BOUNDARY DRIVEN KAWASAKI PROCESS WITH LONG RANGE INTERACTION: DYNAMICAL LARGE DEVIATIONS AND STEADY STATES

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ABSTRACT. A particle system with a single locally-conserved field (density) in a bounded interval with different densities maintained at the two endpoints of the interval is under study here. The particles interact in the bulk through a long range potential parametrized by $\beta \geq 0$ and evolve according to an exclusion rule. It is shown that the empirical particle density under the diffusive scaling solves a quasi-linear integro-differential evolution equation with Dirichlet boundary conditions. The associated dynamical large deviation principle is proved. Furthermore, for β small enough, it is also demonstrated that the empirical particle density obeys a law of large numbers with respect to the stationary measures (hydrostatic). The macroscopic particle density solves a non local, stationary, transport equation.

1. INTRODUCTION

Over the last few years there have been several papers devoted to understanding the macroscopic properties of systems out of equilibrium. Typical examples are systems in contact with two thermostats at different temperature or with two reservoirs at different densities. A mathematical model is provided by interacting particles performing local reversible dynamics (for example reversible hopping dynamics) in the interior of a domain and some external mechanism of creation and annihilation of particles at the boundary of the domain, modeling the reservoirs. The full process is non reversible. The stationary non equilibrium states are characterized by a flow of mass through the system and long range correlations are present. We refer to [3] and [6] for two reviews on this topic. We study a model with such features but in a regime where phase separation might occur at equilibrium for the underlying reversible model. To this aim we consider in the interior of the domain a reversible dynamics (Kawasaki dynamics) constructed trough a mean field interaction (Kac potential), see below for more details. This is the first time, to our knowledge, where both, long range dynamics in the bulk and creation and annihilation of particles at the boundary of the domain, are taken into account.

The particle models we consider are dynamic versions of lattice gases with long-range Kac potentials, i.e. the interaction energy between two particles, say one at x and the other at y (x and y are both in $\mathbb{Z}^d$), is given by $J_N(x,y) = N^{-d}J(\frac{x}{N}, \frac{y}{N})$

where J is a smooth symmetric probability kernel with compact support and N is a positive integer which is sent to ∞. The equilibrium states for these models have been investigated thoroughly [12], [18] and [22], and have provided great mathematical insight into the static aspects of phase transition phenomena. The dynamical version of these lattice gases in domain with periodic boundary conditions (reversible dynamics) has been analyzed in [8], [9], [10] and [1]. This paper starts by studying the dynamics of these systems in a bounded interval with reservoirs (non-reversible dynamics). We investigate their qualitative behavior in the range of the parameter when at equilibrium phase transition is present. Let us describe informally the dynamics. We consider a one dimensional lattice gas with particle reservoirs at the endpoints. The restriction on the dimension is done only for simplicity. Given an integer $N > 1$ at any given time each site of the discret set $\{-N, \ldots, N\}$ is either occupied by one particle or empty. The interaction energy among particles is given by a modified version of the Kac potential J_N and it is tuned by a positive parameter β. The modification of the Kac potential, see (2.1), takes into account that the particles are confined in a bounded domain. In the bulk each particle jumps at random times on the right or on the left nearest neighborhood, if the chosen site is empty, at a rate which depends on the particle configuration through the Kac potential. When $\beta = 0$ we have the simple exclusion process. At the boundary sites, $\pm N$, particles are created and removed for the local density to be $0 < \rho_- \leq \rho_+ < 1$. At rate $\rho_\pm$ a particles is created at $\pm N$ if the site is empty and at rate $1 - \rho_\pm$ the particle at $\pm N$ is removed if the site is occupied.

The dynamics described above defines an *irreducible* Markov jump process on a finite state space, i.e. there is a strictly positive probability to go from any state to another. By the general theory on Markov process [17], the invariant measure μ_N^{stat} is unique and encodes the long time behavior of the system. Let $\mathbb{P}_{\mu_N^{stat}}^{\beta,N}$ be the stationary process. Since μ_N^{stat} is invariant, the measure $\mathbb{P}_{\mu_N^{stat}}^{\beta,N}$ is invariant with respect to time shifts. The measure $\mathbb{P}_{\mu_N^{stat}}^{\beta,N}$ is invariant under time reversal if and only if the measure μ_N^{stat} is reversible for the process, i.e if the generator of the process satisfies the detailed balance condition with respect to μ_N^{stat}. In the case at hand μ_N^{stat} is stationary and reversible only if $\beta = 0$ and $\rho^+ = \rho^- = \rho$. In such case the invariant, reversible measure is the Bernoulli product measure with marginals ρ. When $\beta > 0$ and /or $\rho^+ \neq \rho^-$ then μ_N^{stat} is not reversible. In such case the corresponding process is denoted *non reversible*. We shall consider the latter case. The lattice space is scaled by $\frac{1}{N}$ and the time by N^2 (*diffusive limit*) while the behavior is studied, as $N \uparrow \infty$, of the empirical density of the particles evolving according to the dynamics described above.

To prove the hydrodynamic behavior of the system, we follow the entropy method introduced by Guo, Papanicolaou, and Varadhan [11]. It relies on an estimate on the entropy of the state of the process with respect to a reference invariant state. The main problem in the model considered here is that the stationary state is not explicitly known. We have therefore to consider the entropy of the state of the process with respect to a state which is not invariant, for instance, with respect to a product measure with a slowly varying profile. Since this measure is not invariant, the entropy does not decrease in time and we need to estimate the rate at which it increases. These estimates on the entropy are deduced in Subsection 3.1

It results that the local empirical particle density, in the diffusive limit, is the solution of a boundary value problem for a quasilinear integro differential parabolic equation, see (2.9). In addition to this, it is demonstrated that when β is small enough, then the empirical particle density obeys a law of large numbers with respect to the stationary measures (hydrostatic). The result is obtained characterizing the support of any limit points of $\mathbb{P}^{\beta,N}_{\mu^{stat}_N}$. This intermediate result holds for any $\beta \geq 0$. Then it is shown that with β small enough, the stationary solution of the boundary problem (2.9) is unique, it is a global attractor for the macroscopic evolution with a decay rate uniform with respect to the initial datum. This holds only for $\beta < \beta_0$ where $\beta_0 > 0$ depends on the diameter of the domain and on the chosen interaction J. Namely the quasilinear non local parabolic equation does not satisfy a comparison principle, which is the main tool used in previous papers, see for example [20] and [7], to show the hydrostatic. For value larger than β_0 we are not able to show the uniqueness of the stationary solution of the boundary value problem (2.9). We stress that $\beta_0 < \beta_c$ where β_c is the value above which phase segregation occurs at equilibrium; with the choice made of the parameters $\beta_c = \frac{1}{4}$, see page 1712 of [9]. Further, we prove the dynamical large deviations for the empirical particle density. The large deviation functional is not convex as function of the density, it is lower semicontinuous and has compact level sets. Since the large deviation functional is not convex and the underlying dynamics does not satisfy any comparison principle, care needs to be taken to prove the lower bound. The basic strategy to show the lower bound consists in obtaining this bound for smooth paths and then applying a density argument. The argument goes as follows: Given a path ρ with finite rate functional $I(\rho)$ one constructs a sequence of smooth paths ρ_n so that $\rho_n \to \rho$ in a suitable topology and $I(\rho_n) \to I(\rho)$. When the large deviation functional is convex, this argument is easily implemented. In our case, because of the lack of convexity we modify the definition of the rate functional declaring it infinite if a suitable energy estimate does not hold. In this way the modified rate functional when finite provides the necessary compactness to close the argument. This method has been developed in [21] and we adapted to our model. The modification of the rate functional helps in showing the lower bound but makes more difficult the upper bound. One needs to show that the energy estimate holds with probability super exponentially close to one. A similar strategy was applied in [20] and [4].

In a recent paper, De Masi et alii, [5], constructed, in the phase transition regime, stationary solutions of a boundary value problem equivalent to (2.9) in which the density ρ is replaced by the magnetization $m = 2\rho - 1$. They did not study the derivation of the boundary value problem from the particle system and they did not inquire about the uniqueness of stationary solutions. They investigated the qualitative behavior of constructed stationary solutions of (2.9) as the diameter of the domain goes to infinity. They proved, for the constructed solution, the validity of the Fourier law in the thermodynamic limit showing that, in the phase transition regime, the limit equilibrium profile has a discontinuity (which defines the position of the interface) and satisfies a stationary free boundary Stefan problem.

The paper is organized as follows: The precise feature of the model, notations and results are stated in Section 2, In Section 3, some basic estimates needed along the paper are collected. In Section 4 the hydrodynamic and the hydrostatic limits are shown. Section 5 is split into 5 subsections and deals with dynamical large

deviations. We prove in Section 6 the weak uniqueness of the solution of the quasi-linear integro-differential equation. Furthermore, for β small enough, it is shown that its stationary solution is unique and it is a global attractor in L^2.

2. Notation and Results

Fix an integer $N \geq 1$. Call $\Lambda_N = \{-N, \ldots, N\}$ and $\Gamma_N = \{-N, +N\}$ the boundary points. The sites of Λ_N are denoted by x, y and z. The configuration space is $\mathcal{S}_N \equiv \{0,1\}^{\Lambda_N}$ which we equip with the product topology. Elements of $\mathcal{S}_N$, called configurations, are denoted η so that $\eta(x) \in \{0,1\}$ stands for the number of particles at site x of the configuration η.

We denote $\Lambda = (-1, 1)$ ($\bar{\Lambda} = [-1, 1]$) the macroscopic open (closed) interval, $\Gamma = \{-1, 1\}$ its boundary and $u \in [-1, 1]$ the macroscopic space coordinate.

2.1. **The interaction.** To define the interaction between particles we introduce a smooth, symmetric, translational invariant probability kernel of range 1, i.e $J(u, v) = J(0, v-u)$ for all $u, v \in \mathbb{R}$, $J(0, u) = 0$, for all $|u| > 1$, and $\int J(u, v)dv = 1$, for all $u \in \mathbb{R}$, [1]. When $(u, v) \in \Lambda \times \Lambda$ we define the interaction $J^{neum}(u, v)$ imposing a reflection rule: u interacts with v and with the reflected points of v where reflections are the ones with respect to the left and right boundary of Λ. For this reason it is referred to as "Neuman" interaction. More precisely, we define for u and v in Λ

$$J^{neum}(u, v) := J(u, v) + J(u, 2 - v) + J(u, -2 - v), \tag{2.1}$$

where $2 - v$ is the image of v under reflections on the right boundary $\{1\}$ and $-2 - v$ is the image of v under reflections on the left boundary $\{-1\}$. By the assumption on J, $J^{neum}(u, v) = J^{neum}(v, u)$ and $\int J^{neum}(u, v)dv = 1$ for all $u \in \Lambda$, see Lemma 3.1. We defined the interaction (2.1) by boundary reflections only for convenience. It has the advantage to keep J^{neum} a symmetric probability kernel. This choice of the potential has been done already in [5].

The pair interaction between x and y in Λ_N is given by

$$J_N(x, y) = N^{-1} J^{neum}\left(\frac{x}{N}, \frac{y}{N}\right).$$

The total interaction energy among particles is given by the following Hamiltonian

$$H_N(\eta) = -\frac{1}{2} \sum_{x,y \in \Lambda_N} J_N(x, y)\eta(x)\eta(y) . \tag{2.2}$$

2.2. **The dynamics.** We denote by $\eta^{x,y}$ the configuration obtained from η by interchanging the values at x and y:

$$(\eta^{x,y})(z) := \begin{cases} \eta(y) & \text{if } z = x \\ \eta(x) & \text{if } z = y \\ \eta(z) & \text{if } z \neq x, y, \end{cases}$$

and by $\sigma^x \eta$ the configuration obtained from η by flipping the occupation number at site x:

$$(\sigma^x \eta)(z) := \begin{cases} 1 - \eta(x) & \text{if } z = x \\ \eta(z) & \text{if } z \neq x. \end{cases}$$

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[1] One could take the interaction J so that for all $u \in \mathbb{R}$, $\int J(u, v)dv = a$ with $a > 0$. The only difference, with the case at hand, is that the the underlying reversible particle model has, at equilibrium, phase transition for $\beta \geq \beta_c = \frac{1}{4a}$, see [22].

We denote for $f : \mathcal{S}_N \to \mathbb{R}$, $x, y \in \Lambda_N$ and $\eta \in \mathcal{S}_N$,

$$(\nabla_{x,y} f)(\eta) = f(\eta^{x,y}) - f(\eta).$$

The microscopic dynamics is specified by a continuous time Markov chain on the state space $\mathcal{S}_N$ with infinitesimal generator given by

$$\mathcal{L}_N = \mathbb{L}_{\beta,N} + \mathbb{L}_{-,N} + \mathbb{L}_{+,N} \ , \tag{2.3}$$

where for function $f : \mathcal{S}_N \to \mathbb{R}$

$$(\mathbb{L}_{\beta,N} f)(\eta) = \sum_{\substack{x \in \Lambda_N, y \in \Lambda_N \\ |x-y|=1}} C_N^\beta(x, y; \eta) \left[(\nabla_{x,y} f)(\eta) \right] \ , \tag{2.4}$$

with rate of exchange occupancies C_N^β given by

$$C_N^\beta(x, y; \eta) = \exp \left\{ -\frac{\beta}{2} [H_N(\eta^{x,y}) - H_N(\eta)] \right\} \ , \tag{2.5}$$

where H_N is the Hamiltonian (2.2);

$$\begin{aligned}
(\mathbb{L}_{-,N} f)(\eta) &= c_-\big(\eta(-N)\big) \big[f(\sigma^{-N} \eta) - f(\eta) \big], \\
(\mathbb{L}_{+,N} f)(\eta) &= c_+\big(\eta(N)\big) \big[f(\sigma^{N} \eta) - f(\eta) \big],
\end{aligned}$$

where for any $\rho_\pm \in (0, 1)$, $c_\pm : \{0, 1\} \to \mathbb{R}$ are given by

$$c_\pm(\zeta) := \rho_\pm(1 - \zeta) + (1 - \rho_\pm)\zeta \ .$$

The generator $\mathbb{L}_{\beta,N}$ describes the bulk dynamics which preserves the total number of particles, the so called Kawasaki dynamics, whereas $\mathbb{L}_{\pm,N}$, which is a generator of a birth and death process acting on Γ_N, models the particle reservoir at the boundary of Λ_N. The rate of the bulk dynamics $\{C_N^\beta(x, y; \eta), \quad x \in \Lambda_N, \quad y \in \Lambda_N\}$, see (2.5), satisfies the detailed balance with respect to the Gibbs measure associated to (2.2) with chemical potential $\lambda \in \mathbb{R}$

$$\mu_N^{\beta,\lambda}(\eta) = \frac{1}{Z_N^{\beta,\lambda}} \exp\{-\beta H_N(\eta) + \lambda \sum_{x \in \Lambda_N} \eta(x)\} \ , \qquad \eta \in \mathcal{S}_N \ , \tag{2.6}$$

where $Z_N^{\beta,\lambda}$ is the normalization constant. For the bulk rates this means

$$C_N^\beta(x, y; \eta) = e^{-\beta[H_N(\eta^{x,y}) - H_N(\eta)]} C_N^\beta(y, x; \eta^{x,y}).$$

For the generator it means that $\mathbb{L}_{\beta,N}$ is self-adjoint w.r.t. the Gibbs measure (2.6), for any $\lambda \in \mathbb{R}$. The corresponding process is denoted reversible.

For a positive function $\rho : \Lambda_N \to (0, 1)$ we denote by $\nu_N^\rho(\cdot)$ the Bernoulli measure with marginals

$$\nu_N^{\rho(\cdot)}(\eta(x) = 1) = \rho(x/N) = 1 - \nu_N^\rho(\eta(x) = 0) \ , \qquad x \in \Lambda_N, \eta \in \mathcal{S}_N. \tag{2.7}$$

The Bernoulli measure $\nu_N^{\rho_+}$ (ν^{ρ_-}) is reversible with respect to the boundary generator $\mathbb{L}_{+,N}$ ($\mathbb{L}_{-,N}$).

The Markov process associated to the generator $\mathcal{L}_N$, see (2.3), is *irreducible* and we denote by $\mu_N^{stat} = \mu_N^{stat}(\beta, \rho_-, \rho_+)$ the unique invariant measure. In the notation we stress only the dependence on the parameters relevant to us. This means that for any $f : \mathcal{S}_N \to \mathbb{R}$

$$\int_{\mathcal{S}} \mathcal{L}_N f(\eta) d\mu_N^{stat}(\eta) = 0,$$

but the generator $\mathcal{L}_N$ is *not* self-adjoint with respect to μ_N^{stat}. The corresponding process is called *non reversible*. The only case where the process is reversible is when $\rho_- = \rho_+$ and $\beta = 0$. In such case the product measure associated to $\rho_- = \rho_+$ is invariant and the process with generator $\mathcal{L}_N$ is also reversible.

We denote by $\mathcal{M}$ the space of positive densities bounded by 1:

$$\mathcal{M} := \left\{ \rho \in L_\infty\big([-1,1], du\big) \,:\, 0 \leq \rho \leq 1 \right\},$$

where du stands for the integration with respect to the Lebesgue measure on $[-1,1]$. We equip $\mathcal{M}$ with the topology induced by the weak convergence of measures and denote by $\langle \cdot, \cdot \rangle$ the duality mapping. A sequence $\{\rho^n\} \subset \mathcal{M}$ converges to ρ in $\mathcal{M}$ if and only if

$$\langle \rho^n, G \rangle = \int_\Lambda \rho^n(u)G(u)du \to \langle \rho, G \rangle$$

for any continuous function $G : [-1,1] \to \mathbb{R}$. Note that $\mathcal{M}$ is a compact Polish space that we consider endowed with the corresponding Borel σ-algebra. The empirical density of the configuration $\eta \in \mathcal{S}_N$ is defined as $\pi^N(\eta)$ where the map $\pi^N : \mathcal{S}_N \to \mathcal{M}$ is given by

$$\pi^N(\eta)(u) := \sum_{x=-N+1}^{N-1} \eta(x) \, \mathbf{1}\left\{ \left[\tfrac{x}{N} - \tfrac{1}{2N}, \tfrac{x}{N} + \tfrac{1}{2N} \right) \right\}(u),$$

in which $\mathbf{1}\{A\}$ stands for the indicator function of the set A. Let $\{\eta^N\}$ be a sequence of configurations with $\eta^N \in \mathcal{S}_N$. If the sequence $\{\pi^N(\eta^N)\} \subset \mathcal{M}$ converges to ρ in $\mathcal{M}$ as $N \to \infty$, we say that $\{\eta^N\}$ is *associated* to the macroscopic density profile $\rho \in \mathcal{M}$.

