TRAVELLING FRONTS IN NON LOCAL MODELS FOR PHASE SEPARATION IN EXTERNAL FIELD

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Abstract. We consider the one dimensional, non local, evolution equation derived in [4] for Ising systems with Glauber dynamics, Kac potentials and magnetic field. We prove the existence of travelling fronts, their uniqueness modulo translations among the monotone profiles and their linear stability for all the admissible values of the magnetic field for which the underlying spin system exhibits a stable and a metastable phase.

1. Introduction.

The continuum limit of one dimensional Ising spin systems with Glauber dynamics and Kac potentials gives rise, [4], to the non local evolution equation

$$\frac{\partial m}{\partial t} = -m + \tanh\{\beta[J \star m + h]\} \tag{1.1}$$

where $m = m(x,t)$ represents the magnetization density in $x \in \mathbb{R}$ at time $t \in \mathbb{R}_+$; $\beta > 0$ the inverse temperature of the Ising system; J a non negative, even function on $\mathbb{R}$ which gives the strength of the spin-spin interaction; $h \geq 0$ an external magnetic field and $J \star m$ the convolution between J and m. Non local equations like (1.1) are well studied in physics, see [9], but also in many other fields as neural networks, population dynamics, propagation of diseases, see for instance [12] and [13]. Our interest in the problem comes from the physics of phase separation and interface dynamics, and we will confine to these aspects. In this context a non local equation like (1.1) has been recently studied in [2]. We recall that (1.1) is a rigorous version of a "local mean field limit", which, in equilibrium Statistical Mechanics, gives rise to the Van der Waals

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theory of phase transition, [14]. We refer to [4] for more accurate description of the model. In the following we restrict the analysis of (1.1) to the case in which $J(r) = J(-r)$, $J(\cdot) \in C^2(\mathbb{R})$ and supported in $[-1, 1]$, where it is almost everywhere positive with the normalization condition

$$\int dy J(y) = 1 \tag{1.2}$$

Since the right hand side of (1.1) is uniformly Lipschitz (with respect to the uniform norm $||\cdot||$) the Cauchy problem for (1.1), with initial datum $m, ||m|| \le 1$, has a unique global solution $m(t, \cdot)$, with $||m(t, \cdot)|| \le 1$ for $t \ge 0$ because the unit ball in $C(\mathbb{R})$ is left invariant by the evolution.

In the sequel we assume that

$$\beta > \beta_{cr} = 1, \quad 0 \le h < h^* \tag{1.3}$$

where β_{cr} is the critical inverse temperature, see [4], and $h^* > 0$ is implicitly defined by (2.3) below. In such a situation the underlying spin system exhibits two "locally" stable phases $(m_\beta{}^{\pm}(h))$; our concern is the analysis of the travelling wave solutions of (1.1) connecting these two phases monotonically.

For small values of h the problem has been studied in [7]. Here we partially extend the results of [7], as we consider the existence of the travelling fronts in the full range of the external magnetic field. We focus our attention on their linear stability which comes out from the existence of a spectral "gap" for the linearized evolution operator. Nonlinear global stability could be proved as in [7], and we don't repeat it here. At variance with [7] the linear stability is established through some functional-analytic arguments, exploiting positivity properties of the linearized evolution operator. We prove uniqueness (up to translations) of the travelling fronts only among monotone profiles. Adapting the same dynamical argument as in [7] it is possible to prove uniqueness in a more general class of profiles , precisely those who are definitely strictly bigger than $m^0_\beta(h)$ at$+\infty$, and definitely strictly smaller than $m^0_\beta(h)$ at $-\infty$ where $m^0_\beta(h)$ is defined in (2.4) below.

In the next Section we state the definitions and the main results. The proofs are given in the remaining Sections.

2. Definitions and main results.

We are interested in travelling front solutions of (1.1), i.e. functions $M(\cdot;h) \in C^1(\mathbb{R})$ of the variable $z = x - ct$ for a suitable $c \in \mathbb{R}$ which solve

$$-cM' = -M + \tanh\{\beta[J \star M + h]\} \tag{2.1}$$

with the asymptotic conditions :

$$\lim_{z\to\pm\infty} M(z;h) = m_\beta^\pm(h) \tag{2.2}$$

The values at $\pm\infty$ coincide with the spatially homogeneous solutions described below.

Since $\beta > 1$, there is $h^* > 0$ such that for any $0 \le h < h^*$, there are exactly three stationary and spatially homogeneous solutions, each one identically equal to one of the three roots of the equation

$$m_\beta = \tanh\{\beta[m_\beta + h]\}, \qquad 0 \le h < h^* \tag{2.3}$$

These are denoted by

$$m_\beta^-(h) < m_\beta^0(h) < m_\beta^+(h) \tag{2.4}$$

and we observe that, since $h \ge 0$,

$$m_\beta^-(h) < m_\beta^0(h) \le 0 < m_\beta^+(h) \tag{2.5}$$

$$|m_\beta^-(h)| \le m_\beta^+(h) \tag{2.6}$$

$$m_\beta := m_\beta^+(0) = -m_\beta^-(0) \tag{2.7}$$

The two values, $m_\beta^\pm(h)$, correspond to the two locally stable phases (at the inverse temperature β and with magnetic field h, see [14]). The evolution of profiles $m(x)$ which in different regions of space are approximately equal to $m_\beta^+(h)$ and to $m_\beta^-(h)$, describes the competition between the two phases. When $h = 0$ there is a balance, namely a stationary solution $\bar m(x)$ of (1.1), which statically connects the two phases respectively at $\pm\infty$:

$$\bar m(x) = \tanh\{\beta(J \star \bar m)(x)\} \qquad \text{for all } x \in \mathbb{R} \tag{2.8}$$

