26), each gives rise to two ρ_η functions.

The second term in (3.30a) is also of the form

$$\sum_{\eta'} M_{0,2}(\eta,\eta',m)\rho_{\eta'} \tag{3.31c}$$

with $|\eta'|=|\eta|+1$ and $|\eta'|=|\eta|$, respectively for $\Gamma^i_{3,x}$ and $\Gamma^i_{4,x}$. We also have

$$\sum_{\eta'}|M_{0,2}(\eta,\eta',m)|\le cp \tag{3.31d}$$

because $\Gamma^i_{3,x} = \Gamma^i_{4,x} = 0$ unless $|x - \hat{x}_i| \le \gamma^{-1}$. If $|x - \hat{x}_i| > \gamma^{-1}$, then $f_{\xi_i}(\lambda)$ does not depend on $\lambda(x)$ and $\partial_x f_{\xi_i} = 0$. In any case, as it follows from Lemma 3.3, $\partial_x f_{\xi_i}(m)$ is bounded, hence (3.31b).

We are going to prove that also S_1 and S_2 can be decomposed as the two terms above with coefficients which satisfy (3.25), and this will prove Proposition 3.6.

We start with some easy algebraic computation:
Let $k \ge 1$, then

$$\begin{aligned}
\tilde{\sigma}(x)^k &= \{\frac{1+\sigma(x)}{2}[1-m(x)]^k + \frac{1-\sigma(x)}{2}[-1-m(x)]^k \\
&= \tilde{\sigma}(x)\left\{\frac{[1-m(x)]^k}{2} - \frac{[-1-m(x)]^k}{2}\right\} \\
&\qquad\qquad + \frac{1+m(x)}{2}[1-m(x)]^k + \frac{1-m(x)}{2}[-1-m(x)]^k \\
&= \tilde{\sigma}(x)a(k,m(x)) + b(k,m(x))
\end{aligned} \tag{3.32}$$

with a and b defined by the last equality. Thus $a(1,m(x)) = 1$, $b(1,m(x)) = 0$ and there are constants b_k so that $|a(k,m(x))| \le b_k$ and $|b(k,m(x))| \le b_k$.

Recalling that $k = 1 + |\underline{i}_3| + |\underline{i}_4|$ if $x \in \underline{x}$ and $k = |\underline{i}_3| + |\underline{i}_4|$ if $x \notin \underline{x}$, we have

$$\begin{aligned}
\Delta_{\underline{i}_1,..,\underline{i}_5,\underline{x},x} = \sum_{\underline{i}_1,..,\underline{i}_5} \Big\{ & m(x)^{|\underline{i}_2|} a(k,m(x)) D^{(x)}\tilde{\sigma}(x) + \tilde{\sigma}(x)[m(x)^{|\underline{i}_2|}a(k,m(x))]' \\
& + [m(x)^{|\underline{i}_2|} b\big(k,m(x)\big)]' \Big\} \gamma^{d(|\underline{i}_2|+|\underline{i}_3|+|\underline{i}_4|+2|\underline{i}_5|)} \tilde{\sigma}(\underline{x}-x)\Gamma^{\underline{i}_1}_{1,x}...\Gamma^{\underline{i}_5}_{5,x}
\end{aligned} \tag{3.33}$$

where h' above denotes the derivative of h with respect to $m(x)$ times $\dot{m}(x)$.

We write

$$S_1 = S_{1,1} + S_{1,2} \tag{3.34}$$

where the decomposition arises from taking the term with and respectively without $\tilde{\sigma}(x)$ on the right hand side of (3.33).

We have

$$\begin{aligned}
S_{1,1} = \sum_{x\in\underline{x}} \sum_{\substack{\underline{i}_1,..,\underline{i}_5 \\ |\underline{i}_1|<p}} & \Big\{ m(x)^{|\underline{i}_2|} a(k,m(x)) D^{(x)}\tilde{\sigma}(x) + \tilde{\sigma}(x)[m(x)^{|\underline{i}_2|}a(k,m(x))]' \Big\} \\
& \times \gamma^{d(|\underline{i}_2|+|\underline{i}_3|+|\underline{i}_4|+2|\underline{i}_5|)} \tilde{\sigma}(\underline{x}-x)\Gamma^{\underline{i}_1}_{1,x}...\Gamma^{\underline{i}_5}_{5,x}
\end{aligned} \tag{3.35a}$$

By (3.26),

$$S_{1,1} = \sum_{\eta'} M_{1,1}(\eta,\eta',m)\rho_{\eta'} \tag{3.35b}$$

with $\eta' = |\eta| - (|\underline{i}_4| + |\underline{i}_5|)$, and, for $q \geq 0$,

$$\sum_{|\eta'|=|\eta|-q} |M_{1,1}(\eta,\eta',m)| \leq c|\underline{x}| \sum_{\substack{\underline{i}_1,\dots,\underline{i}_5 \\ |\underline{i}_1|<p}} \mathbf{1}(|\underline{i}_4| + |\underline{i}_5| = q)\gamma^{d(|\underline{i}_2|+|\underline{i}_3|+|\underline{i}_4|+2|\underline{i}_5|)}$$
$$\leq c'|\underline{x}|\gamma^{dq/2}\gamma^{d/2} \tag{3.35c}$$

with c and c' suitable constants.

We next consider $S_{1,2}$, which, with the shorthand notation $k = |\underline{i}_3| + |\underline{i}_4| + 1$, is

$$S_{1,2} = \sum_{x\in\underline{x}} \sum_{\underline{i}_1,\dots,\underline{i}_5} [m(x)^{|\underline{i}_2|} b(k, m(x))]' \gamma^{d(|\underline{i}_2|+|\underline{i}_3|+|\underline{i}_4|+2|\underline{i}_5|)} \tilde{\sigma}(\underline{x} - x)\Gamma^{\underline{i}_1}_{1,x}\dots\Gamma^{\underline{i}_5}_{5,x} \tag{3.36a}$$

which also has the form

$$S_{1,2} = \sum_{\eta'} M_{1,2}(\eta,\eta',m)\rho_{\eta'} \tag{3.36b}$$

with $\eta'| = |\eta| - (|\underline{i}_4| + |\underline{i}_5| + 1)$. These terms are only present when $k = |\underline{i}_3| + |\underline{i}_4| + 1 \geq 2$, because $b(1, m(x)) = 0$. We have, for $q \geq 1$,

$$\sum_{|\eta'|=|\eta|-q} |M_{1,2}(\eta,\eta',m)| \leq c|\underline{x}| \sum_{\underline{i}_1,\dots,\underline{i}_5} \mathbf{1}(|\underline{i}_3| + |\underline{i}_4| \geq 1; |\underline{i}_4| + |\underline{i}_5| + 1 = q)\gamma^{d(|\underline{i}_2|+|\underline{i}_3|+|\underline{i}_4|+2|\underline{i}_5|)}$$
$$\leq c'|\underline{x}|\gamma^{d(q-1)/2}\gamma^{d/2} = c'|\underline{x}|\gamma^{dq/2} \tag{3.36c}$$

with c and c' suitable constants.

Splitting S_2 using (3.33), just as we did for S_1, we get $S_2 = S_{2,1} + S_{2,2}$. Recalling that in this case $k = |\underline{i}_3| + |\underline{i}_4|$, we have

$$S_{2,1} = \sum_{x\notin\underline{x}} \sum_{\substack{\underline{i}_1,\dots,\underline{i}_5 \\ |\underline{i}_1|\leq p-2}} \left\{ m(x)^{|\underline{i}_2|} a(k, m(x)) D^{(x)}\tilde{\sigma}(x) + \tilde{\sigma}(x)[m(x)^{|\underline{i}_2|} a(k, m(x))]' \right\}$$
$$\times \gamma^{d(|\underline{i}_2|+|\underline{i}_3|+|\underline{i}_4|+2|\underline{i}_5|)} \tilde{\sigma}(\underline{x} - x)\Gamma^{\underline{i}_1}_{1,x}\dots\Gamma^{\underline{i}_5}_{5,x} \tag{3.37a}$$

The only contribution to (3.37a) comes from the terms with $|\underline{i}_3|+|\underline{i}_4| \geq 1$ because $a(k, m(x)) = 0$ if $k = |\underline{i}_3| + |\underline{i}_4| = 0$. If $i \in \underline{i}_3$, $|x - \hat{x}_i| \leq \gamma^{-1}$, because of the operator ∂_x in $\Gamma_{i,3}$ and $\Gamma_{i,4}$. Therefore the sum is "only" over $(2\gamma^{-1} + 1)^d$ elements, at most.

$S_{2,1}$ can be written as

$$S_{2,1} = \sum_{\eta'} M_{2,1}(\eta,\eta',m)\rho_{\eta'} \tag{3.37b}$$

with $\eta' = |\eta| - (|\underline{i}_4| + |\underline{i}_5| - 1)$. Recalling that

$$|\underline{i}_2| + |\underline{i}_3| + |\underline{i}_4| + |\underline{i}_5| \geq 2$$

because $|\underline{i}_1| \le p-2$, we have

$$\sum_{|\eta'|=|\eta|+1} |M_{2,1}(\eta,\eta',m)| \le \{(2\gamma^{-1}+1)^d p\} \sum_{\substack{|\underline{i}_4|+|\underline{i}_5|=0\\|\underline{i}_2|+|\underline{i}_3|\ge 2}} c\gamma^{d|\underline{i}_2|+|\underline{i}_3|} \le c'\gamma^d \tag{3.37c}$$

Analogously, for $q \ge 0$:

$$\sum_{|\eta'|=|\eta|-q} |M_{2,1}(\eta,\eta',m)| \le \{(2\gamma^{-1}+1)^d p\} \sum_{\substack{\underline{i}_1,\ldots,\underline{i}_5\\|\underline{i}_1|\le p-2}} \mathbf{1}(|\underline{i}_4|+|\underline{i}_5|=q+1)$$
$$\times\, c\gamma^{d(|\underline{i}_2|+|\underline{i}_3|+|\underline{i}_4|+2|\underline{i}_5|)} \le c'\gamma^{d(q+1)/2} \tag{3.37d}$$

with c and c' suitable constants.

The last term is (again k is a shorthand for $|\underline{i}_3|$, below):

$$S_{2,2} = \sum_{x\notin\underline{x}} \sum_{\substack{\underline{i}_1,\ldots,\underline{i}_5\\|\underline{i}_3|+|\underline{i}_4|\ge 2}} [m(x)^{|\underline{i}_2|} b(k,m(x))]'\gamma^{d(|\underline{i}_2|+|\underline{i}_3|+2|\underline{i}_5|)}\tilde{\sigma}(\underline{x})\Gamma_{1,x}^{\underline{i}_1}\ldots\Gamma_{5,x}^{\underline{i}_5} \tag{3.37a}$$

The condition $k=|\underline{i}_3|+|\underline{i}_4|\ge 2$ follows from the fact that $b(k,m(x))=0$ for $k=0,1$. Like before we have

$$S_{2,2} = \sum_{\eta'} M_{2,2}(\eta,\eta',m)\rho_{\eta'} \tag{3.38a}$$

Here $|\eta'| = |\eta| - (|\underline{i}_4|+|\underline{i}_5|)$ and, for $q\ge 0$,

$$\sum_{|\eta'|=|\eta|-q} |M_{2,2}(\eta,\eta',m)| \le \{(2\gamma^{-1}+1)^d p\} \sum_{\substack{\underline{i}_1,\ldots,\underline{i}_5\\|\underline{i}_3|+|\underline{i}_4|\ge 2}} \mathbf{1}(|\underline{i}_4|+|\underline{i}_5|=q)c\gamma^{d(|\underline{i}_2|+|\underline{i}_3|+|\underline{i}_4|+2|\underline{i}_5|)}$$
$$\le \{(2\gamma^{-1}+1)^d p\}c' \sum_{\substack{\underline{i}_1,\ldots,\underline{i}_5\\|\underline{i}_3|+|\underline{i}_4|\ge 2}} \mathbf{1}(|\underline{i}_4|+|\underline{i}_5|=q)\gamma^{dq/2}\gamma^{d(|\underline{i}_3|+|\underline{i}_4|)/2} \le c''\gamma^{dq/2} \tag{3.38b}$$

with c, c' and c'' suitable constants.

We have thus concluded the proof of the Proposition. $\square$

3.7 Theorem.

For any $\delta > 0$ and $n \ge 1$ there is c so that for all σ:

$$\sup_{\sigma}\ \sup_{|\underline{x}|=n}\ \sup_{t\le\gamma^\delta} |v_n^\gamma(\underline{x},t|\sigma)| \le c\gamma^{dn/2} \tag{3.38}$$

Proof.

By Proposition 3.6, for $\eta \neq \emptyset$

$$\mathbb{E}_\sigma^\gamma\Big(\rho_\eta\big(\sigma_t, m_\gamma(\cdot,t|\sigma)\big)\Big) = \int_0^t ds \sum_{\eta'} M_\gamma(\eta,\eta', m_\gamma(\cdot,s|\sigma))\,\mathbb{E}_\sigma^\gamma\Big(\rho_{\eta'}\big(\sigma_s, m_\gamma(\cdot,s)\big)\Big) \tag{3.39}$$

because $\rho_\eta(\sigma, m_\gamma(\cdot,0|\sigma)) = 0$, since $\sigma = m(\cdot,0|\sigma)$. We then write

$$M_\gamma(\eta,\eta',m) = M'_\gamma(\eta,\eta',m) + M''_\gamma(\eta,\eta',m) \tag{3.40a}$$

where $M'_\gamma(\eta,\eta',m) = 0$ if $\eta' = \emptyset$ and $M''_\gamma(\eta,\eta',m)$ if $\eta' \neq \emptyset$. We then define

$$\hat M'_\gamma(\eta,\eta') = \sup_m |M'_\gamma(\eta,\eta',m)|, \quad \hat M''_\gamma(\eta,\eta') = \sup_m |M''_\gamma(\eta,\eta',m)| \tag{3.40b}$$

We call $\eta_0 = (\underline{x},\emptyset,\emptyset,\emptyset)$, $\underline{x} \in S$, $|\underline{x}| = n$ and set

$$N:\ \delta N > \frac{dn}{2} \tag{3.41}$$

By iterating (3.39) N-times we get:

$$\begin{aligned}\Big|\mathbb{E}_\sigma^\gamma\Big(\rho_{\eta_0}\big(\sigma_t, m(\cdot,t|\sigma)\big)\Big)\Big| \le \sum_{j=0}^{N-1} \frac{\gamma^{\delta j}}{(j+1)!} \sum_{\eta'} (\hat M'_\gamma)^j(\eta_0,\eta')\hat M''_\gamma(\eta',\emptyset) \\ + \frac{\gamma^{\delta N}}{N!}\sum_{\eta'}(\hat M'_\gamma)^N(\eta_0,\eta')\sup_{\sigma,m}\big|\rho_{\eta'}(\sigma,m)\big|\end{aligned} \tag{3.42}$$

By (3.25) there is $C(n,q)$, $(n,q) \in \mathbb{Z}_+^2$, such that

$$\gamma^{-|\eta|d/2}\sum_{\eta'} \hat M_\gamma(\eta,\eta')\gamma^{|\eta'|d/2} \le C\big(|\eta|, d(\eta)\big) \tag{3.43}$$

with $\hat M_\gamma := \hat M'_\gamma + \hat M''_\gamma$. Calling

$$C^\star = \max_{\substack{p \le n+N \\ q \le (n+N)N}} C(p,q) \tag{3.44}$$

we get from (3.43), after telescopic cancellations:

$$\gamma^{-|\eta_0|d/2}\sum_{\eta'} \hat M^j_\gamma(\eta,\eta')\hat M''_\gamma(\eta',\emptyset) \le C^{\star\, j+1} \tag{3.45}$$

so that the sum over j on the right hand side of (3.42) is bounded proportionally to $\gamma^{dn/2}$.

The last term on the right hand side of (3.42) is bounded by

$$\frac{\gamma^{\delta N}}{N!}\sum_{\eta'}(\hat{M}'_\gamma)^N(\eta_0,\eta') \sup_{\substack{|\eta|\le n+N\\ d(\eta)\le N(n+N)}} \sup_{\sigma,m}\big|\rho_\eta(\sigma,m)\big| \tag{3.46}$$

By Lemma 3.3

$$\sup_{\lambda}\sup_{d(\xi)\le N(n+N)}\big|f_\xi(\lambda)\big|\le c \tag{3.47}$$

so that the sup in (3.46) is bounded independently of γ. By (3.25)

$$\sum_{\eta'}(\hat{M}'_\gamma)^N(\eta_0,\eta')\le c'$$

independently of γ, hence by (3.41) the expression in (3.46) is bounded proportionally to $\gamma^{dn/2}$. The Theorem is therefore proven. □

4. Extension to longer times

In this section we prove the Theorems stated in §2.1 and §2.3 of Section 2, extending the estimates on the v functions to times $t\le a\log\gamma^{-1}$ with $a>0$ suitably small.

We postpone to Proposition 4.7 the proof of the existence and of the uniqueness of the Cauchy problems for (1.1) and (2.20a) and of several other properties of these equations, in particular that the sup norms of the solutions of (1.1) and (2.20a) are bounded by 1, if that is so for the initial datum. First we introduce the basic notion of “quasi-solutions” of (2.20a).

4.1 Definition.

Given a configuration $\sigma\in\{-1,1\}^{\mathbb{Z}^d}$ and $s\ge 0$ we denote by

$$m_{\gamma,s}(x,t|\sigma),\quad t\ge s \tag{4.1}$$

the solution of (2.20a) for $t\ge s$, which starts from σ at time s. We set

$$t_k=k\gamma^\delta,\qquad k\in\mathbb{Z}_+,\ \delta>0 \tag{4.2}$$

and given any sequence $\sigma_{t_k}\in\{-1,1\}^{\mathbb{Z}^d}$, $k\in\mathbb{Z}_+$, we define, for $t\ge 0$,

$$m_\gamma(x,t|\{\sigma_{t_k}\})=m_{\gamma,t_k}(x,t|\sigma_{t_k}),\quad t_k\le t<t_{k+1} \tag{4.3}$$

The function $m_\gamma\big(x,t|\{\sigma_{t_k}\}\big)$ is called a "quasi-solution" of (2.20a).

We will study the Glauber dynamics by successively conditioning on the values of the process at the times t_k. The conditioning at t_k fixes a configuration σ_{t_k} and the evolution in the next time step, $[t_k, t_{k+1})$, is well approximated in terms of $m_{\gamma,t_k}(x,t|\sigma_{t_k})$, as it follows from the analysis of the previous Section. At the end of this time interval, i.e. at time t_{k+1}, we proceed by taking, instead of $m_{\gamma,t_k}(x,t_{k+1}|\sigma_{t_k})$ one of the true configurations which appear in the conditioned process, say $\sigma_{t_{k+1}}$. By iterating this procedure we thus construct a quasi solution of (2.20a). See Ch. V of [13] for more comments on this approach.

Clearly for the method to work two conditions must be satisfied: first, for each k, $\sigma_{t_{k+1}}$ and $m_{\gamma,t_k}(x,t_{k+1}|\sigma_{t_k})$ should be close (with large probability) and in a sense to be specified. If this holds we will say that $m_\gamma\big(x,t|\{\sigma_{t_k}\}\big)$ has "small discontinuities". The second condition to prove is that a quasi solution with small discontinuities is close to the true solution, $m_\gamma(x,t|\sigma_0)$. We start from the latter condition.

