Free Energy Inequalities and the Rate of Relaxation to Instantons for Interface Profiles in Glauber Dynamics

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Abstract

In this paper we prove several inequalities for the excess free energy in Glauber dynamics with long range potentials. We apply these results to control the rate of relaxation to local equilibrium along a moving interface, uniformly in the position along the interface.

1. Introduction.

This paper concerns properties of the excess free energy functional relevant to the study of interface motion on the mesoscopic

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scale for Glauber dynamics with Kac potentials at subcritical temperatures $1/\beta \leq 1/\beta_c = 1$ (in the chosen units). At this scale, the scale of the long–range interaction, unit distances are sufficiently large that the microscopic lattice structure $\mathbb{Z}^d$ is replaced by a continuum $\mathbb{R}^d$, and the state of the system is given by a spin density function $m(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^d$ which naturally takes its values in the interval $[-1, 1]$. Now there are two global equilibria: $m(\mathbf{x}) = \pm m_\beta$ where m_β is the fixed point of

$$m_\beta = \tanh(\beta m_\beta) \ .$$

If the spin density is not one of these equilibria, it evolves. The time evolution of the spin density on this scale is given by

$$\frac{\partial m}{\partial t} = -m + \tanh\{\beta J \star m\} \tag{1.1}$$

where J is a non negative, smooth radial function with compact support in $\mathbb{R}^d$ and with integral equal to 1. The $\star$ denotes the convolution product, namely:

$$(J \star m)(x) = \int J(x-y)m(y)\mathrm{d}y \tag{1.2}$$

Under this dynamics, provided that our initial data is not globally close to just one of the equilibrium states, the density very quickly develops large regions in which the density function takes on one or the other equilibrium values with slowly moving interfaces between these regions.

At the *macroscopic* scale, unit distances are so large that all detailed structure about the profile of $m(\mathbf{x})$ across an interface between a $-m_\beta$ region and a m_β region is lost and one simply has surfaces $\mathcal{S}$ dividing these regions. Recent works have shown that these surfaces move by mean curvature, at least under certain regularity conditions, see [6],[7],[11].

On the other hand, the *mesoscopic* scale considered here is characterized by the fact that there is a continuous transition across the interface from regions where $m(\mathbf{x})$ takes on values very close to $-m_\beta$

to regions where it takes on values very close to m_β. The goal of this paper is to establish certain free energy inequalities that can be used to gain good control, where good means uniform in an appropriate class of initial data, of the rate of relaxation to a "local equilibrium" profile. In this we are motivated by an analogy with the Boltzmann equation, as we explain below, and a desire to study quantitatively the sense in which this mesoscopic dynamics gives an approximation to motion by mean curvature.

To be more precise, we return to the original *microscopic* specification of the system, see [7] and [10] for a complete analysis.

Consider an Ising system on $\mathbb{Z}^d$, $d \geq 2$, with a long range potential parametrized by κ, $\kappa \in \mathbb{R}$

$$J^{(\kappa)}(\mathbf{i},\mathbf{j}) = \kappa^{-d} J((|\mathbf{i}-\mathbf{j}|)/\kappa) \qquad \mathbf{i},\mathbf{j} \in \mathbb{Z}^d \tag{1.3}$$

for some positive, smooth radial function J of compact support, $[-1,1]$, with unit total integral. Do this at subcritical temperature $1/\beta$, so that the global equilibria have magnetization $\pm m_\beta$. The parameter κ, which measures the range and the strength of the interaction, is one length scale for our system; another is the size γ of the lattice spacing. These two scales are intrinsic to the dynamics. There is a third length scale L: one associated to the initial data. To be specific, consider initial data that is very close to either m_β or to -m_β in regions that are large compared to the lattice spacing, and let L be a typical length of the transition across the boundary from one type of region to the other. As usual, we distinguish three scales for our problem depending on the relative size of κ:

When $\kappa/\gamma \approx 1$, then we must study the system on the microscopic scale, regardless of what L is. But if $\kappa/\gamma >> 1$, then we have two further cases:

$$\kappa/\gamma >> 1 \qquad \text{and} \qquad \kappa/L \approx 1 \tag{1.4}$$

is the *mesoscopic scale*, while

$$\kappa/\gamma >> 1 \qquad \text{and} \qquad \kappa/L >> 1 \tag{1.5}$$

is the *macroscopic scale.* Thus, the distinction is that at the mesoscopic scale, the transition from m_β to $-m_\beta$ has a smooth "profile". Given any initial spin configuration, let it evolve in this regime under a Glauber spin flip dynamics for the interaction (1.3). Then, in the limit $\gamma \to 0$, $\kappa = 1/\gamma$ the magnetization density $m(\mathbf{x}, t)$, $\mathbf{x} = \gamma \mathbf{i}$, evolves according to the equation (1.1), [7], [10].