2.3. **Functional Spaces.** Fix a positive time T. Let $D([0,T], \mathcal{M})$, respectively $D([0,T], \mathcal{S}_N)$, be the set of right continuous with left limits trajectories $\pi : [0,T] \to \mathcal{M}$, resp. $(\eta_t)_{t\in[0,T]} : [0,T] \to \mathcal{S}_N$, endowed with the Skorohod topology and equipped with its Borel $\sigma-$ algebra. Take μ_N on $\mathcal{S}_N$ and denote by $(\eta_t)_{t\in[0,T]}$ the Markov process with generator $N^2 \mathcal{L}_N$ starting, at time $t = 0$, by η_0 distributed according to μ_N. Notice that the generator of the process has been speeded up by N^2. This corresponds to the diffusive scaling. Denote by $\mathbb{P}_{\mu_N}^{\beta,N}$ the probability measure on the path space $D([0,T], \mathcal{S}_N)$ corresponding to the Markov process $(\eta_t)_{t\in[0,T]}$ and by $\mathbb{E}_{\mu_N}^{\beta,N}$ the expectation with respect to $\mathbb{P}_{\mu_N}^{\beta,N}$. When $\mu_N = \delta_\eta$ for some configuration $\eta \in \mathcal{S}_N$, we write simply $\mathbb{P}_\eta^{\beta,N} = \mathbb{P}_{\delta_\eta}^{\beta,N}$ and $\mathbb{E}_\eta^{\beta,N} = \mathbb{E}_{\delta_\eta}^{\beta,N}$. We denote by π^N the map from $D([0,T], \mathcal{S}_N)$ to $D([0,T], \mathcal{M})$ defined by $\pi^N(\eta.)_t = \pi^N(\eta_t)$ and by $Q_{\mu_N}^{\beta,N} = \mathbb{P}_{\mu_N}^{\beta,N} \circ (\pi^N)^{-1}$ the law of the process $\big(\pi^N(\eta_t)\big)_{t\geq0}$. For $m \in L_\infty([-1,1])$ and $u \in \Lambda$ we denote

$$(J^{\text{neum}} \star m)(u) = \int_\Lambda J^{\text{neum}}(u,v)m(v)dv.$$

We need some more notations. For integers n and m we denote by $C^{n,m}([0,T] \times [-1,1])$ the space of functions $G = G_t(u) : [0,T] \times [-1,1] \to \mathbb{R}$ with n derivatives in time and m derivatives in space which are continuous up to the boundary. We denote by $C_0^{n,m}([0,T] \times [-1,1])$ the subset of $C^{n,m}([0,T] \times [-1,1])$ of functions vanishing at the boundary of $[-1,1]$, i.e. $G_t(-1) = G_t(1) = 0$ for $t \in [0,T]$. We denote by $C_c^{n,m}([0,T] \times (-1,1))$ the subset of $C^{n,m}([0,T] \times (-1,1))$ of functions with compact support in $[0,T] \times (-1,1)$.

2.4. **Results.** We denote $\chi(\rho) = \rho(1 - \rho)$, $\sigma(\rho) = 2\chi(\rho)$ and ∇f, resp. Δf, the gradient, resp. the laplacian, with respect to u of a function f. For $G \in C_0^{1,2}([0, T] \times [-1, 1])$, $\rho \in D([0, T], \mathcal{M})$ denote

$$
\begin{aligned}
\ell_G^\beta(\rho, \rho_0) := \ & \langle \rho_T, G_T \rangle - \langle \rho_0, G_0 \rangle - \int_0^T dt \, \langle \rho_t, \partial_t G_t \rangle \\
& - \int_0^T dt \, \langle \rho_t, \Delta G_t \rangle + \rho_+ \int_0^T dt \, \nabla G_t(1) - \rho_- \int_0^T dt \, \nabla G_t(-1) \quad (2.8) \\
& - \frac{\beta}{2} \int_0^T \langle \sigma(\rho_t), (\nabla G_t) \cdot \nabla (J^{\mathrm{neum}} \star \rho_t) \rangle dt.
\end{aligned}
$$

Denote by $\mathcal{A}_{[0,T]} \subset D([0, T]; \mathcal{M})$ the set

$$
\mathcal{A}_{[0,T]} = \left\{ \rho \in L^2([0, T], H^1(\Lambda)) \ : \quad \forall G \in \mathcal{C}_0^{1,2}([0, T] \times \Lambda), \ \ell_G^\beta(\rho, \rho_0) = 0 \right\}.
$$

Theorem 2.1. *For any sequence of initial probability measures $(\mu_N)_{N \geq 1}$, the sequence of probability measures $(Q_{\mu_N}^{\beta, N})_{N \geq 1}$ is weakly relatively compact and all its converging subsequences converge to some limit $Q^{\beta,*}$ that is concentrated on the absolutely continuous measures whose density $\rho \in \mathcal{A}_{[0,T]}$. Moreover, if for any $\delta > 0$ and for any continuous function $G : [-1, 1] \to \mathbb{R}$*

$$
\lim_{N \to 0} \mu^N \left\{ \left| \frac{1}{N} \sum_{x \in \Lambda_N} \eta(x) G(x/N) - \int_\Lambda \gamma(u) G(u) du \right| \geq \delta \right\} = 0,
$$

for an initial profile $\gamma : \Lambda \to (0, 1)$, then the sequence of probability measures $(Q_{\mu_N}^{\beta, N})_{N \geq 1}$ converges to the Dirac measure concentrated on the unique weak solution of the following boundary value problem on $(t, u) \in (0, T) \times \Lambda$

$$
\begin{cases}
\partial_t \rho_t(u) + \beta \nabla \cdot \left\{ \rho_t(u)(1 - \rho_t(u)) \nabla (J^{neum} \star \rho_t)(u) \right\} = \Delta \rho_t(u) \\
\rho_t(\mp 1) = \rho_\mp \quad for \ 0 \leq t \leq T, \\
\rho_0(u) = \gamma(u) \, .
\end{cases} \quad (2.9)
$$

Remark 2.2. *By weak solution of the boundary value problem (2.9) we mean $\ell_G^\beta(\rho, \gamma) = 0$ for $G \in C_0^{1,2}([0, T] \times [-1, 1])$.*

Theorem 2.3. *There exists β_0 depending on Λ and J^{neum} so that, for any $\beta < \beta_0$, for any $G \in C_0^2([-1, 1])$, for any $\delta > 0$,*

$$
\lim_{N \to \infty} \mu_N^{stat} \left[\left| \langle \pi_N, G \rangle - \langle \bar{\rho}, G \rangle \right| \geq \delta \right] = 0,
$$

where $\bar{\rho}$ is the unique weak solution of the following boundary value problem

$$
\begin{cases}
\Delta \rho(u) - \beta \nabla \cdot \left\{ \rho(u)(1 - \rho(u)) \nabla (J^{neum} \star \rho)(u) \right\} = 0, \quad u \in \Lambda, \\
\rho(\mp 1) = \rho_\mp \quad .
\end{cases} \quad (2.10)
$$

We prove Theorem 2.1 and Theorem 2.3 in Section 4. Recall that the stationary measure μ_N^{stat} depends on β.

Next we state the large deviation principle associated to the law of large numbers stated in Theorem 2.1. Let $\mathcal{J}_G^\beta = \mathcal{J}_{T,G,\gamma}^\beta : D([0, T], \mathcal{M}) \longrightarrow \mathbb{R}$ be the functional

given by

$$\mathcal{J}_G^\beta(\pi) := \ell_G^\beta(\pi, \gamma) \, - \, \frac{1}{2} \int_0^T dt \, \langle \sigma(\pi_t), \left(\nabla G_t \right)^2 \rangle \,, \qquad (2.11)$$

and $\hat{I}_T^\beta(\cdot|\gamma) : D([0,T], \mathcal{M}) \to [0, +\infty]$ the functional defined by

$$\hat{I}_T^\beta(\pi|\gamma) \; := \; \sup_{G \in C_0^{1,2}([0,T] \times [-1,1])} \mathcal{J}_G^\beta(\pi) \,. \qquad (2.12)$$

To define the large deviation rate functional, we introduce the energy functional $\mathcal{Q} : D([0,T], \mathcal{M}) \to [0, +\infty]$ given by

$$\mathcal{Q}(\pi) \; = \; \sup_G \Big\{ \int_0^T dt \langle \pi_t, \nabla G_t \rangle - \frac{1}{2} \int_0^T dt \langle \sigma(\pi_t) G_t, G_t \rangle \Big\} \,, \qquad (2.13)$$

where the supremum is carried over all $G \in C_c^\infty([0,T] \times (-1,1))$. From the concavity of $\sigma(\cdot)$ it follows immediately that $\mathcal{Q}$ is convex and therefore lower semicontinuous. Moreover $\mathcal{Q}(\pi)$ is finite if and only if $\pi \in L^2\big([0,T]; H^1(\Lambda)\big)$, and

$$\mathcal{Q}(\pi) \; = \; \frac{1}{2} \int_0^T dt \int_{-1}^1 du \, \frac{(\nabla \pi_t(u))^2}{\sigma(\pi_t(u))} \,. \qquad (2.14)$$

If (2.14) holds, then an integration by parts and Schwarz inequality imply that (2.13) is finite. The converse needs to be proven, for a proof of it we refer to [4], Subsection 4.1. The rate functional $I_T^\beta(\cdot|\gamma) : D([0,T], \mathcal{M}) \to [0, \infty]$ is given by

$$I_T^\beta(\pi|\gamma) = \begin{cases} \hat{I}_T^\beta(\pi|\gamma) & \text{if} \quad \mathcal{Q}(\pi) < +\infty \,, \\ +\infty & \text{otherwise} \,. \end{cases} \qquad (2.15)$$

We show in Lemma 5.6 that $I_T^\beta(\pi|\gamma) = 0$ if and only if $\pi_t(\cdot)$ solves the problem (2.9) with initial datum $\pi_0(\cdot) = \gamma(\cdot)$.

We have the following dynamical large deviation principle.

Theorem 2.4. *Fix $T > 0$ and an initial profile γ in $\mathcal{M}$. Consider a sequence $\{\eta^N : N \geq 1\}$ of configurations associated to γ. Then, the sequence of probability measures $\{Q_{\eta^N}^{\beta, N} : N \geq 1\}$ on $D([0,T], \mathcal{M})$ satisfies a large deviation principle with speed N and rate function $I_T^\beta(\cdot|\gamma)$, defined in (2.15):*

$$\varlimsup_{N \to \infty} \frac{1}{N} \log Q_{\eta^N}^{\beta, N}\big(\pi^N \in \mathcal{C}\big) \; \leq \; -\inf_{\pi \in \mathcal{C}} I_T^\beta(\pi|\gamma)$$

$$\varliminf_{N \to \infty} \frac{1}{N} \log Q_{\eta^N}^{\beta, N}\big(\pi^N \in \mathcal{O}\big) \; \geq \; -\inf_{\pi \in \mathcal{O}} I_T^\beta(\pi|\gamma) \,,$$

for any closed set $\mathcal{C} \subset D([0,T], \mathcal{M})$ and open set $\mathcal{O} \subset D([0,T], \mathcal{M})$. The functional $I_T^\beta(\cdot|\gamma)$ is lower semi-continuos and has compact level sets.

We prove Theorem 2.4 in Section 5.

3. Basic estimate

Next lemma states some properties of the potential $J^{\mathrm{neum}}(\cdot, \cdot)$ easily obtained by its definition.

Lemma 3.1. *The potential $J^{neum}(\cdot, \cdot)$ is a symmetric probability kernel. Moreover for any regular function $G : \Lambda \to \mathbb{R}$, we have the following:*

$$\left| \nabla \left(\int_\Lambda J^{neum}(u,v)G(v)dv \right) \right| \leq \int_\Lambda J^{neum}(u,v) \left| \nabla G(v) \right| dv. \qquad (3.1)$$

Proof. The symmetry of J^{neum} follows immediately by the one of J. We have

$$J^{\text{neum}}(u,v) = J(0, v-u) + J(0, 2 - (u+v)) + J(0, 2 + (u+v)),$$

which is symmetric in u and v. We now prove that J^{neum} is probabiltity kernel. Fix $u \in \Lambda$, by a change of variables,

$$\int_\Lambda J^{\text{neum}}(u,v)dv = \int_{(-1-u)\vee(-1)}^{(1-u)\wedge 1} J(0,v)dv + \int_{(1-u)\vee(-1)}^{(3-u)\wedge 1} J(0,v)dv$$
$$+ \int_{(-3-u)\vee(-1)}^{(-1-u)\wedge 1} J(0,v)dv.$$

Suppose first that $u \in [0,1]$, then

$$\int_\Lambda J^{\text{neum}}(u,v)dv = \int_{-1}^{1-u} J(0,v)dv + \int_{1-u}^{1} J(0,v)dv$$
$$= \int_{-1}^{1} J(0,v)dv = 1.$$

and thus, $J^{\text{neum}}(u, \cdot)$ is a probability. The proof for $u \in [-1, 0]$ is similar. It remains to prove (3.1):

$$\nabla \left(\int_\Lambda J^{\text{neum}}(u,v)G(v)dv \right) = \int_\Lambda \partial_u J^{\text{neum}}(u,v)G(v)dv$$
$$= \int_\Lambda \left[J(u,v) - J(u, 2-v) - J(u, -2-v) \right] \nabla G(v)dv.$$

The result follows from the following inequality,

$$\left| J(u,v) - \left[J(u, 2-v) + J(u, -2-v) \right] \right| \leq J^{\text{neum}}(u,v)$$

for all $u, v \in \Lambda$. $\qquad \square$

For any $G : \Lambda \to \mathbb{R}$ and $x, x+1 \in \Lambda_N$ denote by $\nabla^N G(\frac{x}{N})$ the discrete gradient:

$$\nabla^N G(x/N) = N \left[G((x+1)/N) - G(x/N) \right]. \qquad (3.2)$$

Next, we show that the rate C_N^β of $\mathbb{L}_{\beta,N}$ is a perturbation of the rate of the symmetric simple exclusion generator.

Lemma 3.2. *For any $x \in \Lambda_N$, with $x \pm 1 \in \Lambda_N$ and $\eta \in \mathcal{S}_N$,*

$$C_N^\beta(x, x \pm 1; \eta) = 1 \mp \frac{\beta}{2} \big(\eta(x+1) - \eta(x) \big) N^{-1} \nabla^N \left[\big(J^{neum} \big) \star \pi(\eta) \right] (x/N) + O(N^{-2}).$$

Proof. By definition of H_N, for all $x, y \in \Lambda_N$ and $\eta \in \mathcal{S}_N$,

$$\big(\nabla_{x,y} H_N \big)(\eta) = \frac{1}{N} \big(\eta(x) - \eta(y) \big)^2 \big(J^{\text{neum}}(\frac{x}{N}, \frac{y}{N}) - J^{\text{neum}}(0,0) \big)$$
$$+ \big(\eta(x) - \eta(y) \big) \frac{1}{N} \sum_{z \in \Lambda_N} \eta(z) \big[J^{\text{neum}}(\frac{x}{N}, \frac{z}{N}) - J^{\text{neum}}(\frac{y}{N}, \frac{z}{N}) \big].$$

This concludes the proof. $\qquad \square$

We start recalling the definitions of relative entropy and Dirichlet form, that are the main tools in the [11] approach. Let $h : \Lambda \to (0,1)$ and $\nu_N^{h(\cdot)}$ be the product Bernoulli measure defined in (2.7). Given μ, a probability measure on $\mathcal{S}_N$, denote by $H(\mu|\nu_N^{h(\cdot)})$ the relative entropy of μ with respect to $\nu_N^{h(\cdot)}$:

$$H(\mu|\nu_N^{h(\cdot)}) \;=\; \sup_f \left\{ \int f(\eta)\mu(d\eta) - \log \int e^{f(\eta)}\nu_N^{h(\cdot)}(d\eta) \right\},$$

where the supremum is carried over all bounded functions on $\mathcal{S}_N$. Since $\nu_N^{h(\cdot)}$ gives a positive probability to each configuration, μ is absolutely continuous with respect to $\nu_N^{h(\cdot)}$ and we have an explicit formula for the entropy:

$$H(\mu|\nu_N^{h(\cdot)}) \;=\; \int \log \left\{ \frac{d\mu}{d\nu_N^{h(\cdot)}} \right\} d\mu . \tag{3.3}$$

Further, since there is at most one particle per site, there exists a constant C, that depends only on $h(\cdot)$, such that

$$H(\mu|\nu_N^{h(\cdot)}) \;\leq\; CN \tag{3.4}$$

for all probability measures μ on $\mathcal{S}_N$ (cf. comments following Remark V.5.6 in [15]).

3.1. **Dirichlet form estimates.** One of the main step for deriving the hydrodynamic limit and the large deviations, is a super exponential estimate which allows the replacement of local functions by functionals of the empirical density. One needs to estimate expression such as $\langle Z, f \rangle_{\mu^N}$ in terms of Dirichlet form $N^2 \langle -\mathcal{L}_N \sqrt{f(\eta)}, \sqrt{f(\eta)} \rangle_{\mu^N}$, where Z is a local function and $\langle \cdot, \cdot \rangle_{\mu^N}$ represents the inner product with respect to some probability measure μ^N. In the context of boundary driven process, the fact that the invariant measure is not explicitly known introduces a technical difficulty. We fix as reference measure a product measure $\nu_N^{\theta(\cdot)}$, see (2.7), where θ is a smooth function with the only requirement that $\theta(\mp 1) \;=\; \rho_\mp$. There is therefore no reasons for $N^2 \langle -\mathcal{L}_N \sqrt{f(\eta)}, \sqrt{f(\eta)} \rangle_{\nu^{\theta(\cdot)}}$ to be positive. Next lemma estimates this quantity.