$\bar m$ is then completely specified by the further request that it is an antisymmetric function. Results on uniqueness and stability of these solutions are presented in [1], [5] and [6]. Notice that when $h > 0$ they do not exist, being replaced by travelling fronts [7]. In the spin system this corresponds to the absence of phase transitions; namely the phase $m_\beta^-(h)$, $0 < h < h^*$, is not thermodynamically stable, but only metastable.

We have conventionally imposed that the phase $m_\beta^+(h)$ is to the right of the origin, the opposite choice is of course completely equivalent. With our choice and since $h > 0$, the velocity $c(h)$ is strictly negative, as we can see later at the end of this section. Clearly in this situation the + phase moves to the left and fills the full space at constant speed $|c(h)|$.

We can now state our result about the existence and uniqueness of monotone travelling fronts for the admissible values of h.

2.1 Theorem. *Existence and uniqueness of fronts.*

For any $0 < h < h^*$ there exists a strictly monotone travelling front , $M = M(\cdot, h) \in C^1(\mathbb{R})$, with velocity $c(h) < 0$, strictly decreasing in h (see (3.25)).

Moreover, any other monotone solution satisfying the asymptotic conditions (2.2) is equal to a translate of M.

The second result is related to the stability properties of the travelling front, with respect to continuous perturbations vanishing at infinity , i.e. belonging to $C_0(\mathbb{R})$. Namely we analyze the linear operator A (densely defined in $C_0(\mathbb{R})$) obtained by linearizing (2.1) around a given travelling front $M(\cdot\,; h)$ moving with velocity $c(h)$:

$$Av = cv' - v + pJ \star v \tag{2.9}$$

where

$$p = p(x; h) = \beta \cosh^{-2} \left\{ \beta[(J \star M)(x; h) + h] \right\} \tag{2.10}$$

Note that A depends on h , but we are not writing explicitly this dependence when the value of h is clear from the context and in the sequel we will use this convention. By a suitable decomposition of A and exploiting positivity properties of the C^0 semigroup generated by A, we prove that the spectrum $\sigma(A)$ lies in $\{ \text{ Re } \lambda \le 0\}$ and for any $0 < h < h^*$ it has a "gap property".

2.2 Theorem. *Gap property*

For any $0 < h < h^*$ the spectrum of the operator A defined in (2.9) has the following properties:

(1) *0 is a simple isolated eigenvalue and M' is the corresponding strictly positive eigenfunction.*

(2) *there exists $\alpha > 0$ such that for any $\lambda \in \sigma(A) \backslash \{0\}$, Re $\lambda \leq -\alpha$.*

From the spectral properties of A and the structure of the nonlinear part a *local* nonlinear stability result follows in a known way [8], [7].

Remark.

The sign of $c(h)$ is easily determined transforming our equation in the form studied in [13]. In fact, defining

$$n = J \star m$$

and applying $J\star$ to (1.1) we get the equation,

$$\partial_t n = -n + J \star \tanh \beta(n + h)$$

and the correspondent travelling wave form

$$-cN' = -N + J \star \tanh \beta(N + h)$$

with $N(x) \to m^{\pm}_{\beta}(h)$ as$x \to \pm\infty$. Denoting by $T(N) := \tanh \beta(N + h)$ and multiplying by the positive quantity $N'T'(N)$ we get

$$-cN'^2 T'(N) = -NN'T'(N) + N'T'(N)J \star T(N)$$

$$= N'T'(N)\big[-N + T(N)\big] + N'T'(N)\big[J \star T(N) - T(N)\big]$$

Integrating on the whole line, we obtain

$$-c \int_{-\infty}^{+\infty} dx N(x)'^2 T'(N(x)) = \int_{m^-_{\beta}(h)}^{m^+_{\beta}(h)} T'(N)\big[-N + T(N)\big] dN$$

In fact the term containing the convolution is zero, being represented by the following integral :

$$\frac{1}{2} \int dxdy[\frac{d}{dx}T(N(x)) - \frac{d}{dy}T(N(y))]J(x-y)[T(N(x)) - T(N(y))] = 0$$

So we get finally

$$\text{sign }(c(h)) = -\text{sign}\Big(\int_{m_\beta^-(h)}^{m_\beta^+(h)} T'(N)\big[-N + T(N)\big]\,dN\Big) = -\text{sign}(h) = -1$$

3. Existence and Uniqueness of the Travelling Fronts

The proof of Theorem 2.1 is based on a Lyapunov-Schmidt argument.