At the macroscopic scale there is no such structure, the transition is given just by a sharp "step function" transition. Because the temperature $1/\beta$ is subcritical, the dynamics is such that the spin configuration quickly stays one in which most of the space is taken up by large regions in which the magnetization is very close to either m_β or to -m_β. These regions are separated by interfaces, which are the "surfaces" S_t^κ given by

$$S_t^\kappa = \{\mathbf{x} \mid m(\mathbf{x}) = 0\} \ . \tag{1.6}$$

As it has been shown in [7] the macroscopic limit in which κ tends to infinity, $\lim_{\kappa \to \infty} S_t^\kappa = S_t$ exists, and these interfaces move by mean curvature, either in the classical sense for as long S_t stays smooth [7], or in more generality, the interface motion corresponds to viscosity solutions [11] of the motion by mean curvature equations.

The interfaces are, however, well defined at finite κ by (1.6),[10], and it is natural to ask: *How large must κ be for the motion of these interfaces to closely approximate motion by mean curvature?*

This question is closely connected with the study of the relaxation to "instantons", i.e., the minimum free energy profiles for transitions across an interface. In more detail, let $\mathbf{u}$ be the unit vector normal to an interface at some point $\mathbf{x_0}$, and define the profile $m : \mathbb{R} \to [-1, 1]$ by

$$m(x) = m(\mathbf{x_0} + x\mathbf{u}) \ . \tag{1.7}$$

Now, keeping $\mathbf{x_0}$ and $\mathbf{u}$ fixed, letting $m(\mathbf{x}, t)$ be the evolution of our initial data $m_0(\mathbf{x})$, derive an equation for the time evolution of the profile. The result is that

$$\frac{\partial m}{\partial t} = -m + \tanh\{\beta \widetilde{J}^{(\kappa)} \star m\} \tag{1.8}$$

where

$$\widetilde{J}^{(\kappa)}(x) = \int_{\mathbb{R}^{d-1}} J^{(\kappa)}((x^2+y^2)^{1/2})\mathrm{d}y$$

and where

$$(\widetilde{J}^{(\kappa)} \star m)(x) = \int \widetilde{J}^{(\kappa)}(x-y)m(y)rmdy \tag{1.9}$$

Since we wish to consider transition profiles, so we seek solutions $m(x,t)$ satisfying

$$\lim_{x\to\pm\infty} m(x,t) = \pm m_\beta \tag{1.10}$$

or, actually, the somewhat stronger property

$$\int_{-\infty}^{\infty} |m(x,t) - \mathrm{sgn}(x)m_\beta|^2 \mathrm{d}x < \infty \tag{1.11}$$

for all t. Stationary solutions of this type are called "instantons" so that $m(x)$ is an instanton for the interaction J if it satisfies

$$-m + \tanh\{\beta\widetilde{J} \star m\} = 0 \tag{1.12}$$

and the boundary conditions (1.11). These instantons minimize a free energy functional, to be specified below, among all profiles satisfying (1.11). There are two one parameter families of these instantons. (The second family consists of reflections of the first). In the following we will denote by $\bar{m}$ the instanton centered in 0, i.e. $\bar{m}(0) = 0$ and by $\bar{m}_b$ the instanton shifted by b, $\bar{m}_b(x) = \bar{m}(x-b)$, $\bar{m}_0 \equiv \bar{m}$.

The equation (1.1) is an analog of the spatially inhomogeneous Boltzmann equation for this system. The equation (1.8) for the local evolution of the profile then is analogous to the spatially homogeneous Boltzmann equation, which governs the local relaxation. The instantons, then, are the analogs for (1.8) of the Maxwellians, which are the equilibrium solutions of the Boltzmann equation.

Entropy production can be used to control the rate of relaxation to Maxwellians for the Boltzmann equation [1],[2]. In this paper we show that the analogy can be extended to this feature as well. We prove several free energy inequalities that can be combined to permit free energy decrease to be used to quantitatively control the relaxation rate, just as entropy production is used to control the relaxation rate for the Boltzmann equation [2],[3].

2. Relaxation to Instantons and Free Energy

An important step, carried out in the DeMasi, Presutti, Orlandi and Triolo papers, in the proof that the macroscopic limit of the interface dynamics is motion by mean curvature, is a study of the relaxation to instantons for interface profiles.

One can obtain a more incisive, quantitative control of this relaxation by getting a strict "free energy destruction" inequality, and a free energy analog of the Cszilar inequality.