4.2 Definition.

Given $\alpha \in (0,1)$ and $k^\star \geq 2$, we define the seminorms $\|f\|^+_{k^\star,\alpha}$ and $\|f\|_{k^\star,\alpha}$ of $f \in \mathcal{M}\big(\{-1,1\}^{\mathbb{Z}^d}\big)$ as

$$\|f\|^+_{k^\star,\alpha} = \sup_{|x|\leq k^\star\gamma^{-2}} \Big|\mathcal{A}_{\gamma^{-\alpha},x}(f)\Big|, \qquad \|f\|_{k^\star,\alpha} = \sup_{|x|\leq (k^\star-1)\gamma^{-2}} \Big|\mathcal{A}_{\gamma^{-\alpha},x}(f)\Big| \tag{4.4a}$$

where, recalling (2.9d),

$$\mathcal{A}_{\gamma^{-\alpha},x}(f) = \frac{1}{|B_{\gamma^{-\alpha}}|}\sum_{y\in B_{\gamma^{-\alpha},x}} f(y) \tag{4.4b}$$

Thus, given $\zeta > 0$, we say that a quasi-solution $m_\gamma\big(x,t|\{\sigma_{t_k}\}\big)$ is ζ-accurate till time T and with respect to the seminorm $\|\cdot\|^+_{k^\star,\alpha}$, if, for all γ small enough,

$$\|m_{\gamma,t_k}(\cdot,t_{k+1}|\sigma_{t_k}) - \sigma_{t_{k+1}}\|^+_{k^\star,\alpha} \leq \gamma^\zeta, \quad \text{for all } t_k \leq T \tag{4.5a}$$

We also say that it is ζ-accurate relative to the initial condition $m_\gamma(\cdot,0)$ if

$$\|m_\gamma(\cdot,0) - \sigma_0\|^+_{k^\star,\alpha} \leq \gamma^\zeta \tag{4.5b}$$

4.3 Proposition.

Let $0<\alpha<1$, $0<\delta<\zeta$, $0<b_2<\min\{\zeta-\delta, 1-\alpha\}$, $\hat c$ as in (4.18) below and

$$0 < a < b_2/\hat c, \quad 0 < b < b_2 - a\hat c \tag{4.6}$$

Then for any $k^\star \geq 2$, any $m_\gamma(\cdot,0) \in \mathcal{M}(\mathbb{Z}^d)$, $|m_\gamma(\cdot,0)| \leq 1$, and any quasi-solution $\{m_\gamma(\cdot,\cdot|\{\sigma_{t_k}\})$ which is ζ-accurate till time $a\log\gamma^{-1}$ with respect to the seminorm $\|\cdot\|^+_{k^\star,\alpha}$, and relative to the initial condition $m_\gamma(\cdot,0)$, the following holds

$$\sup_{t\leq a\log\gamma^{-1}} \|m_\gamma(\cdot,t|\{\sigma_{t_k}\}) - m_\gamma(\cdot,t)\|_{k^\star,\alpha} \leq \gamma^b \tag{4.7}$$

for all γ small enough. $m_\gamma(\cdot,t)$ in (4.7) is the solution of (2.20a) with initial condition $m_\gamma(\cdot,0)$.

Proof.

For $t \geq t_k$ we define

$$\ell_{\gamma,t_k}(x,t) = m_{\gamma,t_k}(x,t|\sigma_{t_k}) - m_\gamma(x,t) \tag{4.8a}$$

We then have

$$\ell_{\gamma,t_k}(x,t) = [\sigma_{t_k}(x) - m_\gamma(x,t_k)] + \int_{t_k}^t ds\,[F_{\gamma,t_k}(x,s) - \ell_{\gamma,t_k}(x,s)] \tag{4.8b}$$

where

$$F_{\gamma,t_k}(x,s) = \tanh\{\beta[(J_\gamma\circ m_{\gamma,t_k})(x,s)+h]\} - \tanh\{\beta[(J_\gamma\circ m_\gamma)(x,s)+h]\} \tag{4.8c}$$

Given any $x\in\mathbb{Z}^d$, and $t \geq t_k$, we denote by

$$L_{\gamma,t_k}(x,t) = \Big|\mathcal{A}_{\gamma^{-\alpha},x}(\ell_{\gamma,t_k}(\cdot,t))\Big| \tag{4.9}$$

By adding and subtracting $m_{\gamma,t_{k-1}}(x,t_k|\sigma_{t_{k-1}})$ to the right hand side of (4.8b), we get

$$\begin{aligned} L_{\gamma,t_k}(x,t) \leq{}& \Big|\mathcal{A}_{\gamma^{-\alpha},x}\Big(\sigma_{t_k} - m_{\gamma,t_{k-1}}(\cdot,t_k|\sigma_{t_{k-1}})\Big)\Big| + L_{\gamma,t_{k-1}}(x,t_k) \\ &+ \int_{t_k}^t ds\Big(\Big|\mathcal{A}_{\gamma^{-\alpha},x}(F_{\gamma,t_k}(\cdot,s))\Big| + L_{\gamma,t_k}(x,s)\Big) \end{aligned} \tag{4.10}$$

We are going to prove that there are c_1 and c_2 so that for all x and s

$$|F_{\gamma,t_k}(x,s)| \leq c_1\sum_y |J_\gamma(x,y)|L_{\gamma,t_k}(y,t) + c_2\gamma^{1-\alpha} \tag{4.11}$$

We have, in fact,

$$|F_{\gamma,t_k}(x,s)| \leq c_1'\Big|(J_\gamma\circ\ell_{\gamma,t_k})(x,s)\Big| \tag{4.12a}$$

and, for any function $f(y)$,

$$\Big|\sum_y J_\gamma(x,y)f(y)\Big| = \Big|\sum_y J_\gamma(x,y)\{f(y) - \mathcal{A}_{\gamma^{-\alpha},y}(f) + \mathcal{A}_{\gamma^{-\alpha},y}(f)\}\Big|$$
$$\le \sum_y |J_\gamma(x,y)||\mathcal{A}_{\gamma^{-\alpha},y}(f)| + \|f\|_\infty \sum_y \Big|J_\gamma(x,y) - \mathcal{A}_{\gamma^{-\alpha},y}\big(J_\gamma(x,\cdot)\big)\Big| \tag{4.12b}$$

Using (4.12) we then obtain (4.11).

Going back to (4.10), using the assumption that $m_\gamma(\cdot,\cdot|\{\sigma_{t_k}\})$ is ζ-accurate, by (4.11), we get, for $|x| \le k^\star\gamma^{-2}$ and all γ small enough,

$$L_{\gamma,t_k}(x,t) \le \gamma^\zeta + L_{\gamma,t_{k-1}}(x,t_k) + c_2\gamma^{\delta+1-\alpha} + \int_{t_k}^t ds \sum_y K_\gamma(x,y)L_{\gamma,t_k}(y,s) \tag{4.13a}$$

where

$$K_\gamma(x,y) = \mathbf{1}_{\{x=y\}} + c_1\frac{1}{|B_{\gamma^{-\alpha}}|}\sum_{z\in B_{\gamma^{-\alpha},x}} |J_\gamma(z,y)| \tag{4.13b}$$

We define

$$L_\gamma(x,t) = L_{\gamma,t_k}(x,t), \qquad \text{if } \ t_k \le t < t_{k+1} \tag{4.14a}$$

We bound $L_{\gamma,t_{k-1}}(x,t_k)$ in (4.13a) using the same (4.13a), then, iterating this procedure, we get for any $t \le a\log\gamma^{-1}$

$$L_\gamma(x,t) \le \gamma^{-\delta}a\log\gamma^{-1}[\gamma^\zeta + c_2\gamma^{\delta+1-\alpha}] + \int_0^t ds\sum_z K_\gamma(x,z)L_\gamma(z,s) \tag{4.14b}$$

Let b_2 be as in the text of the Proposition, then for all γ small enough and all $|x| \le k^\star\gamma^{-2}$

$$L_\gamma(x,t) \le \gamma^{b_2} + \int_0^t ds\sum_z K_\gamma(x,z)\ L_\gamma(z,s) \tag{4.15}$$

We now restrict to $|x| \le (k^\star-1)\gamma^{-2}$. We can then iterate (4.15) $n = [\gamma^{-1}/2]$ times. In fact $K_\gamma(x,z) = 0$ if $|x-z| > 2\gamma^{-1}$ so that all sites z reached in the iteration are such that

$$|z| \le |x| + 2\gamma^{-1}n \le (k^\star-1)\gamma^{-2} + \gamma^{-2} \le k^\star\gamma^{-2} \tag{4.16}$$

We then get, for $|x| \le (k^\star-1)\gamma^{-2}$,

$$L_\gamma(x,t) \le \sum_{j=1}^{n-1}\int_0^t ds_1\int_0^{s_1} ds_2\cdots\int_0^{s_{j-1}} ds_j \sum_{z_1,\dots,z_j} K_\gamma(x,z_1)\dots K_\gamma(z_{j-1},z_j)\gamma^{b_2}$$
$$+\int_0^t ds_1\cdots\int_0^{s_{n-1}} ds_n \sum_{z_1,\dots,z_n} K_\gamma(x,z_1)\dots K_\gamma(z_{n-1},z_n)L_\gamma(z,s_n) \tag{4.17}$$

We set

$$\hat{c} := \sup_{\gamma} \sum_{x} K_{\gamma}(0,x) \tag{4.18}$$

Since $L_{\gamma}(x,t) \le 2$, see Proposition 4.7.1 below, we finally have, for all γ small enough,

$$L_{\gamma}(x,t) \le \gamma^{b_2} e^{\hat{c}t} + 2\frac{(\hat{c}t)^n}{n!} \le \gamma^b \tag{4.19}$$

having used the definition of a and b and observed that for any $b_3 > 0$ there is c so that

$$\frac{(\hat{c}\, a \log \gamma^{-1})^n}{n!} \le \gamma^b, \qquad n = [\gamma^{-1}/2] \tag{4.19b}$$

The Proposition is therefore proven. $\square$

We have the following corollary of the proof of Proposition 4.3:

4.4 Corollary.

Using the same parameters as in Proposition 4.3, let $m_{\gamma}(x,t)$ and $\tilde{m}_{\gamma}(x,t)$, $t \ge 0$, be two solutions of (2.20a) such that

$$\| m_{\gamma}(\cdot,0) - \tilde{m}_{\gamma}(\cdot,0) \|^{+}_{k^\star,\alpha} < \gamma^{\zeta} \tag{4.20a}$$

Then

$$\sup_{t \le a \log \gamma^{-1}} \| m_{\gamma}(\cdot,t) - \tilde{m}_{\gamma}(\cdot,t) \|_{k^\star,\alpha} < \gamma^{b} \tag{4.20b}$$

Proof.

Replace in the proof of Proposition 4.3 $m_{\gamma}(x,t|\{\sigma_{t_k}\})$ by $\tilde{m}_{\gamma}(x,t)$ and σ_{t_k} by $\tilde{m}_{\gamma}(x,t_k)$, for all $t_k > 0$. We then obtain (4.20b) from (4.7). $\square$

We next prove that with large probability the trajectories of the Glauber dynamics give rise to ζ-accurate quasi solutions:

4.5 Proposition.

For any $k^\star \ge 2$, $0 < \alpha < 1$, $\delta > 0$, $a > 0$, $n \ge 1$ and

$$\zeta < \alpha d/2 \tag{4.21}$$

there is c so that, for all σ_0,

$$\mathbb{P}_{\sigma_0}\Big(\sup_{t_k \le a \log \gamma^{-1}} \| \sigma_{t_{k+1}} - m_{\gamma,t_k}(\cdot, t_{k+1}|\sigma_{t_k}) \|^{+}_{k^\star,\alpha} \ge \gamma^{\zeta} \Big) \le c\gamma^{n} \tag{4.22a}$$

and for any product measure μ^γ

$$\mu^\gamma\Big(\|\sigma - m_\gamma(\cdot,0)\|^+_{k^\star,\alpha} \geq \gamma^\zeta\Big) \leq c\gamma^n, \qquad m_\gamma(x,0) := \mathbb{E}_{\mu^\gamma}\left(\sigma(x)\right) \tag{4.22b}$$

Proof.

We first write

$$\mathbb{P}^\gamma_{\sigma_0}\Big(\sup_{t_k \leq a\log\gamma^{-1}} \|\sigma_{t_{k+1}} - m_{\gamma,t_k}(\cdot, t_{k+1}|\sigma_{t_k})\|^+_{k^\star,\alpha} > \gamma^\zeta\Big)$$
$$\leq \gamma^{-\delta} a \log\gamma^{-1}[4k^\star\gamma^{-2}+1]^d \sup_{x,\sigma} \mathbb{P}_\sigma\Big(\big|\Lambda_{\gamma,x,\sigma}\big| > \gamma^\zeta\Big) \tag{4.23a}$$

$$\Lambda_{\gamma,x,\sigma} := \frac{1}{|B_{\gamma^{-\alpha}}|} \sum_{y \in B_{\gamma^{-\alpha},x}} [\sigma(y,t_1) - m_\gamma(y,t_1|\sigma)] \tag{4.23b}$$

Then, using the Chebishev inequality with power $2n$,

$$P_\sigma\Big(\big|\Lambda_{\gamma,x,\sigma}\big| > \gamma^\zeta\Big) \leq \gamma^{-2n\zeta} \frac{1}{|B_{\gamma^{-\alpha}}|^{2n}} \sum_{\substack{x_1,\dots,x_{2n} \\ x_i \in B_{\gamma^{-\alpha},x}}} \mathbb{E}^\gamma_\sigma\Big(\prod_{i=1}^{2n} [\sigma(x_i,t_1) - m_\gamma(x_i,t_1|\sigma)]\Big)$$

If all the x_i are distinct, the expectation is a v-function, so that, by (3.38), it is bounded by γ^{dn}. If the sites are not all distinct we use (3.32). We observe that for each sum which is missing we gain a factor $\gamma^{d\alpha}$ from the normalization, hence we obtain that the last expression is bounded by:

$$c_n\gamma^{-2n\zeta}\max(\gamma^{dn}, \gamma^{\alpha dn})$$

Since $\alpha < 1$, the max is achieved by the second term, which gives the bound $\gamma^{n(d\alpha - 2\zeta)}$, and this proves (4.22a). A completely similar argument gives (4.22b), the Proposition is thus proven. $\Box$

Collecting the above results we have:

4.6 Proposition.

Let $k^\star$, α, δ, ζ, a and b as in Proposition 4.3 and such that (4.21) holds. Then for any n there is c so that for all σ_0:

$$\mathbb{P}^\gamma_{\sigma_0}\Big(\sup_{t \leq a\log\gamma^{-1}} \|m_\gamma(\cdot,t|\sigma_0) - \sigma_t\|_{k^\star,\alpha} > \gamma^b\Big) \leq c\gamma^n \tag{4.24}$$

Furthermore let μ^γ be a product measure, $m_\gamma(x,0)$ its means, $m_\gamma(x,t)$ the solution of (2.20a) with initial condition $m_\gamma(\cdot,0)$, then

$$\mathbb{P}^\gamma_{\mu^\gamma}\Big(\sup_{t\le a\log\gamma^{-1}} \|m_\gamma(\cdot,t)-\sigma_t\|_{k^\star,\alpha} > \gamma^b\Big) \le c\gamma^n \tag{4.25a}$$

$$\mathbb{P}^\gamma_{\mu^\gamma}\Big(\Big\{\sigma_0 : \mathbb{P}^\gamma_{\sigma_0}\Big(\sup_{t\le a\log\gamma^{-1}} \|m_\gamma(\cdot,t)-\sigma_t(\cdot)\|_{k^\star,\alpha} > \gamma^b\Big) \le c\gamma^n\Big\}\Big) \ge 1-c\gamma^n \tag{4.25b}$$

Proof.

The left hand side of (4.24) is bounded by

$$\gamma^{-\delta} a\log\gamma^{-1} \sup_{t_k\le a\log\gamma^{-1}} \mathbb{P}_{\sigma_0}\Big(\sup_{t_k\le t\le t_{k+1}} \|m_\gamma(\cdot,t|\sigma_0)-\sigma_t(\cdot)\|_{k^\star,\alpha} > \gamma^b\Big) \tag{4.26}$$

We add and subtract $m_{\gamma,t_k}(\cdot,t|\sigma_{t_k})$ so that the probability in (4.26) is bounded by

$$P_{\sigma_0}\Big(\sup_{t_k\le t\le t_{k+1}} \|m_\gamma(\cdot,t|\sigma_0)-m_{\gamma,t_k}(\cdot,t|\sigma_{t_k})\|_{k^\star,\alpha} > \frac{1}{2}\gamma^b\Big) + \sup_\sigma P_\sigma\Big(\sup_{0\le t\le t_1} \|\sigma_t(\cdot)-m_\gamma(\cdot,t|\sigma)\|_{k^\star,\alpha} > \frac{1}{2}\gamma^b\Big) \tag{4.27}$$

By Propositions 4.3 and 4.5 we conclude that the first term in (4.27) vanishes faster than any power of γ. Observe that if $b'<b$, then $\gamma^{b'}<\gamma^b/2$, for all γ small enough: by this we take care of the factors $1/2$ in front of γ^b. For the second term in (4.27) we proceed as follows. With a proof similar to that of Proposition 4.5, we have that there for $b<\alpha d/2$ and for any n there is c so that

$$\sup_{0\le t\le t_1}\sup_\sigma P_\sigma\Big(\|\sigma_t(\cdot)-m_\gamma(\cdot,t|\sigma)\|_{k^\star,\alpha} > \frac{1}{4}\gamma^b\Big) \le c\gamma^n \tag{4.28}$$

We then split the time interval $[0,t_1]$ in N subintervals of length ϵ, $\epsilon N = t_1$. Let $\mathcal{O}$ be the set of orbits σ_s, $0\le s\le t_1$, for which at least two spins flip in the same time subinterval and at sites within distance $k^\star\gamma^{-2}$ from the origin. More precisely, there is $n\le N-1$ and there are two times s_1 and s_2 both in $(n\epsilon,(n+1)\epsilon)$ and two sites x_1 and x_2, $|x_i|\le k^\star\gamma^{-2}$ such that $\sigma(x_i,s_i^+) = -\sigma(x_i,s_i^-)$, $i=1,2$. We denote by

$$\mathcal{B} = \bigcap_{n\le N-1}\Big\{ \|\sigma_{n\epsilon}(\cdot)-m_\gamma(\cdot,n\epsilon|\sigma)\|_{k^\star,\alpha} < \frac{1}{4}\gamma^b\Big\} \tag{4.29a}$$

and by $\mathcal{B}^c$ its complement.

We claim that for ϵ small enough

$$\Big\{ \sup_{0\le t\le t_1} \|\sigma_t(\cdot)-m_\gamma(\cdot,t|\sigma)\|_{k^\star,\alpha} > \frac{1}{2}\gamma^b\Big\} \subset \mathcal{B}^c \cup \mathcal{B}\mathcal{O} \tag{4.29b}$$

To prove the claim we will show that $\mathcal{BO}^c$ does not intersect the set on the left hand side of (4.29b). To this end we observe that $m_\gamma(x,t|\sigma)$ has a bounded time derivative so that in a time interval ϵ it varies at most by $c\epsilon$, for some c independent of σ, γ and t. In $\mathcal{O}^c$ there is at most one spin which flips in a time interval $(n\epsilon,(n+1)\epsilon)$ within distance $k^\star\gamma^{-2}$ from the origin, so that if $n\epsilon < t < (n+1)\epsilon$,

$$\begin{aligned}\|\sigma_t(\cdot) - m_\gamma(\cdot,t|\sigma)\|_{k^\star,\alpha} &\le \|\sigma_{n\epsilon}(\cdot) - m_\gamma(\cdot,n\epsilon|\sigma)\|_{k^\star,\alpha} + \frac{2}{|B_{\gamma^{-\alpha}}|} + c\epsilon \\ &\le \frac{1}{4}\gamma^b + c\epsilon + c'\gamma^{\alpha d} \le \frac{1}{2}\gamma^b\end{aligned}$$

if b and ϵ are small enough. (4.29b) is thus proven.

From (4.29b) and (4.28) it follows that

$$\mathbb{P}^\gamma_\sigma\Big(\{\|\sigma_t(\cdot) - m_\gamma(\cdot,n\epsilon|\sigma)\|_{k^\star,\alpha} > \frac{1}{2}\gamma^b\}\Big) \le Nc\gamma^n + \mathbb{P}_\sigma(\mathcal{O}) \tag{4.29c}$$

Since $c_\gamma(x,\sigma) \le 1$, for a suitable constant c',

$$P_\sigma(\mathcal{O}) \le Nc'[(4k^\star\gamma^{-2}+1)^d\epsilon t_1]^2$$

Thus (4.29c) is bounded by

$$\epsilon^{-1}t_1 c\gamma^n + \epsilon^{-1}t_1 c'[(4k^\star\gamma^{-2}+1)^d\epsilon t_1]^2$$

By taking $\epsilon = \gamma^k$, with k sufficiently large, we make the second term vanish as any desired power of γ. Given that k, we then choose n so large that also the first term vanishes as fast as desired. Also the second term on the right hand side of (4.27) vanishes faster than any power of γ, hence (4.24) is proven.