We suppose that for a given value h_0 of the magnetic field there exists the couple $\big(M(\cdot, h_0), c(h_0)\big)$, with $M(\cdot, h_0)$ strictly monotone solution of (2.1) with velocity $c(h_0)$.

Then we write (2.1) in integral form:

$$M(r) = \int_0^\infty dx e^{-x} \tanh\{\beta[(J\star M)(r+cx) + h]\} \equiv T(M; c; h) \tag{3.1}$$

and we prove local existence and uniqueness (up to spatial translations) of the travelling front, for value of h close to h_0 . Using a continuation argument we extend the previous result for all the admissible value of h. Therefore, exploiting the uniqueness of the stationary solution proven in [6], we exclude that for the same values of h there would exist other solution branches of (2.1) among the strictly monotone functions.

Let L be the Fréchet derivative of T calculated in $M(\cdot, h)$, solution of (2.1):

$$(Lv)(r) = \int_0^\infty dx e^{-x} p(r+cx; h)\big(J\star v\big)(r+cx) \tag{3.2}$$

where $v \in C_0(\mathbb{R})$ and p is defined in (2.10). Note that this operator is positivity improving, namely for $v(\cdot) \geq 0$ then $Lv(\cdot) > 0$. This is easily seen observing that $p(\cdot; h) > 0$ and if for some r, $(Lv)(r) = 0$, then $J\star v\big)(r+cx) \equiv 0$ and this implies $v \equiv 0$.

Let $g(\cdot\,; h)$ be any smooth function such that

$$0 < \beta(1 - (m_\beta^+(h))^2) \leq g(r; h) \leq \beta(1 - (m_\beta^-(h))^2) < 1 \quad \forall r \in \mathbb{R} \tag{3.3}$$

$$\lim_{r\to-\infty} g(r; h) = \beta(1 - (m_\beta^-(h))^2) \qquad \lim_{r\to+\infty} g(r; h) = \beta(1 - (m_\beta^+(h))^2)$$

Let $q(r; h)$ be defined as following

$$q(r; h) \equiv p(r; h) - g(r; h) \tag{3.4}$$

Clearly $q(r; h)$ is smooth and goes to 0 when $r \to \pm\infty$. Then we have the following result for the operator L.

3.1 Lemma.

The operator L defined in (3.2) can be decomposed as

$$L = L_1 + L_2 \tag{3.5}$$

where L_1 is the compact operator

$$(L_1 v)(r) = \int_0^\infty dx e^{-x} q(r+cx;h)\big(J \star v\big)(r+cx) \tag{3.6}$$

and L_2 is the following strictly contractive operator :

$$(L_2 v)(r) = \int_0^\infty dx e^{-x} g(r+cx;h)\big(J \star v\big)(r+cx) \tag{3.7}$$

Proof. From (3.3) we obtain

$$||L_2|| \le \beta\big(1-(m_\beta^-(h))^2\big) < 1 \tag{3.8}$$

Finally the properties of the function q defined in (3.4) provide compactness of L_1 in $C_0(\mathbb{R})$. $\square$

In the following Lemma and in the proof of Theorem 2.2 we will use a result due to Šmul'yan [10]. For the reader's ease we recall it here suitably rephrasing .

Theorem - [10].

Let B and K be two linear operator on a Banach space, K being compact. If $\mathcal{G}$ is a connected component of the resolvent set of B, then for the operator $B+K$ one of the following alternatives holds:

i) every point of $\mathcal{G}$ is an eigenvalue

ii) the spectrum of $B+K$ in $\mathcal{G}$ consists of isolated eigenvalues with finite multiplicity.

3.2 Lemma.

If M is a monotone solution of (3.1) then $\lambda = 1$ is an isolated eigenvalue of L with multiplicity one and the corresponding eigenfunction M' is strictly positive.

Proof. By inspection

$$(LM')(r) = \int_0^\infty dx e^{-x} p(r+cx)(J \star M')(r+cx) = M'(r) \tag{3.9}$$

From the monotonicity of M, $M'(r) \geq 0$ and since L is positivity improving, $M'(r) > 0$. Moreover M' is continuous and since M is bounded, $\lim_{r\to\pm\infty} M'(r) = 0$. Therefore M' is an eigenfunction of L in $C_0(\mathbb{R})$ associated to $\lambda = 1$.

From the decomposition of L, see Lemma 3.1, we can apply the previously recalled theorem of Šmul'yan [10]. Since L_1 is compact and the spectrum of L_2 is contained in the set $\{z \in \mathbb{C};\ |z| \leq \beta(1-(m_\beta^-(h))^2)) < 1\}$, so that its complement contains the point at infinity, we have that $\lambda = 1$ is an isolated eigenvalue of finite multiplicity.