The excess free energy functional is

$$\mathcal{F}(m) = \int dx[f(m(x)) - f(m_\beta)] + \frac{1}{4}\int\int dxdy J(|x-y|)[m(x)-m(y)]^2 \tag{2.1}$$

where $f(m)$ is

$$f(m) = -\frac{1}{2}m^2 - \beta^{-1}i(m)$$

and the entropy density $i(m)$ is

$$i(m) = -\frac{1+m}{2}\log\{\frac{1+m}{2}\} - \frac{1-m}{2}\log\{\frac{1-m}{2}\} \ .$$

The bounds we shall prove depend on certain regularity properties of the magnetization. This is necessarily the case since the free energy functional is not convex, and has two distinct minima. Indeed, one can "patch together" two instantons to obtain a "plateau" of arbitrarily large width, but whose free energy is only a bit more than twice the free energy of the instanton. Hence, without some condition to control the width of such plateaus, the inequality of Theorem 1 could not hold.

To obtain this control introduce the sets

$$\mathcal{K}(M,N) = \{m \mid \int_{\mathbb{R}} |m(x) - m_\beta \mathrm{sgn}(x)|^2 \mathrm{d}x \leq M, \|m\|_{\mathrm{H}^1} \leq N\} \tag{2.2}$$

where

$$\|m\|_{\mathrm{H}^1}^2 = \int_{\mathbb{R}} |m'(x)|^2 \mathrm{d}x \tag{2.3}$$

and where m' denotes the derivative of m.

Theorem 3 below says that for initial data in one of these sets $\mathcal{K}(M,N)$, there are other M' and N' such that the solution of (1.8) stays in $\mathcal{K}(M',N')$ for all time. This is what allows us to apply Theorems 1 and 2 to the long–time behavior of solutions of (1.8).

Having explained the nature of the hypotheses, we now state our analog of the Cszilar [5] type inequality.

Theorem 1: *For any $\beta > 1$, and for any function m with $\|m(\cdot)\|_\infty \leq c < 1$, such that $m \in \mathcal{K}(M,N)$ for some finite constants N and M, there are constants C_1 and C_2 depending only on β, c, N and M so that*

$$C_1\|m - \bar{m}_b\|_{L^2}^2 \geq \mathcal{F}(m) - \mathcal{F}(\bar{m}_b) \geq C_2||m - \bar{m}_b||_{L^2}^2 \tag{2.4}$$

where by $\|\cdot\|_{L^2}$ we denote the L^2 norm, and where $\bar{m}_b$ is any instanton which minimizes the L^2 distance to m among all instantons.

We also have the following strict free energy destruction bounds:

Theorem 2: *For all initial data m_0 with $\|m(\cdot)\|_\infty \leq c < 1$, such that $m_0 \in \mathcal{K}(M,N)$ for some finite constants N and M, there is a constant C_3 depending only on β, c, N and M so that the corresponding solution of (1.8) satisfies*

$$\frac{d}{dt}\big(\mathcal{F}(m(\cdot,t)) - \mathcal{F}(\bar{m}_{b(t)}(\cdot))\big) \leq -C_3\|m(t) - \bar{m}_{b(t)}\|_{L^2}^2 \tag{2.5}$$

for all $t > 0$ where $\bar{m}_{b(t)}$is any instanton minimizing the L^2 distance to $m(x,t)$ among all instantons.

Notice that by the maximum principle for solutions of (1.8), if $\|m_0(\cdot)\|_\infty \leq c$, then $\|m(\cdot,t)\|_\infty \leq c$ for all time, and in fact, by results of [7], even if the intial data does not have this property for any $c < 1$, for any $t_0 > 0$, there will be a $c < 1$ so that $\|m(\cdot,t)\|_\infty \leq c$ for all $t > t_0$.

Moreover, there is at least one instanton $\bar{m}_b$ minimizing $||m(\cdot,t) - \bar{m}_b||^2$ since clearly this quantity is differentiable as a function of b, and tends to infinity as $b \to \pm\infty$. Hence the distance is minimized at one value of b at least, and at this value of b, $\bar{m}'_b$ is orthogonal to $m - \bar{m}_b$. Let $b(t)$ denote any such value. Note that nothing is said about the dependence on t of $b(t)$ in Theorem 2.

To apply these results we also need:

Theorem 3 *For all initial data $m_0 \in \mathcal{K}(M,N)$, and for any finite M and N, there are finite M' and N' such that if $m(\cdot,t)$ is the solution of (1.8) with m_0 as initial data, then*

$$m(\cdot,t) \in \mathcal{K}(M',N') \tag{2.6}$$

for all $t \geq 0$.

Thus, combining the theorems, one sees that

$$\frac{d}{dt}\left(\mathcal{F}\left(m(\cdot,t)\right) - \mathcal{F}\left(\bar{m}_{b(t)}(\cdot)\right)\right) \leq -\frac{C_3}{C_1}\left(\mathcal{F}\left((m(\cdot,t)\right) - \mathcal{F}\left((\bar{m}_{b(t)}(\cdot)\right)\right)$$

so that the free energy decays exponentially fast. The Czislar type inequality then implies that $||m(t)-\bar{m}_{b(t)}||^2$ decays to zero exponentially fast.