Define the following functionals from $h \in L^2(\nu)$ to $\mathbb{R}^+$:

$$\mathcal{D}_{0,N}(h, \nu) = \sum_{x=-N}^{N-1} \int \left(h(\eta^{x,x+1}) - h(\eta) \right)^2 d\nu(\eta),$$

$$\mathcal{D}_{+,N}(h, \nu) = \frac{1}{2} \int c_+\left(\eta(N) \right) \left(h(\sigma^{N-1}\eta) - h(\eta) \right)^2 d\nu(\eta), \tag{3.5}$$

$$\mathcal{D}_{-,N}(h, \nu) = \frac{1}{2} \int c_-\left(\eta(-N) \right) \left(h(\sigma^{-N+1}\eta) - h(\eta) \right)^2 d\nu(\eta).$$

Lemma 3.3. *Let $\theta : \overline{\Lambda} \to (0,1)$ be a smooth function such that $\theta(\mp 1) \;=\; \rho_\mp$. There exists a positive constant $C_0 \equiv C_0(\|\nabla\theta\|_\infty)$ so that for any $a > 0$ and for $f \in L^2(\nu_N^{\theta(\cdot)})$,*

$$\int_{\mathcal{S}_N} f(\eta)\mathbb{L}_{\beta,N}f(\eta)d\nu_N^{\theta(\cdot)}(\eta) \leq -\left(1 - a\right)\mathcal{D}_{0,N}\left(f, \nu_N^{\theta(\cdot)}\right) + \frac{C_0}{a}N^{-1}\|f\|^2_{L^2(\nu_N^{\theta(\cdot)})}, \tag{3.6}$$

$$\int_{\mathcal{S}_N} f(\eta)\mathbb{L}_{\pm,N}f(\eta)d\nu_N^{\theta(\cdot)}(\eta) = -\mathcal{D}_{\pm,N}\left(f, \nu_N^{\theta(\cdot)}\right). \tag{3.7}$$

Proof. The proof of (3.7) follows from the reversibility of the Bernoulli measure $\nu_N^{\theta(\cdot)}$ with respect to $\mathbb{L}_{\pm,N}$. Next, we show (3.6). By Lemma 3.2,

$$
\int_{\mathcal{S}_N} f(\eta) \mathbb{L}_{\beta,N} f(\eta) d\nu_N^{\theta(\cdot)}(\eta) \leq \sum_{x=-N}^{N-1} \int \left[(\nabla^{x,x+1} f)(\eta) \right] f(\eta) d\nu_N^{\theta(\cdot)}(\eta)
$$
$$
+ \frac{A_1}{N} \sum_{x=-N}^{N-1} \int \left| (\nabla^{x,x+1} f)(\eta) \right| f(\eta) d\nu_N^{\theta(\cdot)}(\eta) \tag{3.8}
$$

for some positive constant A_1 depending only on β and J. We write the first term of the right hand side of (3.8) as

$$
- \sum_{x=-N}^{N-1} \int \left[(\nabla^{x,x+1} f)(\eta) \right]^2 d\nu_N^{\theta(\cdot)}(\eta)
$$
$$
+ \sum_{x=-N}^{N-1} \int R_N(x, x+1; \theta, \eta) \left[(\nabla^{x,x+1} f)(\eta) \right] f(\eta^{x,x+1}) d\nu_N^{\theta(\cdot)}(\eta) , \tag{3.9}
$$

where

$$
R_N(x, x+1; \theta, \eta) = \left[1 - e^{-N^{-1} \nabla^N \lambda(\theta(x/N)) \left(\nabla^{x,x+1} \eta(x) \right)} \right] , \tag{3.10}
$$

λ is the chemical potential defined by

$$
\lambda(r) = \log \left[r/(1-r) \right] \tag{3.11}
$$

and ∇^N stands for the discrete derivative defined in (3.2). By the inequality

$$
\text{for all } A, B \in \mathbb{R} \quad \text{and} \quad a > 0 , \quad AB \leq \frac{a}{2} A^2 + \frac{1}{2a} B^2 \tag{3.12}
$$

and Taylor expansion, the formula (3.9) is bounded by

$$
-\left(1 - \frac{a}{2}\right) \sum_{x=-N}^{N-1} \int \left[(\nabla^{x,x+1} f)(\eta) \right]^2 d\nu_N^{\theta(\cdot)}(\eta) + \frac{A_2}{a} N^{-1} \|f\|_{L^2(\nu_N^{\theta(\cdot)})}^2 \tag{3.13}
$$

for all $a > 0$. Here A_2 is a positive constant.

The second term on the right hand side of (3.8) is handled in the identical way. It is bounded by

$$
\frac{a}{2} \sum_{x=-N}^{N-1} \int \left[(\nabla^{x,x+1} f)(\eta) \right]^2 d\nu_N^{\theta(\cdot)}(\eta) + \frac{A_3}{a} N^{-1} \|f\|_{L^2(\nu_N^{\theta(\cdot)})}^2 . \tag{3.14}
$$

The lemma follows from (3.8),(3.9), (3.13) and (3.14). $\square$

Denote for $h \in L^2(\nu)$

$$
\mathcal{D}_{\beta,N}(h, \nu) = \sum_{x=-N}^{N-1} \int C_N^\beta(x, x+1; \eta) \left(h(\eta^{x,x+1}) - h(\eta) \right)^2 d\nu(\eta) .
$$

Lemma 3.4. *There exists a positive constant $C_1 = C_1(\beta, J)$ such that, for any measure ν and for $h \in L^2(\nu)$,*

$$
\left(1 - \frac{C_1}{N}\right) \mathcal{D}_{0,N}(h, \nu) \leq \mathcal{D}_{\beta,N}(h, \nu) \leq \left(1 + \frac{C_1}{N}\right) \mathcal{D}_{0,N}(h, \nu).
$$

Proof. The proof is elementary since $\left| C_N^\beta(x, x+1, \eta) - 1 \right|$ is uniformly bounded in N, x and η. $\square$

Lemma 3.5. *Let $\rho, \rho_0 : \overline{\Lambda} \to (0,1)$ be two smooth functions. There exists a positive constant $C_0' \equiv C_0'(\|\nabla \rho_0\|_\infty, \|\nabla \rho\|_\infty)$ such that for any probability measure μ^N on $\mathcal{S}_N$,*

$$\mathcal{D}_{0,N}\left(\sqrt{\frac{d\mu^N}{d\nu_N^{\rho(\cdot)}}}, \nu_N^{\rho(\cdot)}\right) \leq 2\,\mathcal{D}_{0,N}\left(\sqrt{\frac{d\mu^N}{d\nu_N^{\rho_0(\cdot)}}}, \nu_N^{\rho_0(\cdot)}\right) + C_0' N^{-1}. \tag{3.15}$$

Proof. Denote by $f(\eta) = \frac{d\mu^N}{d\nu_N^{\rho(\cdot)}}(\eta)$ and $h(\eta) = \frac{d\mu^N}{d\nu_N^{\rho_0(\cdot)}}(\eta)$. Since $f(\eta) = h(\eta)\frac{d\nu_N^{\rho_0(\cdot)}(\eta)}{d\nu_N^{\rho(\cdot)}(\eta)}$ we obtain for $-N \leq x \leq N-1$ the following

$$\int_{\mathcal{S}_N} \left[\nabla_{x,x+1}\sqrt{f}(\eta)\right]^2 d\nu_N^{\rho(\cdot)}(\eta)$$

$$= \int_{\mathcal{S}_N} \left[\sqrt{h}(\eta^{x,x+1})R_2(x, x+1; \eta) + \nabla_{x,x+1}\sqrt{h}(\eta)\right]^2 d\nu_N^{\rho_0(\cdot)}(\eta)$$

$$\leq 2\int_{\mathcal{S}_N} \left[\nabla_{x,x+1}\sqrt{h}(\eta)\right]^2 d\nu_N^{\rho_0(\cdot)}(\eta)$$

$$+ 2\int_{\mathcal{S}_N} h(\eta^{x,x+1})\left[R_N(x, x+1; \rho, \eta)\right]^2 d\nu_N^{\rho_0(\cdot)}(\eta),$$

where

$$R_2(x, x+1; \eta) = \exp\left\{(1/2)N^{-1}\nabla^N[\lambda(\rho(x/N)) - \lambda(\rho_0(x/N))]\nabla_{x,x+1}\eta(x)\right\} - 1$$

and λ is the chemical potential defined by (3.11). We conclude the proof using Taylor expansion and integration by parts. $\square$

3.2. **Superexponential estimates.** For a positive integer ℓ and $x \in \Lambda_N$ denote

$$\Lambda_\ell(x) = \Lambda_{N,\ell}(x) = \{y \in \Lambda_N : |y - x| \leq \ell\}.$$

When $x = 0$, we shall denote $\Lambda_\ell(0)$ simply by Λ_ℓ, that is, for all $1 \leq \ell \leq N$,

$$\Lambda_\ell \equiv \Lambda_{N,\ell}(0) = \{-\ell, \cdots, \ell\}.$$

Denote the empirical mean density on the box $\Lambda_\ell(x)$ by $\eta^\ell(x)$:

$$\eta^\ell(x) = \frac{1}{|\Lambda_\ell(x)|} \sum_{y \in \Lambda_\ell(x)} \eta(y). \tag{3.16}$$

For a cylinder function Ψ, that is a function on $\{0,1\}^{\mathbb{Z}}$ depending on $\eta(x)$, $x \in \mathbb{Z}$, only trough finitely many x, denote by $\widetilde{\Psi}(\rho)$ the expectation of Ψ with respect to ν^ρ, the Bernoulli product measure with density ρ:

$$\widetilde{\Psi}(\rho) = E^{\nu^\rho}[\Psi]. \tag{3.17}$$

Further, denote for $G \in \mathcal{C}([0,T] \times [-1,1])$ and $\varepsilon > 0$

$$V_{N,\varepsilon}^{G,\Psi}(s, \eta) = \frac{1}{N} \sum_{x \in \Lambda_N} G_s(x/N)\left[\tau_x\Psi(\eta) - \widetilde{\Psi}(\eta^{[\varepsilon N]}(x))\right], \tag{3.18}$$

where the sum is carried over all x such that the support of $\tau_x\Psi$ belongs to Λ_N and $[\cdot]$ denotes the lower integer part.

Proposition 3.6. *Let $\{\mu_N : N \geq 1\}$ be a sequence of probability measures on $\mathcal{S}_N$. For every $\delta > 0$,*

$$\varlimsup_{\varepsilon \to 0} \varlimsup_{N \to \infty} \frac{1}{N} \log \mathbb{P}_{\mu_N}^{\beta,N} \left[\left| \int_0^T V_{N,\varepsilon}^{G,\Psi}(s,\eta_s) \, ds \right| > \delta \right] = -\infty \, .$$

Proof. Fix $c > 0$ that will decreases to 0 after ε and a smooth function $\rho_c \colon \Lambda \to (0,1)$ which is constant in $\Lambda_{(1-c)} = [-1+c, 1-c]$ and equal to $\rho_\pm$ at the boundary, i.e $\rho_c(\pm 1) = \rho_\pm$. The constant can be arbitrarily chosen and we denote it γ_0. Divide Λ_N in two subsets, $\Lambda_{[(1-2c)N]}$ and $\Lambda_N \setminus \Lambda_{[(1-2c)N]}$ and split the sum over x in the definition of $V_{N,\varepsilon}^{G,\Psi}$ into the sum over these two sets. Since

$$\sup_{\eta,\varepsilon,N,x\in\Lambda_N} \left\{ G_s(x/N) \left[\tau_x \Psi(\eta) - \widetilde{\Psi}(\eta^{[\varepsilon N]}(x)) \right] \right\} < \infty \, ,$$

we have that

$$\left| \frac{1}{N} \sum_{x \in \Lambda_N \setminus \Lambda_{[(1-2c)N]}} \int_0^T ds \, G_s(x/N) \left[\tau_x \Psi(\eta) - \widetilde{\Psi}(\eta^{[\varepsilon N]}(x)) \right] \right| \leq cTK_0 \qquad (3.19)$$

for some positive constant K_0 which depends on G. By Chebyshev exponential inequality, for all $a > 0$,

$$\frac{1}{N} \log \mathbb{P}_{\mu_N}^{\beta,N} \left[\left| \int_0^T V_{N,\varepsilon}^{G,\Psi}(s,\eta_s) \, ds \right| > \delta \right]$$

$$\leq -a\big(\delta - TK_0 c\big) + \frac{1}{N} \log \mathbb{P}_{\mu_N}^{\beta,N} \left[\exp \left(aN \left| \int_0^T \mathbb{V}_{N,\varepsilon}^{c,G,\Psi}(s,\eta_s) \, ds \right| \right) \right] \, , \tag{3.20}$$

where

$$\mathbb{V}_{N,\varepsilon}^{c,G,\Psi}(s,\eta) = \frac{1}{N} \sum_{x \in \Lambda_{[(1-2c)N]}} G_s(x/N) \left[\tau_x \Psi(\eta) - \widetilde{\Psi}(\eta^{\varepsilon N}(x)) \right] \, . \tag{3.21}$$

It is immediate to see that the Radon-Nikodym derivative

$$\frac{d\mathbb{P}_{\mu_N}^{\beta,N}}{d\mathbb{P}_{\nu_N^{\rho_c(\cdot)}}^{\beta,N}} \big((\eta_t)_{t \in [0,T]} \big) = \frac{d\mu_N}{d\nu_N^{\rho_c(\cdot)}} \leq e^{NK_1(c)}$$

for some positive $K_1(c)$ that depends on c. The right hand side of (3.20) is bounded by

$$-a\big(\delta - TK_0 c\big) + K_1(c) + \frac{1}{N} \log \mathbb{P}_{\nu_N^{\rho_c(\cdot)}}^{\beta,N} \left[\exp \left(aN \left| \int_0^T \mathbb{V}_{N,\varepsilon}^{c,G,\Psi}(s,\eta_s) \, ds \right| \right) \right] \, . \tag{3.22}$$

Since $e^{|x|} \leq e^x + e^{-x}$ and

$$\varlimsup N^{-1} \log\{a_N + b_N\} \leq \max\{\varlimsup N^{-1} \log a_N \, , \, \varlimsup N^{-1} \log b_N\}, \tag{3.23}$$

we may remove the absolute value in the third term of (3.22), provided our estimates remain in force if we replace G by $-G$. Denote by

$$(\mathcal{L}_N)^s = \frac{1}{2}(\mathcal{L}_N + \mathcal{L}_N^\star)$$

where $\mathcal{L}_N^\star$ is the adjoint of $\mathcal{L}_N$ in $L^2(\nu_N^{\rho_c(\cdot)})$. By the Feynman-Kac formula,

$$\frac{1}{N} \log \mathbb{P}_{\nu_N^{\rho_c(\cdot)}}^{\beta,N} \left[exp \left(aN \int_0^T \mathbb{V}_{N,\varepsilon}^{c,G,\Psi}(s,\eta_s)\,ds \right) \right] \leq \frac{1}{N} \int_0^T \lambda_{N,\varepsilon}(G_s)\,ds \,, \quad (3.24)$$

where $\lambda_{N,\varepsilon}(G_s)$ is the largest eigenvalue of $\{N^2(\mathcal{L}_N)^s + Na\mathbb{V}_{N,\varepsilon}^{c,G,\Psi}(G_s,\cdot)\}$. By the variational formula for the largest eigenvalue, for each $s \in [0,T]$,

$$\frac{1}{N}\lambda_{N,\varepsilon}(G_s) = \sup_f \left\{ \int a\mathbb{V}_{N,\varepsilon}^{c,G,\Psi}(G_s,\eta)f(\eta)\nu_N^{\rho_c(\cdot)}(d\eta) \;+\; N\langle \mathcal{L}_N\sqrt{f},\sqrt{f}\rangle_{\nu_N^{\rho_c(\cdot)}} \right\}. \tag{3.25}$$

In this formula the supremum is carried over all densities f with respect to $\nu_N^{\rho_c(\cdot)}$. By Lemma 3.3, since (3.7) gives a negative contribution, it is enough, to get the result, to choose c such that $c < \frac{\delta}{TK_0}$ and to show that, there exists $M > 0$ that depends only on G and c, such that, for all $a > 0$

$$\varlimsup_{\varepsilon \to 0} \varlimsup_{N \to \infty} \sup_f \left\{ \int a\mathbb{V}_{N,\varepsilon}^{c,G,\Psi}(G_s,\eta)f(\eta)\nu_N^{\rho_c(\cdot)}(d\eta) \;-\; N\mathcal{D}_{0,N}(\sqrt{f},\nu_N^{\rho_c(\cdot)}) \right\} \leq M \,.$$

We then let $a \uparrow \infty$. Notice that for N large enough the function $\mathbb{V}_{N,\varepsilon}^{c,G,\Psi}(G_s,\eta)$ depends on the configuration η only through the variables $\{\eta(x),\ x \in \Lambda_{(1-c)N}\}$. Since ρ_c is equal to γ_0, in Λ_c, we replace $\nu_N^{\rho_c(\cdot)}$ in the previous formula by $\nu_N^{\gamma_0}$ with respect to which the operator $\mathbb{L}_{0,N}$ is reversible. Therefore $\mathcal{D}_{0,N}(\cdot,\nu_N^{\gamma_0})$ is the Dirichlet form associated to the generator $\mathbb{L}_{0,N}$. Since the Dirichlet form is convex, it remains to show that

$$\varlimsup_{\varepsilon \to 0} \varlimsup_{N \to \infty} \sup_f \left\{ \int a\mathbb{V}_{N,\varepsilon}^{c,G,\Psi}(G_s,\eta)f(\eta)\nu_N^{\gamma_0}(d\eta) \;-\; N\mathcal{D}_{0,N}(\sqrt{f},\nu_N^{\gamma_0(\cdot)}) \right\} = 0,$$

for any $a > 0$. This follows from the usual one block and two blocks estimates (cf. Chap 5 of [15]). $\qquad\square$

For $x = \pm N$, a configuration η and $\ell \geq 1$, let

$$W_N^{\pm,\ell}(\eta) = \left| \eta^\ell(\pm N) - \rho_\pm \right|, \tag{3.26}$$

where, see (3.16), $\eta^\ell(N) = \frac{1}{\ell+1}\{\eta(N-\ell)+\cdots+\eta(N)\}$ and $\eta^\ell(-N) = \frac{1}{\ell+1}\{\eta(-N+\ell)+\cdots+\eta(-N)\}$.