To prove that the multiplicity is one we apply the same argument as in [13]. Namely we suppose that there exists an other eigenfunction Ψ associated to $\lambda = 1$. Without loss of generality we can always suppose that Ψ has a negative minimum (otherwise we consider $-\Psi$). Set $\rho_\mu = M' + \mu\Psi$, since $\rho_0 = M'$ is strictly positive there exists a $\bar{\mu}$, $\bar{\mu} \geq 0$, such that for $\mu > \bar{\mu}$, ρ_μ has a negative minimum. Let r_μ be a point where $\rho_\mu(r)$ attains the absolute (negative) minimum. Then if $\{r_\mu\}$ is bounded as $\mu \to \bar{\mu}$, there exists a subsequence of points $\{r_{\mu_n}\}$ converging to r' as $\mu_n \to \bar{\mu}$ such that $\lim_{\mu_n\to\bar{\mu}} \rho_\mu(r_{\mu_n}) = \rho_{\bar{\mu}}(r')$ with $\rho_{\bar{\mu}}(r') = 0$ and $\rho_{\bar{\mu}}(r) \geq 0$. This contradicts the positivity-improving property of the operator L, according to which a non-negative eigenfunction belonging to a positive eigenvalue has to be strictly positive. On the other hand, if the set $\{r_\mu\}$ is unbounded, we select a diverging sequence $\{\bar{r}_{\mu_n}\}$.

Then from (3.2)

$$\rho_{\mu_n}(\bar{r}_{\mu_n}) = \int_0^\infty dx e^{-x}(J\star\rho_{\mu_n})(\bar{r}_{\mu_n}+cx)p(\bar{r}_{\mu_n}+cx;h) \tag{3.10}$$

We decompose the integral in (3.10) as follows :

$$\int_0^\infty dx... = \int_0^k dx... + \int_k^\infty dx... \tag{3.11}$$

where k is a suitably chosen constant. It is easy to see that the "tail" integral $\int_k^\infty dx..$ can be made less than an arbitrary $\epsilon > 0$ if k is sufficiently large (uniformly in μ). We set $A_\mu = \{x \in \mathbb{R}^+ : (J\star\rho_\mu)(\bar{r}_\mu + cx) \leq 0\}$.

From (2.2) and the definition of p in (2.10)

$$p(\bar{r}_{\mu_n} + cx; h) \leq 1 - \delta < 1$$

for some positive δ, n sufficiently large and for any $x \in A_\mu \cap [0,k]$.

Since $\rho_\mu(r_\mu)$ is a negative minimum $|(J \star \rho_\mu)(r_\mu + cx)| < |\rho_\mu(r_\mu)|$ for all $x \in A_\mu$. Therefore the negative contribution of the right part of (3.11) (i.e. the first term with the integration variable x restricted to $A_\mu \cap [0,k]$) would be less or equal to $(1-\delta)|\rho_\mu(\bar{r}_{\mu_n})|$ and this is absurd.

$\square$

3.3 Lemma.

The adjoint operator $L^\star$ has 1 as a simple isolated eigenvalue , with non negative eigenfunction ψ.

Proof. From general theory [3] $\lambda = 1$ is still a simple isolated eigenvalue of $L^\star$. We denote by ψ the associated eigenfunction. Since ψ solves $L^\star\psi = \psi$; ψ is a $C_b(\mathbb{R}) \cap L^1(\mathbb{R})$ function. To prove that ψ does not change sign we can use the same argument as [13].

Namely let us suppose that ψ changes sign. We consider its positive part ψ^+

$$\psi^+(r) = \begin{cases} \psi(r) & \text{if } \psi(r) \geq 0 \\ 0 & \text{if } \psi(r) < 0 \end{cases}$$

and we have that $L^\star(\psi^+)(r) \geq L^\star(\psi)(r)$. Since $L^\star$ is positivity improving $L^\star(\psi^+) > \psi^+$ in a set of positive measure. This would imply the following contradiction

$$0 = \langle \psi^+ \, , \, (L - I)M' \rangle = \langle (L^\star - I)\psi^+ \, , \, M' \rangle > 0$$

where $\langle \, , \, \rangle$ denotes the duality bracket. $\square$

3.4 Lemma. *(Lyapunov-Schmidt argument)*

Suppose that for a given h_0, $0 < h_0 < h^*$ there exists a continuous monotone solution to (2.1) *$M(\cdot\,; h_0)$ with velocity $c(h_0)$, satisfying the asymptotic boundary conditions* (2.2).

Then given $\epsilon > 0$ there exists $\delta(h_0, \epsilon) > 0$ such that for $|h - h_0| < \delta$ there exists in $||M(\cdot\,; h) - M(\cdot\,; h_0)|| < \epsilon$, $|c(h) - c(h_0)| < \epsilon$ modulo spatial translations of $M(\cdot\,; h)$, an unique pair $(M(\cdot\,; h), c(h))$, $M(\cdot\,; h) \in C(\mathbb{R})$, $c(h) \in \mathbb{R}$ satisfying (2.1) *and* (2.2). *Moreover $M(\cdot\,; h)$ is a strictly monotone function.*

Proof.

To apply the Lyapunov-Schmidt method we note that any solution of (3.1) can be suitably decomposed as follows: $M(r;h) = \phi(r) + M(r;h_0) + \Gamma(r;h)$ where

$$\phi(r) = M(r;h) - M(r;h_0) - \Gamma(r;h) \tag{3.12}$$

with $\phi(\cdot) \in C_0(\mathbb{R})$ and $\Gamma(r;h) = \big(J \star [\chi_h - \chi_{h_0}]\big)(r)$ taking care of the different asymptotic boundary conditions when the parameter h is changed.