The point here is that we get estimates on how fast this is happening which are *uniform in the profile* within the classes $\mathcal{K}(M,N)$, so that the relaxation is taking place at a uniform rate everywhere on the interface. This is what is needed to put together the local profile changes and understand the motion of the interface as a whole. It is hoped that the better control obtained this way will lead to a derivation of motion by mean curvature without assumptions on the behavior of the macroscopic equations.

We now turn to proving these results.

3. Proof of Theorems 1, 2 and 3

A key ingredient in the proof is a spectral analysis of the linear operator associated to the second Frechet derivative of the free energy $\mathcal{F}$. This analysis has been done in [8]. The relevant results are these:

Let $\bar{m}(x)$ be an instanton, let $\mathrm{d}\nu$ be the measure equivalent to the Lebesgue measure

$$\mathrm{d}\nu = \frac{\mathrm{d}x}{(1-\bar{m}^2(x))} \tag{3.1}$$

and let $\mathcal{H}$ be the Hilbert space $L^2(\mathrm{I\!R}, \mathrm{d}\nu)$ with the $\|\cdot\|_{\mathcal{H}}$ norm. Let $\mathcal{L}$ be the operator

$$\mathcal{L}v = -v + (1-\bar{m}^2)\beta J \star v \tag{3.2}$$

on $\mathcal{H}$.

The connection with the free energy is as follows: Let us denote by $\mathcal{F}'(m)v$ and $\mathcal{F}''(m)(v,w)$ respectively the first and the second Frechet derivative of $\mathcal{F}$ computed at m in the directions $v,w\in\mathcal{H}$.

It is easy to see that for $\|m(\cdot)\|_\infty \leq c < 1$ repeated Frechet derivativesexist with

$$\mathcal{F}'(m)v = \int_{\mathbb{R}} \big[\frac{1}{\beta} \tanh^{-1} m(x) - J * m(x)\big] v(x) \mathrm{d}x$$

and

$$\mathcal{F}''(m)(v,w) = \int_{\mathbb{R}} \big[\frac{1}{\beta} \frac{v(x)w(x)}{(1-m^2(x))} - J * w(x)v(x)\big] \mathrm{d}x$$

Note that $\mathcal{F}'(\bar{m}) = 0$ and

$$\mathcal{F}''(\bar{m})(v,w) = -\frac{1}{\beta} \int_{\mathbb{R}} (\mathcal{L}w)(x)v(x)\mathrm{d}\nu \tag{3.3}$$

Hence the spectral properties of $\mathcal{L}$ govern the local convexity of the free energy. The result concerning these, proved in [8], is that

Lemma 1 (proved in [8]) *Let $\mathcal{L}$ be the operator on $\mathcal{H}$ defined in (3.2). Then:*
i) $\mathcal{L}$ is a bounded symmetric operator;
ii) $\mathcal{L}$ is negative semidefinite, 0 is a simple eigenvalue and $(\bar{m})'$ its eigenfunction;
iii) (gap property) 0 is an isolated eigenvalue, namely there exists $\alpha > 0$ such that

$$< v, \mathcal{L}v >_{\mathcal{H}} \leq \ -\alpha \|v\|^2_{\mathcal{H}}, \qquad \text{for all } v \in (\mathrm{Null}\mathcal{L})^{\perp} \tag{3.4}$$

In the proofs that follow, we apply Lemma 1 to (3.3) with $v = w = m - \bar{m}$. Unfortunately, $m - \bar{m}$ does not belong to $(\mathrm{Null}\mathcal{L})^{\perp}$ since $m - \bar{m}$ is only orthogonal to $\bar{m}'$ in L^2, the metric in which we minimized the distance to the instanton $\bar{m}$. However, it does have a significant component that is orthogonal to $\mathrm{Null}\mathcal{L}$ in $\mathcal{H}$. That is the content of the following lemma:

Lemma 2 *Suppose that* $\int (m-\bar{m})\bar{m}' \mathrm{d}x = 0$. *Let* P *be the orthogonal projection in* $\mathcal{H}$ *onto the orthogonal complement, in* $\mathcal{H}$, *of* $\mathrm{Null}\mathcal{L}$. *Then*

$$\|P(m-\bar{m})\|_{\mathcal{H}} \geq \sqrt{1-m_\beta^4}\|m-\bar{m}\|_{\mathcal{H}}$$

It is a direct consequence of this lemma that

$$\mathcal{F}''(\bar{m})(m-\bar{m}, m-\bar{m}) \geq \frac{\alpha}{\beta}(1-m_\beta^4)\|m-\bar{m}\|_{\mathcal{H}}^2 \tag{3.5}$$