Proposition 3.7. *Fix a sequence $\{\mu_N : N \geq 1\}$ of probability measures on $\mathcal{S}_N$. For every $\delta > 0$,*

$$\varlimsup_{\ell \to \infty} \varlimsup_{N \to \infty} \frac{1}{N} \log \mathbb{P}_{\mu_N}^{\beta,N}\left[\int_0^T W_N^{\pm,\ell}(\eta_s)ds > \delta \right] = -\infty \,.$$

Proof. Consider first the limit with the term $W_N^{+,\ell}$. Fix a smooth function $\gamma : \overline{\Lambda} \to (0,1)$ such that $\gamma(-1) = \rho_-$, and $\gamma(u) = \rho_+$ for $u \in [0,1]$. Since the Radon-Nikodym derivative $\frac{d\mu_N}{d\nu_N^{\gamma(\cdot)}}$ is bounded by $\exp(NK_1)$ for some positive constant K_1, it is enough to show that

$$\varlimsup_{\ell \to \infty} \varlimsup_{N \to \infty} \frac{1}{N} \log \mathbb{P}_{\nu_N^{\gamma(\cdot)}}^{\beta,N}\left[\int_0^T W_N^{+,\ell}(\eta_s)ds > \delta \right] = -\infty \,.$$

We follow the same steps as in Proposition 3.6. Applying Chebyshev exponential inequality and Feynman-Kac formula, the expression in the last limit is bounded for all $a > 0$ by

$$-a\delta \; + \; \frac{T}{N} \, \widetilde{\lambda}_{N,\varepsilon}(a) \; , \tag{3.27}$$

where for all $a > 0$, $\frac{1}{N}\widetilde{\lambda}_{N,\varepsilon}(a)$ is the largest eigenvalue of the $\nu_N^{\gamma(\cdot)}$-reversible operator

$$f \to N(\mathcal{L}_N)^s(f) + a\Big(W_N^{+,\ell}\Big)f.$$

Here $(\mathcal{L}_N)^s$ is the symmetric part of the operator $\mathcal{L}_N$ in $L^2(\nu_N^{\gamma(\cdot)})$. By the variational formula for the largest eigenvalue, we have

$$\frac{1}{N}\widetilde{\lambda}_{N,\varepsilon}(a) \; = \; \sup_f \Big\{ \int aW_N^{+,\ell}(\eta)f(\eta)\nu_N^{\gamma(\cdot)}(d\eta)$$

$$+ \; N <\mathcal{L}_N \sqrt{f}, \sqrt{f} >_{\nu_N^{\gamma(\cdot)}} \Big\} \; .$$

In this formula the supremum is carried over all densities f with respect to $\nu_N^{\gamma(\cdot)}$. By Lemma 3.3, we just need to show that, there exists $M > 0$, such that, for all $a > 0$

$$\varlimsup_{\ell \to \infty} \varlimsup_{N \to \infty} \sup_f \Big\{ \int aW_N^{+,\ell}(\eta)f(\eta)\nu_N^{\gamma(\cdot)}(d\eta)$$

$$- \; N\mathcal{D}_{0,N}(\sqrt{f}, \nu_N^{\gamma(\cdot)}) - N\mathcal{D}_{+,N}(\sqrt{f}, \nu_N^{\gamma(\cdot)}) \Big\} \; \leq \; M \; .$$

Recall that the profile γ is constant equal to ρ_+ on $[0, 1]$. Since $W_N^{+,\ell}(\eta)$ depends only on coordinates in a box $\Lambda_\ell(N)$, we replace $\nu_N^{\gamma(\cdot)}$ in the previous formula by $\nu_N^{\rho_+}$. On the other hand, $\nu_N^{\rho_+}$ is reversible for $\mathbb{L}_{0,N} + \mathbb{L}_{+,N}$ and therefore $\mathcal{D}_{0,N}(\cdot, \nu_N^{\rho_+}) + \mathcal{D}_{+,N}(\cdot, \nu_N^{\rho_+})$ is the Dirichlet form associated to the generator $\mathbb{L}_{0,N} + \mathbb{L}_{+,N}$. Since the Dirichlet form is convex, it remains to show that

$$\varlimsup_{\ell \to \infty} \varlimsup_{N \to \infty} \sup_f \Big\{ \int aW_N^{+,\ell}(\eta)f(\eta)\nu_N^{\rho_+}(d\eta)$$

$$- \; N\mathcal{D}_{0,N}(\sqrt{f}, \nu_N^{\rho_+}) - N\mathcal{D}_{+,N}(\sqrt{f}, \nu_N^{\rho_+}) \Big\} \; = \; 0.$$

for any $a > 0$. This follows from the law of large numbers by applying the same device used in the proof of the one block and two blocks estimates, (cf. Chap 5 of [15], and Lemma 3.12, Lemma 3.13 in [20]). $\qquad\square$

3.3. **Energy estimate.** We prove in this subsection an energy estimate which is one of the main ingredient in the proof of large deviations and hydrodynamic limit. It allows to prove Lemma 4.3 and to exclude paths with infinite energy in the large deviation regime. For $\delta > 0$, $G \in C_c^\infty([0, T] \times \Lambda)$ define

$$\tilde{\mathcal{Q}}_G^\delta(\pi) \; = \; \int_0^T dt\langle \pi_t, \nabla G_t \rangle - \delta \int_0^T dt\langle \sigma(\pi_t)G_t, G_t \rangle \, , \tag{3.28}$$

$$\tilde{\mathcal{Q}}^\delta(\pi) \; = \; \sup_{G \in C_c^\infty([0,T]\times\Lambda)} \Big\{ \tilde{\mathcal{Q}}_G^\delta(\pi) \Big\} \, . \tag{3.29}$$

Notice that

$$\tilde{\mathcal{Q}}^\delta(\pi) = \frac{1}{2\delta}\mathcal{Q}(\pi),$$

where $\mathcal{Q}(\cdot)$ is defined in (2.13).

For a function m in $\mathcal{M}$, let $m^\varepsilon : \Lambda \to \mathbb{R}_+$ be given by

$$m^\varepsilon(u) \;=\; \frac{1}{2\varepsilon} \int_{[u-\varepsilon, u+\varepsilon]\cap\Lambda} m(v)\, dv \;.$$

When $u \in [-1+\varepsilon, 1-\varepsilon]$, $m^\varepsilon(u) = (m * \iota_\varepsilon)(u)$, where ι_ε is the approximation of the identity defined by

$$\iota_\varepsilon(u) \;=\; \frac{1}{2\varepsilon}\, \mathbb{1}\{[-\varepsilon, \varepsilon]\}(u) \;.$$

Lemma 3.8. *There exists a positive constant C_1 depending only on $\rho_\pm$ so that for any given $\delta_0 > 0$, for any δ, $0 \le \delta \le \delta_0$, for any sequence $\{\eta^N \in \mathcal{S}_N : N \ge 1\}$ and for any $G \in \mathcal{C}_c^\infty([0,T] \times \Lambda)$, we have*

$$\varlimsup_{\varepsilon\to 0}\; \varlimsup_{N\to\infty}\; \frac{1}{N}\log \mathbb{P}^{\beta,N}_{\eta^N}\Big[\exp\big(\delta\, N\, \tilde{\mathcal{Q}}^{\delta_0}_G(\pi^N * \iota_\varepsilon)\big)\Big] \;\le\; C_1(T+1)\;.$$

Proof. Assume without loss of generality that ε is small enough so that the support of $G(\cdot,\cdot)$ is contained in $[0,T] \times [-1+\varepsilon, 1-\varepsilon]$. Let $\theta : \overline{\Lambda} \to (0,1)$ be a smooth function such that $\theta(\mp 1) = \rho_\mp$. Since $\nu^{\theta(\cdot)}_N(\eta^N) \ge \exp\{-C_1'N\}$ for some finite constant C_1' depending only on θ, it is enough to prove the lemma with $\mathbb{P}^{\beta,N}_{\nu^{\theta(\cdot)}_N}$ in place of $\mathbb{P}^{\beta,N}_{\eta^N}$.

Set $\Psi_1(\eta) = [\eta(1) - \eta(0)]^2$ and note that $\widetilde{\Psi}_1(a) = E^{\nu^a}[\Psi_1] = \sigma(a) = 2a(1-a)$, where ν^a is the Bernoulli measure with parameter $a \in [0,1]$. Denote $B^{G,\Psi_1}_{N,\varepsilon,\delta_0}$ the set of trajectories $(\eta_t)_{t\in[0,T]}$ so that

$$B^{G,\Psi_1}_{N,\varepsilon,\delta_0} \;=\; \Big\{ \eta_\cdot \in D([0,T], \mathcal{S}_N) : \Big|\int_0^T V^{G^2, \Psi_1}_{N,\varepsilon}(t, \eta_t)dt\Big| \le \frac{1}{\delta_0^2}\Big\}\;,$$

where $V^{G^2, \Psi_1}_{N,\varepsilon}$ is defined in (3.18). By (3.23) and Proposition 3.6, it is enough to show

$$\varlimsup_{\varepsilon\to 0}\; \varlimsup_{N\to\infty}\; \frac{1}{N}\log \mathbb{P}^{\beta,N}_{\nu^{\theta(\cdot)}_N}\Big[e^{\big(\delta\, N \tilde{\mathcal{Q}}^{\delta_0}_G(\pi^N * \iota_\varepsilon)\big)}\, \mathbb{1}\{B^{G,\Psi_1}_{N,\varepsilon,\delta_0}\}\Big] \;\le\; C_1(T+1)\;.$$

Recalling the definition $\tilde{\mathcal{Q}}^\delta_G$, see (3.28), we have

$$\int_0^T dt\langle\pi^N_t * \iota_\varepsilon, \nabla G\rangle = \int_0^T dt\, \sum_{x=-N+1}^{N-1}\{\eta_t(x) - \eta_t(x+1)\}\, G_t(x/N) + O_G(\varepsilon).$$

Further on the set $B^{G,\Psi_1}_{N,\varepsilon,\delta_0}$

$$\delta_0 \int_0^T dt\langle\sigma(\pi^N_t * \iota_\varepsilon), G^2_t\rangle \ge \delta_0 \int_0^T dt\, \frac{1}{N}\sum_{x=-N+1}^{N-1} G^2_t(x/N)\, \tau_x \Psi_1(\eta_t)$$

$$-\,\delta_0 O_{G^2}(N,\varepsilon) - \frac{1}{\delta_0}\;,$$

where $O_G(\varepsilon)$ is absolutely bounded by a constant which vanishes as $\varepsilon \downarrow 0$ and $O_{G^2}(N,\varepsilon)$ is is absolutely bounded by a constant which vanishes as $N \uparrow \infty$. Therefore to conclude the proof it is enough to show that

$$\varlimsup_{N\to\infty}\; \frac{1}{N}\log \mathbb{P}^{\beta,N}_{\nu^{\theta(\cdot)}_N}\Big[\exp\Big(N\int_0^T dt\, V^\delta_G(t,\eta_t)\Big)\Big] \;\le\; C_1 T \qquad (3.30)$$

for any $\delta \leq \delta_0$, where

$$V_G^\delta(t, \eta) = \delta \sum_{x=-N+1}^{N-1} G_t(x/N)[\eta(x) - \eta(x+1)] - \frac{\delta^2}{N} \sum_{x=-N+1}^{N-1} G_t^2(x/N)\, \tau_x \Psi_1(\eta).$$

Now, observe that $V_G^\delta = V_{\delta G}^1$. Therefore, to prove the lemma, we need to show that for any smooth function G,

$$\varlimsup_{N\to\infty} \frac{1}{N} \log \mathbb{P}_{\nu_N^{\theta(\cdot)}}^{\beta,N} \left[\exp\left(N \int_0^T dt\, V_G^1(t, \eta_t) \right) \right] \leq C_1 T \qquad (3.31)$$

for some constant C_1 that not depends on G. By Feynman-Kac formula and the same arguments used in the proof of Proposition 3.6, the expression of the limit in the right hand side of (3.31) is bounded above by

$$\int_0^T dt\, \sup_f \left\{ \int V_G^1(t, \eta) f^2(\eta) \nu_N^{\theta(\cdot)}(d\eta) \;+\; N\langle \mathcal{L}_N f, f \rangle_{\nu_N^{\theta(\cdot)}} \right\},$$

where the supremum is over all functions f in $L^2(\nu_N^{\theta(\cdot)})$ such that $\langle f, f \rangle_{\nu_N^{\theta(\cdot)}} = 1$. By Lemma 3.3, we replace $N\langle \mathcal{L}_N f, f \rangle_{\nu_N^{\theta(\cdot)}}$ by $-N(1-b)\mathcal{D}_{0,N}\big(f, \nu_N^{\theta(\cdot)}\big) + \frac{C_0}{b}$, where $b \in (0,1)$ is arbitrarily chosen and C_0 is a constant depending only on $\rho_\pm$. It remains, therefore, to show that

$$\varlimsup_{N\to\infty} \int_0^T dt\, \sup_f \left\{ \int V_G^1(t, \eta) f^2(\eta) \nu_N^{\theta(\cdot)}(d\eta) \;-\; N(1-b)\mathcal{D}_{0,N}\big(f, \nu_N^{\theta(\cdot)}\big) \right\} \;\leq\; C_1 T. \qquad (3.32)$$

We split

$$\int V_G^1(t, \eta) f^2(\eta) \nu_N^{\theta(\cdot)}(d\eta) = I_1 - I_2,$$

where

$$I_1 = \sum_{x=-N+1}^{N-1} G_t(x/N) \int \{\eta(x) - \eta(x+1)\}\, f^2(\eta)\, d\nu_N^{\theta(\cdot)}(\eta),$$

$$I_2 = \frac{1}{N} \sum_{x=-N+1}^{N-1} G_t^2(x/N) \int f^2(\eta)\, \tau_x \Psi_1(\eta)\, d\nu_N^{\theta(\cdot)}(\eta).$$

We estimate I_1 in term of I_2 and $\mathcal{D}_{0,N}\big(f, \nu_N^{\theta(\cdot)}\big)$. By changing variables $\eta' = \eta^{x,x+1}$, we have that

$$I_1 = \frac{1}{2} \sum_{x=-N+1}^{N-1} G_t(x/N) \int \{\eta(x) - \eta(x+1)\} \{f^2(\eta) - f^2(\eta^{x,x+1})\}\, d\nu_N^{\theta(\cdot)}(\eta)$$

$$+ \frac{1}{2} \sum_{x=-N+1}^{N-1} G_t(x/N) \int \{\eta(x) - \eta(x+1)\}\, R_N(x, x+1; \theta, \eta)\, f^2(\eta)\, d\nu_N^{\theta(\cdot)}(\eta)\,,$$

$$(3.33)$$

where $R_N(x, x+1; \theta, \eta)$ is defined in (3.10). For the first term of (3.33), by inequality (3.12) and Taylor expansion, we have

$$
\frac{1}{2} \sum_{x=-N+1}^{N-1} G_t(x/N) \int \{\eta(x) - \eta(x+1)\} \{f^2(\eta) - f^2(\eta^{x,x+1})\} \, d\nu_N^{\theta(\cdot)}(\eta)
$$

$$
\leq \frac{aN}{4} \mathcal{D}_{0,N}\big(f, \nu_N^{\theta(\cdot)}\big)
$$

$$
+ \frac{1}{4aN} \sum_{x=-N+1}^{N-1} G_t^2(x/N) \int \tau_x \Psi_1(\eta) \left[f(\eta) + f(\eta^{x,x+1})\right]^2 d\nu_N^{\theta(\cdot)}(\eta) \quad (3.34)
$$

$$
\leq \frac{aN}{4} \mathcal{D}_{0,N}\big(f, \nu_N^{\theta(\cdot)}\big) + \frac{1}{aN} C(G)
$$

$$
+ \frac{1}{aN} \sum_{x=-N+1}^{N-1} G_t^2(x/N) \int \tau_x \Psi_1(\eta) \, f^2(\eta) \, d\nu_N^{\theta(\cdot)}(\eta)
$$

where $C(G)$ is some constant that depends on G. For the second term of (3.33), by (3.12) and Taylor expanding R_N we have that

$$
\left| \frac{1}{2} \sum_{x=-N+1}^{N-2} G_t(x/N) \int \{\eta(x) - \eta(x+1)\} R_N(x, x+1; \theta, \eta) \, f^2(\eta) \, d\nu_N^{\theta(\cdot)}(\eta) \right|
$$

$$
\leq C\, a + \frac{1}{N\, a} \sum_{x=-N+1}^{N-1} \int G_t(x/N)^2 \, \tau_x \Psi_1(\eta) \, f^2(\eta) \, d\nu_N^{\theta(\cdot)}(\eta) \,,
$$

$$
= C\, a + \frac{1}{a} I_2
$$

$$
(3.35)
$$

for all $a > 0$, for some positive constant C depending only on $\rho_\pm$. Taking into account (3.33), (3.35) and (3.34) we have

$$
I_1 \leq \frac{2}{a} I_2 + \frac{aN}{4} \mathcal{D}_{0,N}\big(f, \nu_N^{\theta(\cdot)}\big) + C\, a + \frac{1}{aN} C(G) \tag{3.36}
$$

We conclude the proof, by taking $a = 2$ and $b = \frac{1}{2}$ in (3.32). $\square$

The following corollary allows to show Lemma 4.3.

Corollary 3.9. *Fix a sequence $\{G_j : j \geq 1\} \subset \mathcal{C}_c^\infty([0,T] \times \Lambda)$, $\delta_0 > 0$ and a sequence $\{\eta^N \in \mathcal{S}_N : N \geq 1\}$ of configurations. There exists a positive constant C_1 depending only on the values $\rho_\mp$, such that for any $0 < \delta \leq \delta_0$ and any $k \geq 1$*

$$
\overline{\lim_{\varepsilon \to 0}} \, \overline{\lim_{N \to \infty}} \, \frac{1}{N} \log \mathbb{P}_{\eta^N}^{\beta, N} \Big[\exp \big(\delta N \sup_{1 \leq j \leq k} \tilde{\mathcal{Q}}_{G_j}^{\delta_0}(\pi^N * \iota_\varepsilon) \big) \Big] \leq C_1(T+1) \,. \tag{3.37}
$$

Proof. From (3.23), the limit in (3.37) is bounded above by

$$
\max_{1 \leq j \leq k} \Big\{ \overline{\lim_{\varepsilon \to 0}} \, \overline{\lim_{N \to \infty}} \, \frac{1}{N} \log \mathbb{P}_{\eta^N}^{\beta, N} \Big[\exp \big(\delta \, N \tilde{\mathcal{Q}}_{G_j}^{\delta_0}(\pi^N * \iota_\varepsilon) \big) \Big] \Big\}.
$$

By Lemma 3.8 the thesis follows. $\square$

4. Hydrodynamic and hydrostatic limits

We prove in this section the hydrodynamic and hydrostatic limit for our system. The proof is based on the method introduced in [11] for the hydrodynamic limit and in [7] for hydrostatic, taking into account, as explained in the introduction, the lack of comparison and maximum principle of (2.9).