The function $\chi_h(r)$ is the following step function

$$\chi_h(r) = \begin{cases} m_\beta^+(h) \text{ when } r > 0 \\ m_\beta^-(h) \text{ when } r \le 0 \end{cases}$$

Then from (3.1) we obtain

$$\phi(r) + \Gamma(r;h) + M(r\,;h_0) = T(\phi + M(\cdot, h_0) + \Gamma(\cdot\,, h); c; h) \tag{3.13}$$

Let us compute the derivatives of $T(M;c;h)$ in
$(M(\cdot\,, h_0), c(h_0), h_0)$ denoting them by $(D_cT)_0(\cdot)$ and $(D_hT)_0(\cdot)$ respectively:

$$(D_cT)_0(r) = \int_0^\infty dx e^{-x} x p(r + c(h_0)x)\big(J \star M'\big)(r + c(h_0)x; h_0) \tag{3.14}$$

$$(D_hT)_0(r) = \int_0^\infty dx e^{-x} p(r + c(h_0)x) \tag{3.15}$$

Differentiating $T(M;c;h)$ with respect to its arguments we obtain from (3.13) the following

$$\begin{aligned} (I - L_0)\phi &= (D_cT)_0[c - c(h_0)] + (D_hT)_0[h - h_0] + \\ &\big[L_0 - I\big](D_h\Gamma)(\cdot, h_0)[h - h_0] + R(\phi, c, h) \end{aligned} \tag{3.16}$$

where we denoted by $(D_h\Gamma)(\cdot; h_0)$ the derivative of $\Gamma(\cdot;h)$ with respect to h computed in h_0 and used the following identity

$$\Gamma(r;h) = D_h\Gamma(r;h_0)(h - h_0) + \big[\Gamma(r;h) - (D_h\Gamma)(r;h_0)(h - h_0)\big] \tag{3.17}$$

and, see (3.2),

$$(L_0\phi)(r) = \int_0^\infty dx e^{-x} p(r + c(h_0)x)\big(J \star \phi\big)(r + c(h_0)x) \tag{3.18}$$

$$R(\phi, c, h) = T(\phi + M(h_0) + \Gamma(h); c; h) - L_0\phi - (D_c T)_0[c + c(h_0)] -$$
$$(D_h T)_0[h - h_0] + \big[L_0 - I\big]\big[\Gamma(h) - (D_h\Gamma)(h_0)(h - h_0)\big] \tag{3.19}$$

is infinitesimal at least of order 2 with respect to its arguments. Since the operator $I - L_0$ is not invertible (see Lemma 3.2), we split the space $C_0(\mathbb{R})$ in two subspaces and project the equation in each of them. One equation is solved via the implicit function theorem and the other is one dimensional.

Since $< \psi, (I - L_0)\phi >= 0$ where ψ is the eigenfunction of $L^\star$ associated to $\lambda = 1$, see Lemma 3.3, multiplying in the duality sense both part of the (3.16) we obtain

$$c = \gamma(\phi, c, h) \tag{3.20}$$

where

$$\gamma(\phi, c, h) = c(h_0) - \big[< \psi, (D_h T)_0 > (h - h_0) + < \psi, R(\phi, c, h) > \big] / < \psi, (D_c T)_0 > \tag{3.21}$$

We decompose $\phi = \phi_p + \phi_\perp$, ϕ_p the projection of ϕ along M': $\phi_p =< \psi, \phi > M'$ and $\phi_\perp = \phi - \phi_p$ We then have from (3.16)

$$\phi_\perp = G(\phi, c, h) \tag{3.22}$$

where we set

$$G(\phi, c, h) = \big(I - L_0\big)^{-1}\big\{(D_c T)_0[c - c(h_0)] +$$
$$(D_h T)_0[h - h_0] - (D_h\Gamma)_0[h - h_0] + R(\phi, h, c)\big\} \tag{3.23}$$

Since $R(\phi, c, h)$ is infinitesimal of order bigger or equal to 2 in its argument, $\gamma(\phi, c, h)$ and $G(\phi, c, h)$ are contraction maps with respect to c and ϕ_o provided $|h - h_0|$ is suitably small. Namely they map the set $R_\epsilon = \{c, \phi; |c - c(h_0)| < \epsilon, \quad ||\phi + \Gamma(h)|| < \epsilon\}$ in itself. Therefore we have in R_ϵ an unique solution $(\phi_0(\cdot\,; h), c(h))$ which depends continuously on h. To prove that $M(\cdot\,; h)$ is unique modulo translations we first show that $M(\cdot\,; h)$ is a strictly monotone function. We note that since $||M(\cdot\,; h) - M(\cdot\,; h_0)|| < \epsilon$ for $|h - h_0| < \delta$ we also have that $||M'(\cdot\,; h) - M'(\cdot\,; h_0)|| < \kappa\epsilon$ where κ is a positive constant. Therefore since $M'(\cdot\,; h_0)$ is positively bounded away from zero on the compact sets, we can find an arbitrarily large interval where $M'(\cdot\,;h)$ is strictly positive, if $|h-h_0|$ is sufficiently small . In order to exclude that $M'(\cdot\,;h)$ is negative outside the chosen large interval we can proceed as previously estimating the negative contribution of the integral in the following formula

$$M'(\bar r;h) = \int_0^\infty dx e^{-x} p(\bar r + c(h)x;h)\big(J\star M'\big)(\bar r + c(h)x;h) \tag{3.24}$$

where $\bar r$ denotes the negative minimum outside the chosen large interval; as before (Lemma 3.2) we would obtain a contradiction .