Proof of Lemma 2: Clearly, by the hypothesis,

$$< m-\bar{m}, \bar{m}' >_{\mathcal{H}} = \int_{\mathbb{R}} (m-\bar{m})\frac{1-\bar{m}^2+\bar{m}^2}{1-\bar{m}^2}\bar{m}'\mathrm{d}x =$$
$$< \bar{m}^2(m-\bar{m}), \bar{m}' >_{\mathcal{H}}$$

Hence, by the Schwarz inequality,

$$|< m-\bar{m}, \bar{m}' >_{\mathcal{H}}| =< \bar{m}^2(m-\bar{m}), \bar{m}' >_{\mathcal{H}} \leq m_\beta^2\|m-\bar{m}\|_{\mathcal{H}}\|\bar{m}'\|_{\mathcal{H}}$$

Therefore, if $P^\perp$ is the orthogonal projection in $\mathcal{H}$ onto the span of $\bar{m}'$,

$$\|P^\perp(m-\bar{m})\|_{\mathcal{H}} \leq m_\beta^2\|m-\bar{m}\|_{\mathcal{H}} \ .$$

The lower bound on $\|P(m-\bar{m})\|_{\mathcal{H}}$ follows directly from this. $\square$

Using these results, we now prove Theorem 1.

Proof of Theorem 1. We can represent

$$\mathcal{F}(m)-\mathcal{F}(\bar{m}) = \int_0^1 \mathcal{F}'(\bar{m}+\tau(m-\bar{m}))(m-\bar{m})\mathrm{d}\tau$$

$$= \int_0^1 \int_0^\tau \mathcal{F}''(\bar{m}+s(m-\bar{m}))(m-\bar{m}, m-\bar{m})\mathrm{d}s\mathrm{d}\tau$$

In order to get a lower bound for the last term above we expand $\mathcal{F}''\big(\bar{m}+s(m-\bar{m})\big)(m-\bar{m},m-\bar{m})$ about $s=0$ obtaining

$$\mathcal{F}''\big(\bar{m}+s(m-\bar{m})\big)(m-\bar{m},m-\bar{m}) =$$

$$\mathcal{F}''\big(\bar{m})(m-\bar{m},m-\bar{m}) + \mathcal{F}'''\big(\widetilde{m}\big)(m-\bar{m},m-\bar{m},m-\bar{m})$$

where $\widetilde{m} = \bar{m} + s_0(m-\bar{m})$ for some s_0 between 0 and 1 by the mean value theorem. By (3.5)

$$\mathcal{F}''\big(\bar{m}+s(m-\bar{m})\big)(m-\bar{m},m-\bar{m})$$

$$\geq \frac{\alpha}{\beta}(1-m_\beta^4)\|m-\bar{m}\|_{\mathcal{H}}^2 + \mathcal{F}'''\big(\widetilde{m}\big)(m-\bar{m},m-\bar{m},m-\bar{m})$$

Now we need a lower bound on the term involving the third derivative of the free energy. But by direct computation,

$$\begin{aligned} &\big|\mathcal{F}'''\big(\widetilde{m}\big)(m-\bar{m},m-\bar{m},m-\bar{m})\big| = \\ &\frac{2}{\beta}\Big|\int_{\mathbb{R}} \frac{\widetilde{m}}{(1-\widetilde{m}^2)^2}\big(m(x)-\bar{m}(x)\big)^3 \mathrm{d}x\Big| \\ &\leq K_{\beta,c}\int_{\mathbb{R}} |m-\bar{m}|^3 \mathrm{d}x \end{aligned} \tag{3.6}$$

for some constant $K_{\beta,c}$ depending only on β and c, the bound on $\|m(\cdot)\|_\infty$.

To bound this in terms of $\|m-\bar{m}\|_{L^2}$, and hence $\|m-\bar{m}\|_{\mathcal{H}}^2$, note that for any x,

$$\big(m-\bar{m}\big)^2(x) = 2\int_{-\infty}^{x}\big(m-\bar{m}\big)\big(m-\bar{m}\big)'\mathrm{dr} \leq$$

$$2\|m-\bar{m}\|_{L^2}\|m-\bar{m}\|_{\mathrm{H}^1}$$

Since membership in $\mathcal{K}(M,N)$ provides a bound on $\|m-\bar{m}\|_{\mathrm{H}^1}$, we have that the right hand side of (3.6) can be replaced by

$$\widetilde{K}_{\beta,c}\big(\|m-\bar{m}\|_{\mathcal{H}}^2\big)^{5/2}\ .$$

Now if $\|m-\bar{m}\|_{\mathcal{H}}$ is sufficiently small, since $5/2>2$, it is clear that

$$\mathcal{F}''\big(\bar{m}+s(m-\bar{m})\big)(m-\bar{m},m-\bar{m}) \quad \geq \quad D_{\beta,c}\alpha\|m-\bar{m}\|^2_{\mathcal{H}}$$

where $D_{\beta,c}$ is a constant depending only on β and c. Suppose the $\epsilon>0$ is a number so that this is true whenever $\|m-\bar{m}\|_{\mathcal{H}}\leq\epsilon$. Then we have the desired lower bound for such m.