4.1. **The steps to prove Theorem 2.1.** Following [11] we divide the proof of the hydrodynamic behavior in three steps: tightness of the measures $(Q_{\mu_N}^{\beta,N})$, an energy estimate to provide the needed regularity for functions in the support of any limit point of the sequence $(Q_{\mu_N}^{\beta,N})$, and identification of the support of limit point of the sequence $(Q_{\mu_N}^{\beta,N})$ as weak solution of (2.9). We then refer to [16], Chapter 4, that present arguments, by now standard, to deduce the hydrodynamic behavior of the empirical measures from the preceding results and the uniqueness of the weak solution to equation (2.9).

Lemma 4.1. (Tightness) *The sequence $(Q_{\mu_N}^{\beta,N})$ is tight and all its limit points $Q^{\beta,*}$ are concentrated on the following set*

$$Q^{\beta,*}\Big\{\pi \,:\, 0 \leq \pi_t(u) \leq 1, \quad t \in [0,T], \quad u \in [-1,1]\Big\} = 1 \,. \qquad (4.1)$$

We then show that $Q^{\beta,*}$ is supported on densities ρ that satisfy (2.9) in the weak sense.

We start defining for $G \in \mathcal{C}_0^{1,2}([0,T] \times \Lambda)$ and $\varepsilon > 0$

$$
\begin{aligned}
\mathcal{B}_\varepsilon^{G,N} &= \int_\Lambda G_T(u)\pi^N(\eta_T)(u)du - \int_\Lambda G_0(u)\pi^N(\eta_0)(u)du \\
&\quad - \int_0^T \int_\Lambda \partial_s G_s(u)\pi^N(\eta_s)(u)duds - \int_0^T \int_\Lambda \Delta G_s(u)\pi^N(\eta_s)(u)duds \\
&\quad - \frac{\beta}{2}\frac{1}{N}\sum_{x \in \Lambda_N}\int_0^T (\nabla G_s)(x/N)\Big\{\sigma\big(\eta_s^{[\varepsilon N]}(x)\big)\nabla^N\big(J^{\mathrm{neum}} \star \pi^N(\eta_s)\big)(x/N)\Big\}ds \\
&\quad + \int_0^T dt\,[\rho_+(\nabla G_t)(+1) - \rho_-(\nabla G_t)(-1)] \,,
\end{aligned}
$$
$$(4.2)$$

where $\eta_s^{[\varepsilon N]}(x)$ is the local mean defined in (3.16) and $\nabla^N G_s(x/N)$ stands for the discrete gradient of $G_s(x/N)$ defined in (3.2).

Proposition 4.2. (Identification of the limit equation). *For any function G in $\mathcal{C}_0^{1,2}([0,T] \times \Lambda)$ and any $\delta > 0$ we have*

$$\varlimsup_{\varepsilon \to 0}\varlimsup_{\mathbb{N} \to \infty} \mathbb{P}_{\mu_N}^{\beta,N}\big(\big|\mathcal{B}_\varepsilon^{G,N}\big| \geq \delta\big) = 0 \,. \qquad (4.3)$$

The last statement is an energy estimate. Every limit point $Q^{\beta,*}$ of the sequence $(Q_{\mu_N}^{\beta,N})$ is concentrated on paths whose densities $\rho \in L^2\big(0,T;H^1(\Lambda)\big)$.

Lemma 4.3. (Energy estimate) *Let $Q^{\beta,*}$ be a limit point of the sequence $(Q_{\mu_N}^{\beta,N})$. Then,*

$$Q^{\beta,*}\Big[L^2\big(0,T;H^1(\Lambda)\big)\Big] = 1 \,. \qquad (4.4)$$

4.2. **Proof of Proposition 4.2.** Let $Q^{\beta,*}$ be a limit point of the sequence $(Q^{\beta,N}_{\mu^N})$ and assume, without loss of generality, that $Q^{\beta,N}_{\mu^N}$ converges to $Q^{\beta,*}$. Fix a function G in $\mathcal{C}^{1,2}_0([0,T] \times \Lambda)$. Consider the $\mathbb{P}^{\beta,N}_{\mu^N}$ martingales with respect to the natural filtration associated with $(\eta_t)_{t \in [0,T]}$, $M^G_t \equiv M^{G,N,\beta}_t$ and $\mathcal{N}^G_t \equiv \mathcal{N}^{G,N,\beta}_t$, $t \in [0,T]$, defined by

$$M^G_t = < \pi^N_t, G_t > - < \pi^N_0, G_0 > - \int_0^t \left(< \pi^N_s, \partial_s G_s > + N^2 \mathcal{L}^\beta_N < \pi^N_s, G_s > \right) ds,$$

$$\mathcal{N}^G_t = \left(M^G_t \right)^2$$
$$- \int_0^t \left\{ N^2 \mathcal{L}^\beta_N \left(< \pi^N_s, G_s > \right)^2 - 2 < \pi^N_s, G_s > N^2 \mathcal{L}^\beta_N < \pi^N_s, G_s > \right\} ds.$$
$$(4.5)$$

A computation of the integral term of $\mathcal{N}^G_t$ shows that the expectation of the quadratic variation of M^G_t vanishes as $N \uparrow 0$. Therefore, by Doob's inequality, for every $\delta > 0$,

$$\lim_{N \to \infty} \mathbb{P}^{\beta,N}_{\mu^N} \left[\sup_{0 \le t \le T} |M^G_t| > \delta \right] = 0. \qquad (4.6)$$

Since for any $s \in [0,T]$ the function G_s vanishes at the boundary of Λ, a summation by parts permits to rewrite the integral term of M^G_t as

$$\int_0^t < \pi^N_s, \partial_s G_s > ds$$
$$- \int_0^t N \left\{ \sum_{x=-N+1}^{N-1} (\nabla^N G_s)(x/N) C^\beta_N(x, x+1, \eta_s)\left(\nabla_{x,x+1} \eta_s(x) \right) \right\} ds,$$

where ∇^N is defined in (3.2).

From Lemma 3.2, a summation by parts and Taylor expansion permit to rewrite the last expression as

$$O(N) + \int_0^t < \pi^N_s, \partial_s G_s > ds + \int_0^t < \pi^N_s, \Delta G_s > ds$$

$$+ \int_0^t \left\{ - \nabla G_s(1) \eta_s(N) + \nabla G_s(-1) \eta_s(-N) \right\} ds$$

$$+ \frac{\beta}{2N} \int_0^t \left\{ \sum_{x \in \Lambda_N} (\nabla G_s)(x/N)\left(\nabla_{x,x+1} \eta_s(x) \right)^2 \nabla^N (J^{\mathrm{neum}} \star \pi^N(\eta_s))(x/N) \right\} ds.$$

Next, we use the replacement lemma stated in Proposition 3.6 and Proposition 3.7. We obtain that the integral term of the martingal M^G_t can be replaced by

$$\int_0^t < \pi^N_s, \partial_s G_s > ds + \int_0^t < \pi^N_s, \Delta G_s > ds$$

$$+ \int_0^t \left\{ - \nabla G_s(1) \rho_+ + \nabla G_s(-1) \rho_- \right\} ds$$

$$+ \frac{\beta}{2} \int_0^t \frac{1}{N} \left\{ \sum_{x \in \Lambda_N} (\nabla G_s)(x/N) \sigma \left(\eta^{\varepsilon N}(x) \right) \nabla^N (J^{\mathrm{neum}} \star \pi^N(\eta_s))(x/N) \right\} ds.$$

This concludes the proof of the lemma. $\qquad \square$

4.3. **Steps to prove Theorem 2.3.** Let $\mu_N^{stat} = \mu_N^{stat}(\beta, \rho_-, \rho_+)$ be the unique stationary measure of the irreducible Markov process $(\eta_t)_{t \geq 0}$ with generator $\mathcal{L}_N$. From Tchebyshev's unequality, we need to show that

$$\lim_{N \to \infty} E^{\mu_N^{stat}} \left[\left| \langle \pi^N, G \rangle - \langle \bar{\rho}, G \rangle \right| \right] = 0 , \tag{4.7}$$

where $E^{\mu_N^{stat}}$ stands for the expectation with respect to the stationary measure μ_N^{stat}. It is enough to prove that any subsequence of the sequence of real numbers in the limit (4.7) vanishes. Without loss of generality we consider a sequence in (4.7) as a subsequence along which the limit exists.

Denote by $Q^{\beta,N,stat} := Q^{\beta,N}_{\mu_N^{stat}}$ the probability measure on the Skorohod space $D([0,T]; \mathcal{M})$ induced by the Markov process $(\pi_t^N) \equiv (\pi_N(\eta_t))$, when the initial measure is μ_N^{stat}.

By the first part of Theorem 2.1 we have that all limit points of the sequence $Q^{\beta,N,stat}$ are concentrated on $\mathcal{A}_{[0,T]}$ for any $T > 0$, i.e all its limit points are concentrated on the weak solutions of the hydrodynamic equation for some unknown initial profile.

Let $(Q^{\beta,N_k,stat})$ be a sub-sequence converging to a limit point which we denote by $Q^{\beta,*,stat}$. Note that different subsequences might have different limit points. Let β small enough and denote by $\bar{\rho}$ the unique stationary solution of (2.9), see Theorem 6.2. By stationarity we have for any $\delta > 0$,

$$E^{\mu_{N_k}^{stat}} \left[\left| \langle \pi^N, G \rangle - \langle \bar{\rho}, G \rangle \right| \right] = \mathbb{E}^{\beta,N_k}_{\mu_{N_k}^{stat}} \left[\left| \langle \pi_T^N, G \rangle - \langle \bar{\rho}, G \rangle \right| \right] .$$

Since the integrand is bounded we have the following:

$$\begin{aligned}
\lim_{k \to \infty} \mathbb{E}^{\beta,N_k}_{\mu_{N_k}^{stat}} &\left\{ \left| \langle \pi_T^N, G \rangle - \langle \bar{\rho}, G \rangle \right| \right\} \\
&= E^{Q^{\beta,*,stat}} \left\{ \left(\left| \langle \rho_T, G \rangle - \langle \bar{\rho}, G \rangle \right| \mathbb{1}_{\{\mathcal{A}_{[0,T]}\}}(\rho) \right) \right\} \\
&\leq \|G\|_2 \, E^{Q^{\beta,*,stat}} \left\{ \|\rho_T - \bar{\rho}\|_2 \, \mathbb{1}_{\{\mathcal{A}_{[0,T]}\}}(\rho) \right\} \\
&\leq \|G\|_2 e^{-c(\beta)T}
\end{aligned} \tag{4.8}$$

by Theorem 6.2 and $\|v\|_2$ denotes the $L^2(\Lambda)$ norm of v. Then letting $T \to \infty$ we show the thesis. $\qquad \square$

5. Large deviations

In this section we prove some properties of the rate function and we present the main steps to derive the large deviations results.

Let $L^2(\Lambda)$ be the Hilbert space of functions $G : \Lambda \to \mathbb{R}$ such that $\int_\Lambda |G(u)|^2 du < \infty$ equipped with the inner product

$$\langle G, J \rangle_2 = \int_\Lambda G(u) \, J(u) \, du .$$

The norm of $L^2(\Lambda)$ is denoted by $\| \cdot \|_2$.

Let $H^1(\Lambda)$ be the Sobolev space of functions G with generalized derivatives ∇G in $L^2(\Lambda)$. $H^1(\Lambda)$ endowed with the scalar product $\langle \cdot, \cdot \rangle_{H^1}$, defined by

$$\langle G, J \rangle_{H^1} = \langle G, J \rangle_2 + \langle \nabla G, \nabla J \rangle_2 ,$$

is a Hilbert space. The corresponding norm is denoted by $\|\cdot\|_{H^1}$. Denote by $H_0^1(\Lambda)$ the closure of $C_c^\infty(\Lambda)$ (the set of infinitely differentiable functions from Λ to $\mathbb{R}$ with compact support in Λ) in $H^1(\Lambda)$. Denote by $H^{-1}(\Lambda)$ the Hilbert space, dual of $H_0^1(\Lambda)$, equipped with the norm $\|\cdot\|_{-1}$

$$\|v\|_{-1}^2 = \sup_{G\in\mathcal{C}_c^\infty(\Lambda)} \left\{ 2\langle v, G\rangle_{-1,1} - \int_\Lambda |\nabla G(u)|^2 du \right\},$$

where $\langle v, G\rangle_{-1,1}$ stands for the duality between H_0^1 and H^{-1}. Fix $T > 0$. For a Banach space $(\mathbb{B}, \|\cdot\|_\mathbb{B})$ we denote by $L^2([0,T],\mathbb{B})$ the Banach space of measurable functions $U : [0,T] \to \mathbb{B}$ for which

$$\|U\|_{L^2([0,T],\mathbb{B})}^2 \;=\; \int_0^T \|U_t\|_\mathbb{B}^2\, dt \;<\; \infty$$

holds.

5.1. **Properties of the rate function.** Denote

$$\mathcal{B}_\gamma^{\rho\pm} = \{\pi \in D([0,T],\mathcal{M}) : \pi_0(\cdot) = \gamma(\cdot); \quad \pi_t(\pm 1) = \rho_\pm, t \in (0,T]\}.$$

Lemma 5.1. *Let π be a trajectory in $D([0,T],\mathcal{M})$ such that $\hat{I}_T^\beta(\pi|\gamma) < \infty$. Then π belongs to $\mathcal{B}_\gamma^{\rho\pm}$.*

The proof is similar to the one of Lemma 3.5 in [2]. To prove the lower-semicontinuity of the rate function, we need the next results

Lemma 5.2. *For any $\beta \geq 0$, there exists a constant $C_0 = C_0(\beta)$ such that*

$$\int_0^T \|\partial_t \pi_t\|_{-1}^2 \;\leq\; C_0\Big\{ I_T^\beta(\pi|\gamma) + \mathcal{Q}(\pi) \Big\}, \quad \mathcal{Q}(\pi) \;\leq\; C_0\big\{1 + I_T^\beta(\pi|\gamma)\big\}$$

for all π in $D([0,T],\mathcal{M})$.

Proof. The proof is the same as in Proposition 4.3. [21] or Theorem 3.3. in [19], or Lemma 4.9. in [4] $\square$

Lemma 5.3. *Let $\{\rho^n : n \geq 1\}$ be a sequence of functions in $L^2([0,T]\times\Lambda)$ such that*

$$\int_0^T dt\, \|\rho_t^n\|_{H^1}^2 \;+\; \int_0^T dt\, \|\partial_t \rho_t^n\|_{-1}^2 \;\leq\; C_0$$

for some finite constant C_0 and all $n \geq 1$. Suppose that the sequence ρ^n converges weakly in $L^2([0,T]\times[-1,1])$ to some ρ. Then, ρ^n converges strongly in $L^2([0,T]\times[-1,1])$ to ρ.

Proof. Recall that $H^1(\Lambda) \subset L^2(\Lambda) \subset H^{-1}(\Lambda)$. By [25, Theorem 21.A], the embedding $H^1(\Lambda) \subset L^2(\Lambda)$ is compact. Hence, by [23, Lemma 4, Theorem 5], the sequence $\{\rho^n : n \geq 1\}$ is relatively compact in $L^2(0,T;L^2(\Lambda))$. In particular, weak convergence of the sequence $\{\rho^n : n \geq 1\}$ implies strong convergence. $\square$

Theorem 5.4. *The functional $I_T^\beta(\cdot|\gamma)$ is lower semicontinuous and has compact level sets.*

Proof. Theorem 5.4 is proven applying Lemma 5.2 and Lemma 5.3. See Theorem 3.4. in [19] or Lemma 4.2. in [4]. $\square$

We provide an explicit representation for the rate function $I_T^\beta(\cdot|\gamma)$ when it is finite. For $\pi \in D([0,T], \mathcal{M})$, denote by $H_0^1(\sigma(\pi))$ the Hilbert space induced by $\mathcal{C}_0^{1,2}([0,T] \times [-1,1])$ endowed with the inner product $\langle \cdot, \cdot \rangle_{\sigma(\pi)}$ defined by

$$\langle F, G \rangle_{\sigma(\pi)} = \int_0^T dt \, \langle \sigma(\pi_t), \nabla F_t \cdot \nabla G_t \rangle.$$

Induced means that we first declare two functions F, G in $\mathcal{C}_0^{1,2}([0,T] \times [-1,1])$ to be equivalent if $\langle F - G, F - G \rangle_{\sigma(\pi)} = 0$ and then we complete the quotient space with respect to the inner product $\langle \cdot, \cdot \rangle_{\sigma(\pi)}$. The norm of $H_0^1(\sigma(\pi))$ is denoted by $\| \cdot \|_{\sigma(\pi)}$.

Lemma 5.5. *Take $\pi \in D([0,T], \mathcal{M})$ with $I_T^\beta(\pi|\gamma) < \infty$. Then, it is uniquely determined a function F in $H_0^1(\sigma(\pi))$ such that π is the weak solution of the following boundary value problem:*

$$\begin{cases} \partial_t \pi & = \Delta \pi - \nabla \cdot \left\{ \sigma(\pi) \left[\frac{\beta}{2} \nabla(J^{neum} \star \pi) + \nabla F \right] \right\} \quad in \quad \Lambda \times (0,T), \\ \pi_0(\cdot) & = \gamma(\cdot) \quad in \quad \Lambda, \\ \pi_t(\pm 1) & = \rho_\pm \quad for \ 0 \leq t \leq T. \end{cases} \tag{5.1}$$

Moreover,

$$I_T^\beta(\pi|\gamma) = \frac{1}{2} \|F\|_{\sigma(\pi)}^2 = \frac{1}{2} \int_0^T dt \, \langle \sigma(\pi_t) \nabla F_t \cdot \nabla F_t \rangle. \tag{5.2}$$

Proof. By assumption $I_T^\beta(\pi|\gamma) = \hat{I}_T^\beta(\pi|\gamma) < \infty$, defined in (2.12). Then one proceeds as in [16] with the only difference that because the boundary conditions the space is $H_0^1(\sigma(\pi))$. $\qquad \square$

Lemma 5.6. *Let $\rho \in L^2([0,T], H^1(\Lambda))$ be the weak solution of the boundary value problem (2.9) then*

$$I_T^\beta(\rho|\gamma) = \hat{I}_T^\beta(\rho|\gamma) = 0, \quad and \quad \mathcal{Q}(\rho) < \infty.$$

Further if $I_T^\beta(\rho|\gamma) = 0$, then $\rho \in L^2([0,T], H^1(\Lambda))$ is the weak solution of the boundary value problem (2.9).