Moreover from (3.21) we have that $c(h)$ is a decreasing function of h, in fact

$$\lim_{h\to h_0}\{c(h)-c(h_0)\}/(h-h_0) = -\langle\psi,(D_hT)_0\rangle/\langle\psi,(D_cT)_0\rangle < 0 \tag{3.25}$$

□

In Lemma 3.5 stated below we extend Lemma 3.4 to any h, $0<h<h^*$, and moreover we prove uniqueness, modulo spatial translations, of $M(\cdot\,;h)$ among the monotone functions.

3.5 Lemma. *Continuation argument*

For any $0<h<h^*$ there exists an unique couple $(c(h), M(\cdot\,;h))$ (modulo translation of $M(\cdot\,;h)$) among the monotone profiles such that

a) $M(\cdot\,;h)$ is a solution of (2.1) *and satisfies the asymptotic boundary conditions* (2.2)

b) $M'(\cdot\,;h)$ is strictly positive

c) $c(h)$ is bounded

Proof. We prove first that it is possible to apply the preceding method for all values of h up to h^*. Namely, suppose that Lemma 3.4 holds for $h<\bar h$ with $0<\bar h<h^*$. Then we show that it is possible to apply Lemma 3.4 also in $(c(\bar h), M(\cdot\,;\bar h))$. In fact, the $\{M(\cdot\,;h)\}$ are equibounded and equicontinuous functions for any $0\le h<\bar h$. Therefore by the Ascoli-Arzelá Theorem we can found a subsequence $\{h_n\}$ such that when $\lim_{n\to\infty} h_n = \bar h$, $M(\cdot\,;h_n)$ converges uniformly on the compact sets of $\mathbb{R}$. From the continuity of the second right hand of (3.1) we obtain for the limit function

$$M(r;\bar h) = \int_0^\infty dx e^{-x}\tanh\{\beta[(J\star M)(r+c(\bar h)x;\bar h)+\bar h]\} \tag{3.26}$$

provided that $c(\bar{h})$ is finite. To apply Lemma (3.4) up to $\bar{h}$, we should exclude that

(a) $\lim_{n\to\infty} |c(h_n)| = \infty$

(b) $M'(\cdot\,;\bar{h}) = 0$ for some $r \in \mathbb{R}$

(c) $M(\cdot\,;\bar{h})$ does not satisfy the asymptotic boundary conditions (2.2).

To exclude (a) we use the translation invariance of (2.1). Namely if $M(\cdot\,;h_n)$ is the solution of (2.1) satisfying the asymptotic boundary conditions (2.2) we can always set $M(0;h_n) = 0$. From the uniform convergence on the compact sets of $\mathbb{R}$, since

$$M(0;h_n) = \int_0^\infty dx e^{-x} \tanh\{\beta[(J \star M(c(h_n)x;h_n) + h_n]\} = 0 \tag{3.27}$$

and the asymptotic boundary conditions ($m_\beta^\pm(\bar{h})$ are different from zero), we get an absurd if $\lim_{n\to\infty} |c(h_n)| = \infty$.

To prove (b) note that $M'(\cdot\,;\bar{h})$ is either strictly positive or identical equal to 0 since L is positivity improving. Suppose that $M'(\cdot\,;\bar{h}) > 0$ then we must verify that the asymptotic boundary conditions (2.2) are satisfied. The only alternative possibilities since $M'(\cdot\,;\bar{h}) > 0$ are:

$$(i) \qquad \lim_{r\to-\infty} M(r;\bar{h}) = m_\beta^-(\bar{h}) < 0 \qquad \lim_{r\to\infty} M(r;\bar{h}) = m_\beta^0(\bar{h}) \le 0$$

$$(ii) \qquad \lim_{r\to-\infty} M(r;\bar{h}) = m_\beta^0(\bar{h}) \le 0 \qquad \lim_{r\to\infty} M(r;\bar{h}) = m_\beta^+(\bar{h}) > 0$$

They are both excluded since (3.27). Moreover it cannot happen that for a given value of $h < h^*$ could exist two monotone solutions. Namely let us suppose that for $h = h_1$ there exists another monotone solution. Then, since Lemma 3.4, it belongs to a local branch that cannot have bifurcation points. Therefore, the only possibility we have is to continue back from this given value up to $h = 0$. Since there exists a unique solution for $h = 0$, see [5] we would get a contradiction.

□

The proof of Theorem 2.2 follows from Lemma 3.4 and Lemma 3.5.