Next consider the functional

$$m \mapsto \frac{\mathcal{F}(m)-\mathcal{F}(\bar{m}_a)}{\|m-\bar{m}_a\|^2_{\mathcal{H}}} \tag{3.7}$$

on the set $\mathcal{K}(M,N)\cap\{m \mid \|m-\bar{m}\|^2_{\mathcal{H}}\geq\epsilon\}$. Then the functional in (3.7) is continuous on this compact set, and therefore has a minimizer $\widehat{m}$ within it. $\mathcal{F}(\widehat{m})$ cannot be zero, since we are excluding in $\mathcal{K}(M,N)$ a neighborhood of $\bar{m}$, and therefore we have

$$\inf\left\{\frac{\mathcal{F}(m)-\mathcal{F}(\bar{m}_a)}{\|m-\bar{m}_a\|^2_{\mathcal{H}}} \mid m\in\mathcal{K}(M,N)\cap\{m \mid \|m-\bar{m}\|^2_{\mathcal{H}}\geq\epsilon\}\right\} =$$

$$B>0$$

From this we conclude that

$$\mathcal{F}(m)-\mathcal{F}(\bar{m}_a)\geq\min(B,D_{\beta,c}\alpha)\|m-\bar{m}\|^2_{\mathcal{H}} \tag{3.8}$$

for all $m\in\mathcal{K}(M,N)$.

The other side of the inequality in Theorem 1 is relatively simple – it follows by the same integration of second derivatives simply using the boundedness of $\mathcal{L}$. $\square$

Proof of Theorem 2 Again, recall, as we have quoted from [7], that, for all $t_0\geq0$, $m(\cdot,t_0)$ is continuous with $\|m(\cdot,t_0)\|_\infty\leq c<1$, and (1.11) is satisfied for all $t>0$. Then the Cauchy problem for (1.8) is well posed, and the solution continues to satisfy (1.11), and $\|m(\cdot,t)\|_\infty\leq c<1$ for all $t>t_0$. Hence, even if our bound on the sup norm of the intitial data didn't hold, we could obtain it simply by waiting. Nonetheless, we include this bound explicitly in the conditions since the constants depend on it.

Moreover, $\mathcal{F}\big(m(\cdot,t)\big)$ is well defined for all $t \geq 0$, it is differentiable with respect to t if $t > 0$ and

$$\frac{d}{dt}\mathcal{F}\big(m(\cdot,t)\big) = -I\big(m(\cdot,t)\big) \leq 0 \tag{3.9}$$

where, for any continuous function $h \in C_b(\mathbb{R})$ with $\|h\|_\infty < 1$,

$$I\big(h(\cdot)\big) = \int_{\mathbb{R}} [(J\star h)(x) - \beta^{-1}\tanh^{-1}h(x)][\tanh\beta(J\star h)(x) - h(x)]\mathrm{d}x \tag{3.10}$$

The integrand in $I(h)$ is a non negative function which is in $L_1(dx)$ when h satisfies (1.11). Finally, for all $t \geq t_0$

$$\mathcal{F}\big(m(\cdot,t)\big) - \mathcal{F}\big(m(\cdot,t_0)\big) = -\int_{t_0}^{t} I\big(m(\cdot,s)\big)\mathrm{d}s \leq 0 \tag{3.11}$$

Let $I'(m)v$ and $I''(m)(v,w)$ denote respectively the first and the second Frechet derivatives of I as before with $\mathcal{F}$. One readily calculates that

$$\begin{aligned} &I'(m)v = \\ &\int_{\mathbb{R}} \big[-\frac{1}{\beta}\tanh^{-1} m(r) + (J\star m)(r)\big]\big[\frac{\beta(J*v)(r)}{\cosh^2\beta(J*m)(r)} - v\big]\mathrm{d}r + \\ &\int_{\mathbb{R}} \big[\tanh\beta(J*m)(r) - m\big]\big[(J*v)(r) - v(r)/(\beta(1-m^2(r))\big]\mathrm{d}r \end{aligned} \tag{3.12}$$

and

$$I''(m)(v,w) =$$

$$
\begin{aligned}
&2\int_{\mathbb{R}}\big[-\frac{1}{\beta}\tanh^{-1}m(r)+(J*m)(r)\big]\times\\
&\qquad\big[\frac{\beta^2 J*v(r)(J*w)(r)}{\cosh^2\beta(J*m)(r)}\tanh\beta(J*m)(r)\big]\,\mathrm{d}r+\\
&\int_{\mathbb{R}}\big[\frac{\beta(J*w)(r)}{\cosh^2\beta(J*m)(r)}-w(r)\big]\,\big[(J*v)(r)-\frac{v(r)}{\beta(1-m^2(r))}\big]\,\mathrm{d}r+\\
&\int_{\mathbb{R}}\big[\frac{\beta(J*v)(r)}{\cosh^2\beta(J*m)(r)}-v(r)\big]\,\big[(J*w)(r)-\frac{w(r)}{\beta(1-m^2(r))}\big]\,\mathrm{d}r\\
&-2\int_{\mathbb{R}}\big[\tanh\beta(J*m)(r)-m(r)\big]\,\frac{m(r)v(r)w(r)}{\beta(1-m^2(r))^2}\mathrm{d}r
\end{aligned}
$$