Proof. We start showing that if $\rho \in L^2([0,T], H^1(\Lambda))$ is the weak solution of the boundary value problem (2.9) then $\mathcal{Q}(\rho) < \infty$. Take $F(\rho) = \rho \log \rho + (1-\rho) \log(1-\rho)$, for $\rho \in [0,1]$. Since $\int_\Lambda F(\rho_t(u)) du$ is a bounded quantity for all $t \in R^+$, we have that

$$\int_0^T dt \frac{d}{dt} \int_\Lambda F(\rho_t(u)) du = \int_\Lambda [F(\rho_T(u)) - F(\rho_0(u))] du.$$

Notice that

$$F'(\rho) = \log \frac{\rho}{(1-\rho)} \quad and \quad F''(\rho) = \frac{1}{\rho(1-\rho)} = \frac{1}{\chi(\rho)}$$

are not uniformly bounded for $\rho \in (0,1)$. Therefore we need some care to derive $F(\rho_t(u))$ with respect to t. We consider a sequence of smooth functions

$$F_n(\rho) = \left(1 + \frac{2}{n}\right)^{-1}(\rho + \frac{1}{n}) \log(\rho + \frac{1}{n}) + (1 - \rho + \frac{1}{n}) \log(1 - \rho + \frac{1}{n})$$

so that $\lim_{n\to\infty} F_n(a) = F(a)$. We have

$$\int_0^T dt \frac{d}{dt} \int_\Lambda F_n(\rho_t(u)) du = \int_0^T dt \int_\Lambda F_n'(\rho_t(u)) \frac{d}{dt} \rho_t(u). \qquad (5.3)$$

To avoid boundary terms, take a smooth function $b(\cdot)$ defined on a neighborhood of $[-1, 1]$ such that $b(\mp 1) = \rho^\mp$ and $0 < \rho^- \le b(\cdot) \le \rho^+ < 1$. Denote

$$U_n(t, u) = F_n'(\rho_t(u)) - F_n'(b(u)).$$

We have

$$\int_0^T dt \int_\Lambda F_n'(\rho_t(u)) \frac{d}{dt} \rho_t(u)$$

$$= \int_0^T dt \int_\Lambda U_n(\rho_t(u)) \frac{d}{dt} \rho_t(u) + \int_0^T dt \int_\Lambda F_n'(b(u)) \frac{d}{dt} \rho_t(u) \qquad (5.4)$$

$$= \int_0^T dt \int_\Lambda U_n(\rho_t(u)) \frac{d}{dt} \rho_t(u) du + \int_\Lambda F_n'(b(u)) [\rho_T(u) - \rho_0(u)] du.$$

Taking into account (5.3), (5.4) and $U_n \in L^2([0, T], H_0^1)$, we get

$$\int_0^T dt \frac{d}{dt} \int_\Lambda F_n(\rho_t(u)) du - \int_\Lambda F_n'(b(u)) \big[\rho_T(u) - \rho_)(u)\big] du$$

$$\qquad (5.5)$$

$$= -\int_0^T dt \int_\Lambda \nabla U_n(t, u) \left[\nabla \rho_t(u) - \beta \rho_t(u)(1 - \rho_t(u))(J^{\text{neum}} \star \nabla \rho_t)(u)\right].$$

Denote $\chi_n(a) = (a + \frac{1}{n})(1 + \frac{1}{n} - a)$. We have that

$$\nabla U_n(t, u) = \frac{\nabla \rho_t(u)}{\chi_n(\rho_t(u))} - \frac{\nabla b(u)}{\chi_n(b(u))}.$$

Taking this into account and collecting the above estimates, we obtain

$$\int_0^T dt \int_\Lambda \frac{(\nabla \rho_t(u))^2}{\chi_n(\rho_t(u))} du$$

$$\le -\int_\Lambda [F(\rho_T(u)) - F(\rho_0(u))] du + \int_\Lambda F_n'(b(u)) \big[\rho_T(u) - \rho_0(u)\big] du$$

$$+ \int_0^T dt \int_\Lambda \frac{\nabla b(u) \cdot \nabla \rho_t(u)}{\chi_n(b(u))} du \qquad (5.6)$$

$$+ \beta \int_0^T dt \int_\Lambda \nabla \rho_t(u) \frac{\chi(\rho_t(u))}{\chi_n(\rho_t(u))} (J^{\text{neum}} * \nabla \rho_t)(u) du$$

$$- \beta \int_0^T dt \int_\Lambda \nabla b(u) \frac{\chi(\rho_t(u))}{\chi_n(b(u))} (J^{\text{neum}} * \nabla \rho_t)(u) du.$$

Since $b(\cdot)$ is bounded below by a strictly positive constant and above by a constant strictly smaller than 1, and since

$$\int_0^T dt \int_\Lambda du \nabla \rho_t(u)(J^{\text{neum}} \star \nabla \rho_t)(u) \le C$$

for some constant C, we obtain, uniformly in n

$$\int_0^T dt \int_\Lambda \frac{(\nabla \rho_t(u))^2}{\chi_n(\rho_t(u))} du \le C'$$

for some finite constant C' which depends only on b and T. To conclude the proof it remains to apply Fatou's Lemma and recall the definition of $\mathcal{Q}(\rho)$ given in (2.14). We have shown that $\mathcal{Q}(\rho) < \infty$. By Lemma 5.5 we conclude that $I_T^\beta(\rho|\gamma) = 0$. Similar arguments allow to prove the second statement of the lemma. $\qquad\square$

5.2. **Comparison between $\hat{I}_T^\beta(\cdot|\gamma)$ and $\hat{I}_T^0(\cdot|\gamma)$.** Next, we compare the rate functional $\hat{I}_T^\beta(\cdot|\gamma)$ with the rate functional $\hat{I}_T^0(\cdot|\gamma)$ of the symmetric simple exclusion process (i.e. $\beta = 0$).

Lemma 5.7. *For $\pi \in D([0,T], \mathcal{M})$, with finite energy $\mathcal{Q}(\pi) < \infty$, we have*

$$
\begin{aligned}
\frac{1}{2}\hat{I}_T^0(\pi|\gamma) - \frac{\beta^2}{16}\int_0^T dt \int_\Lambda \left(\nabla\pi_t\right)^2 &\leq \hat{I}_T^\beta(\pi|\gamma) \\
&\leq 2\hat{I}_T^0(\pi|\gamma) + \frac{\beta^2}{8}\int_0^T dt \int_\Lambda \left(\nabla\pi_t\right)^2 .
\end{aligned}
\tag{5.7}
$$

Proof. Fix $\pi \in D([0,T], \mathcal{M})$ with finite energy and $G \in C_0^{1,2}([0,T] \times [-1,1])$. Recall from (2.11), (2.8) and (2.12) the definitions of $\mathcal{J}_G^\beta(\pi)$, ℓ_G^β and $I_T^0(\pi)$. By the inequality $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$ we obtain

$$
\begin{aligned}
\left|\ell_G^\beta(\pi,\gamma) - \ell_G^0(\pi,\gamma)\right| &= \left|\frac{\beta}{2}\int_0^T dt \int_\Lambda \sigma(\pi_t)(\nabla G_t) \cdot \nabla(J^{\mathrm{neum}} \star \pi_t)\right| \\
&\leq \frac{1}{4}\int_0^T dt \int_\Lambda \sigma(\pi_t)(\nabla G_t)^2 + \frac{\beta^2}{4}\int_0^T dt \int_\Lambda \sigma(\pi_t)[\nabla(J^{\mathrm{neum}} \star \pi_t)]^2 .
\end{aligned}
$$

Since for each u, $J^{\mathrm{neum}}(u,v)dv$ is a probability density on Λ and $\sigma(\cdot) \leq 1/2$, by Lemma 3.1, Jensen inequality and Fubini's Theorem,

$$
\begin{aligned}
\frac{\beta^2}{4}\int_0^T dt \int_\Lambda &\sigma(\pi_t)[\nabla(J^{\mathrm{neum}} \star \pi_t)]^2 \\
&\leq \frac{\beta^2}{8}\int_0^T dt \int_\Lambda J^{\mathrm{neum}} \star (\nabla\pi_t)^2 = \frac{\beta^2}{8}\int_0^T dt \int_\Lambda (\nabla\pi_t)^2 .
\end{aligned}
$$

Hence

$$
\begin{aligned}
\mathcal{J}_G^\beta(\pi) &\leq \ell_G^0(\pi) - \frac{1}{4}\int_0^T dt\langle\sigma(\pi_t),(\nabla G_t)^2\rangle + \frac{\beta^2}{8}\int_0^T dt \int_\Lambda (\nabla\pi_t)^2 \\
&\leq \frac{1}{2}\sup_{G \in C_0^{1,2}([0,T]\times\Lambda)} \left\{2\ell_G^0(\pi) - \frac{1}{2}\int_0^T dt\langle\sigma(\pi_t),(\nabla G_t)^2\rangle\right\} \\
&\quad + \frac{\beta^2}{8}\int_0^T dt \int_\Lambda (\nabla\pi_t)^2 \\
&= 2\hat{I}_T^0(\pi) + \frac{\beta^2}{8}\int_0^T dt \int_\Lambda (\nabla\pi_t)^2 .
\end{aligned}
$$

Now, it is enough to take the supremum over $G \in C_0^{1,2}([0,T] \times [-1,1])$ to obtain

$$
\widehat{I}_T^\beta(\pi) \leq 2\widehat{I}_T^0(\pi) + \frac{\beta^2}{8}\int_0^T dt \int_\Lambda (\nabla\pi_t)^2 .
$$

The inequality in the left hand side of the statement is obtained in the same way. $\qquad\square$

Setting $\beta = 0$ in the boundary value problem (2.9) one gets the following boundary value problem for the heat equation:

$$\begin{cases} \partial_t \rho & = & \Delta \rho & \text{in} & \Lambda \times (0,T), \\ \rho_0(\cdot) & = & \gamma(\cdot) & \text{in} & \Lambda, \\ \rho_t(\pm 1)| & = & \rho_\pm & \text{for } 0 \le t \le T. \end{cases} \tag{5.8}$$

Lemma 5.8. *Let $\rho^{(0)}$ be the solution of (5.8), we have*

$$\hat{I}_T^\beta(\rho^{(0)}|\gamma) \le \frac{\beta^2}{8} \int_0^T dt \langle \sigma(\rho_t^{(0)}), J^{neum} \star (\nabla \rho_t^{(0)})^2 \rangle$$
$$\le \frac{\beta^2}{16} \int_0^T dt \int_\Lambda |\nabla \rho_t^{(0)}|^2. \tag{5.9}$$

Proof. For any G in $C_0^{1,2}([0,T] \times [-1,1])$, see (2.11), we have

$$\mathcal{J}_G^\beta(\rho^{(0)}) = -\frac{\beta}{2} \int_0^T dt \left\langle \sigma(\rho_t^{(0)}) \nabla(J^{\text{neum}} \star \rho_t^{(0)}), \nabla G_t \right\rangle$$
$$- \frac{1}{2} \int_0^T dt \langle \sigma(\rho_t^{(0)}), (\nabla G_t)^2 \rangle \tag{5.10}$$
$$\le \frac{\beta^2}{8} \int_0^T dt \langle \sigma(\rho_t^{(0)}), [\nabla(J^{\text{neum}} \star \rho_t^{(0)})]^2 \rangle,$$

by inequality (3.12), taking $a = 1$. The solution of (5.8) belongs to $L^2([0,T], H^1(\Lambda))$ and its time derivative belongs to $L^2([0,T], H^{-1}(\Lambda))$. Therefore,

$$\hat{I}_T^\beta(\rho^0|\gamma) = \sup_{G \in C_0^{1,2}([0,T] \times [-1,1])} \left\{ \frac{\beta}{2} \int_0^T dt \left\langle \sigma(\rho_t^0) \nabla(J^{\text{neum}} \star \rho_t^0), \nabla G_t \right\rangle \right.$$
$$\left. - \frac{1}{2} \int_0^T dt \langle \sigma(\rho_t^0), (\nabla G_t)^2 \rangle \right\}.$$

By inequality (3.12), this last expression is bounded by

$$\frac{\beta^2}{8} \int_0^T dt \langle \sigma(\rho_t^0), [\nabla(J^{\text{neum}} \star \rho_t^0)]^2 \rangle.$$

We conclude the proof by applying Lemma 3.1 and Jensen inequality. $\qquad \square$

5.3. $I_T^\beta(\cdot|\gamma)$-Density. In this section we show that any trajectory $\pi \in D([0,T], \mathcal{M})$, with finite rate function, $I_T^\beta(\pi|\gamma) < \infty$, can be approximated by a sequence of smooth trajectories $\{\pi^n : n \ge 1\}$ such that

$$\lim_{n \to \infty} \pi^n = \pi \quad \text{in} \quad D([0,T], \mathcal{M}) \quad \text{and} \quad \lim_{n \to \infty} I_T^\beta(\pi^n|\gamma) = I_T^\beta(\pi|\gamma).$$

Definition 5.9. *A subset $\mathcal{A}$ of $D([0,T], \mathcal{M})$ is said to be $I_T^\beta(\cdot|\gamma)$-dense if for every π in $D([0,T], \mathcal{M})$ such that $I_T^\beta(\pi|\gamma) < \infty$, there exists a sequence $\{\pi^n : n \ge 1\}$ in $\mathcal{A}$ such that π^n converges to π in $D([0,T], \mathcal{M})$ and $I_T^\beta(\pi^n|\gamma)$ converges to $I_T^\beta(\pi|\gamma)$.*

Definition 5.10. *Let $\mathcal{A}_1$ be the subset of $D([0,T], \mathcal{M})$ consisting of trajectories π such that $I_T^\beta(\pi|\gamma) < \infty$ and for which there exists $\delta > 0$ such that π is a weak solution of the equation (5.8) in the time interval $[0, \delta]$.*

Lemma 5.11. *The set $\mathcal{A}_1$ is $I_T^\beta(\cdot|\gamma)$-dense.*

Proof. Fix a path π such that $I_T^\beta(\pi|\gamma) < \infty$ and let $\rho^{(0)}$ be the solution of the heat equation (5.8). For $\varepsilon > 0$, define π^ε as

$$
\pi_t^\varepsilon(\cdot) = \begin{cases} \rho_t^{(0)}(\cdot) & \text{for } 0 \le t \le \varepsilon, \\ \rho_{2\varepsilon-t}^{(0)}(\cdot) & \text{for } \varepsilon \le t \le 2\varepsilon, \\ \pi_{t-2\varepsilon}(\cdot) & \text{for } 2\varepsilon \le t \le T. \end{cases}
$$

Since $\lim_{\varepsilon \to 0} \pi^\varepsilon = \pi$ in $D([0,T], \mathcal{M})$ and $I(\cdot|\gamma)$ is lower semicontinuous, it is enough to prove that $\forall \varepsilon > 0$, $I_T^\beta(\pi^\varepsilon|\gamma) < \infty$ and that $\underline{\lim}_{\varepsilon \to 0} I_T^\beta(\pi^\varepsilon|\gamma) \le I_T^\beta(\pi|\gamma)$. From Lemma 5.1, for each $\varepsilon > 0$, $\pi_0^\varepsilon(\cdot) = \gamma(\cdot)$ and $\pi_t^\varepsilon(\pm 1) = \rho_\pm$. Decompose the rate function $I_T^\beta(\pi^\varepsilon|\gamma)$ as the sum of the contribution on each interval $[0, \varepsilon]$, $[\varepsilon, 2\varepsilon]$ and $[2\varepsilon, T]$. Since on $[0, \varepsilon]$ the path π satisfies equation (5.8), by Lemma 5.8, the contribution to the first interval is bounded by

$$
\frac{\beta^2}{8} \int_0^\varepsilon dt \int_\Lambda (\nabla \rho_t^{(0)})^2(v) dv\,.
$$

This converges to 0 when $\varepsilon \downarrow 0$. On the time interval $[\varepsilon, 2\varepsilon]$, π^ε satisfies

$$
\partial_t \pi_t^\varepsilon = -\partial_t \rho_{2\varepsilon-t}^{(0)} = -\Delta \rho_{2\varepsilon-t}^{(0)} = -\Delta \pi^\varepsilon\,.
$$

In particular, the contribution to $[\varepsilon, 2\varepsilon]$ is equal to

$$
\sup \Big\{ 2 \int_0^\varepsilon dt\, \langle \nabla \rho_t^{(0)}, \nabla G \rangle + \frac{\beta}{2} \int_0^\varepsilon dt \Big\langle \sigma(\rho_t^{(0)}) \nabla(J^{\text{neum}} \star \rho_t^{(0)}), \nabla G_t \Big\rangle \\ - \frac{1}{2} \int_0^\varepsilon dt \big\langle \sigma(\rho_t^{(0)}), (\nabla G_t)^2 \big\rangle \Big\}, \tag{5.11}
$$

where the supremum in taken over all $G \in C_0^{1,2}([0,T] \times [-1,1])$. We apply inequality (3.12) to the first and second term inside the supremum, then apply Lemma 5.8. By Lemma 3.1, the supremum (5.11) is bounded by

$$
4 \int_0^\varepsilon dt \int_\Lambda du\, \frac{(\nabla \rho_t^{(0)})^2}{\sigma(\rho_t^{(0)})} + \frac{\beta^2}{4} \int_0^\varepsilon dt \big\langle \sigma(\rho_t^{(0)}) [\nabla(J^{\text{neum}} \star \rho_t^{(0)})]^2 \big\rangle
$$

$$
\le 4 \int_0^\varepsilon dt \int_\Lambda du\, \frac{(\nabla \rho_t^{(0)})^2}{\sigma(\rho_t^{(0)})} + \frac{\beta^2}{8} \int_0^\varepsilon dt \int_\Lambda (\nabla \rho_t^{(0)})^2\,.
$$

This last expression converges to zero as $\varepsilon \downarrow 0$. Finally, the contribution on $[2\varepsilon, T]$ is bounded by $I_T^\beta(\pi|\gamma)$. $\qquad\square$

Definition 5.12. *Denote by $\mathcal{A}_2$ the subset of $\mathcal{A}_1$ of all trajectories π such that for all $0 < \delta \le T$, there exists $\varepsilon > 0$ such that $\varepsilon \le \pi_t(u) \le 1 - \varepsilon$ for $(t, u) \in [\delta, T] \times [-1, 1]$.*