As a final remark note that the analysis done so far breaks down for $h = h^\star$. Namely $m_\beta^-(h^*) = m_\beta^0(h^*)$ and $\beta\big(1 - (m_\beta^-(h))^2\big) = 1$ so that the estimate in (3.8) does not hold.

4. Linear stability of the travelling fronts

Local stability properties of the travelling front $M(\cdot;h)$ are obtained by analyzing the spectrum of the linear operator A already defined in (2.9):

$$Av \equiv cv' - v + pJ \star v$$

We consider A with domain densely defined in $C_0(\mathbb{R})$ and we decompose it in the following way:

$$A = A_1 + A_2 \tag{4.1}$$

where

$$\begin{cases} A_1 v = cv' - v + gJ \star v \\ \\ A_2 v = qJ \star v \end{cases} \tag{4.2}$$

and the functions g and q are those previously defined in (3.3), (3.4), $g(x) + q(x) = p(x)$.

Proof of Theorem 2.2. The proof goes on several steps exploiting in different ways the properties of the operator A.

Step 1: Analysis of A_1.

Let $\lambda^\star \equiv 1-\beta\big(1-(m_{\bar\beta}^-(h))^2\big) > 0$ and let us first prove that $\sigma(A_1) \subset \{z; -\infty < \text{ Re } z \le -\lambda^\star\}$. Namely the resolvent equation

$$(A_1 - \lambda)v \equiv cv' - (1+\lambda)v + gJ \star v = f \tag{4.3}$$

in $C_0(\mathbb{R})$ can be solved for $\text{ Re } \lambda > -\lambda^\star$. More precisely the resolvent operator $R_\lambda(A_1) \equiv (\lambda - A_1)^{-1}$ is holomorphic in the half plane $\{z; -\lambda^\star < \text{ Re } z < +\infty\}$ and moreover

i) $||R_\lambda(A_1)|| \le 1/(\text{ Re } \lambda + \lambda^\star)$

ii) $R_{\alpha+i\beta}(A_1) \to 0$ strongly as $\beta \to \infty$, uniformly in α for $\alpha + \lambda^\star$ positively bounded away from zero.

This is shown by writing (4.3) in integral form as follows

$$v(r) = \int_0^\infty e^{-(1+\lambda)x} g(r+cx)(J\star v)(r+cx)dx + \int_0^\infty e^{-(1+\lambda)x} f(r+cx)dx \tag{4.4}$$

that is

$$v = G_\lambda v + E_\lambda f \tag{4.5}$$

where these operators are defined below in $C_0(\mathbb{R})$ for $\text{Re } \lambda > -1$

$$G_\lambda v = \int_0^\infty e^{-(1+\lambda)x} g(r+cx)(J \star v)(r+cx)dx$$

and

$$E_\lambda f = \int_0^\infty e^{-(1+\lambda)x} f(r+cx)dx \tag{4.6}$$

It is easy to see that

$$\begin{aligned} ||G_\lambda|| &\leq (1-\lambda^\star)/(1+\text{ Re } \lambda) \\ ||E_\lambda|| &\leq 1/(1+\text{ Re } \lambda) \end{aligned} \tag{4.7}$$

so that $||G_\lambda|| < 1$ for $\text{Re } \lambda + \lambda^\star > 0$.

Then from (4.5) $R_\lambda(A_1)$ can be represented through the Neumann series:

$$v = R_\lambda(A_1)f = \sum_{n=0}^{\infty} {G_\lambda}^n E_\lambda f \quad , \ \text{Re } \lambda + \lambda^\star > 0$$

Moreover for the operator $S_\lambda \equiv \sum_{n=0}^{\infty} {G_\lambda}^n$ the following estimate is valid:

$$||S_\lambda|| \leq \sum_{n=0}^{\infty} ||G_\lambda||^n \leq (1+\text{ Re } \lambda)/(\lambda^\star + \text{ Re } \lambda \quad , \ \text{Re } \lambda + \lambda^\star > 0$$

It turns out from (4.7) that

$$||R_\lambda(A_1)|| \leq 1/(\lambda^\star + \text{ Re } \lambda) \tag{4.8}$$

Furthermore with a Riemann-Lebesgue type argument we have for (4.6)

$$E_{\alpha+i\beta} f(r) = \int_0^\infty e^{-i\beta x} e^{-(1+\alpha)x} f(r+cx)dx \to 0\,,\ \beta \to \infty \tag{4.9}$$

uniformly in r for $\alpha + 1 \geq \delta$, with δ any positive constant. Then for any positive ϵ and $f \in C_0(\mathbb{R})$, there exists a $\sigma(\epsilon, f)$ such that for

$$||R_{\alpha+i\beta}(A_1)f|| = ||S_{\alpha+i\beta}E_{\alpha+i\beta}f|| \leq (1+\alpha)||E_{\alpha+i\beta}f||/(\alpha+\lambda^\star) < \epsilon \quad , \text{ if } |\beta| > \sigma(\epsilon, f) \tag{4.10}$$

Step 2. Use of compactness of A_2.