Note that $I(\bar{m})=I'(\bar{m})=0$ and

$$
I''(\bar{m})<v,v>=2\int_{\mathbb{R}}(\mathcal{L}v)(x)(\mathcal{L}v)(x)\mathrm{d}\nu\geq\alpha^2\|v\|^2_{\mathcal{H}} \tag{3.13}
$$

for all $v\in\mathrm{Null}(\mathcal{L})^{\perp}$.

As in our previous analysis of $\mathcal{F}$, we have

$$
I(m)=\int_0^1 I'\big(\bar{m}+\tau(m-\bar{m})\big)(m-\bar{m})\mathrm{d}\tau
$$

$$
=\int_0^1\int_0^\tau I''\big(\bar{m}+s(m-\bar{m})\big)(m-\bar{m},m-\bar{m})\mathrm{d}s\mathrm{d}\tau
$$

As before with $\mathcal{F}$, one can directly compute the third derivative I''', and again, one finds that as between (3.6) and (3.7),

$$
\big|\mathcal{I}'''\big(\widetilde{m}\big)(m-\bar{m},m-\bar{m},m-\bar{m})\big|\leq\widetilde{K}_\beta\big(\|m-\bar{m}\|_{\mathcal{H}}\big)^{5/2} \tag{3.14}
$$

for some constant $\widetilde{K}_{\beta,c}$ depending only on β and c.

As before, if $\|m-\bar{m}\|_{\mathcal{H}}$ is sufficiently small, it is clear from (3.13) that

$$
I''\big(\bar{m}+s(m-\bar{m})\big)(m-\bar{m},m-\bar{m})\quad\geq\quad(d_{\beta,c}\alpha^2)\|m-\bar{m}\|^2_{\mathcal{H}}
$$

where $d_{\beta,c}$ is a constant depending only on β and c. Fix an $\epsilon > 0$ such that this is true whenever $\|m - \bar{m}\|_{\mathcal{H}} \leq \epsilon$. Then we have the result for such m.

Next, the functional

$$m \mapsto I(m)/\|m - \bar{m}\|^2_{\mathcal{H}} \tag{3.15}$$

is continuous on the compact set

$$\mathcal{K}(M,N) \cap \{m \mid \|m - \bar{m}\|^2_{\mathcal{H}} \geq \epsilon\} \ ,$$

and hence has a minimizer $\widehat{m}$ in it. Clearly $I(\widehat{m}) > 0$.

We therefore have that

$$\inf\left\{ \frac{I(m)}{\|m - \bar{m}\|^2_{\mathcal{H}}} \mid m \in \mathcal{K}(M,N) \cap \{m \mid \|m - \bar{m}\|^2_{\mathcal{H}} \geq \epsilon\} \ \right\} = B_1 > 0$$

We conclude from this that

$$I(m) \geq \min(B_1, (d_\beta \alpha^2))\|m - \bar{m}\|^2_{\mathcal{H}}$$

for all $m \in \mathcal{K}(M,N)$. This completes the proof. □

Proof of Theorem 3. We begin with the bound on the on the L^2 norm. In reference [9] such a bound is proved for *compact time intervals*. If one keeps track of the constants in that proof, one sees that it actually provides the desired bound *uniformly* in t. However, since this is not pointed out there, we include an explicit proof here for completeness.

First observe that if f, g and h are any square integrable real valued functions such that

$$f(x) \leq g(x) \leq h(x)$$

almost everywhere, then

$$|g(x)| \leq \max\{|f(x)|, |h(x)|\}$$

so that

$$\|g\|_{L^2} \leq \|f\|_{L^2} + \|h\|_{L^2} \ .$$

Now observe that

$$\lim_{b\to\infty} \int_{\mathbb{R}} |(\bar{m}_b(x) - m_0(x))_+|^2 \mathrm{d}x = 0$$

where $(\cdot)_+$ denotes the positive part function. Hence, for any fixed $\epsilon > 0$ we can choose a finite b such that