Lemma 5.13. *The set $\mathcal{A}_2$ is $I_T^\beta(\cdot|\gamma)$-dense.*

Proof. By the previous lemma, it is enough to show that each trajectory π in $\mathcal{A}_1$ can be approximated by trajectories in $\mathcal{A}_2$. Fix π in $\mathcal{A}_1$ and let ρ^0 be the solution of the equation (5.8). For each $0 < \varepsilon \le 1$, let $\pi^\varepsilon = (1-\varepsilon)\pi + \varepsilon\rho^0$. We have that π^ε converges to π as $\varepsilon \downarrow 0$ a.e. in $\Lambda \times (0, T)$. By the lower semicontinuity of $I_T^\beta(\cdot|\gamma)$, it is enough to show that

$$
\sup_{\varepsilon > 0} I_T^\beta(\pi^\varepsilon|\gamma) < \infty \quad \text{and} \quad \overline{\lim_{\varepsilon \to 0}} I_T^\beta(\pi^\varepsilon|\gamma) \le I_T^\beta(\pi|\gamma)\,. \tag{5.12}
$$

Fix $\varepsilon > 0$, by construction $\pi_0^\varepsilon(\cdot) = \gamma$ and $\pi_t^\varepsilon(\pm 1) = \rho_\pm$ for almost all $t \in [0, T]$. From the convexity of $\mathcal{Q}(\cdot)$, for each $0 \le \varepsilon \le 1$,

$$\mathcal{Q}(\pi_\varepsilon) \le (1 - \varepsilon)\mathcal{Q}(\pi) + \varepsilon\mathcal{Q}(\rho^0) \le \mathcal{Q}(\pi) + \mathcal{Q}(\rho^0) < \infty\,.$$

Since $\partial_t \pi^\varepsilon = \varepsilon \partial_t \rho^0 + (1 - \varepsilon)\partial_t \pi$ and, by the assumption, $I_T^\beta(\pi|\gamma)$ is finite, it follows from Lemma 5.2, that $\pi^\varepsilon \in L^2(0, T; H^1(\Lambda))$ and $\partial_t \pi^\varepsilon \in L^2(0, T; H^{-1}(\Lambda))$. Next we show that $I_T^\beta(\pi^\varepsilon|\gamma)$ is finite uniformly on ε. We decompose the rate $I_T^\beta(\pi^\varepsilon|\gamma)$ in two terms:

$$I_T^\beta(\pi^\varepsilon|\gamma) \ \le\ A_1 + A_2 \tag{5.13}$$

where

$$A_1 = \sup\Big\{ \int_0^T dt\, \langle \partial_t\pi^\varepsilon, G_t \rangle \ +\ \int_0^T dt\, \langle \nabla\pi_t^\varepsilon, \nabla G_t \rangle \ -\ \frac{1}{4}\int_0^T dt \langle \sigma(\pi_t^\varepsilon), (\nabla G_t)^2 \rangle \Big\} \tag{5.14}$$

$$A_2 = \sup\Big\{ \frac{\beta}{2}\int_0^T dt \Big\langle \sigma(\pi_t^\varepsilon)\nabla(J^{\mathrm{neum}} \star \pi_t^\varepsilon)\,, \nabla G_t \Big\rangle \ -\ \frac{1}{4}\int_0^T dt \langle \sigma(\pi_t^\varepsilon), (\nabla G_t)^2 \rangle \Big\}\,, \tag{5.15}$$

and the supremum is taken over $G \in C_0^{1,2}([0, T] \times [-1, 1])$. By concavity of $\sigma(\cdot)$, the term A_1 is bounded above by

$$(1 - \varepsilon)\sup\Big\{ \int_0^T dt\, \langle \partial_t\pi_t, G_t \rangle \ +\ \int_0^T dt\, \langle \nabla\pi_t, \nabla G_t \rangle \ -\ \frac{1}{4}\int_0^T dt \langle \sigma(\pi_t), (\nabla G_t)^2 \rangle \Big\}$$

$$+\ \varepsilon \sup\Big\{ \int_0^T dt\, \langle \partial_t\rho^0, G_t \rangle \ +\ \int_0^T dt\, \langle \nabla\rho_t^0, \nabla G_t \rangle \ -\ \frac{1}{4}\int_0^T dt \langle \sigma(\rho_t^0)\nabla G_t, \nabla G_t \rangle \Big\}\,. \tag{5.16}$$

Since ρ^0 solves the heat equation, the second line of the last expression is equal to zero, while the first line is bounded above by $2 I_T^0(\pi|\gamma)$ which is bounded by Lemma 5.7. By Schwartz inequality, Lemma 3.1 and Jensen inequality

$$A_2 \le \frac{\beta^2}{4}\int_0^T dt \langle \sigma(\pi_t^\varepsilon)\big(J^{\mathrm{neum}} \star |\nabla\pi_t^\varepsilon|\big)^2 \rangle \le \frac{\beta^2}{4}\int_0^T dt \int_\Lambda \big(|\nabla\pi_t|^2 + |\nabla\rho_t^0|^2\big)\,. \tag{5.17}$$

By (5.16) and (5.17) we have that $\sup\limits_{\varepsilon > 0} I_T^\beta(\pi^\varepsilon|\gamma) < \infty$.

We are now going to prove $\varlimsup_{\varepsilon \to 0} I_T^\beta(\pi^\varepsilon|\gamma) \ \le\ I_T^\beta(\pi|\gamma)$. By definition of π^ε, we have that, for any $G \in C_0^{1,2}([0, T] \times [-1, 1])$,

$$\int_0^T dt \langle \partial_t\pi^\varepsilon, G_t \rangle = (1 - \varepsilon)\int_0^T dt \left\{ \langle \partial_t\pi, G_t \rangle + \varepsilon \langle \partial_t\rho^0, G_t \rangle \right\}\,. \tag{5.18}$$

By Lemma 5.5, there exists $F \in H_0^1(\sigma(\pi))$ such that π solves the boundary value problem (5.1). Taking this into account and (5.18) we can write

$$\int_0^T dt \langle \partial_t\pi^\varepsilon, G_t \rangle$$

$$= \int_0^T dt \Big\{ \langle -\nabla\pi^\varepsilon, \nabla G_t \rangle + (1 - \varepsilon)\Big\langle \sigma(\pi_t)\Big[\frac{\beta}{2}\nabla(J^{\mathrm{neum}} \star \pi) + \nabla F_t\Big], \nabla G_t \Big\rangle \Big\}$$

$$= \int_0^T dt \Big\{ \langle -\nabla\pi^\varepsilon, \nabla G_t \rangle + \Big\langle \mathbb{F}_t^\beta(\pi^\varepsilon), \nabla G_t \Big\rangle + \Big\langle \frac{\beta}{2}\sigma(\pi_t^\varepsilon)\nabla(J^{\mathrm{neum}} \star \pi_t^\varepsilon), \nabla G_t \Big\rangle \Big\}, \tag{5.19}$$

where

$$\mathbb{F}_t^\beta(\pi^\varepsilon) = (1 - \varepsilon)\sigma(\pi_t)\Big[\frac{\beta}{2}\nabla(J^{\mathrm{neum}} \star \pi_t) + \nabla F\Big] - \frac{\beta}{2}\sigma(\pi_t^\varepsilon)\nabla(J^{\mathrm{neum}} \star \pi_t^\varepsilon) .$$

By the definition of $\hat{I}_T^\beta$, see (2.12), we have that

$$\hat{I}_T^\beta(\pi^\varepsilon|\gamma) = \sup\Big\{ \int_0^T dt\langle \mathbb{F}_t^\beta(\pi^\varepsilon), \nabla G_t \rangle - \frac{1}{2}\int_0^T dt\langle \sigma(\pi_t^\varepsilon), (\nabla G_t)^2 \rangle\Big\} = I_T^\beta(\pi^\varepsilon|\gamma),$$
(5.20)

where the supremum is taken over all $G \in \mathcal{C}_0^{1,2}([0,T] \times \Lambda)$. The last equality in (5.20) holds because $\mathcal{Q}(\pi_\varepsilon)$ is bounded for any $\varepsilon > 0$ and then (2.15) applies. Since for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ so that $\sigma(\pi^\varepsilon) \geq \delta(\varepsilon)$ we can apply to the first term inside the argument of the supremum of (5.20) inequality (3.12) to cancel the contribution of the second term inside the argument of the supremum, obtaining

$$I_T^\beta(\pi^\varepsilon|\gamma) \leq \frac{1}{2}\int_0^T dt \int_\Lambda du\, \frac{(\mathbb{F}_t^\beta(\pi^\varepsilon))^2}{\sigma(\pi_t^\varepsilon)} .$$

On the other hand, from (5.2),

$$I_T^\beta(\pi|\gamma) = \frac{1}{2}\int_0^T dt\, \langle \sigma(\pi_t)\nabla F_t, \nabla F_t \rangle .$$

Therefore, to conclude the proof, it is enough to show that

$$\lim_{\varepsilon \to 0} \frac{1}{2}\int_0^T dt \int_\Lambda du\, \frac{(\mathbb{F}_t^\beta(\pi^\varepsilon))^2}{\sigma(\pi_t^\varepsilon)} = \frac{1}{2}\int_0^T dt\, \langle \sigma(\pi_t)\nabla F_t, \nabla F_t \rangle .$$
(5.21)

By the continuity of σ and the definition of $\mathbb{F}_t^\beta(\pi^\varepsilon)$,

$$\lim_{\varepsilon \to 0} \frac{(\mathbb{F}_t^\beta(\pi^\varepsilon))^2}{\sigma(\pi_t^\varepsilon)} = \sigma(\pi_t)(\nabla F_t)^2, \quad a.e. \quad \mathrm{in} \quad \Lambda \times (0,T).$$

By the convexity of $a \to a^2$, the inequality (3.12), the following inequality

$$(a + b + c)^3 \leq 3(a^2 + b^2 + c^2), \quad \forall a, b, c \in \mathbb{R},$$

the concavity of $\sigma(\cdot)$ and Lemma 3.1, for any $0 < \varepsilon < 1$,

$$\frac{(\mathbb{F}_t^\beta(\pi^\varepsilon))^2}{\sigma(\pi^\varepsilon)} \leq \frac{3\beta^2}{4}\sigma(\pi)\big[J^{neum} \star (\nabla\pi)^2 + (\nabla F)^2\big]$$

$$+ \frac{3\beta^2}{2}\big[\sigma(\pi) + \sigma(\rho^0)\big]\big[J^{neum} \star ((\nabla\pi)^2 + (\nabla\rho^0)^2)\big] .$$

Therefore (5.21) follows by Lebesgue dominated convergence theorem. □

Definition 5.14. *Denote by $\mathcal{A}_3$ the trajectories $\pi \in D([0,T],\mathcal{M})$ such that π is the solution of the boundary value problem* (5.1) *for some $F \in \mathcal{C}_0^{1,2}([0,T] \times \Lambda)$.*

The last step is to prove that $\mathcal{A}_3$ is $I_T^\beta(\cdot|\gamma)$-dense. We follow the strategy adopted in [7]: given a trajectory π in $D([0,T],\mathcal{M})$ with finite rate function $I_T^\beta(\pi|\gamma) < \infty$, from Lemma 5.5, there exists a function F in $H_0^1(\sigma(\pi))$ such that π is a weak solution to the equation (5.1). Instead of approximating π by a sequence of smooth trajectories in $D([0,T],\mathcal{M})$ (cf. [4],[19],[21]), we will approximate F by smooth functions (F_n) and we then show that the corresponding smooth solutions (π_n) of (5.1) converge to π in $D([0,T],\mathcal{M})$ and $I_T^\beta(\pi^n|\gamma)$ converges to $I_T^\beta(\pi|\gamma)$.

Lemma 5.15. *The set $\mathcal{A}_3$ is $I_T^\beta(\cdot|\gamma)$-dense.*

Proof. In view of the previous lemma, it is enough to show that for each π in $\mathcal{A}_2$, we can exhibit a sequence $\{\pi_n : n > 0\}$ in $\mathcal{A}_3$ which converges to π in $D([0,T], \mathcal{M})$ and such that $I_T^\beta(\pi_n|\gamma)$ converges to $I_T^\beta(\pi|\gamma)$. Fix $\pi \in \mathcal{A}_2$. Since $I_T^\beta(\pi|\gamma)$ is finite, by Lemma 5.5, there exists a function $F \in H_0^1(\sigma(\pi))$ such that π is the weak solution to the boundary value problem (5.1). We claim hat $F \in L^2([0,T], H^1(\Lambda))$ and then, can be approximated by a sequence of smooth functions $(F^n)_{n \geq 1}$. Indeed, let $0 < \delta < T$ be such that, π is the solution of the heat equation (5.8) in the time interval $[0, \delta]$. We have that $\nabla F = -\frac{\beta}{2}\nabla(J^{\mathrm{neum}} \star \pi)$ in $[0, \delta] \times \Lambda$ and

$$
\int_0^T dt \int_\Lambda \left|\nabla F_t(u)\right|^2 du = \int_0^\delta dt \int_\Lambda \frac{\beta^2}{4}\left|\nabla(J^{\mathrm{neum}} \star \pi)(t,u)\right|^2 du
$$
$$
+ \int_\delta^T dt \int_\Lambda \left|\nabla F_t(u)\right|^2 du. \tag{5.22}
$$

On the other hand, since $\pi \in \mathcal{A}_2$, there exists $0 < \varepsilon < 1$ such that $\varepsilon \leq \pi_t(\cdot) \leq 1 - \varepsilon$ for $\delta \leq t \leq T$. Therefore

$$
\int_\delta^T dt \int_\Lambda \left|\nabla F_t(u)\right|^2 du \leq \frac{1}{\sigma(\varepsilon)}\|F\|_{\sigma(\pi)}^2
$$
$$
= \frac{2}{\sigma(\varepsilon)}I_T^\beta(\pi|\gamma) < \infty. \tag{5.23}
$$

It follows from (5.22) and (5.23) that $F \in L^2([0,T], H^1(\Lambda))$. Let $(F^n)_{n>0}$ be a sequence of functions in $C_0^{1,2}([0,T] \times \Lambda)$ such that $\lim_{n \to +\infty} F^n = F$ in $L^2([0,T], H^1(\Lambda))$.

For each integer $n > 0$, let π^n be the weak solution of (5.1) with F^n in place of F. By (5.2)

$$
I_T^\beta(\pi^n|\gamma) = \frac{1}{2}\int_0^T dt \langle \nabla F_t^n \cdot \sigma(\pi_t^n)\nabla F_t^n \rangle \leq \frac{1}{4}\int_0^T dt \int_\Lambda du \|\nabla F_t^n(u)\|^2.
$$

Since the sequence $(F_n)_{n>0}$ converges to F in $L^2([0,T], H^1(\Lambda))$, it follows from the last inequality that $I_T^\beta(\pi^n|\gamma)$ is uniformly bounded. Thus, by Theorem 5.4, the sequence π^n is relatively compact in $D([0,T], \mathcal{M})$. Let $\{\pi^{n_k} : k \geq 1\}$ be a subsequence of π^n converging to some π^0 in $D([0,T], \mathcal{M})$, then $\{\pi^{n_k} : k \geq 1\}$ converges weakly to π^0 in $L^2([0,T] \times [-1,1])$. Since $I_T^\beta(\pi^n|\gamma)$ is uniformly bounded, by Lemma 5.2 and Lemma 5.3, π^{n_k} converges to π^0 strongly in $L^2([0,T] \times [-1,1])$. For every G in $\mathcal{C}_0^{1,2}([0,T] \times [-1,1])$, we have

$$
\langle \pi_T^{n_k}, G_T \rangle - \langle \gamma, G_0 \rangle = \int_0^T dt \langle \pi_t^{n_k}, \partial_t G_t \rangle
$$
$$
+ \int_0^T dt \langle \pi_t^{n_k}), \Delta G_t \rangle - \rho_+ \int_0^T dt \nabla G_t(1) + \rho_- \int_0^T dt \nabla G_t(-1)
$$
$$
+ \int_0^T \langle \nabla G_t, \sigma(\pi_t^{n_k})\left[\frac{\beta}{2}\nabla(J^{\mathrm{neum}} \star \pi_t^{n_k}) + \nabla F_t^{n_k}\right]\rangle dt.
$$

Letting $k \to \infty$, we obtain

$$\langle \pi_T^0, G_T \rangle - \langle \gamma, G_0 \rangle = \int_0^T dt \, \langle \pi_t^0, \partial_t G_t \rangle$$

$$+ \int_0^T dt \, \langle \pi_t^0, \Delta G_t \rangle \, - \rho_+ \int_0^T dt \, \nabla G_t(1) \, + \, \rho_- \int_0^T dt \, \nabla G_t(-1)$$

$$+ \int_0^T \langle \nabla G_t, \sigma(\pi_t^0) \big[\frac{\beta}{2} \nabla (J^{\text{neum}} \star \pi_t^0) + \nabla F_t \big] \rangle \, dt.$$

That is π^0 is a weak solution of equation (5.1). Thus, by uniqueness of weak solutions of (5.1), $\pi^0 = \pi$.

To conclude the proof of the lemma it remains to prove that $\lim_{n \to \infty} I_T^\beta(\pi^n | \gamma) = I_T^\beta(\pi | \gamma)$. The sequence $(\pi^n)_{n>0}$ converges to π strongly in $L^2([0,T] \times [-1,1])$ and the sequence $(F^n)_{n>0}$ converges to F in $L^2([0,T], H^1(\Lambda))$. Taking into account that π is bounded and σ is Lipschitz, we obtain

$$\lim_{n \to \infty} I_T^\beta(\pi^n | \gamma) = \lim_{n \to \infty} \frac{1}{2} \int_0^T dt \, \langle \nabla F_t^n \cdot \sigma(\pi_t^n) \nabla F_t^n \rangle$$

$$= \frac{1}{2} \int_0^T dt \, \langle \nabla F_t \cdot \sigma(\pi_t) \nabla F_t \rangle \, = I_T^\beta(\pi | \gamma) \, .$$

$$\square$$

5.4. **Upper bound.** Let $\tilde{\mathcal{Q}} = \tilde{\mathcal{Q}}^2$ be the functional defined in (3.29) with $\delta_0 = 2$. For $0 \le a \le 1$, denote by $\mathcal{E}_a : D([0,T], \mathcal{M}) \to [0, +\infty]$ the following functional

$$\mathcal{E}_a(\pi) \, = \, \hat{I}_T^\beta(\pi | \gamma) \, + \, a(1+a)\tilde{\mathcal{Q}}(\pi) \, .$$

The proof of the upper bound relies on the following proposition.