To study the spectrum of the operator $A_1 + A_2$ we take advantage of the compactness of A_2 (the integral kernel of A_2, $g(x)J(x-y)$ is smooth and decays at infinity in x). So the resolvent equation

$$(A_1 + A_2)\phi - \lambda\phi = f \tag{4.11}$$

can be written in the following way:

$$\phi - R_\lambda(A_1)A_2\phi = -R_\lambda(A_1)f \quad , \qquad \text{Re}\,\lambda + \lambda^\star > 0 \tag{4.12}$$

Remark that the half-plane $\{\ \text{Re}\,\lambda > -\lambda^\star + ||A_2||\}$ belongs to the resolvent set of $(A_1 + A_2)$ since

$$||R_\lambda(A_1)A_2|| \leq ||R_\lambda(A_1)||\,||A_2|| \leq ||A_2||/(\ \text{Re}\,\lambda + \lambda^\star) < 1 \quad , \qquad \text{Re}\,\lambda + \lambda^\star > ||A_2||$$

We know on the other side that 0 is an eigenvalue of $A_1 + A_2$ with a positive eigenfunction M' and applying the mentioned theorem of Šmul'yan [10], we find that $\sigma(A_1 + A_2) \cap \{z; -\lambda^\star < \text{Re}\, z < -\lambda^\star + ||A_2||\}$ consists of isolated eigenvalues with finite multiplicity. Namely, from (4.8), (4.10) we have that in the vertical strip $\{z; -\lambda^\star + \delta < \ \text{Re}\, z < -\lambda^\star + ||A_2||\}$, with $0 < \delta < \lambda^\star$, do not exist eigenvalues with "too big" imaginary parts. This is readily seen as follows.

Suppose that there exists a sequence of (normalized) eigenvectors $\{\phi_{\lambda_n}\}$, with $\text{Im}\,\lambda_n \to \infty$ when $n \to \infty$; i.e. the relation

$$\phi_{\lambda_n} = R_{\lambda_n}(A_1)A_2\phi_{\lambda_n}$$

holds for any n. We can extract a subsequence, called again $\{\phi_{\lambda_n}\}$, such that $A_2\phi_{\lambda_n} \to \psi$ so that

$$\phi_{\lambda_n} = R_{\lambda_n}(A_1)A_2\phi_{\lambda_n} = R_{\lambda_n}(A_1)\psi + R_{\lambda_n}(A_1)(A_2\phi_{\lambda_n} - \psi)$$

and therefore we get a contradiction since

$$1 = ||\phi_{\lambda_n}|| \leq ||R_{\lambda_n}(A_1)\psi|| + ||A_2\phi_{\lambda_n} - \psi||/(\ \text{Re}\,\lambda_n + \lambda^\star) < \epsilon(1 + \delta^{-1})$$

for $\text{Re}\,\lambda_n + \lambda^\star > \delta$, and $\text{Im}\,\lambda_n \to \infty$. We conclude that for a positive $\delta < \lambda^\star$ there is a finite (non zero) number of eigenvalues in the strip $\{z; -\lambda^\star + \delta < \ \text{Re}\, z < -\lambda^\star + ||A_2||\}$; in particular 0 belongs to this set.

Step 3. Use of positiveness of the semigroup generated by $A = A_1 + A_2$, (see [11], BIII).

The linear operator A generates a positive C^0 semigroup $\{T_t\}$ in the space $C_0(\mathbb{R})$. This can be seen directly or by a perturbative argument representing the semigroup operator T_t as follows

$$T_t = T_t^0 + \int_0^t T_{t-s}^0(pJ\star)T_s ds$$

where T_t^0 is given by $(T_t^0 u)(x) = e^{-t}u(x+ct)$. We know, from step 2, that there is a finite number of eigenvalues of A in $\{z; -\lambda^\star + \delta < \text{ Re } z < -\lambda^\star + ||A_2||\}$. Therefore there will be at least one λ_0 with the largest real part. Since A generates a positive semigroup then [11], (see BIII, Theorem 1.1), then the nonnegative spectral bound $s(A) := sup\{ \text{ Re}\lambda | \lambda \in \sigma_(A)\}$ is a simple eigenvalue. It could exist other eigenvalues of the kind $\lambda = s(A) + i\alpha$, with $\alpha \neq 0$. To exclude such a situation we note that if there exists λ then $\lambda = s(A) + in\alpha$ will belong to $\sigma_(A)$, (ciclicity of the boundary spectrum [11], (see BIII, Theorem 2.9)). But this is impossible since there is a finite number of eigenvalues of A in $\{z; -\lambda^\star + \delta < \text{ Re } z < -\lambda^\star + ||A_2||\}$. We show now that indeed $s(A) = 0$. We observe that the resolvent $R_\lambda(A)$ has a simple pole in $s(A)$, with a positive residue Q. In fact

$$Q = \lim_{\lambda \to s(A)} (\lambda - s(A)) R_\lambda(A)$$

and we have that

$$AQ = QA = s(A)Q, \text{ and } Q \geq 0$$

We therefore conclude observing that $AM' = 0\,, M' > 0$ so that if $s(A) > 0\,, QM' = \frac{1}{s(A)}QAM' = 0$ obtaining a contradiction since the positiveness of Q.

$\square$

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