Proof of (4.25b). We write

$$\begin{aligned}\Big\{\sup_{t\le a\log\gamma^{-1}} \|m_\gamma(\cdot,t) - \sigma_t\|_{k^\star,\alpha} > \gamma^b\Big\} \subset &\Big\{\sup_{t\le a\log\gamma^{-1}} \|m_\gamma(\cdot,t|\sigma_0) - \sigma_t\|_{k^\star,\alpha} > \frac{\gamma^b}{2}\Big\} \\ &\bigcup\Big\{\sup_{t\le a\log\gamma^{-1}} \|m_\gamma(\cdot,t) - m_\gamma(\cdot,t|\sigma_0)\|_{k^\star,\alpha} > \frac{\gamma^b}{2}\Big\}\end{aligned}$$

By (4.24) we then get for all γ small enough

$$\mathbb{P}^\gamma_{\sigma_0}\Big(\sup_{t\le a\log\gamma^{-1}} \|m_\gamma(\cdot,t|\sigma_0) - \sigma_t\|_{k^\star,\alpha} > \frac{\gamma^b}{2}\Big) \le c\gamma^n$$

Furthermore, if

$$\|\sigma_0 - m_\gamma(\cdot,0)\|^+_{k^\star,\alpha} < \gamma^\zeta \tag{4.30}$$

by Corollary 4.4, for all γ small enough,

$$\sup_{t\le a\log\gamma^{-1}} \|m_\gamma(\cdot,t)-m_\gamma(\cdot,t|\sigma_0)\|_{k^\star,\alpha}\le\frac{\gamma^b}{2}$$

We have thus shown that the set in curly brackets on the left hand side of (4.25b) is contained in the complement of the set (4.30). (4.25b) then follows from (4.22b).

Since (4.25a) follows immediately from (4.25b) the Proposition is proven. □

Proposition 4.6 "almost" proves (2.10b) and (2.13b). The "almost" is because $m_\gamma(\cdot,t|\sigma_0)$ and $m_\gamma(\cdot,t)$ solve (2.20a) and not (1.1), as required in (2.10b) and (2.13b). We fill this gap in the next Proposition, where we also prove several properties of (1.1) and (2.20a). For ease of reference we split the Proposition in several distinct statements.

4.7.1 Proposition.

For any $\gamma>0$ and any $m\in\mathcal{M}(\mathbb{Z}^d)$, $\|m\|_\infty\le 1$, there is a unique bounded function $m_\gamma(x,t)$ on $\mathbb{Z}^d\times\mathbb{R}_+$ which for each x is differentiable in t, solves (2.20a) for all x and all $t\ge 0$ and satisfies the initial condition $m_\gamma(x,0)=m(x)$, for all x. Moreover $\|m_\gamma(\cdot,t)\|_\infty\le 1$ for all $t\ge 0$.

Analogously, for any $m\in\mathcal{M}(\mathbb{R}^d)$, $\|m\|_\infty\le 1$, there is a unique bounded function $m(r,t)$, differentiable in t for all r, which solves (1.1) and satisfies the initial condition $m(\cdot,0)=m(\cdot)$. Moreover $\|m(r,t)\|_\infty\le 1$ for all $t\ge 0$.

Proof.

We consider the statement relative to (1.1), the argument for (2.20a) is completely analogous and omitted.

We fix $T>0$ and for $0\le t\le T$ we write the integral version of (1.1)

$$m(r,t)=e^{-t}m(r,0)+\int_0^t dse^{-(t-s)}\tanh\Big(\beta[J\star m(r,s)+h]\Big) \tag{4.31a}$$

which can be thought of as a fixed point problem for the map

$$K:\Big\{u\in\mathcal{M}(\mathbb{R}^d\times[0,T]):\|u\|_\infty<\infty\Big\}\to\Big\{u\in\mathcal{M}(\mathbb{R}^d\times[0,T]):\|u\|_\infty\le 1\Big\}$$

defined as

$$Ku(r,t)=e^{-t}m(r,0)+\int_0^t dse^{-(t-s)}\tanh\Big(\beta[J\star u(r,s)+h]\Big) \tag{4.31b}$$

For T small enough K is a strict contraction in the sup norm, with therefore a unique fixed point, $m(r,t)$. Since $Ku(r,t)$ is for all r a differentiable function of t, the fixed point

$m(r,t)$ is differentiable in t for all r, hence $m(r,t)$ solves (1.1) for $t \le T$. Observe also that $m(r,t) - e^{-t} m(r,0)$ is in $C^1(\mathbb{R}^d \times [0,T])$.

As the solution at time T is in the same class of the datum at time 0, by iterating the previous argument we prove global existence. Uniqueness follows because any solution of (1.1) is also a solution of its integral version, and we have seen that this is a contraction. □

4.7.2 Proposition.

There are c_2' and c_2 so that the following holds. Let $u_\gamma(x,t)$ be for each γ a solution of (2.20a) and denote by $u^\star_\gamma(r,t)$ the Γ_γ image of $u_\gamma(x,t)$. Let $m_\gamma(r,t)$ solve (1.1) with initial datum $m_\gamma(r,0) = u^\star_\gamma(r,0)$ and suppose $\|u^\star_\gamma(\cdot,0)\|_\infty \le 1$. Then

$$\|m_\gamma(\cdot,t) - u^\star_\gamma(\cdot,t)\|_\infty \le c_2' e^{c_2 t} \gamma \tag{4.32}$$

Proof.

Observe that $u^\star_\gamma(r,t)$ solve (1.1) with $J(r-r')$ replaced by

$$\Delta_\gamma(r,r') := J([r]_\gamma - [r']_\gamma)$$

Let

$$L_\gamma(r,t) := |u^\star_\gamma(r,t) - m_\gamma(r,t)|$$

then

$$L_\gamma(r,t) \le \int_0^t ds \Big\{ \beta[|J| \star L_\gamma(r,s) + 2c\gamma] \Big\}$$

because, for a suitable c,

$$|J(r-r') - \Delta_\gamma(r,r')| \le c\gamma$$

and, by Proposition 4.7.1, $\|u^\star_\gamma\|_\infty \le 1$, $\|m_\gamma\|_\infty \le 1$. Hence (4.32). □

4.7.3 Proposition.

Let $m_\gamma(r,t)$ solve (1.1), assume that $\|m_\gamma(\cdot,0)\|_\infty \le 1$ and that $m_\gamma(r,0) \to m(r,0)$ Lebesgue almost everywhere. Then, for any t, $m_\gamma(r,t) \to m(r,t)$ Lebesgue almost everywhere, $m(r,t)$ being the solution of (1.1) with initial condition $m(r,0)$.

Proof.

Let

$$L_\gamma(r,t) := |m_\gamma(r,t) - m(r,t)|$$

then

$$
\begin{aligned}
L_\gamma(r,t) &\le e^{-t} L_\gamma(r,0) + \beta \int_0^t ds e^{-(t-s)} |J| \star L_\gamma(r,s) \\
&\le e^{-t} L_\gamma(r,0) + e^{-t} \sum_{n=1}^{N} \frac{(\beta t)^n}{n!} |J|^{n-1} \star L_\gamma(r,0) + 2 \frac{(\beta \|J\|_\infty t)^N}{N!} \\
&\le e^{-t} L_\gamma(r,0) + e^{-t} \sum_{n=1}^{N} \frac{(\beta t \|J\|_\infty)^n}{n!} \int_{|r-r'|\le N} dr' L_\gamma(r',0) + 2 \frac{(\beta \|J\|_\infty t)^N}{N!}
\end{aligned}
\tag{4.33}
$$

Given any $\epsilon > 0$, we choose N so large that the last term is less than ϵ. On the other hand, by the Lebesgue theorem, for any bounded region Λ,

$$
\lim_{\gamma \to 0} \int_\Lambda dr L_\gamma(r,0) = 0
$$

It then follows that for all γ small enough, also the second term on the right hand side of (4.33) is less than ϵ. By the arbitrarity of ϵ we then conclude that $L_\gamma(r,t) \to 0$ for all r for which this happens at $t=0$. The Proposition is thus proven. $\square$

4.7.4 Proposition.

There is c_4 so that for any $V > 2\beta\|J\|_\infty$, for any $T > 0$ and any pair $m(r,t)$ and $\tilde{m}(r,t)$ of solutions of (1.1) such that $m(r,0) = \tilde{m}(r,0)$ for all $|r| \le VT$, ($\|m(\cdot,0)\|_\infty \le 1$, $\|\tilde{m}(\cdot,0)\|_\infty \le 1$)

$$
\left| m(0,t) - \tilde{m}(0,t) \right| \le c_4 e^{-TV/2 \log\{V/(2\beta\|J\|_\infty)\}}, \qquad \textit{for all } t \le T \tag{4.34}
$$

Proof.

Define $L(r,t) = \left| m(r,t) - \tilde{m}(r,t) \right|$, then $L(r,t)$ satisfies the same inequality (4.33) as $L_\gamma(r,t)$. We take $r=0$ in (4.33) and N the largest integer smaller than VT. Let T_0 be such that $N \ge VT/2$ for all $T \ge T_0$. Then (4.34) holds for $T \le T_0$ if c_4 is such that

$$
c_4 \ge 2e^{-T_0 V/2 \log\{V/(2\beta\|J\|_\infty)\}}
$$

The bound in (4.34) for $T \ge T_0$ then follows by using the Stirling formula to estimate the last term in (4.33) and bounding from below N by $TV/2$. $\square$

4.7.5 Proposition.

There is c_5 so that if $m(r,t)$ and $\tilde{m}(r,t)$ solve (1.1), $\|m(\cdot,0)\|_\infty \le 1$, $\|\tilde{m}(\cdot,0)\|_\infty \le 1$ and $\|m(\cdot,0) - \tilde{m}(\cdot,0)\|_\infty < \epsilon$, then,

$$
\|m(\cdot,t) - \tilde{m}(\cdot,t)\|_\infty \le e^{c_5 t} \epsilon, \qquad \textit{for all } t \ge 0
$$

Proof.

Defining $L(r,t)$ as in the proof of Proposition 4.7.4, we obtain the result from the first inequality in (4.33). $\Box$

4.7.6 Proposition.

There are c_6' and c_6 so that the following holds. Let $m(r,t)$ and $\tilde{m}(r,t)$ solve (1.1), $\|m(\cdot,0)\|_\infty \le 1$, $\|\tilde{m}(\cdot,0)\|_\infty \le 1$, and suppose that there are $0 < \alpha < 1$, $b > 0$, $u > 0$ and c' so that

$$\sup_{r\in\mathbb{R}^d} \Big| D \cap \{r' : |r-r'| \le 1\} \Big| \le c'\gamma^u$$

where

$$D := \Big\{ r \in \mathbb{R}^d : \|m^{(\alpha,\gamma)}(\cdot,0) - \tilde{m}^{(\alpha,\gamma)}(\cdot,0)\|_\infty > \gamma^b \Big\}$$

(see 2.1.5 for notation). Then

$$\sup_{r\notin D} |m^{(\alpha,\gamma)}(r,t) - \tilde{m}^{(\alpha,\gamma)}(r,t)| \le c_6' e^{c_6 t}[\gamma^b + \gamma^{1-\alpha} + c'\gamma^u]$$

Proof.

We have

$$\Big| m(r,t) - \tilde{m}(r,t) - e^{-t}[m(r,0) - \tilde{m}(r,0)] \Big| \le \int_0^t ds e^{-(t-s)} \beta \Big| \big(J \star m(\cdot,s)\big)(r) - \big(J \star \tilde{m}(\cdot,s)\big)(r) \Big|$$

We fix r and, setting $j(r') \equiv J(r-r')$, we write the first convolution as

$$\int dr' j(r') m(r',s) = \int dr' j(r') \Big\{ m^{(\alpha,\gamma)}(r',s) + [m(r',s) - m^{(\alpha,\gamma)}(r',s)] \Big\}$$
$$= \int dr' j(r') m^{(\alpha,\gamma)}(r',s) + \int dr' m(r',s)[j(r') - j^{(\alpha,\gamma)}(r')]$$

An analogous expression holds for $\tilde{m}$. We then set $L_\gamma(r,t) = |m^{(\alpha,\gamma)}(r,t) - \tilde{m}^{(\alpha,\gamma)}(r,t)|$, and, for $r \notin D$,

$$L_\gamma(r,t) \le e^{-t}\gamma^b + \int_0^t ds e^{-(t-s)} \beta \Big\{ \|J\|_\infty \Big[2c'\gamma^u + \sup_{r'\notin D} |L_\gamma(r',s)| \Big] + 2\|J'\|_\infty \gamma^{1-\alpha} \Big\}$$

hence the Proposition is proven. $\Box$

4.7.7 Proposition.

There is c_7 so that the following holds. Let $m(\cdot,0) \in C^1(\mathbb{R}^d)$, assume that $\|m(\cdot,0)\|_\infty \le 1$ and that $\|m^{(1)}(\cdot,0)\|_\infty < \infty$, $m^{(1)}$ being the derivative with respect to r of m. Let $m(r,t)$ be the solution of (1.1) with initial condition $m(r,0)$, then $m(\cdot,t) \in C^1(\mathbb{R}^d)$ and

$$\|m^{(1)}(\cdot,t)\|_\infty \le e^{-t}\|m^{(1)}(\cdot,0)\|_\infty + c_7 \qquad (4.35)$$

Proof.

The statement follows from differentiating (4.31a) with respect to r and using that $\|J^{(1)}\|_\infty < \infty$ and that $\|m(\cdot,t)\|_\infty \le 1$. We omit the details. $\square$

4.7.8 Proposition.

There is a constant c_8 so that for any solution $m(r,t)$ of (1.1) with $\|m(\cdot,0)\|_\infty \le 1$, for any $r \in \mathbb{R}^d$ and any $\ell \in \mathbb{R}^d$:

$$|m(r,t) - m(r+\ell,t)| \le c_8|\ell| + 2e^{-t} \tag{4.36}$$

Proof.

Let

$$D(\ell,t) = \sup_r |m(r,t) - m(r+\ell,t)|$$

Then, for a suitable c,

$$D(\ell,t) \le e^{-t}D(\ell,0) + \int_0^t dse^{-(t-s)}c|\ell|$$

because

$$\beta|J\star m(r,s) - J\star m(r+\ell,s)| \le \beta\int dr'|J(r-r') - J(r+\ell-r')| \le c|\ell|$$

Recalling that $|D(\ell,0)| \le 2$, we then obtain (4.36). $\square$

Proof of Theorem 2.1.6.

We observe that

$$\Gamma_\gamma\Big(\mathcal{A}_{\gamma^{-\alpha},\cdot}(\sigma_t)\Big) = \sigma^{(\alpha)}_{\gamma,t}$$

as defined in (2.9b). We then use 4.7.2 with $u_\gamma(x,t) = m_\gamma(x,t|\sigma_0)$ and $m_\gamma(r,t) = \psi_\gamma(r,t)$, defined as the solution of (1.1) with initial datum $\sigma_{gam,0}(r)\big)$. Then, by (4.32),

$$\|\Gamma_\gamma(\psi_\gamma(\cdot,t|\sigma_0)) - m_\gamma(\cdot,t)\|_\infty \le c_2' e^{c_2 t}\gamma \tag{4.37a}$$

By Proposition 4.7.6 and recalling (2.10a),

$$\|\psi^{(\alpha,\gamma)}_\gamma(\cdot,t) - m^{(\alpha,\gamma)}_\gamma(\cdot,t)\|_\infty \le c_6' e^{c_6 t}[\gamma^\zeta + \gamma^{1-\alpha}] \tag{4.37b}$$

with $m^{(\alpha,\gamma)}_\gamma$ as in (2.10c). (2.10b) then follows from (4.24) and (4.37), having chosen a small enough. $\square$

Proof of Theorem 2.1.8.

(2.13a) is proven by (4.25a) and Proposition 4.7.2. (2.13b) is a straight consequence of (2.10b) and (2.13a). $m_\gamma(r,t)$ converges Lebesgue almost everywhere to $m(r,t)$ by Proposition 4.7.3. The statements in Theorem 2.1.8 relative to $m_0 \in C^1(\mathbb{R}^d)$ are proved using that $|m_\gamma(\cdot,0) - m(\cdot,0)|_\infty \le C\gamma$. Then, by Proposition 4.7.5,

$$\|m_\gamma(\cdot,t) - m(\cdot,t)\|_\infty \le e^{c_5 t} C\gamma, \qquad \|m_\gamma^{(\alpha,\gamma)}(\cdot,t) - m^{(\alpha,\gamma)}(\cdot,t)\|_\infty \le e^{c_5 t} C\gamma$$

hence (2.13a) with $m^{(\alpha,\gamma)}$. Then, by Proposition 4.7.7, the sup norm of $m - m^{(\alpha,\gamma)}$ vanishes as γ^b, for some $b > 0$. The proof of Theorem 2.1.8 is thus concluded. $\square$

We next extend the bounds on the v-functions proven in Section 3 to finite times and to times $t \le a \log \gamma^{-1}$, when a is sufficiently small, proving Theorem 2.3.4.

There are two key observations: first, by using Proposition 4.6, it follows that the average of $v_n^\gamma(\underline{x}, t|\sigma)$ when each x_i independently varies in the box $B_{\gamma^{-\alpha}, y_i}$ is bounded, for any given $\underline{y} = (y_1, .., y_n)$, by $c\gamma^{bn}$. Second observation is that while the v-functions do not obey a closed set of equations, the new terms appear as averages or, more precisely, can be reduced to averages. The right algebra to evidentiate this finer structure is that introduced, but not fully used, in Section 3, to which we refer also for the notation that we will be using in the sequel. The main conclusions of our analysis are summarized in a generalization of Proposition 3.6:

4.8 Proposition.

There are real valued coefficients $M_\gamma(\eta,\eta',m)$ and $M_\gamma^{\rm dg}(\eta,\eta',m)$, η and η' in H, (see Definition 3.4), $m \in [-1,1]^{\mathbb{Z}^d}$, such that for all γ, η, m and $\sigma \in \{-1,1\}^{\mathbb{Z}^d}$

$$\sum_{x\in\mathbb{Z}^d} D^{(x)} \rho_\eta(\sigma,m) = \sum_{\eta'\in H} \left\{ M_\gamma(\eta,\eta',m) + M_\gamma^{\rm dg}(\eta,\eta',m)\right\} \rho_{\eta'}(\sigma,m) \tag{4.38}$$

(ρ_η as in Definition 3.5). The coefficients M_γ and $M_\gamma^{\rm dg}$ satisfy the following support properties and bounds.

Support properties.

$M_\gamma^{\rm dg}(\eta,\eta',m) = 0$ unless $|\eta| = |\eta'|$, $g(\eta) = g(\eta')$, $d(\eta) = d(\eta')$ and

$$\max_{\xi'\in\eta'} |V_{\xi'}| \le \max_{\xi\in\eta} |V_\xi| \tag{4.39a}$$

with V_ξ as in Definition 3.2; $\xi \in \eta$ meaning that ξ is an entry of η.

$M_\gamma(\eta,\eta',m) = 0$ unless $|\eta'| \le |\eta| + 2$, $d(\eta') - d(\eta) \le |\eta|$ and

$$\max_{\xi'\in\eta'} |V_{\xi'}| \le \max_{\xi\in\eta} |V_\xi| + 2 \tag{4.39b}$$

Bounds.