Proposition 5.16. *Let $\mathcal{K}$ be a compact set of $D([0,T], \mathcal{M})$. There exists a positive constants C, such that for any $0 \le a \le 1$,*

$$\overline{\lim_{N \to \infty}} \frac{1}{N^d} \log Q_{\eta^N}^{\beta, N}(\mathcal{K}) \, \le \, - \frac{1}{1+a} \inf_{\pi \in \mathcal{K}} \mathcal{E}_a(\pi) \, + \, aC(T+1) \, .$$

Proof. Fix a density profile $\theta : [-1,1] \to (0,1)$, a function G in $C_0^{1,2}([0,T] \times [-1,1])$ and a function H in $C_c^\infty([0,T] \times \Lambda)$. For a local function $\Psi : \{0,1\}^{\mathbb{Z}} \to \mathbb{R}$, $c > 0$ and $\varepsilon > 0$, denote $B_{N,\varepsilon,c}^{G,\Psi}$ and $E_{N,c}^G$ the set of trajectories $(\eta_t)_{t \in [0,T]}$ defined by

$$B_{N,\varepsilon,c}^{G,\Psi} \quad = \quad \Big\{ \eta_\cdot \in D([0,T], \mathcal{S}_N) : \Big| \int_0^T V_{N,\varepsilon}^{G,\Psi}(t, \eta_t) dt \Big| \le c \Big\} \, ,$$

$$E_{N,c}^G \quad = \quad \Big\{ \eta_\cdot \in D([0,T], \mathcal{S}_N) : \Big| \int_0^T W_N^G(t, \eta_t) dt \Big| \le c \Big\} \, ,$$

where $V_{N,\varepsilon}^{G,\Psi}$ and W_N^G are defined in (3.18) and (3.26).

Define $\Psi_1(\eta) = [\eta(1) - \eta(0)]^2$, let $A_{N,\varepsilon,c}^{G,H}$ be the set

$$A_{N,\varepsilon,c}^{G,H} \quad = \quad B_{N,\varepsilon,c}^{\nabla G, \Psi_1} \cap E_{N,c}^{\nabla G} \cap B_{N,\varepsilon,c}^{\nabla H, \Psi_1} \, .$$

By the superexponential estimates stated in Proposition 3.6 and Proposition 3.7, it is enough to prove that, for every $0 < a \le 1$,

$$\overline{\lim_{c \to 0}} \, \overline{\lim_{\varepsilon \to 0}} \, \overline{\lim_{N \to \infty}} \frac{1}{N} \log Q_{\eta^N}^{\beta, N} \Big(\mathcal{K} \cap A_{N,\varepsilon,c}^{G,H} \Big) \, \le \, - \frac{1}{1+a} \inf_{\pi \in \mathcal{K}} \mathcal{E}_a(\pi) \, + \, aC(T+1) \, .$$

For $H \in \mathcal{C}_c^\infty([0,T] \times \Lambda)$, recall from (3.28) the definition of $\tilde{\mathcal{Q}}_H(\pi) = \tilde{\mathcal{Q}}_H^2(\pi)$, with $\delta_0 = 2$ and write

$$\frac{1}{N} \log Q_{\eta^N}^{\beta,N}\left(\mathcal{K} \cap A_{N,\varepsilon,c}^{G,H}\right) =$$

$$\frac{1}{N} \log \mathbb{E}_{\eta^N}^{\beta,N}\left[\mathbb{1}\{\mathcal{K} \cap A_{N,\varepsilon,c}^{G,H}\} e^{aN\tilde{\mathcal{Q}}_H(\pi^N * \iota_\varepsilon)} e^{-aN\tilde{\mathcal{Q}}_H(\pi^N * \iota_\varepsilon)} \right] .$$

By Hölder inequality the right hand side of the last equality is bounded above by

$$\frac{1}{(1+a)N} \log \mathbb{E}_{\eta^N}^{\beta,N}\left[\mathbb{1}\{\mathcal{K} \cap A_{N,\varepsilon,c}^{G,H}\} e^{-a(1+a)N\tilde{\mathcal{Q}}_H(\pi^N * \iota_\varepsilon)} \right] \qquad (5.24)$$

$$+ \frac{a}{(1+a)N} \log \mathbb{E}_{\eta^N}^{\beta,N}\left[e^{(1+a)N\tilde{\mathcal{Q}}_H(\pi^N * \iota_\varepsilon)} \right] .$$

From Lemma 3.8, the second term of this inequality is bounded by $aC_1(T+1)$. Consider the exponential martingale M_t^G defined by

$$M_t^G = \exp\left\{ N\left[\langle \pi_t^N, G_t \rangle - \langle \pi_0^N, G_0 \rangle \right.\right.$$

$$\left.\left. - \frac{1}{N} \int_0^t e^{-N\langle \pi_s^N, G_s \rangle} (\partial_s + N^2 \mathcal{L}_N) e^{N\langle \pi_s^N, G_s \rangle} \, ds \right] \right\} .$$

$$(5.25)$$

Since the sequence $\{\eta^N : N \geq 1\}$ is associated to γ, an elementary computation shows that on the set $A_{N,\varepsilon,c}^{G,H}$

$$M_T^G = \exp N\left\{ \mathcal{J}_G^\beta(\pi^N * \iota_\varepsilon) + O_G(\varepsilon) + O(c) \right\} , \qquad (5.26)$$

where $O_G(\varepsilon)$ (resp. $O(c)$) is a quantity which vanishes as $\varepsilon \downarrow 0$ (resp. $c \downarrow 0$) and $\mathcal{J}_G^\beta(\cdot)$ is the functional defined in (2.11). Consider the first term of (5.24) and rewrite it as

$$\frac{1}{(1+a)N} \log \mathbb{E}_{\eta^N}^{\beta,N}\left[M_T^G (M_T^G)^{-1} \mathbb{1}\{\mathcal{K} \cap A_{N,\varepsilon,c}^{G,H}\} e^{-a(1+a)N\tilde{\mathcal{Q}}_H(\pi^N * \iota_\varepsilon)} \right]$$

Optimizing over π^N in $\mathcal{K}$, since M_t^G is a mean one positive martingale, the previous expression is bounded above by

$$-\frac{1}{1+a} \inf_{\pi \in \mathcal{K}} \left\{ \mathcal{J}_G^\beta(\pi * \iota_\varepsilon) + a(1+a)\tilde{\mathcal{Q}}_H(\pi * \iota_\varepsilon) \right\} + O_G(\varepsilon) + O(c) .$$

Optimize the previous expression with respect to G and H. Since the set $\mathcal{K}$ is compact and $\mathcal{J}_G^\beta(\cdot * \iota_\varepsilon)$ and $\tilde{\mathcal{Q}}_H(\cdot * \iota_\varepsilon)$ are lower semi-continuous for every G, H, ε, we may apply the arguments presented in [24, Lemma 11.3] to exchange the supremum with the infimum. In this way we obtain that the last expression is bounded above by

$$\frac{1}{1+a} \sup_{\pi \in \mathcal{K}} \inf_{G,H,\varepsilon} \left\{ -\mathcal{J}_G^\beta(\pi * \iota_\varepsilon) - a(1+a)\tilde{\mathcal{Q}}_H(\pi * \iota_\varepsilon) \right\} + O_G(\varepsilon) + O(c) .$$

Letting first $\varepsilon \downarrow 0$ and then $c \downarrow 0$, we obtain that the limit of the previous expression is bounded above by

$$\frac{1}{1+a} \sup_{\pi \in \mathcal{K}} \inf_{G,H} \left\{ -\mathcal{J}_G^\beta(\pi) - a(1+a)\tilde{\mathcal{Q}}_H(\pi) \right\}$$

This concludes the proof of the proposition because $\sup_G \mathcal{J}_G^\beta(\pi) = \hat{I}_T^\beta(\pi|\gamma)$. $\qquad \square$

Proof of the upper bound. Let $\mathcal{K}$ be a compact set of $D([0,T],\mathcal{M})$. If for all $\pi \in \mathcal{K}$, $\tilde{\mathcal{Q}}(\pi) = \infty$ then the upper bound is trivially satisfied. Suppose that $\inf_{\pi \in \mathcal{K}} \{\tilde{\mathcal{Q}}(\pi)\} < \infty$, from Proposition 5.16, for any $0 < a \leq 1$,

$$\varlimsup_{N \to \infty} \frac{1}{N} \log Q_{\eta^N}^{\beta,N}(\mathcal{K}) \leq -\frac{1}{1+a} \inf_{\pi \in \mathcal{K}, \tilde{\mathcal{Q}}(\pi) < \infty} \mathcal{E}_a(\pi) + aC(T+1)$$

$$= -\frac{1}{1+a} \inf_{\pi \in \mathcal{K}} \left\{ I_T^\beta(\pi|\gamma) + a(1+a)\tilde{\mathcal{Q}}(\pi) \right\} + aC(T+1)$$

$$\leq -\frac{1}{1+a} \inf_{\pi \in \mathcal{K}} \left\{ I_T^\beta(\pi|\gamma) \right\} - a \inf_{\pi \in \mathcal{K}} \tilde{\mathcal{Q}}(\pi) + aC(T+1) .$$

To conclude the proof of the upper bound for compact sets, it remains to let $a \downarrow 0$.

To pass from compact sets to closed sets, we have to obtain exponential tightness for the sequence $Q_{\eta^N}^{\beta,N}$. The proof presented in [15, Section 10.4.] is easily adapted to our context.

$\square$

5.5. Lower bound. In this section we establish the large deviations lower bound.

The strategy of the proof of the lower bound consists of two steps. We first prove that for each $\pi \in \mathcal{A}_3$, recall its definition in 5.14, and each neighborhood $\mathcal{N}_\pi$ of π in $D([0,T],\mathcal{M})$

$$\lim_{N \to \infty} \frac{1}{N} \log Q_{\eta^N}^{\beta,N} \{\mathcal{N}_\pi\} \geq -I_T^\beta(\pi|\gamma) . \tag{5.27}$$

The proof of the lower bound is then accomplished by showing, see Subsection 5.3, that for any $\pi \in D([0,T],\mathcal{M})$ with $I_T^\beta(\pi|\gamma) < \infty$ we can find a sequence of $\pi^k \in \mathcal{A}_3$ such that $\lim_{k \to \infty} \pi^k = \pi$ in $D([0,T],\mathcal{M})$ and $\lim_{k \to \infty} I_T^\beta(\pi^k|\gamma) = I_T^\beta(\pi|\gamma)$.

The proof of (5.27) is similar to the one in the periodic case, see [15, Section 10.5.]. It depends on establishing laws of large numbers, in hydrodynamic scaling, for weak perturbations of the original process, and controlling by the Girsanov formula the relative entropies of the processes that go with these perturbations. Fix a path $\pi \in \mathcal{A}_3$. Then, by construction, there exists $F \in \mathcal{C}_0^{1,2}([0,T] \times [-1,1])$ so that π is the weak solution of the equation (5.1). Recall from (5.25) the definition of the exponential martingale M_T^F. Let $\mathbb{P}_{\eta^N}^F$ be the probability measure on the path space $D([0,T],\mathcal{S}_N)$ with density M_T^F with respect to $\mathbb{P}_{\eta^N}^{\beta,N}$: $\mathbb{P}_{\eta^N}^F[A] = \mathbb{E}_{\eta^N}^{\beta,N}[M_T^F \mathbb{1}\{A\}]$. Let $(\eta_t)_{t \in [0,T]}$ be the process with law $\mathbb{P}_{\eta^N}^F$ on $D([0,T],\mathcal{S}_N)$. Let $(\pi_t^N)_{t \in [0,T]}$ be the corresponding empirical measure. Then $(\pi_t^N)_{t \in [0,T]}$ converges weakly in probability to $(\pi_t)_{t \in [0,T]}$. From the super exponential estimates, Proposition 3.6 and Proposition 3.7, we have

$$\lim_{N \to \infty} \frac{1}{N} \log Q_{\eta^N}^{\beta,N} \{\mathcal{N}_\pi\} \geq - \lim_{N \to \infty} \frac{1}{N} H\left(\mathbb{P}_{\eta^N}^F \big| \mathbb{P}_{\eta^N}^{\beta,N}\right) ,$$

where $H\left(\mathbb{P}_{\eta^N}^F \big| \mathbb{P}_{\eta^N}^{\beta,N}\right)$ stands for the relative entropy given by

$$H\left(\mathbb{P}_{\eta^N}^F \big| \mathbb{P}_{\eta^N}^{\beta,N}\right) = \int \log \left\{ \frac{d\mathbb{P}_{\eta^N}^F}{d\mathbb{P}_{\eta^N}^{\beta,N}} \right\} d\mathbb{P}_{\eta^N}^F .$$

To conclude the proof, it remains to show that

$$\lim_{N \to \infty} \frac{1}{N} H\left(\mathbb{P}_{\eta^N}^F \big| \mathbb{P}_{\eta^N}^{\beta,N}\right) = I_T^\beta(\pi|\gamma) .$$

The proof of [15, Theorem 10.5.4] is easily adapted to our model.   □

6. APPENDIX

In this section we summarize the properties of the equation (2.9) needed to prove the main results of the paper. The proofs of these results are based on applying standard tools in partial differential equations, although some care need to be taken because of the presence of the nonlocal term. Notice that because of the nonlocal term the comparison property does not hold for this equation, so tools based on maximum principle will not work for (2.9).

We recall the notion of weak solution of (2.9). A function $\rho(\cdot, \cdot) : [0, T] \times \Lambda \to [0, 1]$ is a weak solution of the initial-boundary value problem (2.9) if $\rho \in L^2\big([0, T], H^1(\Lambda)\big)$ and for every $G \in C_0^{1,2}([0, T] \times [-1, 1])$ one has $\ell_G^\beta(\rho, \gamma) = 0$, where ℓ_G^β was defined in (2.8).

Theorem 6.1. *For any $\beta \geq 0$ there exists an unique weak solution of (2.9).*

The existence of a weak solution of (2.9) is a consequence of the the tightness of $(Q_{\mu_N}^\beta)_{N \geq 1}$ and the characterization of the support of its limit points, see Lemma 4.3. The uniqueness can be easily proven performing estimates as in Theorem 6.2 for all β. A proof of existence without invoking the hydrodynamic limit can be done applying in our setting the argument done in [9], Section 4.

Theorem 6.2. *There exists β_0 depending on J^{neum} and Λ, so that for $\beta \leq \beta_0$ there exists an unique weak stationary solution $\bar\rho$ of (2.10). Further, let $\rho_t(\rho_0)$ be the weak solution of (2.9) with initial datum $\rho_0 \in \mathcal{M}$. For $\beta < \beta_0$, there exists $c(\beta) > 0$ so that*

$$\|\rho_t(\rho_0) - \bar\rho\|_{L^2(\Lambda)} \leq e^{-c(\beta)t}\|\rho_0 - \bar\rho\|_{L^2(\Lambda)}.$$

Proof. Let $\rho_{i,0} \in \mathcal{M}$ and $\rho_{i,t}$ be the solution of (2.9) for $t \geq 0$, with initial datum $\rho_{i,0}$, $i = 1, 2$. Set $v = \rho_1 - \rho_2$, we have

$$\frac{1}{2}\frac{d}{dt}\|v_t\|_{L^2}^2 = -\int_\Lambda \nabla v\left[\nabla v - \beta\chi(\rho_1)\nabla(J^{neum} \star \rho_1) + \beta\chi(\rho_2)\nabla(J^{neum} \star \rho_2)\right]$$

$$= -\int_\Lambda \nabla v\left[\nabla v - \beta\chi(\rho_1)\nabla(J^{neum} \star v) + \beta[\chi(\rho_2) - \chi(\rho_1)]\nabla(J^{neum} \star \rho_2)\right].$$

$$(6.1)$$

Since $\chi(a) \leq \frac{1}{4}$ for $a \in [0, 1]$ and $|\chi(\rho_2) - \chi(\rho_1)| = |[\rho_2 - \rho_1][1 - (\rho_2 + \rho_1)]| \leq |v|$ we have that

$$\int_\Lambda \nabla v\left[\nabla v - \beta\chi(\rho_1)\nabla(J^{neum} \star v)\right] \geq \|\nabla v\|_{L^2}^2[1 - \frac{\beta}{4}] \qquad (6.2)$$

$$\left|\int_\Lambda \nabla v\left[\chi(\rho_2) - \chi(\rho_1)\right]\nabla(J^{neum} \star \rho_2)\right| \leq \sup|\nabla J^{neum}|\|\nabla v\|_{L^2}\|v\|_{L^2}$$

$$\leq \frac{a}{2}\sup|\nabla J^{neum}|^2\|\nabla v\|_{L^2}^2 + \frac{1}{2a}\|v\|_{L^2}^2,$$

$$(6.3)$$

for any $a > 0$. Taking into account (6.2) and (6.3) we can estimate (6.1) as following:

$$\frac{1}{2}\frac{d}{dt}\|v\|_{L^2}^2 \leq -\|\nabla v\|_{L^2}^2[1 - \beta(\frac{1}{4} + \frac{a}{2}\sup|\nabla J^{neum}|^2)] + \frac{\beta}{2a}\|v\|_{L^2}^2. \qquad (6.4)$$

and we choose a so that
$$\frac{1}{4} + \frac{a}{2}\sup|\nabla J^{\mathrm{neum}}|^2 \le \frac{1}{3}$$
Since we are in a bounded domain we can use the Poincaré inequality
$$\|v\|_{L^2}^2 \le C(\Lambda)\|\nabla v\|_{L^2}^2$$
obtaining
$$\frac{1}{2}\frac{d}{dt}\|v\|_{L^2}^2 \le -\|v\|_{L^2}^2\frac{[1-\frac{\beta}{3}]}{C(\Lambda)} + \frac{\beta}{2a}\|v\|_{L^2}^2. \qquad (6.5)$$
Take β_0 so that
$$\frac{[1-\frac{\beta_0}{3}]}{C(\Lambda)} - \frac{\beta_0}{2a} = 0$$
Then for $\beta < \beta_0$ there exists $c(\beta) > 0$, $\frac{[1-\frac{\beta}{3}]}{C(\Lambda)} - \frac{\beta}{2a} = c(\beta) > 0$, so that
$$\frac{1}{2}\frac{d}{dt}\|v\|_{L^2}^2 \le -c(\beta)\|v\|_{L^2}^2. \qquad (6.6)$$
This implies immediately that the stationary solution is unique and that it is exponential attractive in L^2. $\qquad\square$

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