$$\|(\bar{m}_b - m_0)_+\|_{L^2} \leq \epsilon \ . \tag{3.16}$$

In the same way, we can choose a finite a so that

$$\|(m_0 - \bar{m}_a)_+\|_{L^2} \leq \epsilon \tag{3.17}$$

Then we have

$$\bar{m}_b - (\bar{m}_b - m_0)_+ \leq m \leq \bar{m}_a + (m_0 - \bar{m}_a)_+$$

Next we use the fact that the time evolution specified by (1.8) is order preserving. Hence if $m(x,t)$ is the solution of (1.8) with initial data $m_0(x)$, and if $m_1(x,t)$ and $m_2(x,t)$ are the solutions of (1.8) with initial data $\bar{m}_b(x) - (\bar{m}_b(x) - m_0(x))_+$ and $\bar{m}_a + (m_0(x) - \bar{m}_a(x))_+$ respectively, then

$$m_1(x,t) \leq m(x,t) \leq m_2(x,t)$$

almost everywhere in x for all $t \geq 0$, and hence

$$m_1(x,t) - m_\beta \mathrm{sgn}(x) \leq m(x,t) - m_\beta \mathrm{sgn}(x) \leq m_2(x,t) - m_\beta \mathrm{sgn}(x)$$

By the argument recalled at the beginning of the proof

$$\begin{aligned} &\|m(\cdot,t) - m_\beta \mathrm{sgn}(\cdot)\|_{L^2} \leq \\ &\|m_1(\cdot,t) - m_\beta \mathrm{sgn}(\cdot)\|_{L^2} + \|m_2(\cdot,t) - m_\beta \mathrm{sgn}(\cdot)\|_{L^2} \end{aligned} \tag{3.18}$$

Now, by the local non-linear stability of the instantons [8], there is a value of $\epsilon > 0$ so that the conditions (3.16) and (3.17) imply that

$$\sup_{t\geq 0} \|m_1(\cdot,t) - \bar{m}_b\|_{L^2} \leq 1 \qquad \text{and} \qquad \sup_{t\geq 0} \|m_2(\cdot,t) - \bar{m}_a\|_{L^2} \leq 1$$

This together with (3.18) yields

$$\|m(\cdot,t) - m_\beta \mathrm{sgn}(\cdot)\|_{L^2} \leq \|\bar{m}_a - m_\beta \mathrm{sgn}(\cdot)\|_{L^2} + \|\bar{m}_b - m_\beta \mathrm{sgn}(\cdot)\|_{L^2} + 2$$

for all $t \geq 0$, which is the bound we seek.

Next, we note that in Prop. 2.2 of [9] it is proved that the Lipschitz norm of $m(\cdot,t)$, actually $\|m'(\cdot,t)\|_{L^\infty}$, is bounded uniformly in t. Using this result and similar methods, we now prove a uniform bound on $\|m'(\cdot,t)\|_{L^2}$.

Let $\bar{m}$ denote any fixed instanton, and define

$$\phi(x,t) = m(x,t) - \bar{m}(x) \ .$$

Then rewriting (1.8) as an integral equation one obtains

$$\phi(x,t) = e^{-t}\phi(x,0) + \int_0^t e^{s-t}\Big[\tanh(\beta(J\star m)(x,s) - \bar{m}(x)\Big]\,\mathrm{d}s$$

By (1.12), the integral on the right is equal to

$$\int_0^t e^{s-t}\Big[\tanh(\beta(J\star m)(x,s) - \tanh(\beta(J\star \bar{m})(x)\Big]\,\mathrm{d}s \tag{3.19}$$

Then applying the identity

$$\tanh(A) - \tanh(B) = (1 - \tanh(A)\tanh(B))\tanh(A-B)$$

to the integrand in (3.19) and differentiating, we obtain

$$\Big|\frac{\mathrm{d}}{\mathrm{d}x}(\tanh(\beta(J\star m)(x,s) - \tanh(\beta(J\star \bar{m})(x))\Big| \leq$$

$$(|\beta J\star m'| + |\beta J\star \bar{m}'|)|\beta J\star\phi| + 2|\beta J'\star\phi|$$

Now by the uniform bound on $\|m'(\cdot,t)\|_{L^\infty}$ from Proposition 2.2 of [9],

$$\sup_{t>0}\left\{|\beta J \star m'(\cdot,t)| + |\beta J \star \bar{m}'(\cdot)|\right\} \leq K$$

for some finite constant K. Also clearly

$$\|\beta J' \star \phi\|_{L^2} \leq \beta \|J'\|_{L^1}\|\phi\|_{L^2}$$

and thus by the Minkowski inequality

$$\|\phi'(\cdot,t)\|_{L^2} \leq e^{-t}\|\phi'(\cdot,0)\|_{L^2} + (1-e^{-t})\beta(K + 2\|J'\|_{L^1})\|\phi(\cdot,t)\|_{L^2}$$

By the first part of the proof, $\|\phi(\cdot,t)\|_{L^2}$ is bounded uniformly in t, and hence we have the desired bound. $\square$

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