There is C so that for all $n \geq 1$:

$$\sup_{m \in [-1,1]^{\mathbb{Z}^d}} \sup_{|\eta|=n} \sum_{\eta'} \left| M_\gamma^{\mathrm{dg}}(\eta, \eta', m) \right| \leq Cn \tag{4.40a}$$

There is $c_0(n, n')$, $(n, n') \in \mathbb{Z}_+^2$, so that

$$\sup_{m \in [-1,1]^{\mathbb{Z}^d}} \sup_{|\eta|=n, d(\eta)=n'} \sum_{\eta'} \left| M_\gamma(\eta, \eta', m) \right| \leq c_0(n, n') \tag{4.40b}$$

Moreover for any $b > 0$ there are coefficients $c(n, n')$, $(n, n') \in \mathbb{Z}_+^2$, and positive parameters $b_2^\star$, $b_3^\star$, $\zeta^\star$ such that $b > b_2^\star > b_3^\star > \zeta^\star > 0$ and the following holds. Let (b_1, b_2, b_3, ζ) be either equal to $(d/2, b_2^\star, b_3^\star, \zeta^\star)$ or equal to $(0, 0, 0, 0)$, then

$$\sup_{m \in [-1,1]^{\mathbb{Z}^d}} \sup_{|\eta|=n, d(\eta)=n'} N_\gamma(\eta) \sum_{\eta'} |M_\gamma(\eta, \eta', m)| N_\gamma(\eta')^{-1} \leq c(n, n') \gamma^\zeta \tag{4.40c}$$

$$N_\gamma(\eta) := \gamma^{-(d/2 - b_1)|\eta| - b_2 g(\eta) + b_3 d(\eta)} \tag{4.40d}$$

The Proposition is proven in the Appendix by computing the action of $D^{(x)}$ on the left hand side of (4.38) and by a very careful classification of all the terms which are then produced. The only way to make the proof really accessible to the reader is to report it with most of the details, because of its lemgth we have shifted it to the Appendix. We proceed with the following corollary of Proposition 4.7.6 which explains the reason for the different bounds which appear in its statement.

Proof of (2.23).

Let $\eta_0 \neq \emptyset$, then, using (4.38) and denoting by $m_t = m_\gamma(\cdot, t|\sigma)$,

$$\mathbb{E}_\sigma^\gamma \Big(\rho_{\eta_0}(\sigma_t, m_t) \Big) = \int_0^t ds \sum_{\eta'} [M_\gamma(\eta, \eta', m_s) + M_\gamma^{\mathrm{dg}}(\eta, \eta', m_s)] \mathbb{E}_\sigma^\gamma \Big(\rho_{\eta'}(\sigma_s, m_s) \Big) \tag{4.41}$$

The term at $t = 0$ is missing because $m_0 = \sigma$, hence $\rho_\eta(\sigma, m) = 0$ since $\eta_0 \neq \emptyset$.

As our bounds on the coefficients M_γ and M_γ^{dg} do not distinguish η's with the same values of $\big(|\eta|, g(\eta), d(\eta)\big)$, we introduce the following notation:

$$\underline{n} = (n, m, \ell) \tag{4.42a}$$

$$\psi_{\underline{n}}(t) = \sup_{\substack{|\eta|=n, d(\eta)=m \\ g(\eta)=\ell}} \left| \mathbb{E}_\sigma^\gamma \Big(\rho_{\eta_0}(\sigma_t, m_t) \Big) \right| \tag{4.42b}$$

$$K_\gamma(\underline{n}, \underline{n}') = \sup_{\substack{|\eta|=n, d(\eta)=m \\ g(\eta)=\ell}} \sum_{\substack{|\eta'|=n', d(\eta')=m' \\ g(\eta')=\ell'}} \sup_{m(\cdot) \in [-1,1]^{\mathbb{Z}^d}} \left| M_\gamma(\eta, \eta', m) \right| \tag{4.43}$$

Then, using (4.40a), we get from (4.41)

$$\left|\mathbb{E}_\sigma^\gamma\Big(\rho_{\eta_0}(\sigma_t,m_t)\Big)\right| \le \int_0^t ds[Cn_0\psi_{\underline{n}_0}(s)+\sum_{\underline{n}_1} K_\gamma(\underline{n}_0,\underline{n}_1)\psi_{\underline{n}_1}(s) \tag{4.44a}$$

having denoted by $\underline{n}_0=(n_0,m_0,\ell_0)$ the triple $(|\eta_0|,d(\eta_0)g(\eta_0))$.

From (4.44a) we get

$$\psi_{\underline{n}_0}(t) \le \int_0^t dse^{Cn_0(t-s)}\sum_{\underline{n}_1} K_\gamma(\underline{n}_0,\underline{n}_1)\psi_{\underline{n}_1}(s) \tag{4.44b}$$

As in the proof of Theorem 3.6, we split $K_\gamma=K'_\gamma+K''_\gamma$, where $K'_\gamma(\underline{n},\underline{n}')=0$ if $\underline{n}'=(0,0,0)$ $\equiv\emptyset$ and $K''_\gamma(\underline{n},\underline{n}')=0$ if $\underline{n}'\neq\emptyset$. We then iterate (4.44b) N times, N as in (4.53) below, and get:

$$\psi_{\underline{n}_0}(t) \le \sum_{p=1}^n I_p + L \tag{4.45a}$$

where I_p is a shorthand for

$$I_p = \int_0^t ds_1 ... \int_0^{s_{p-1}} ds_p \sum_{\underline{n}_1 .. \underline{n}_{p-1}} \exp\Big\{C\sum_{i=1}^p n_{i-1}(s_{i-1}-s_i)\Big\} \times \{\prod_{i=1}^{p-1} K'_\gamma(\underline{n}_{i-1},\underline{n}_i)\}K''_\gamma(\underline{n}_{p-1},\emptyset) \tag{4.45b}$$

where $s_0\equiv t$ and

$$L = \int_0^t ds_1 ... \int_0^{s_{N-1}} ds_N \sum_{\underline{n}_1 .. \underline{n}_N} \exp\Big\{C\sum_{i=1}^N n_{i-1}(s_{i-1}-s_i)\Big\}\{\prod_{i=1}^N K'_\gamma(\underline{n}_{i-1},\underline{n}_i)\}\psi_{\underline{n}_N}(s_N) \tag{4.45c}$$

Observe that

$$n_i \le \bar n_i \equiv n_0+2i, \quad m_i \le \bar m_i \equiv m_0 + i\bar n_i, \quad \ell_i \le \bar\ell_i \equiv n_i \tag{4.46}$$

where the first two inequalities follow from the support properties of $M_\gamma(\eta,\eta',m)$ (see Proposition 4.8), the third one is true by definition, as $g(\eta)\le|\eta|$.

We start by bounding I_p. We divide the terms in the sum over $\underline{n}_1 .. \underline{n}_{p-1}$ according to the maximum value u attained by the first entry n_i of $\underline{n}_i$ and to j, the label where it is attained, i.e. $n_j=u$, and j is the first label where this happens. Then

$$I_p \le \sum_{u=n_0}^{n_0+2p}\sum_{j=0}^{p-1} e^{Cut}\frac{t^p}{p!}\sum_{\underline{n}_j,\underline{n}_{p-1}} \mathbf{1}_{n_j=u}(K'_\gamma)^j(\underline{n}_0,\underline{n}_j) \times (K'_\gamma)^{p-1-j}(\underline{n}_j,\underline{n}_{p-1})\, K''_\gamma(\underline{n}_{p-1},\emptyset) \tag{4.47a}$$

For $(K'_\gamma)^j(\underline{n}_0,\underline{n}_j)$ we use the bound (4.40b), for $(K')^{p-1-j}$ we use (4.40c). We have

$$\sum_{\underline{n}_{p-1}} (K'_\gamma)^{p-1-j}(\underline{n}_j,\underline{n}_{p-1})\, K''_\gamma(\underline{n}_{p-1},\emptyset) = N_\gamma(\underline{n}_j)^{-1}N_\gamma(\emptyset)$$
$$\times \Big\{ N_\gamma(\underline{n}_j) \sum_{\underline{n}_{p-1}} (K'_\gamma)^{p-1-j}(\underline{n}_j,\underline{n}_{p-1})\, K''_\gamma(\underline{n}_{p-1},\emptyset) N_\gamma(\emptyset)^{-1} \Big\}$$

having noticed that N_γ, defined in (4.40d), depends on η only via the triple $\underline{n}$ associated to η. Since $N_\gamma(\emptyset)=1$, recalling that by (4.47c) $(n_j,m_j,\ell_j)\le(\bar n_j,\bar m_j,\bar\ell_j)$, we have from (4.40c) with $b_1=b_2=b_3=0$

$$\le c^\star(u,m_0+pu)^{p-j}N_\gamma(\underline{n}_j)^{-1} = c^\star(u,m_0+pu)^{p-j}\gamma^{du/2} \tag{4.47b}$$

where we have used the notation

$$c^\star(n,m) = \max_{n'\le n, m'\le m} c(n',m') \tag{4.48a}$$

with $c(n,m)$ as in (4.40c). We also set

$$c_0^\star(n,m) = \max_{n'\le n, m'\le m} c_0(n',m') \tag{4.48b}$$

with $c_0(n,m)$ as in (4.40b) and have

$$I_p \le \sum_{u=n_0}^{n_0+2p}\sum_{j=0}^{p-1} e^{Cut}\frac{t^p}{p!}c_0^\star(u,m_0+ju)^j c^\star(u,m_0+pu)^{p-j}\gamma^{du/2} \tag{4.49}$$

The leading term in (4.49), as $\gamma\to 0$, is the one with $u=n_0$. There is therefore a constant $c(N,\underline{n}_0)$ such that

$$\sum_{p=1}^N I_p \le c(N,\underline{n}_0)e^{Cn_0t}t^N\gamma^{dn_0/2} \tag{4.50}$$

We suppose that $aC<d/2$ (and as usual $t\le a\log\gamma^{-1}$), then the right hand side vanishes satisfying the bound (2.23).

The bound of L, in (4.45a), is analogous. We use (4.40c), in the version with $b_1=d/2$ and (4.46) to get

$$L \le e^{C(n_0+2N)t}\int_0^t ds_1 ... \int_0^{s_{N-1}} ds_N N_\gamma(\underline{n}_0)^{-1}c^\star\big(n_0+2N, m_0+N(n_0+2N)\big)^N\gamma^{\zeta^\star N}$$
$$\times \sup_{\underline{n}_N}{}^\star\{N_\gamma(\underline{n}_N)\psi_{\underline{n}_N}(s_N)\}$$

where the $\sup^\star$ is over all those $\underline{n}_N$ whose entries are bounded as in (4.46).

We will prove afterwards that there is $c_{\underline{n}_0,N}$ such that

$$\sup_{t\le a\log\gamma^{-1}}\ \sup_{\underline{n}_N}{}^{\star} N_\gamma(\underline{n}_N)\psi_{\underline{n}_N}(t)\le c_{\underline{n}_0,N} \tag{4.51}$$

By (4.51)

$$L\le e^{C(n_0+2N)t}\frac{t^N}{N!}c^\star\big(n_0+2N,m_0+N(n_0+2N)\big)^N c_{\underline{n}_0,N}N_\gamma(\underline{n}_0)^{-1}\gamma^{\zeta^\star N} \tag{4.52}$$

We then restrict to $\underline{n}_0=(0,0,0)$, as we are interested in the case $\rho_{\eta_0}=\tilde\sigma(\underline{x},t)$. With this choice $N_\gamma(\underline{n}_0)=1$. Then with a such that $2aC<\zeta^\star$, we can finally specify the choice of N, namely such that

$$\zeta^\star N>aC(n+2N)+dn/2 \tag{4.53}$$

Then, from (4.52).

$$L\le c\gamma^{dn/2}$$

with c a suitable constant. This bound together with (4.50) proves (2.23) which is therefore proven modulo (4.51).

Proof of (4.51).

Let $\eta=(\underline{x},\underline{y},\xi_1,..,\xi_p)$, then

$$\big|\rho_\eta(\sigma,m)\big|\le 2^{|\underline{x}|}\Big\{\prod_{i=1}^p|f_{\xi_i}(\sigma)-f_{\xi_i}(m)|\Big\}\Big\{\prod_{i=1}^{|\underline{y}|}\Big(\big|\mathcal{A}_{\gamma^{-\alpha},y_i}(\tilde\sigma(\cdot))\big|+2|\eta|\gamma^{d\alpha}\Big)\Big\} \tag{4.54a}$$

(4.54a) is straightforward consequence of the definition (3.21), with the last factor obtained as follows.

We write

$$\underline{z}=(z_1,..,z_\ell);\ \underline{y}=(y_1,..,y_\ell);\ \underline{y}'=(y_1,..,y_{\ell-1});\ \underline{z}'=(z_1,..,z_{\ell-1})$$

then, recalling (3.22),

$$\chi_{\underline{x},\underline{y}}(\underline{z})=\chi_{\underline{x},\underline{y}'}(\underline{z}')\chi_{\underline{x}+\underline{z}',y_\ell}(z_\ell)$$

so that

$$\sum_{\underline{z}}\chi_{\underline{x},\underline{y}}(\underline{z})\tilde\sigma(\underline{z})=\sum_{\underline{z}'}\chi_{\underline{x},\underline{y}'}(\underline{z}')\tilde\sigma(\underline{z}')\sum_z\chi_{\underline{x}+\underline{z}',y_\ell}(z)$$

and

$$\Big|\sum_z\chi_{\underline{x}+\underline{z}',y_\ell}(z)\tilde\sigma(\underline{z})\Big|\le\big|\mathcal{A}_{\gamma^{-\alpha},y_\ell}(\tilde\sigma(\cdot))\big|+(|\underline{x}|+\ell)2\gamma^{d\alpha}$$

hence (4.54a).

We next show that the differences $|f_{\xi_i}(\sigma)-f_{\xi_i}(m)|$ can be also bounded in terms of the averages $\mathcal{A}_{\gamma^{-\alpha},\cdot}(\tilde\sigma(\cdot))$.

By Lemma 3.3 there is $c(d(\xi))$ so that, for any ξ,

$$|f_\xi(\sigma)-f_\xi(m)|\le c(d(\xi))\sup_{\epsilon\in\mathbb{R}}\Big|\tanh^{(d(\xi))}(\psi_\sigma+\epsilon)-\tanh^{(d(\xi))}(\psi_m+\epsilon)\Big| \tag{4.54b}$$

where

$$\psi_\lambda=\beta\sum_{y\notin V_\xi}J_\gamma(x,y)\lambda(y)$$

Since in (4.51) we have a $\sup^\star$, we may restrict in (4.54b) to $d(\xi)\le\bar m_N=N(n_0+2N)$, by (4.46), and to $|V_\xi|\le 2N$, because in η_0 no ξ is present and, by Proposition 4.8, the transitions $\eta\to\eta'$ in $M^{\rm dg}_\gamma(\eta,\eta',m)$ do not increase the max of $|V_\xi|$, see (4.39a), while those due to M_γ make the max of $|V_\xi|$ increase at most by 2, see (4.39b).

There is therefore a constant c, which depends on n_0 and N, such that

$$|f_\xi(\sigma)-f_\xi(m)|\le c\Big|\sum_{y\notin V_\xi}J_\gamma(x,y)\tilde\sigma(y,t)\Big|\le c\gamma^d[2|V_\xi|\|J\|_\infty]+c\Big|\sum_y J_\gamma(x,y)\tilde\sigma(y,t)\Big|$$

and

$$\Big|\sum_y J_\gamma(x,y)\tilde\sigma(y,t)\Big|\le\Big|\sum_y\{\sum_{z\in B_{\gamma^{-\alpha}},y}\frac{1}{|B_{\gamma^{-\alpha}}|}J_\gamma(x,z)\}\tilde\sigma(y,t)\Big|+2\gamma^{1-\alpha}\|J'\|_\infty$$

The first term on the right hand side is bounded by

$$\sum_z|J_\gamma(x,z)|\Big|\mathcal{A}_{\gamma^{-\alpha},z}(\tilde\sigma(\cdot,t))\Big|\le c\gamma^b \tag{4.54c}$$

by Proposition 3.6, with probability that vanishes faster than any power of γ. The same argument applies to the other averages in (4.54a), we thus conclude that with probability that vanishes faster than any power of γ, $\rho_\eta(\sigma_t,m_t)$ is bounded proportionally to $\gamma^{bg(\eta)}$ with a proportionality constant which depends, via (4.46), only on n_0 and N. The factor $N_\gamma(\eta)$ in (4.51) has an exponential factor $\gamma^{-b_2^\star g(\eta)}$, so that by choosing $b_2^\star$ equal to b in (4.54c) we obtain a bound independent of γ, in the set where (4.54c) holds. The complement of this set has a probability which vanishes faster than any power of γ, we thus complete the proof of (4.51) by recalling that $N_\gamma(\eta)\le\gamma^{-b_2^\star g(\eta)}$ and that, by Lemma 3.3, $|\rho_\eta|$ is also bounded. $\square$

Proof of Theorem 2.3.4.

We have already proven (2.23) and it only remains to prove (2.24). Since (2.23) holds as well starting from an arbitrary product measure, we have

$$\lim_{\gamma\to0}\sup_{0\le t\le a\log\gamma^{-1}}\sup_{|\underline{x}|=n}\Big|\mathbb{E}^\gamma_{\mu^\gamma}\Big(\prod_{i=1}^n\sigma(x_i,t)\Big)-\prod_{i=1}^n m_\gamma(x_i,t|\mu^\gamma)\Big|=0 \tag{4.55}$$

By Proposition 4.7.2

$$\|\Gamma_\gamma\Big(m_\gamma(\cdot,t|\mu^\gamma)\Big) - m_\gamma(\cdot,t)\|_\infty \le c_2' e^{c_2 t}\gamma \tag{4.56a}$$

where $m_\gamma(r,t)$ solves (1.1) with initial datum $m_{\gamma,0}(r)$ as in (2.12a). If $m_{\gamma,0}(r) = m_0([r]_\gamma)$ and m_0 as in (ii) of Theorem 2.3.4. Since $m_0 \in C^1(\mathbb{R}^d)$, $\|m_0'\|_\infty < \infty$, we have that, for some c,

$$\|m_\gamma(\cdot,0) - m_0(\cdot)\|_\infty \le c\gamma$$

Then, by Proposition 4.7.5

$$\|m_\gamma(\cdot,t) - m(\cdot,t)\|_\infty \le e^{c_5 t} c\gamma \tag{4.56b}$$

Here $m(r,t)$ denotes the solution of (1.1) with initial datum m_0. (2.24) then follows from (4.55) and (4.56). Theorem 2.3.4 is thus proven. □

We actually have results stronger than those proven so far:

Theorem 4.9.

Let (m_0,μ^γ) be an initial state in the sense of Definition 2.1.7 and assume that there are $0<\alpha<1$ and $b'>0$ such that, for all γ small enough

$$\|m_{\gamma,0}^{(\alpha,\gamma)} - m_0^{(\alpha,\gamma)}\|_\infty \le \gamma^{b'} \tag{4.57}$$

where $m_{\gamma,0}^{(\alpha,\gamma)}$ is the block spin transform of $m_{\gamma,0}$, the latter being as in (2.12a), while $m_0^{(\alpha,\gamma)}$ is the block spin image of m_0.

Then there are $a>0$ and given any $\tau>0$ there is $b>0$ so that the following holds. For any n and $k^\star$ there is c so that

$$\mathbb{P}_{\mu^\gamma}^\gamma\left(\Big\{\sigma_0 : \mathbb{P}_{\sigma_0}^\gamma\Big(\sup_{|r|\le k^\star\gamma^{-1}}\Big|\sigma_{\gamma,\tau\log\gamma^{-1}}^{(\alpha)}(r) - m(r,\tau\log\gamma^{-1})\Big| > \gamma^b\Big) \le c\gamma^n\Big\}\right) \ge 1 - c\gamma^n \tag{4.58}$$

where $m(r,t)$ is the solution of (1.1) starting from m_0.

Remark: the two examples mentioned in the remarks following Definition 2.1.7 fulfill the conditions of Theorem 4.9.

Proof.

By (2.13b), (4.58) holds with $m(\tau\log\gamma^{-1})$ replaced by $m_\gamma^{(\alpha,\gamma)}(r,\tau\log\gamma^{-1})$ which is the block spin transform of $m_\gamma(\cdot,\tau\log\gamma^{-1})$, solution of (1.1) starting from $m_{\gamma,0}$. Denoting by $m^{(\alpha,\gamma)}(r,t)$ the block spin transform of $m(\cdot,t)$, by Proposition 4.7.6 and (4.57) we can then replace $m_\gamma^{(\alpha,\gamma)}$ by $m^{(\alpha,\gamma)}(r,\tau\log\gamma^{-1})$, their difference being bounded by $c_6' e^{c_6\tau\log\gamma^{-1}} c\gamma$, where $c\gamma$ bounds their difference at time 0. By Proposition 4.7.8, $m^{(\alpha,\gamma)}(r,\tau\log\gamma^{-1})$ is, for any given τ, $\gamma^{b''}$, $b''>0$, close to $m(r,\tau\log\gamma^{-1})$ (if $\tau<a$ and a is suitably small). Theorem 4.9 is therefore proven. $\square$

5. Motion by mean curvature

In this Section we restrict to ferromagnetic interactions, $J\geq 0$, in the absence of magnetic fields, $h=0$, and below the critical temperature, $\beta>1$, having imposed the normalization condition $\int dr J=1$.

The proof of Theorem 2.2.4 is based on an iterative procedure similar to that used in [18] to prove convergence to the motion by mean curvature in the Glauber + Kawasaki spin dynamics. The single steps of the iteration exploit the results in [9] and the previous estimates on the Glauber dynamics valid for $t\leq a\log\gamma^{-1}$.

We start by recalling some definitions and results in [9] that we will use in the sequel.

5.1 Definition. *The instanton.*

The instanton $\bar m:\mathbb{R}\to[-1,1]$ is an antisymmetric, strictly increasing function such that $m^{(d)}(r):=\bar m(r_1)$ (r_1 the first coordinate of r) is a stationary solution of (1.1). The instanton solution exists and it is unique, see [7], [9], [10] and [11], furthermore

$$\lim_{r_1\to\pm\infty}\bar m(r_1)=\pm m_\beta,\quad \textit{exponentially fast} \tag{5.1}$$

with m_β as in (2.14).

In [9] it is proven that under the macroscopic scaling defined in (2.17) the mesoscopic equation (1.1) gives rise to the motion by mean curvature of Definition 2.2.3 with parameter

$$\theta=N\int dr_1\bar m'(r_1)^2\beta\int dr_1'dr_\perp J\big(|(r_1'-r_1)^2+r_\perp^2|^{1/2}\big)\frac{\bar m'(r_1')}{\bar m'(r_1)}r_\perp^2/2 \tag{5.2a}$$

$r_\perp$ denoting the vector in the plane perpendicular to the first coordinate axis; N is the normalization constant:

$$N^{-1}=\int_{\mathbb{R}}dr_1\frac{\bar m'(r_1)^2}{1-\bar m(r_1)^2} \tag{5.2b}$$

As in [9] and, before, in the studies on the Allen-Cahn equation, see for instance [4], a fundamental role in the proof of Theorem 2.2.4 will be played by the ferromagnetic inequalities.

At the level of the mesoscopic equation (1.1), they say that if $\tilde{m}(r,t)$ and $m(r,t)$ solve (1.1) and $\tilde{m}(r,0) \ge m(r,0)$ for all r, then $\tilde{m}(r,t) \ge m(r,t)$ for all r and t. Here is essential the assumption that $J \ge 0$. A variant of the above inequality is based on the notion of super and subsolutions of (1.1), namely functions which satisfy (1.1) with $\ge$ instead of $=$, for the supersolutions, and with $\le$ for the subsolutions. Such functions are then respectively upper and lower bounds for the true solution, if they are so at time 0. The whole game is then to find super and subsolutions which, in the limit $\gamma \to 0$, squeeze (in macroscopic coordinates) the true solution toward a function which describes the interface moving by mean curvature. This is what done in [9], here we describe the supersolution, as the subsolution is defined symmetrically.

5.2.1 Definition. *The signed distance.*

We will consider regular C^∞ surfaces Σ which are boundary of compact, connected regions $\Lambda \subset \mathbb{R}^d$. We then denote by $d(\xi,\Sigma)$, $\xi \in \mathbb{R}^d$, the distance of ξ from Σ, if $\xi \in \Lambda$, and minus the distance, if $\xi \notin \Lambda$. Analogously, the signed distance in mesoscopic coordinates is $d(r,\lambda^{-1}\Sigma) = \lambda^{-1} d(\xi,\Sigma)$, where $\xi = \lambda r$.

5.2.2 Definition. *The function $m^+_{(\Sigma,\lambda)}$.*

For each λ we set

$$m^+_{(\Sigma,\lambda)}(r) = \bar{m}\big(d(r,\lambda^{-1}\Sigma)\big), \qquad \text{for } |d(r,\lambda^{-1}\Sigma)| \le R_0 \log \lambda^{-1} \tag{5.3a}$$

with $R_0 > 0$ the constant defined in (34a) of [9]. The definition of $m^+_{(\Sigma,\lambda)}(r)$ is then completed by setting

$$m^+_{(\Sigma,\lambda)}(r) = \pm m_\beta + \lambda^{3/2}, \qquad \text{for } d(r,\lambda^{-1}\Sigma) \gtrless \pm R_0 \log \lambda^{-1} \tag{5.3b}$$

The definition of $m^-_{(\Sigma,\lambda)}(r)$ differs from the above one only in (5.3b), where we subtract instead of adding $\lambda^{3/2}$. The super and subsolutions are then the functions $m^\pm_{(\Sigma_\tau,\lambda)}(r)$ provided Σ_τ moves according to the following:

5.3 Definition. *A biased motion by mean curvature.*

Given λ and $\bar\tau > 0$, we consider the two biased motions by mean curvature $\Sigma^\pm_{(\tau,\lambda)}$, in the time interval $0 \le \tau \le \bar\tau$, $\bar\tau > 0$, defined as follows. For each τ, $\Sigma^\pm_{(\tau,\lambda)}$ is a connected C^∞ surface which has a C^∞ parametrization

$$\xi^\pm = \xi^\pm(\tau,\xi_0,\lambda), \quad \xi_0 \in S_0, \quad S_0 \text{ a } C^\infty,\ d-1 \text{ dimensional, compact manifold}$$

which satisfies the equation

$$\frac{d\xi^\pm}{d\tau} = \theta\kappa\nu \mp h\nu \tag{5.4}$$

We will eventually restrict to θ as in (5.2) and $h = \lambda^{\delta/2}$, $\delta > 0$ as in §8 of [9] and in Theorem 5.5 below. We next state a classical result on parabolic equations, see [2] and references therein:

5.4 Theorem.

Let Σ_τ, $0 \le \tau \le \tau^\star$, be a motion by mean curvature as in Definition 2.2.3 and let $\xi = \xi(\tau, \xi_0)$, $\xi_0 \in S_0$, be the corresponding parametrization. Then, given any $R_0 > 0$ (we shall use the result with R_0 as in 5.2.2), for all λ small enough,

$$\Sigma^{\pm}_{(0,\lambda)} := \{\xi : d(\xi, \Sigma_0) = \mp 2R_0 \lambda \log \lambda^{-1}\}$$

is also a C^∞ surface. Moreover, for any $\bar\tau < \tau^\star$, for all h small enough and for all λ small enough, there is a h-biased motion by curvature $\Sigma^{\pm}_{(\tau,\lambda)}$, $0 \le \tau \le \bar\tau$, in the sense of Definition 5.3, starting at $\tau = 0$ from $\Sigma^{\pm}_{(0,\lambda)}$ and with parametrization $\xi^{\pm}(\tau, \xi_0, \lambda)$, where $\xi_0 \in S_0$ and S_0 is the same manifold used for the parametrization of Σ_τ. Furthermore there is c, that depends only on $\bar\tau$ and Σ_τ, so that

$$\left|\xi^{\pm}(\xi_0, \tau, \lambda) - \xi(\xi_0, \tau)\right| \le ch, \qquad 0 \le \tau \le \bar\tau \tag{5.5}$$

The statement on the regularity of $\Sigma^{\pm}_{(0,\lambda)}$ is not contained in [9], but it is a direct consequence of the regularity of Σ_0 for λ small enough.

We next recall a result proven in [9]:

5.5 Theorem.

Let Σ_τ, $0 \le \tau \le \tau^\star$, be the motion by mean curvature considered in Theorem 5.4. Then there are $\delta \in (0, 1/2)$, $\omega > 0$ and given any $\tau \in [0, \bar\tau]$, (see Theorem 5.4) there is c so that for all $u \in [1, 2]$ the following holds.

Recalling Definition 5.2.2, we define, for all λ small enough and all $0 \le s \le \lambda^{-2}\bar\tau$,

$$m^{\pm}_\lambda(r, s) = m^{\pm}_{(\Sigma,\lambda)}(r), \qquad \Sigma \equiv \Sigma^{\pm}_{(\lambda^2 s,\lambda)} \tag{5.6a}$$

where $\Sigma^{\pm}_{(\tau,\lambda)}$ are motion by curvature with bias $h = \lambda^{\delta/2}$. Then, if $m_\lambda(r, t)$ solves (1.1) for $t \ge s$ and

$$m^{-}_\lambda(r, s) \le m_\lambda(r, s) \le m^{+}_\lambda(r, s), \qquad \text{for all } r \in \mathbb{R}^d \tag{5.6b}$$

then,

$$m^{-}_\lambda(r, s + u\lambda^{-\delta}) + \lambda^\omega \le m_\lambda(r, s + u\lambda^{-\delta}) \le m^{+}_\lambda(r, s + u\lambda^{-\delta}) - \lambda^\omega \tag{5.7a}$$

By repeated use of Theorem 5.5 with suitably different values of u we also have that for all t such that $s + \lambda^{-\delta} \le t \le \lambda^{-2}\bar\tau$,

$$m^{-}_\lambda(r, t) + \lambda^\omega \le m_\lambda(r, t) \le m^{+}_\lambda(r, t) - \lambda^\omega \tag{5.7b}$$

We have now all the ingredients for the proof of Theorem 2.2.4. Let $\tau > 0$ be as in Theorem 2.2.4. We then take $\bar\tau = \tau$ in Theorem 5.5. Let α, ζ, a and b as in Theorem 2.1.6, with $a > 0$ so small that (5.21) below holds and such that $\tau/a =: N$ is a positive integer. Setting

$$s_n = na\log\gamma^{-1}, \quad R_n = (N+1-n)\gamma^{-1}, \qquad n \text{ a non negative integer} \tag{5.8}$$

we first define for any $n \ge 0$ and any spin trajectory $(\sigma_t)_{t\ge 0}$ $m_{(n)}(r,t)$ to be the solution of (1.1) for $t \ge s_n$ and such that $m_{(n)}(r, s_n) = \sigma_{\gamma,s_n}(r)$, for all $r \in \mathbb{R}^d$, see (2.8c) for notation. Then we introduce the set

$$\mathcal{G}_n = \Big\{ (\sigma_t)_{t\ge 0} : \Big| \sigma^{(\alpha)}_{\gamma,t}(r) - m^{(\alpha,\gamma)}_{(n)}(r,t) \Big| \le \gamma^b, \quad s_n \le t \le s_{n+1}, \ |r| \le R_n \Big\} \tag{5.9a}$$

with $m^{(\alpha,\gamma)}_{(n)}$ as in Definition 2.1.5. We finally set

$$\mathcal{G} = \Big\{ \bigcap_{n=0}^{N-1} \mathcal{G}_n \Big\} \bigcap \Big\{ \Big| \sigma^{(\alpha)}_{\gamma,0}(r) - m^{(\alpha,\gamma)}_{\gamma,0}(r) \Big| \le \gamma^b, \quad |r| \le R_0 \Big\} \tag{5.9b}$$

with $m_{\gamma,0}$ as in (2.12a).

By Theorem 2.1.6, for any k there is c so that

$$\mathbb{P}^{\gamma}_{\mu^\epsilon}(\mathcal{G}) \ge 1 - c\gamma^k \tag{5.10}$$

We will prove Theorem 2.2.4 by showing that for all γ small enough, any spin trajectory in $\mathcal{G}$ is in the set appearing on the right hand side of (2.19a).

We first prove a weaker version with the sup over t in (2.19a) restricted to $t \notin (s_n, s_n + \lambda^{-\delta})$, for any $0 \le n \le N-1$. For $n \ge 1$ we define

$$\psi_{(n)}(r) = \begin{cases} m_{(n-1)}(r, s_n) & \text{if } |r| \le R_n \\ -m_\beta & \text{otherwise} \end{cases} \tag{5.11}$$

and $\psi_{(n)}(r,t)$, $t \ge s_n$, as the solution of (1.1) such that $\psi_{(n)}(r, s_n) = \psi_{(n)}(r)$, for all $r \in \mathbb{R}^d$. We also define $\psi_{(0)}(r) = m_{\gamma,0}(r)$ for all $r \in \mathbb{R}^d$ and $\psi_{(0)}(r,t)$ the solution of (1.1) for $t \ge 0$ with initial datum $\psi_{(0)}(\cdot)$.

Observe that the definition of the $\psi_{(n)}$'s depends upon the trajectory $(\sigma_t)_{t\ge 0}$. We start by proving that in $\mathcal{G}$, for all $0 \le n \le N-1$,

$$m^-_\lambda(\cdot, s_n) \le \psi_{(n)}(\cdot) \le m^+_\lambda(\cdot, s_n) \tag{5.12}$$

We prove (5.12) by induction. Since it is evidently true for $n = 0$, we need only show that if (5.12) holds for $n < N-1$, then it also holds for $n+1$. By (5.7b)

$$m^-_\lambda(r, s_{n+1}) + \lambda^\omega \le \psi_{(n)}(r, s_{n+1}) \le m^+_\lambda(r, s_{n+1}) - \lambda^\omega \tag{5.13}$$

Then (5.12) with $n+1$ follows from (5.13) and the following bound, that we will prove next. There are $\vartheta > 0$ and c so that

$$\left| m_{(n)}(r, s_{n+1}) - \psi_{(n)}(r, s_{n+1}) \right| \le c\gamma^{\vartheta}, \qquad \text{for all } |r| \le R_{n+1} \tag{5.14}$$

Analogously to (5.11) we define:

$$z_{(n)}(r) = \begin{cases} \sigma_{\gamma, s_n}(r) & \text{if } |r| \le R_n \\ -m_\beta & \text{otherwise} \end{cases} \tag{5.15}$$

and $z_{(n)}(r,t)$, $t \ge s_n$, as the solution of (1.1) with $z_{(n)}(\cdot, s_n) = z_{(n)}(\cdot)$. Then, by Proposition 4.7.4, for any k there is c so that

$$\left| z_{(n)}(r,t) - m_{(n)}(r,t) \right| \le c\gamma^k, \qquad |r| \le R_{n+1}, \quad s_n \le t \le s_{n+1} \tag{5.16a}$$

$$\left| z_{(n)}^{(\alpha,\gamma)}(r,t) - m_{(n)}^{(\alpha,\gamma)}(r,t) \right| \le c\gamma^k, \qquad |r| \le R_{n+1}, \quad s_n \le t \le s_{n+1} \tag{5.16b}$$

By the definition of $\mathcal{G}$,

$$\left| z_{(n)}^{(\alpha,\gamma)}(r) - \psi_{(n)}^{(\alpha,\gamma)}(r) \right| \le \gamma^b, \qquad r \notin D, \quad D = \{r : R_n - \gamma^{1-\alpha} \le |r| \le R_n + \gamma^{1-\alpha}\} \tag{5.17}$$

We then apply Proposition 4.7.6 to $z_{(n)}$ and $\psi_{(n)}$, with α and b and D as above, $u = 1-\alpha$ and c' a suitable constant. Therefore

$$\left| z_{(n)}^{(\alpha,\gamma)}(r,t) - \psi_{(n)}^{(\alpha,\gamma)}(r,t) \right| \le c_6' e^{c_6(t-s_n)}[\gamma^b + \gamma^{1-\alpha} + c'\gamma^{1-\alpha}], \qquad |r| \le R_{n+1}, \quad s_n \le t \le s_{n+1} \tag{5.18}$$

On the other hand, by Proposition 4.7.8, for all $r \in \mathbb{R}^d$,

$$\left| z_{(n)}^{(\alpha,\gamma)}(r, s_{n+1}) - z_{(n)}(r, s_{n+1}) \right| \le c_8 \gamma^{1-\alpha} + 2\gamma^a \tag{5.19}$$

The analogous inequality holds for $\psi_{(n)}$ as well. Hence, from (5.18), (5.19) and (5.16a)

$$\left| m_{(n)}(r, s_{n+1}) - \psi_{(n)}(r, s_{n+1}) \right| \le c\gamma^k + 2[c_8\gamma^{1-\alpha} + 2\gamma^a] + c_6' e^{c_6 a \log \gamma^{-1}}[\gamma^b + \gamma^{1-\alpha} + c'\gamma^{1-\alpha}] \qquad |r| \le R_{n+1} \tag{5.20}$$

We choose a as

$$a \le c_6^{-1} \min\{1-\alpha, b\} \tag{5.21}$$

Then (5.14) holds with $\vartheta > 0$ and with $c\gamma^{\vartheta}$ an upper bound for the right hand side of (5.20). We have thus completed the proof of the induction and of (5.12).

By (5.16) and (5.18), we have for any $0 \le n \le N-1$,

$$\left| m^{(\alpha,\gamma)}_{(n)}(r,t) - \psi^{(\alpha,\gamma)}_{(n)}(r,t) \right| \le c\gamma^k + c_6' e^{c_6(t-s_n)}[\gamma^b + \gamma^{1-\alpha} + c'\gamma^{1-\alpha}],$$
$$|r| \le R_{n+1}, \quad s_n \le t \le s_{n+1} \tag{5.22}$$

By (5.12) and (5.7b)

$$m^-_\lambda(r,t) + \lambda^\omega \le \psi_{(n)}(r,t) \le m^+_\lambda(r,t) - \lambda^\omega, \qquad s_n + \lambda^{-\delta} \le t \le s_{n+1} \tag{5.23}$$

Hence, by (5.22) and (5.23), there are c' and $b' > 0$ so that

$$\left(m^-_\lambda\right)^{(\alpha,\gamma)}(r,t) + \lambda^\omega - c'\gamma^{b'} \le m^{(\alpha,\gamma)}_{(n)}(r,t) \le \left(m^+_\lambda\right)^{(\alpha,\gamma)}(r,t) - \lambda^\omega + c'\gamma^{b'},$$
$$s_n + \lambda^{-\delta} \le t \le s_{n+1}, \ \ |r| \le R_{n+1} \tag{5.24a}$$

and, by (5.9),

$$\left(m^-_\lambda\right)^{(\alpha,\gamma)}(r,t) + \lambda^\omega - c'\gamma^{b'} - \gamma^b \le \sigma^{(\alpha)}_{\gamma,t}(r) \le \left(m^+_\lambda\right)^{(\alpha,\gamma)}(r,t) - \lambda^\omega + c'\gamma^{b'} + \gamma^b,$$
$$s_n + \lambda^{-\delta} \le t \le s_{n+1}, \ \ |r| \le R_{n+1} \tag{5.24b}$$

For γ small enough, $t \le \lambda^{-2}\bar\tau$ and $\left|d\big(r, \lambda^{-1}\Sigma^-_{(\lambda^2 t,\lambda)}\big)\right| > R_0 \log \lambda^{-1}$

$$\left(m^-_\lambda\right)^{(\alpha,\gamma)}(r,t) + \lambda^\omega - c'\gamma^{b'} - \gamma^b \ge \pm m_\beta - \lambda^{3/2} \tag{5.25}$$

for r inside, respectively outside, $\lambda^{-1}\Sigma^-_{(\lambda^2 t,\lambda)}$.

By (5.5), we have that (5.25) holds as well for t and r as in (5.24b) and r such that

$$|d\big(r, \lambda^{-1}\Sigma_\tau\big)| > R_0 \log \lambda^{-1} + ch \ge c' \log \lambda^{-1} \tag{5.26a}$$

having recalled that $h = \lambda^{\delta/2}$, see Theorem 5.5. We finally have:

$$\pm m_\beta - \lambda^{3/2} \le \sigma^{(\alpha)}_{\gamma,t}(r) \le \pm m_\beta + \lambda^{3/2} \tag{5.26b}$$

for r inside, respectively outside, $\lambda^{-1}\Sigma_{\lambda^2 t}$. The upper bound in (5.26b) is proven by the same arguments used for the lower bound.

By repeating the previous procedure with suitably different values of a, we prove the bound (5.26b) for all $\lambda^{-\delta} \le t \le \lambda^{-2}\bar\tau$. In this "short" time interval we can, however, use a very rough estimate: by Proposition 4.7.4, in fact, for any k there is c, so that ($\psi_{(0)}(r,t)$ being defined below (5.11))

$$\left|\psi_{(0)}(r,t) \pm m_\beta\right| \le c\gamma^k, \qquad t \le \lambda^{-\delta}, \ \ d(r, \lambda^{-1}\Sigma_0) \ge \lambda^{-2\delta} \tag{5.27}$$

for r respectively outside and inside $\lambda^{-1}\Sigma_0$. By choosing ζ in (2.19a) so that $1 - \zeta > 2\delta$, recalling (5.9b) and using Proposition 4.7.4 we then complete the proof of Theorem 2.2.4.

Appendix

In this Appendix we prove Proposition 4.8. The notation are those of Section 3, see in particular Proposition 3.6, whose proof is for many aspects analogous to the present one, but simpler in several respects.

We shall see that the expression $\sum_x D^{(x)}\rho_\eta(\sigma,m)$ has many terms that may be divided into groups, giving rise to an expression of the form $\sum_i T_i$, with i in a finite set. The terms in each T_i are in also manipulated and regrouped: $T_i=\sum_j T_{(i,j)}$. We keep doing that till we have a decomposition where each term T_κ, κ a multi-index, such that

$$T_\kappa=\sum_{\eta'} M_\kappa^{\mathrm{dg}}(\eta,\eta',m)\rho_{\eta'} \tag{A.1a}$$

$$T_\kappa=\sum_{\eta'} M_\kappa(\eta,\eta',m)\rho_{\eta'} \tag{A.1b}$$

with M_κ^{dg} and M_κ having the right support properties as in Proposition 4.8 and satisfying the bounds (4.40). Since the number of subcases necessary to reach the situations (A.1a) and (A.1b) will be finite, we will then have proved Proposition 4.8.

The above classification has a branching structure, each branch ends when we reach an expression compatible with (A.1). Observe that we are not making explicit the dependence on γ which is however present in M_κ^{dg} and M_κ.

Notation.

We write

$$\eta=(\underline{x},\underline{y},\xi_1,..,\xi_p);\quad \xi_i=\left(\hat{x}_i,\underline{\hat{y}}_{i,1},\underline{\hat{y}}_{i,2},\underline{\hat{y}}_{i,3},\underline{\hat{y}}_i^\star\right) \tag{A.2}$$

and denote by V_{ξ_i} the collection of all the sites in ξ_i, see Definition 3.2.

The set $\underline{y}$, which is the second entry in η, will be written as $\underline{y}=(y_1,..,y_\ell)$; we also denote $\underline{z}=(z_1,..,z_\ell)$ and recall that

$$\chi_{\underline{x},\underline{y}}(\underline{z})=\prod_{i=1}^{\ell}\left\{\frac{1}{|B_{\gamma^{-\alpha}}|}\mathbf{1}(z_i\in B_{\gamma^{-\alpha},y_i}-\underline{x})\right\} \tag{A.3}$$

with $A\pm B$ being the union, respectively the difference of the sets A and B.

We will also use the shorthand notation

$$F_i:=f_{\xi_i}(\sigma)-f_{\xi_i}(m) \tag{A.4}$$

The initial step of the branching is

$$\sum_x D^{(x)}\rho_\eta(\sigma,m)=\sum_{i=1}^{3}T_{(i)} \tag{A.5a}$$

where the terms $T_{(i)}$ are identified by the following identity, the labelling corresponds to the order of appearance:

$$\sum_x D^{(x)}\rho_\eta = \sum_{x\in\underline{x}} D^{(x)}\rho_\eta + \tilde\sigma(\underline{x})\sum_{\underline{z}}\chi_{\underline{x},\underline{y}}(\underline{z})\sum_{x\in\underline{z}} D^{(x)}[\tilde\sigma(\underline{z})\prod_{i=1}^p F_i]$$
$$+\tilde\sigma(\underline{x})\sum_{\underline{z}}\chi_{\underline{x},\underline{y}}(\underline{z})\sigma(\underline{z})\sum_{x\notin\underline{x}+\underline{z}} D^{(x)}\prod_{i=1}^p F_i \tag{A.5b}$$

Analysis of $T_{(1)}$.

We have

$$T_{(1)} = T_{(1,1)} + T_{(1,2)} \tag{A.6a}$$

where, recalling (3.28) and the notation used in the proof of Proposition 3.6,

$$T_{(1,1)} = \sum_{\underline{z}}\chi_{\underline{x},\underline{y}}(\underline{z})\sum_{x\in\underline{x}}[\prod_{i=1}^p\Gamma^i_{1,x}]\tilde\sigma(\underline{x}+\underline{z}-x)D^{(x)}\tilde\sigma(x) \tag{A.6b}$$

$$T_{(1,2)} = \sum_{x\in\underline{x}}\left\{\tilde\sigma(\underline{x}-x)\sum_{\underline{z}}\chi_{\underline{x},\underline{y}}(\underline{z})\tilde\sigma(\underline{z})\sum_{\substack{\underline{i}_1,..\underline{i}_5\\|\underline{i}_1|<p}}\Delta_{\underline{i}_1,..\underline{i}_5,\underline{x},x}\right\} \tag{A.6c}$$

with Δ as in (3.30d)-(3.30e).

We start from $T_{(1,1)}$. We write $D^{(x)}\sigma(x)$ using (3.26) and we get

$$T_{(1,1)} =: T_{(1,1,1)} + T_{(1,1,2)} \tag{A.7a}$$

$$T_{(1,1,1)} = -\sum_{x\in\underline{x}}\tilde\sigma(\underline{x})\sum_{\underline{z}}\chi_{\underline{x},\underline{y}}(\underline{z})\tilde\sigma(\underline{z})[\prod_{i=1}^p\Gamma^i_{1,x}] \tag{A.7b}$$

while $T_{(1,1,2)}$ is defined in (A.8) below. The right hand side of (A.7b) has already the form

$$T_{(1,1,1)} = \sum_{\eta'} M^\star_{(1,1,1)}(\eta,\eta',m)\rho_{\eta'}$$

The sum is restricted to $|\eta'| = |\eta|$, $g(\eta') = g(\eta)$ and $d(\eta') = d(\eta)$. Moreover

$$\sum_{\eta'}|M^\star_{(1,1,1)}(\eta,\eta',m)| \le |\underline{x}|$$

but

$$\eta' = (\underline{x},\underline{y},\xi'_1,..\xi'_p),\quad \text{with } \xi'_i = \Gamma^i_{1,x}\text{, hence it may be that } |V_{\xi'_i}| = |V_{\xi_i}|+1$$

so that $M^\star$ is neither of the form M_κ nor $M_\kappa^{\rm dg}$.

This problem is solved by going back to (3.28a) that we now read as a relation which expresses $\Gamma^i_{1,x}$ in terms of F_i and $\Gamma^i_{2,x}$,..,$\Gamma^i_{5,x}$. We then have

$$T_{(1,1,1)} = T_{(1,1,1,1)} + T_{(1,1,1,2)} \tag{A.7d}$$

$$T_{(1,1,1,1)} = -|\underline{x}|\rho_\eta, \qquad T_{(1,1,1,2)} = T_{(1,1,1,2,1)} + T_{(1,1,1,2,2)} \tag{A.7e}$$

The last decomposition distinguishes the terms with and without $\tilde\sigma(x)$; using the notation $k := |\underline{i}_3| + |\underline{i}_4| + 1$, we get

$$T_{(1,1,1,2,1)} = \sum_{x\in\underline{x}} \tilde\sigma(\underline{x}) \sum_{\underline{z}} \chi_{\underline{x},\underline{y}}(\underline{z})\tilde\sigma(\underline{z}) \sum_{\substack{\underline{i}_1,..,\underline{i}_5\\ |\underline{i}_1|<p}} (-1)^{p-|\underline{i}_1|+1}\gamma^{d(|\underline{i}_2|+|\underline{i}_3|+|\underline{i}_4|+2|\underline{i}_5|)} \times a\big(k, m(x)\big)[\prod_{j\in\underline{i}_1} F_j]\Gamma^{\underline{i}_2}_{2,x}...\Gamma^{\underline{i}_5}_{5,x} \tag{A.7f}$$

$$T_{(1,1,1,2,2)} = \sum_{x\in\underline{x}} \tilde\sigma(\underline{x}-x) \sum_{\underline{z}} \chi_{\underline{x},\underline{y}}(\underline{z})\tilde\sigma(\underline{z}) \sum_{\substack{\underline{i}_1,..,\underline{i}_5\\ |\underline{i}_1|<p}} (-1)^{p-|\underline{i}_1|+1}\gamma^{d(|\underline{i}_2|+|\underline{i}_3|+|\underline{i}_4|+2|\underline{i}_5|)} \times b\big(k, m(x)\big)[\prod_{j\in\underline{i}_1} F_j]\Gamma^{\underline{i}_2}_{2,x}...\Gamma^{\underline{i}_5}_{5,x} \tag{A.7g}$$

$T_{(1,1,1,1)}$, see (A.7e), can be written as in (A.1a) with

$$M^{\rm dg}_{(1,1,1,1)} = -|\underline{x}|\delta_{\eta,\eta'}$$

thus satisfying the conditions of Proposition 4.8.

The two other terms $T_{(1,1,1,2,1)}$ and $T_{(1,1,1,2,2)}$ will be examined together with some of the terms arising from $T_{(1,2)}$.

We next consider

$$T_{(1,1,2)} = \sum_{x\in\underline{x}} \tilde\sigma(\underline{x}-x) \sum_{\underline{z}} \chi_{\underline{x},\underline{y}}(\underline{z})\tilde\sigma(\underline{z})[\prod_{i=1}^p \Gamma^i_{1,x}][f_x(\sigma) - f_x(m)] \tag{A.8}$$

This term has the "wrong" function $\chi_{\underline{x},\underline{y}}$ because in (A.8) there is $\tilde\sigma(\underline{x}-x)$ instead of $\tilde\sigma(\underline{x})$. We thus add and subtract $\chi_{\underline{x}-x,\underline{y}}(\underline{z})$, getting

$$T_{(1,1,2)} = T_{(1,1,2,1)} + T_{(1,1,2,2)} \tag{A.9}$$

with $T_{(1,1,2,2)}$ as in (A.11b) below and

$$T_{(1,1,2,1)} = \sum_{x\in\underline{x}} \tilde\sigma(\underline{x}-x) \sum_{\underline{z}} \tilde\sigma(\underline{z})\chi_{\underline{x}-x,\underline{y}}(\underline{z})[\prod_{i=1}^p \Gamma^i_{1,x}][f_x(\sigma) - f_x(m)] \tag{A.10a}$$

Thus (A.1b) holds when $\kappa = (1,1,2,1)$ and

$$\sum_{\eta'} |M_{(1,1,2,1)}(\eta,\eta',m)| \le |\underline{x}| \tag{A.10b}$$

Moreover, since $|\eta| = |\eta'|$, $d(\eta) = d(\eta')$ and $g(\eta') = g(\eta)+1$,

$$N_\gamma(\eta)\sum_{\eta'} |M_{(1,1,2,1)}(\eta,\eta',m)| N_\gamma(\eta')^{-1} \le |\underline{x}|\gamma^{b_2} \le |\underline{x}|\gamma^{\zeta} \tag{A.10c}$$

Thus $M_{(1,1,2,1)}$ satisfies the bounds (4.40) and the support properties mentioned in Proposition 4.8.

Recalling that $\ell = |\underline{y}|$,

$$\chi_{\underline{x},\underline{y}}(\underline{z}) - \chi_{\underline{x}-x,\underline{y}}(\underline{z}) = -\sum_{i=1}^{\ell} \mathbf{1}(z_i = x)\frac{1}{|B_{\gamma^{-\alpha}}|}\mathbf{1}\big(x \in B_{\gamma^{-\alpha},y_i}\big)\chi_{\underline{x},\underline{y}-y_i}(\underline{z}-z_i) \tag{A.11a}$$

we then have

$$T_{(1,1,2,2)} = -\sum_{x\in\underline{x}}\sum_{i=1}^{\ell}\frac{1}{|B_{\gamma^{-\alpha}}|}\mathbf{1}\big(x \in B_{\gamma^{-\alpha},y_i}\big)\tilde{\sigma}(\underline{x})\sum_{\underline{z}-z_i}\chi_{\underline{x},\underline{y}-y_i}(\underline{z}-z_i)\tilde{\sigma}(\underline{z}-z_i)$$

$$\times [\prod_{i=1}^{p}\Gamma^{i}_{1,x}][f_x(\sigma) - f_x(m)] \tag{A.11b}$$

Then (A.1b) holds when $\kappa = (1,1,2,2)$. For a suitable constant c, (proportional to $|\underline{x}||\underline{y}|$, this is why these terms are not $M^{\rm dg}$ terms),

$$N_\gamma(\eta)\sum_{\eta'} |M_{(1,1,2,2)}(\eta,\eta',m)| N_\gamma(\eta')^{-1} \le c\gamma^{d\alpha} \le c\gamma^{\zeta^\star} \tag{A.11c}$$

Since $|\eta| = |\eta'|$, $g(\eta) = g(\eta')$ and $d(\eta) = d(\eta')$, $N_\gamma(\eta)N_\gamma(\eta')^{-1} = 1$, hence $M_{(1,1,2,2)}$ satisfies both (4.40b) and (4.40c). For the last inequality to hold we must choose $b < d\alpha$, which implies $\zeta^\star < d\alpha$.

We next examine $T_{(1,2)}$ together with the remaining terms $T_{(1,1,1,2,1)}$ and $T_{(1,1,1,2,2)}$, thus completing the analysis of $T_{(1)}$. By comparing (A.6c) and (3.30b), we see that

$$T_{(1,2)} = \sum_{\underline{z}} \chi_{\underline{x},\underline{y}}(\underline{z})\tilde{\sigma}(\underline{z})S_1 \tag{A.12a}$$

with S_1 as in (3.30b). The decomposition (3.34) gives rise to

$$T_{(1,2)} = T_{(1,2,1)} + T_{(1,2,2)} \tag{A.12b}$$

where $T_{(1,2,1)}$, given in (A.12c)-(A.12d) below, and $T_{(1,2,2)}$, see (A.17) below, are obtained from (A.12a) with S_1 replaced respectively by $S_{1,1}$, see (3.35a), and by $S_{1,2}$, see (3.36a).

Using (3.26) for $D^{(x)}\tilde{\sigma}(x)$ in (3.35a), we obtain $T_{(1,2,1)} = T_{(1,2,1,1)} + T_{(1,2,1,2)}$, with

$$T_{(1,2,1,1)} = \sum_{x\in\underline{x}} \tilde{\sigma}(\underline{x}) \sum_{\underline{z}} \chi_{\underline{x},\underline{y}}(\underline{z})\tilde{\sigma}(\underline{z}) \sum_{\substack{\underline{i}_1,..,\underline{i}_5 \\ |\underline{i}_1|<p}} \phi_{x,...} \times \gamma^{d(|\underline{i}_2|+|\underline{i}_3|+|\underline{i}_4|+2|\underline{i}_5|)} \Gamma^{\underline{i}_1}_{1,x} ... \Gamma^{\underline{i}_5}_{5,x} \tag{A.12c}$$

$$T_{(1,2,1,2)} = \sum_{x\in\underline{x}} \tilde{\sigma}(\underline{x}-x) \sum_{\underline{z}} \chi_{\underline{x},\underline{y}}(\underline{z})\tilde{\sigma}(\underline{z}) \sum_{\substack{\underline{i}_1,..,\underline{i}_5 \\ |\underline{i}_1|<p}} \psi_{x,...} \times \gamma^{d(|\underline{i}_2|+|\underline{i}_3|+|\underline{i}_4|+2|\underline{i}_5|)} \Gamma^{\underline{i}_1}_{1,x} ... \Gamma^{\underline{i}_5}_{5,x} [f_x(\sigma) - f_x(m)] \tag{A.12d}$$

where, recalling that in the following expressions $k = |\underline{i}_3| + |\underline{i}_4| + 1$,

$$\phi_{x,...} = -m(x)^{|\underline{i}_2|} a(k, m(x)) + [m(x)^{|\underline{i}_2|} a(k, m(x))]' \tag{A.12e}$$

$$\psi_{x,...} = m(x)^{|\underline{i}_2|} a(k, m(x)) \tag{A.12f}$$

By changing

$$\Gamma^{i}_{1,x} \to F_i, \qquad \phi_{x,...} \to (-1)^{p-|\underline{i}_1|+1} a\big(k, m(x)\big)$$

we obtain $T_{(1,1,1,2,1)}$ from $T_{(1,2,1,1)}$. The bounds for these terms will be identical, and we will only consider $T_{(1,2,1,1)}$ in the sequel.

The right hand side of (A.12c) has the form (A.1b) with

$$|\eta'| = |\eta| - (|\underline{i}_4| + |\underline{i}_5|), \quad g(\eta') = g(\eta) - (|\underline{i}_4| + |\underline{i}_5|), \quad d(\eta') = d(\eta) + (|\underline{i}_2| + |\underline{i}_3|)$$

and

$$V(\eta') \le V(\eta) + (|\underline{i}_1| + |\underline{i}_2| + |\underline{i}_3|)$$

so that the support properties stated in Proposition 4.8 are satisfied by $M_{(1,2,1,1)}$. Moreover

$$\sum_{\eta'} |M_{(1,2,1,1)}(\eta, \eta', m)| \le c|\underline{x}| \tag{A.13a}$$

with c a consant dependent on η, and

$$N_\gamma(\eta) \sum_{\eta'} |M_{(1,2,1,1)}(\eta, \eta', m)| N_\gamma(\eta')^{-1} \le c \sum_{\substack{\underline{i}_1,..,\underline{i}_5 \\ |\underline{i}_1|<p}} \gamma^{A(\underline{i}_2,..,\underline{i}_5)} \tag{A.13b}$$

where

$$
\begin{aligned}
A(\underline{i}_2,..,\underline{i}_5) &= -(d/2-b_1)(|\underline{i}_4|+|\underline{i}_5|) - b_2(|\underline{i}_4|+|\underline{i}_5|) - b_3(|\underline{i}_2|+|\underline{i}_3|) + d(|\underline{i}_2|+|\underline{i}_3|+|\underline{i}_4|+2|\underline{i}_5|)\\
&= (d/2+b_1-b_2)(|\underline{i}_4|+|\underline{i}_5|) + (d-b_3)(|\underline{i}_2|+|\underline{i}_3|) + d|\underline{i}_5| \qquad \text{(A.13c)}
\end{aligned}
$$

If $b_1 = b_2 = b_3 = 0$, then $A \geq d/2(|\underline{i}_4|+|\underline{i}_5|)$ and (4.40c) is satisfied. If on the other hand $b_1 = d/2$, then (4.40c) is still satisfied, choosing $\zeta^\star < d - b_2^\star - b_3^\star$: the left hand side of (A.13b) is then bounded by $c\gamma^{\zeta^\star}$.

Proceeding as in (A.11), we decompose $T_{(1,2,1,2)}$ into the sum of the following two terms:

$$
\begin{aligned}
T_{(1,2,1,2,1)} = \sum_{x\in\underline{x}} \tilde\sigma(\underline{x}-x) \sum_{\underline{z}} \chi_{\underline{x}-x,\underline{y}}(\underline{z})\tilde\sigma(\underline{z}) &\sum_{\substack{\underline{i}_1,..,\underline{i}_5\\ |\underline{i}_1|<p}} \psi_{x,...}\gamma^{d(|\underline{i}_2|+|\underline{i}_3|+|\underline{i}_4|+2|\underline{i}_5|)}\\
&\times \Gamma^{\underline{i}_1}_{1,x}...\Gamma^{\underline{i}_5}_{5,x}[f_x(\sigma)-f_x(m)]
\end{aligned}
\qquad \text{(A.14a)}
$$

$$
\begin{aligned}
T_{(1,2,1,2,2)} = -\sum_{x\in\underline{x}}\sum_{i=1}^{\ell} \frac{1}{|B_{\gamma^{-\alpha}}|}\mathbf{1}(x\in B_{\gamma^{-\alpha},y_i})\tilde\sigma(\underline{x}) &\sum_{\underline{z}-z_i} \chi_{\underline{x},\underline{y}-y_i}(\underline{z}-z_i)\tilde\sigma(\underline{z}-z_i)\\
\times \sum_{\substack{\underline{i}_1,..,\underline{i}_5\\ |\underline{i}_1|<p}} \psi_{x,...}\gamma^{d(|\underline{i}_2|+|\underline{i}_3|+|\underline{i}_4|+2|\underline{i}_5|)}&\Gamma^{\underline{i}_1}_{1,x}...\Gamma^{\underline{i}_5}_{5,x}[f_x(\sigma)-f_x(m)]
\end{aligned}
\qquad \text{(A.14b)}
$$

The representation (A.1b) holds as well for $\kappa = (1,2,1,2,1)$ and $= (1,2,1,2,2)$, the coefficients M_κ satisfying the support properties of Proposition 4.8 and the bound (4.40b). Moreover, recalling (A.13c),

$$
N_\gamma(\eta)\sum_{\eta'} |M_{(1,2,1,2,1)}(\eta,\eta',m)|N_\gamma(\eta')^{-1} \leq c \sum_{\substack{\underline{i}_1,..,\underline{i}_5\\ |\underline{i}_1|<p}} \gamma^{A(\underline{i}_2,..,\underline{i}_5)+b_2} \leq c\gamma^{\zeta^\star} \qquad \text{(A.15)}
$$

We have, in a similar way,

$$
N_\gamma(\eta)\sum_{\eta'} |M_{(1,2,1,2,2)}(\eta,\eta',m)|N_\gamma(\eta')^{-1} \leq c \sum_{\substack{\underline{i}_1,..,\underline{i}_5\\ |\underline{i}_1|<p}} \gamma^{A(\underline{i}_2,..,\underline{i}_5)+d\alpha} \leq c\gamma^{\zeta^\star} \qquad \text{(A.16)}
$$

The term

$$
\begin{aligned}
T_{(1,2,2)} = \sum_{\underline{z}} \chi_{\underline{x}-x,\underline{y}}(\underline{z}) \sum_{x\in\underline{x}} \tilde\sigma(\underline{x}-x)\tilde\sigma(\underline{z}) &\sum_{\substack{\underline{i}_1,..,\underline{i}_5\\ |\underline{i}_1|<p}} [m(x)^{|\underline{i}_2|}b(k,m(x))]'\\
&\times \gamma^{d(|\underline{i}_2|+|\underline{i}_3|+|\underline{i}_4|+2|\underline{i}_5|)}\Gamma^{\underline{i}_1}_{1,x}...\Gamma^{\underline{i}_5}_{5,x}
\end{aligned}
\qquad \text{(A.17)}
$$

is obtained from (A.12a) with S_1 replaced by $S_{1,2}$, the latter having been defined in (3.36a).

By changing in (A.17)

$$\Gamma^i_{1,x} \to F_i, \qquad [m(x)^{\underline{i}_2} b(k,m(x))]' \to (-1)^{p-|\underline{i}_1|+1} b(k,m(x))$$

we obtain $T_{(1,1,1,2,2)}$ from $T_{(1,2,2)}$ so that the bound for the former is reduced to that for the latter.

By using again the decomposition (A.11a), we get $T_{(1,2,2)} = T_{(1,2,2,1)} + T_{(1,2,2,2)}$ and (A.1b) holds with $\kappa \in \{(1,2,2,1),(1,2,2,2)\}$. The support properties and (4.40b) are verified and

$$N_\gamma(\eta) \sum_{\eta'} |M_{(1,2,2,1)}(\eta,\eta',m)| N_\gamma(\eta')^{-1} \le c \sum_{|\underline{i}_3|+|\underline{i}_4|\ge 1} \gamma^{A(\underline{i}_2,..,\underline{i}_5)-(d/2-b_1)} \tag{A.18a}$$

Recalling (A.13c) we have that in each term of the sum there is γ raised to the power

$$d/2(|\underline{i}_3|+|\underline{i}_4|-1)+(b_1-b_2)|\underline{i}_4|+(3d/2-b_2)|\underline{i}_5|+(d/2-b_3)|\underline{i}_3|+(d-b_3)|\underline{i}_2|+b_1(1+|\underline{i}_5|) \tag{A.18b}$$

Since $|\underline{i}_3|+|\underline{i}_4| \ge 1$, the left hand side of (A.18a) is bounded proportionally to $\gamma^{\zeta^\star}$ if $b_1 = d/2$ and to a constant if $b_1 = 0$ (recall that if $b_1 = 0$ then also $b_2 = b_3 = \zeta = 0$). This is due to the term $|\underline{i}_4| = 1$, $|\underline{i}_2| = |\underline{i}_3| = |\underline{i}_5| = 0$; all the other terms have a factor γ raised to some positive power.

With similar arguments, we also get

$$\begin{aligned} N_\gamma(\eta) \sum_{\eta'} |M_{(1,2,2,2)}(\eta,\eta',m)| N_\gamma(\eta')^{-1} &\le c \sum_{|\underline{i}_3|+|\underline{i}_4|\ge 1} \gamma^{A(\underline{i}_2,..,\underline{i}_5)-(d/2-b_1)+d\alpha-b_2} \\ &\le c'\gamma^{\zeta^\star} \end{aligned} \tag{A.18c}$$

which is bounded proportionally to $\gamma^{d\alpha}$, because the exponent in (A.18c) can be written as

$$d/2(|\underline{i}_3|+|\underline{i}_4|-1)+(b_1-b_2)(1+|\underline{i}_4|)+(3d/2-b_2)|\underline{i}_5|+(d/2-b_3)|\underline{i}_3|+b_1|\underline{i}_5|+d\alpha \tag{A.18d}$$

Analysis of $T_{(2)}$.

We have

$$T_{(2)} = T_{(2,1)} + T_{(2,2)} \tag{A.19}$$

where

$$T_{(2,1)} = \tilde\sigma(\underline{x}) \sum_{\underline{z}} \chi_{\underline{x},\underline{y}}(\underline{z}) \sum_{i=1}^{\ell} [\prod_{j=1}^{p} \Gamma^j_{1,z_i}] \tilde\sigma(\underline{z}-z_i) D^{(z_i)} \tilde\sigma(z_i) \tag{A.20a}$$

$$T_{(2,2)} = \tilde\sigma(\underline{x}) \sum_{\underline{z}} \chi_{\underline{x},\underline{y}}(\underline{z}) \sum_{i=1}^{\ell} \tilde\sigma(\underline{z}-z_i) \sum_{\substack{\underline{i}_1,..\underline{i}_5 \\ |\underline{i}_1|<p}} \Delta_{i_1..\underline{i}_5,\underline{z},z_i} \tag{A.20b}$$

with Δ as in (3.30d)-(3.30e).

The analysis of $T_{(2,2)}$ will be reduced to that of $T_{(1,2)}$, so we start with $T_{(2,1)}$. Like in (A.7) we write

$$T_{(2,1)} = T_{(2,1,1)} + T_{(2,1,2)} \tag{A.21}$$

$$T_{(2,1,1)} = \tilde\sigma(\underline{x}) \sum_{\underline{z}} \chi_{\underline{x},\underline{y}}(\underline{z}) \sum_{i=1}^{\ell} \tilde\sigma(\underline{z} - z_i)[\prod_{j=1}^{p} F_j] D^{(z_i)}\tilde\sigma(z_i) \tag{A.22a}$$

and calling $z_i = x$ below

$$T_{(2,1,2)} = \sum_{i=1}^{\ell}\sum_{x} \chi_{\underline{x},y_i}(x)\Big\{\tilde\sigma(\underline{x}) \sum_{\underline{z}-z_i} \chi_{\underline{x}+x,\underline{y}-y_i}(\underline{z}-z_i)\tilde\sigma(\underline{z}-z_i) \sum_{\substack{\underline{i}_1,..\underline{i}_5 \\ |\underline{i}_1|<p}} (-1)^{p-|\underline{i}_1|+1}$$
$$\times \gamma^{d(|\underline{i}_2|+|\underline{i}_3|+|\underline{i}_4|+2|\underline{i}_5|)}[\prod_{i\in\underline{i}_1} F_i]\Gamma^{\underline{i}_2}_{2,z_i}..\Gamma^{\underline{i}_5}_{5,z_i}\tilde\sigma(x)^{|\underline{i}_3|+|\underline{i}_4|}D^{(x)}\tilde\sigma(x)\Big\} \tag{A.22b}$$

The term $T_{(2,1,2)}$ will be estimated together with $T_{(2,2)}$.

After using (3.26) for $D^{(z_i)}\tilde\sigma(z_i)$ in (A.22a), we get

$$T_{(2,1,1)} = T_{(2,1,1,1)} + T_{(2,1,1,2)} \tag{A.23a}$$

$$T_{(2,1,1,1)} = -\ell\rho_\eta, \qquad M^{\rm dg}_{(2,1,1,1)}(\eta,\eta',m) = -\ell\delta_{\eta,\eta'} \tag{A.23b}$$

As in (A.8), $T_{(2,1,1,2)}$ has the wrong χ function, we then proceed as in (A.11) and have

$$T_{(2,1,1,2)} = T_{(2,1,1,2,1)} + T_{(2,1,1,2,2)} \tag{A.24a}$$

where

$$T_{(2,1,1,2,1)} = \sum_{i=1}^{\ell}\sum_{z} \chi_{\underline{x},y_i}(z)\Big\{\tilde\sigma(\underline{x}) \sum_{\underline{z}-z_i} \chi_{\underline{x},\underline{y}-y_i}(\underline{z}-z_i)\tilde\sigma(\underline{z}-z_i)$$
$$\times [\prod_{j=1}^{p} F_j][f_z(\sigma) - f_z(m)]\Big\} \tag{A.24b}$$

$$T_{(2,1,1,2,2)} = \sum_{i=1}^{\ell}\sum_{z} \chi_{\underline{x},y_i}(z)\sum_{j\neq i}\frac{1}{|B_{\gamma^{-\alpha}}|}\Big\{\tilde\sigma(\underline{x}+z) \sum_{\underline{z}-z_i-z_j} \chi_{\underline{x}+z,\underline{y}-y_i-y_j}(\underline{z}-z_i-z_j)$$
$$\times [\prod_{j=1}^{p} F_j][f_z(\sigma) - f_z(m)]\Big\} \tag{A.24c}$$

We have that (A.1a) holds when $\kappa = (2,1,1,2,1)$ and

$$\eta' = (\underline{x},\underline{y}',\xi_1,..,\xi_p,\xi'_{p+1}, \quad \underline{y}' = \underline{y} - y_i, \quad \xi'_{p+1} = (z,\emptyset,\emptyset,\emptyset,\emptyset) \tag{A.25a}$$

$$\sum_{\eta'} |M^{\mathrm{dg}}_{(2,1,1,2,1)}(\eta,\eta',m)| \le \ell \tag{A.25b}$$

The representation (A.1b) is valid for $\kappa = (2,1,1,2,2)$ and we have

$$\sum_{\eta'} |M_{(2,1,1,2,2)}(\eta,\eta',m)| \le \ell^2\gamma^{d\alpha} \tag{A.26a}$$

$$N_\gamma(\eta)\sum_{\eta'} |M_{(2,1,1,2,2)}(\eta,\eta',m)| N_\gamma(\eta')^{-1} \le \ell^2\gamma^{-b_2+d\alpha} \tag{A.26b}$$

bounded by $c\gamma^{\zeta^\star}$ when $b_1 = d/2$. If $b_1 = 0$, (4.40) is still satisfied because $|\eta'| = |\eta|$.

We next consider the term $T_{(2,2)}$, see (A.20b). This can be written as

$$T_{(2,2)} = \sum_{i=1}^{\ell}\sum_{x} \chi_{\underline{x},y_i}(x)\Big\{\tilde\sigma(\underline{x})\sum_{\underline{z}-x}\chi_{\underline{x}+x,\underline{y}-y_i}(\underline{z}-z_i)\tilde\sigma(\underline{z}-z_i)\sum_{\substack{\underline{i}_1..\underline{i}_5\\ |\underline{i}_1|<p}}\Delta_{\underline{i}_1,..\underline{i}_5,\underline{x}+x,x}\Big\} \tag{A.27}$$

For any fixed i in (A.27) we write

$$\underline{x}' = \underline{x}+x,\quad \underline{y}' = \underline{y}-y_i;\ \eta' = (\underline{x}',\underline{y}',\xi_1,..,\xi_p) \tag{A.28}$$

Then the curly bracket in (A.27) is equal to the curly bracket in (A.6c), when the latter is defined starting from η' instead of η. The same remark applies to $T_{(2,1,2)}$ which, via (A.28), is identified to the sum of the terms (A.7g) and (A.7h). The bounds for these terms are thus reduced to those for $T_{(2,2)}$, just like the bounds for (A.7g) and (A.7h) were reduced to those of (A.12c) and (A.17), respectively. We will thus restrict to the bound of $T_{(2,2)}$. Proceeding as in (A.12b)-(A.18) we have

$$T_{(2,2)} = T_{(2,2,1,1)} + T_{(2,2,1,2,1)} + T_{(2,2,1,2,2)} + T_{(2,2,2)} \tag{A.29}$$

The decomposition (A.1b) then holds for $\kappa = (2,2,1,1)$, $(2,2,1,2,1)$, $(2,2,1,2,2)$ and $(2,2,2)$. The bound (4.40b) holds for each of these terms and

$$N_\gamma(\eta)\sum_{\eta'} |M_{(\kappa)}(\eta,\eta',m)| N_\gamma(\eta)^{-1} \le c\gamma^{-b_2}\begin{cases}\sum\gamma^{A} & \text{if } \kappa = 2,2,1,1 \text{ see (A.13b)}\\ \sum\gamma^{A+b_2} & \text{if } \kappa = 2,2,1,2,1 \text{ see (A.15)}\\ \sum\gamma^{A+d\alpha} & \text{if } \kappa = 2,2,1,2,2 \text{ see (A.16)}\end{cases} \tag{A.30}$$

For $\kappa = 2,2,2$ we get γ^{-b_2} times the bound in (A.18c).

We have thus completed the analysis of $T_{(2)}$.

Analysis of $T_{(3)}$.

We write

$$T_{(3)} = T_{(3,1)} + T_{(3,2)} \tag{A.31a}$$

$$T_{(3,1)} = \tilde\sigma(\underline{x}) \sum_{\underline{z}} \chi_{\underline{x},\underline{y}}(\underline{z})\tilde\sigma(\underline{z}) \sum_{x\notin(\underline{x}+\underline{z})} \sum_{i=1}^{p} [\prod_{j\neq i} \Gamma^j_{1,x}]\gamma^d(\Gamma^i_{3,x} + \Gamma^i_{4,x})D^{(x)}\tilde\sigma(x) \tag{A.31b}$$

$$T_{(3,2)} = \sum_{i=1}^{p}\sum_{x\notin\underline{x}} \Big\{\tilde\sigma(\underline{x}) \sum_{\underline{z}} \chi_{\underline{x}+x,\underline{y}}(\underline{z})\tilde\sigma(\underline{z})\gamma^d(\Gamma^i_{3,x} + \Gamma^i_{4,x}) \times \sum_{\substack{|\underline{i}_3|+|\underline{i}_4|\geq 1\\ |\underline{i}_1|<p-1}} \Delta_{\underline{i}_1,..\underline{i}_5,\underline{x}+x,x}\Big\} \tag{A.31c}$$

where the sum is over all the disjoint sets $\underline{i}_1 \dots \underline{i}_5$ which cover $\{1,..,p\}$ with the condition stated in (A.31c); Δ is as in (3.30d) with $k = |\underline{i}_3| + |\underline{i}_4| + 1$.

We start from $T_{(3,1)}$ that we write as

$$T_{(3,1)} = T_{(3,1,1)} + T_{(3,1,2)} \tag{A.32a}$$

$$T_{(3,1,1)} = \sum_{i=1}^{p} \gamma^d \sum_{x\notin\underline{x}} \tilde\sigma(\underline{x}) \sum_{\underline{z}} \chi_{\underline{x}+x,\underline{y}}(\underline{z})\tilde\sigma(\underline{z}) \times [\prod_{j\neq i} F_j](\Gamma^i_{3,x} + \Gamma^i_{4,x})D^{(x)}\tilde\sigma(x) \tag{A.32a}$$

$$T_{(3,1,2)} = \sum_{i=1}^{p} \gamma^d \sum_{x\notin\underline{x}}\tilde\sigma(\underline{x}) \sum_{\underline{z}} \chi_{\underline{x}+x,\underline{y}}(\underline{z})\tilde\sigma(\underline{z}) \times \sum_{\substack{|\underline{i}_3|+|\underline{i}_4|\geq 1\\ |\underline{i}_1|<p-1}} (-1)^{p-1-|\underline{i}_1|}\gamma^{d(|\underline{i}_2|+|\underline{i}_3|+|\underline{i}_4|+2|\underline{i}_5|}[\prod_{i\in\underline{i}_1} F_i]\Gamma^{\underline{i}_2}_{2,x}..\Gamma^{\underline{i}_5}_{5,x}D^{(x)}\tilde\sigma(x) \tag{A.32b}$$

By using (3.26) we write

$$T_{(3,1,1)} = T_{(3,1,1,1)} + T_{(3,1,1,2)} \tag{A.33a}$$

where

$$T_{(3,1,1,1)} = -\sum_{i=1}^{p} \gamma^d \sum_{x\notin\underline{x}} \tilde\sigma(\underline{x}+x) \sum_{\underline{z}} \chi_{\underline{x}+x,\underline{y}}(\underline{z})\tilde\sigma(\underline{z})[\prod_{j\neq i} F_j](\Gamma^i_{3,x} + \Gamma^i_{4,x}) \tag{A.33a}$$

$$T_{(3,1,1,2)} = -\sum_{i=1}^{p} \gamma^d \sum_{x\notin\underline{x}} \tilde\sigma(\underline{x}) \sum_{\underline{z}} \chi_{\underline{x}+x,\underline{y}}(\underline{z})\tilde\sigma(\underline{z})[\prod_{j\neq i} F_j](\Gamma^i_{3,x} + \Gamma^i_{4,x})[f_x(\sigma) - f_x(m)] \tag{A.33b}$$

We write

$$\partial_x f_{\xi_i}(\sigma) = \frac{1}{|B_{\gamma^{-\alpha}}|} \sum_{u \in B_{\gamma^{-\alpha},x}} \partial_u f_{\xi_i}(\sigma) + \frac{1}{|B_{\gamma^{-\alpha}}|} \sum_{u \in B_{\gamma^{-\alpha},x}} \{\partial_x f_{\xi_i}(\sigma) - \partial_u f_{\xi_i}(\sigma)\} \tag{A.34}$$

and

$$T_{(3,1,1,1)} = T_{(3,1,1,1,1)} + T_{(3,1,1,1,2)} + T_{(3,1,1,1,3)} \tag{A.35a}$$

where

$$T_{(3,1,1,1,1)} = -\sum_{i=1}^{p} \gamma^d \sum_u \Big\{ \tilde\sigma(\underline{x}) \sum_{\underline{z}+x} \chi_{\underline{x},\underline{y}+u}(\underline{z}+x) \tilde\sigma(\underline{z}+x) [\prod_{j\neq i} F_j] \partial_u f_{\xi_i}(\sigma) \Big\} \tag{A.35b}$$

and, recalling the definition of V_ξ in Definition 3.2,

$$\begin{aligned} T_{(3,1,1,1,2)} = -\sum_{i=1}^{p} \gamma^d \sum_{x \notin \underline{x}} \frac{1}{|B_{\gamma^{-\alpha}}|} \sum_{u \in B_{\gamma^{-\alpha},x} - V_i} \tilde\sigma(\underline{x}) \sum_{\underline{z}} \chi_{\underline{x}+x,\underline{y}}(\underline{z}) \tilde\sigma(\underline{z}) \\ \times [\prod_{j\neq i} F_j] \tilde\sigma(x) [\partial_x - \partial_u] f_{\xi_i}(\sigma) \end{aligned} \tag{A.35c}$$

$$\begin{aligned} T_{(3,1,1,1,3)} = -\sum_{i=1}^{p} \gamma^d \sum_{x \notin \underline{x}} \frac{1}{|B_{\gamma^{-\alpha}}|} \sum_{u \in B_{\gamma^{-\alpha},x} \cap V_i} \tilde\sigma(\underline{x}) \sum_{\underline{z}} \chi_{\underline{x}+x,\underline{y}}(\underline{z}) \tilde\sigma(\underline{z}) \\ \times [\prod_{j\neq i} F_j] \tilde\sigma(x) \partial_x f_{\xi_i}(\sigma) \end{aligned} \tag{A.35d}$$

because $\partial_u f_{\xi_i} = 0$ if $u \in V_i$.

We write $T_{(3,1,1,1,1)} = T_{(3,1,1,1,1,1)} + T_{(3,1,1,1,1,2)}$, by adding and subtracting $f_{\xi_i}(m)$ to $f_{\xi_i}(\sigma)$, obtaining

$$T_{(3,1,1,1,1,1)} = -\sum_{i=1}^{p} \gamma^d \sum_u \tilde\sigma(\underline{x}) \sum_{\underline{z}+x} \chi_{\underline{x},\underline{y}+u}(\underline{z}+x) \tilde\sigma(\underline{z}+x) [\prod_{j\neq i} F_j] \partial_u F_i \tag{A.36a}$$

$$T_{(3,1,1,1,1,2)} = -\sum_{i=1}^{p} \gamma^d \sum_u \tilde\sigma(\underline{x}) \sum_{\underline{z}+x} \chi_{\underline{x},\underline{y}+u}(\underline{z}+x) \tilde\sigma(\underline{z}+x) [\prod_{j\neq i} F_j] \partial_u f_{\xi_i}(m) \tag{A.36b}$$

Then (A.1b) holds with $\kappa = (3,1,1,1,1,1)$ and, for a suitable constant c, which depends on η,

$$\sum_{\eta'} |M_{(3,1,1,1,1,1)}(\eta,\eta',m)| \le c \tag{A.36c}$$

$$N_\gamma(\eta) \sum_{\eta'} |M_{(3,1,1,1,1,1)}(\eta,\eta',m)| N_\gamma(\eta')^{-1} \le c\gamma^{(d/2-b_1)+b_2-b_3} \tag{A.36d}$$

which, for $b_1 = d/2$ is bounded by $c\gamma^{\zeta^\star}$, provided $b_2 - b_3 > \zeta^\star$.

We next write $T_{(3,1,1,1,1,2)} = T_{(3,1,1,1,1,2,1)} + T_{(3,1,1,1,1,2,2)}$, collecting in the first term the labels i such that $d(\xi_i) = 0$ and in the latter the other values of i:

$$T_{(3,1,1,1,1,2,1)} = -\sum_{i:d(\xi_i)=0} \gamma^d \sum_u \partial_u f_{\xi_i}(m)\tilde\sigma(\underline{x}) \sum_{\underline{z}+x} \chi_{\underline{x},\underline{y}+u}(\underline{z}+x)\tilde\sigma(\underline{z}+x)[\prod_{j\neq i} F_j] \tag{A.37}$$

$$T_{(3,1,1,1,1,2,2)} = -\sum_{i:d(\xi_i)>0} \gamma^d \sum_u \partial_u f_{\xi_i}(m)\tilde\sigma(\underline{x}) \sum_{\underline{z}+x} \chi_{\underline{x},\underline{y}+u}(\underline{z}+x)\tilde\sigma(\underline{z}+x)[\prod_{j\neq i} F_j] \tag{A.38}$$

We thus have that (A.1a) holds with $\kappa = (3,1,1,1,1,2,1)$ and (A.1b) with $\kappa = (3,1,1,1,1,2,2)$ ∎ We have

$$\sum_{\eta'} |M^{\rm dg}_{(3,1,1,1,1,2,1)}(\eta,\eta',m)| \le (2\gamma^{-1}+1)^d\gamma^d pC(0) \tag{A.39a}$$

where $C(k)$ is such that, for any ξ,

$$\sup_{m\in[-1,1]^{\mathbb{Z}^d}} \sup_u |\partial_u f_\xi(m)| \le C\big(d(\xi)\big) \tag{A.39b}$$

$$\sum_{\eta'} |M_{(3,1,1,1,1,2,2)}(\eta,\eta',m)| \le (2\gamma^{-1}+1)^d\gamma^d p \max_{i=1,p} C(d(\xi_i)) \tag{A.39c}$$

$$N_\gamma(\eta)\sum_{\eta'} |M_{(3,1,1,1,1,2,2)}(\eta,\eta',m)| N_\gamma(\eta')^{-1} \le \gamma^{b_3}[(2\gamma^{-1}+1)^d\gamma^d p \max_{i=1,p} C(d(\xi_i))] \tag{A.39d}$$

To estimate $T_{(3,1,1,1,2)}$, we write

$$\begin{aligned}
\partial_x - \partial_u &= \partial_x\Big\{\frac{1+\sigma(u)}{2}a_u^+ + \frac{1-\sigma(u)}{2}a_u^-\Big\} - \partial_u\Big\{\frac{1+\sigma(x)}{2}a_x^+ + \frac{1-\sigma(x)}{2}a_x^-\Big\}\\
&= \partial_x\frac{a_u^+ + a_u^-}{2} - \partial_u\frac{a_x^+ + a_x^-}{2} + \gamma^d\sigma(u)\partial_u\partial_x - \gamma^d\sigma(x)\partial_x\partial_u\\
&= \frac{1}{2\gamma^d}[a_x^+a_u^- - a_u^+a_x^-] + \gamma^d[\sigma(u)-\sigma(x)]\partial_u\partial_x\\
&= \gamma^{1-\alpha}\delta_{x,u} + \gamma^d[\sigma(u)-\sigma(x)]\partial_u\partial_x
\end{aligned} \tag{A.40a}$$

Thus

$$\tilde\sigma(x)[\partial_x-\partial_u] = \tilde\sigma(x)\gamma^{1-\alpha}\delta_{x,u} + \gamma^d\big\{\tilde\sigma(x)\tilde\sigma(u) + \tilde\sigma(x)[m(u)+m(x)] - 1 + m(x)^2\big\}\partial_u\partial_x \tag{A.40b}$$

We then have

$$T_{(3,1,1,1,2)} = \sum_{i=1}^4 T_{(3,1,1,1,2,i)} \tag{A.41}$$

where i labels the four terms on the right hand side of (A.40b), in the order of appearence. By adding and subtracting $f_{\xi_i}(m)$ to $f_{\xi_i}(\sigma)$, we get

$$T_{(3,1,1,1,2,i)} = T_{(3,1,1,1,2,i,1)} + T_{(3,1,1,1,2,i,2)} \tag{A.42a}$$

the first term being the one with F_i.

Again the decomposition (A.1b) holds with $\kappa = (3,1,1,1,2,i,j)$, $i = 1,..,3$ and $j = 1,2$. For all such κ and for a suitable constant c:

$$\sum_{\eta'} |M_{(3,1,1,1,2,i,j)}(\eta,\eta',m)| \le c \tag{A.42b}$$

and

$$N_\gamma(\eta) \sum_{\eta'} |M_{(3,1,1,1,2,i,1)}(\eta,\eta',m)| N_\gamma(\eta')^{-1} \le \begin{cases} c\gamma^{1-\alpha+(d/2-b_1)-b_3} & \text{if } i=1 \\ c\gamma^{d+2(d/2-b_1)-2b_3} & \text{if } i=2 \\ c\gamma^{d+(d/2-b_1)-2b_3} & \text{if } i=3 \end{cases} \tag{A.42c}$$

For $\kappa = (3,1,1,1,2,i,2)$ we obtain the previous bounds times the factor

$$\gamma^{-(d/2-b_1)-b_2+b_3}, \quad \text{for } i=1, \text{ and } \gamma^{-(d/2-b_1)-b_2+2b_3}, \quad \text{for } i=2,3 \tag{A.42d}$$

When $i = 4$, we have $\tilde\sigma(\underline{x})$ instead of $\tilde\sigma(\underline{x}+x)$, hence the wrong χ function. We write, as in (A.11),

$$\chi_{\underline{x}+x,\underline{y}}(\underline{z}) = \chi_{\underline{x},\underline{y}}(\underline{z}) - \frac{1}{|B_{\gamma^{-\alpha}}|} \sum_{j=1}^{\ell} \mathbf{1}(z_j = x)\chi_{\underline{x}+x,\underline{y}-y_j}(\underline{z}-z_j) \tag{A.43}$$

and we correspondingly write

$$T_{(3,1,1,1,2,4,j)} = T_{(3,1,1,1,2,4,j,1)} + T_{(3,1,1,1,2,4,j,2)} \tag{A.44}$$

where

$$T_{(3,1,1,1,2,4,1,1)} = \sum_{i=1}^{p} \gamma^d \sum_{x\notin\underline{x}} \frac{1}{|B_{\gamma^{-\alpha}}|} \sum_{u\in B_{\gamma^{-\alpha},x}} \tilde\sigma(\underline{x}) \sum_{\underline{z}} \chi_{\underline{x},\underline{y}}(\underline{z})\tilde\sigma(\underline{z}) \times [\prod_{j\neq i} F_j][\gamma^d(1-m(x)^2)\partial_u\partial_x F_i] \tag{A.45a}$$

$$T_{(3,1,1,1,2,4,1,2)} = \sum_{i=1}^{p}\sum_{j=1}^{\ell} \gamma^d \sum_x \chi_{\underline{x},y_i}(x) \frac{1}{|B_{\gamma^{-\alpha}}|} \sum_{u\in B_{\gamma^{-\alpha},x}} \tilde\sigma(\underline{x}+x) \sum_{\underline{z}-z_i} \chi_{\underline{x}+x,\underline{y}-y_i}(\underline{z}-z_i)\tilde\sigma(\underline{z}-z_i) \times [\prod_{j\neq i} F_j][\gamma^d(1-m(x)^2)\partial_u\partial_x F_i] \tag{A.45b}$$ ∎

The terms $T_{(3,1,1,1,2,4,2,i)}$ are obtained from (A.45) with F_i replaced by $f_{\xi_i}(m)$. Thus we have that (A.1b) holds for $\kappa = (3,1,1,1,2,4,i,j)$ and, for a suitable constant c,

$$\sum_{\eta'} |M_{(3,1,1,1,2,4,i,j)}(\eta,\eta',m)| \le c \tag{A.46a}$$

$$N_\gamma(\eta)\sum_{\eta'}|M_{(3,1,1,1,2,4,i,j)}(\eta,\eta',m)|N_\gamma(\eta')^{-1}\le\begin{cases} c\gamma^{d-2b_3} & \text{if } i=1,\ j=1\\ c\gamma^{d-b_2-2b_3} & \text{if } i=2,\ j=1\\ c\gamma^{d-(d/2-b_1)} & \text{if } i=1,\ j=2\\ c\gamma^{d-(d/2-b_1)-b_2} & \text{if } i=2,\ j=2\end{cases} \tag{A.46b}$$

We next consider $T_{(3,1,1,1,3)}$, see (A.35d), that we rewrite as the sum of $T_{(3,1,1,1,3,1)}+T_{(3,1,1,1,3,2)}$, by adding and subtracting $f_{\xi_i}(m)$, ($T_{(3,1,1,1,3,1)}$ corresponds to F_i). The representation (A.1b) then holds for $\kappa=(3,1,1,1,3,i)$, and, for a suitable constant c, and for $i=1,2$,

$$\sum_{\eta'}|M_{(3,1,1,1,3,i)}(\eta,\eta',m)|\le c \tag{A.47a}$$

$$N_\gamma(\eta)\sum_{\eta'}|M_{(3,1,1,1,3,i)}(\eta,\eta',m)|N_\gamma(\eta')^{-1}\le\begin{cases} c\gamma^{d/2-b_1-b_3+d\alpha} & \text{if } i=1\\ c\gamma^{-b_2+d\alpha} & \text{if } i=2\end{cases} \tag{A.47b}$$

We next go back to $T_{(3,1,1,2)}$ that we write as

$$T_{(3,1,1,2)}=\sum_{h=1}^{2}\sum_{j=1}^{2}T_{(3,1,1,2,i,j)} \tag{A.48}$$

where $h=1$ comes from selecting $\Gamma^i_{3,x}$ in (A.33b), $h=2$ from $\Gamma^i_{4,x}$; $j=1$ corresponds to the first term on the right hand side of (A.43) and $j=2$ to the second one. Therefore

$$T_{(3,1,1,2,1,1)}=\sum_{i=1}^{p}\gamma^d\sum_{x\notin\underline{x}}\Big\{\tilde\sigma(\underline{x})\sum_{\underline{z}}\chi_{\underline{x},\underline{y}}(\underline{z})\tilde\sigma(\underline{z})[\prod_{j\neq i}F_j]\Gamma^i_{3,x}[f_x(\sigma)-f_x(m)]\Big\} \tag{A.49a}$$

$$\begin{aligned}T_{(3,1,1,2,1,2)}=-\sum_{i=1}^{p}\sum_{j=1}^{\ell}\gamma^d\sum_{x}\chi_{\underline{x},y_j}(x)\Big\{\tilde\sigma(\underline{x}+x)\sum_{\underline{z}-z_j}\chi_{\underline{x}+x,\underline{y}-y_j}(\underline{z}-z_j)\tilde\sigma(\underline{z}-z_j)\\ \times[\prod_{j\neq i}F_j]\Gamma^i_{3,x}[f_x(\sigma)-f_x(m)]\Big\}\end{aligned} \tag{A.49b}$$

$$T_{(3,1,1,2,2,1)}=\sum_{i=1}^{p}\gamma^d\sum_{x\notin\underline{x}}\Gamma^i_{4,x}\Big\{\tilde\sigma(\underline{x})\sum_{\underline{z}}\chi_{\underline{x},\underline{y}}(\underline{z})\tilde\sigma(\underline{z})[\prod_{j\neq i}F_j][f_x(\sigma)-f_x(m)]\Big\} \tag{A.49c}$$

$$\begin{aligned}T_{(3,1,1,2,2,2)}=-\sum_{i=1}^{p}\sum_{j=1}^{\ell}\gamma^d\sum_{x}\chi_{\underline{x},y_j}(x)\Gamma^i_{4,x}\Big\{\tilde\sigma(\underline{x}+x)\sum_{\underline{z}-z_j}\chi_{\underline{x}+x,\underline{y}-y_j}(\underline{z}-z_j)\tilde\sigma(\underline{z}-z_j)\\ \times[\prod_{j\neq i}F_j][f_x(\sigma)-f_x(m)]\Big\}\end{aligned} \tag{A.49d}$$

The representation (A.1b) holds for $\kappa = (3,1,1,2,1,i), (3,1,1,2,2,2)$ and

$$\sum_{\eta'} |M_\kappa(\eta,\eta',m)| \le c \tag{A.50}$$

for a suitable constant c: moreover

$$N_\gamma(\eta)\sum_{\eta'} |M_\kappa(\eta,\eta',m)|N_\gamma(\eta')^{-1} \le c\begin{cases} \gamma^{(d/2-b_1)+b_2-b_3} & \text{when } \kappa=(3,1,1,2,1,1) \\ \gamma^{(d/2-b_1)-b_3+d} & \text{when } \kappa=(3,1,1,2,1,2) \\ \gamma^{-b_2+d} & \text{when } \kappa=(3,1,1,2,2,2) \end{cases} \tag{A.51}$$

We write

$$T_{(3,1,1,2,2,1)} = T_{(3,1,1,2,2,1,1)} + T_{(3,1,1,2,2,1,2)} \tag{A.52a}$$

$$T_{(3,1,1,2,2,1,1)} = \sum_{d(\xi_i)=0} \gamma^d \sum_{x\notin \underline{x}} \Gamma^i_{4,x}\Big\{\tilde\sigma(\underline{x})\sum_{\underline{z}} \chi_{\underline{x},\underline{y}}(\underline{z})\tilde\sigma(\underline{z})[\prod_{j\neq i} F_j][f_x(\sigma)-f_x(m)]\Big\} \tag{A.52b}$$

$$T_{(3,1,1,2,2,1,1)} = \sum_{d(\xi_i)>0} \gamma^d \sum_{x\notin \underline{x}} \Big\{\tilde\sigma(\underline{x})\sum_{\underline{z}} \chi_{\underline{x},\underline{y}}(\underline{z})\tilde\sigma(\underline{z})[\prod_{j\neq i} F_j]\Gamma^i_{4,x}[f_x(\sigma)-f_x(m)]\Big\} \tag{A.52c}$$

Therefore (A.1a) holds with $\kappa = (3,1,1,2,2,1,1)$ and also for this term the bound (4.40a) holds. The representation (A.1b) holds for $\kappa = (3,1,1,2,2,1,2)$, (A.40b) is satisfied and

$$N_\gamma(\eta)\sum_{\eta'} |M_{(3,1,1,2,2,1,2)}(\eta,\eta',m)|N_\gamma(\eta)^{-1} \le c\gamma^{b_3} \le c\gamma^{\zeta^*} \tag{A.53}$$

The only terms which need still to be estimated are $T_{(3,2)}$ and $T_{(3,1,2)}$. Except for constant factors, the terms in $T_{(3,1,2)}$ are identified with terms which are present in $T_{(3,2)}$, in analogy with similar identifications in the analysis of $T_{(1)}$ and $T_{(2)}$, we omit the details and just consider $T_{(3,2)}$.

We fix x in (A.31c), then the curly bracket term is identified with the terms in (A.6c) where $\underline{i}_1..\underline{i}_5$ have the same restrictions as in (A.31c), provided we replace in (A.6c) $\underline{x}$ by $\underline{x}' = \underline{x} + x$. The analysis of $T_{(3,2)}$ is then obtained from that of $T_{(1,2)}$. The bounds for this latter need to be multiplied by γ^{-d}, to take into account the sum over x in (A.31c), and by $\gamma^{(d/2-b_1)}$ because of the extra $\tilde\sigma(x)$.

We then have

$$T_{(3,2)} = T_{(3,2,1,1)} + T_{(3,2,1,2,1)} + T_{(3,2,1,2,2)} + T_{(3,2,2)} \tag{A.54}$$

The representation (A.1b) holds for all such κ and

$$\sum_{\eta'} |M_\kappa(\eta,\eta',m)| \le c \tag{A.54}$$

for a suitable constant c. Moreover the sum

$$N_\gamma(\eta)\sum_{\eta'}|M_\kappa(\eta,\eta',m)|N_\gamma(\eta')^{-1}$$

is bounded by

$$\begin{cases} \sum_{|\underline{i}_1|<p-1}\gamma^{A-d/2-b_1} & \text{for } \kappa=(3,2,1,1), \text{ see (A.1b)} \\ \sum_{|\underline{i}_1|<p-1}\gamma^{A+b_2-d/2-b_1} & \text{for } \kappa=(3,2,1,2,1), \text{ see (A.15)} \\ \sum_{|\underline{i}_1|<p-1}\gamma^{A+d\alpha-d/2-b_1} & \text{for } \kappa=(3,2,1,2,2), \text{ see (A.16)} \\ \sum_{|\underline{i}_1|<p-1}\gamma^{A-(d/2-b_1)+d\alpha-b_2+d\alpha-d/2-b_1} & \text{for } \kappa=(3,2,2), \text{ see (A.18c)} \end{cases} \tag{A.55}$$

A is given in (A.13b), and can be rewritten as

$$A=[d/2-b_3](|\underline{i}_2|+|\underline{i}_3|)+[b_1-b_2](|\underline{i}_4|+|\underline{i}_5|)+(|\underline{i}_2|+|\underline{i}_3|+|\underline{i}_4|+2|\underline{i}_5|)d/2 \tag{A.56a}$$

and, since $|\underline{i}_2|+\cdots+|\underline{i}_5|\geq 2$, we have that

$$A-d/2-b_1\geq d/2-b_1+2\min\{[d/2-b_3],[b_1-b_2]\} \tag{A.56b}$$

Therefore, if $b_1=0$ all the bounds in (A.55) vanish as $\gamma\to 0$; if $b_1=d/2$, the first three are bounded by $c\gamma^{\zeta^\star}$, provided $\zeta^\star<2\min\{[d/2-b_3],[b_1-b_2]\}$; the exponent in the last term is bounded by $2d\alpha>b_2^\star+\zeta^\star$.

Proposition 4.8 is therefore proven. □

Acknowledgments.

Two of us, A.DM and E.P, acknowledge very kind hospitality at the Isaac Newton Institute for Mathematical Sciences, University of Cambridge.

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