SPECTRAL PROPERTIES OF INTEGRAL OPERATORS IN BOUNDED, LARGE INTERVALS.

ENZA ORLANDI

Abstract. We study the spectrum of one dimensional integral operators in bounded real intervals of length $2L$, for value of L large. The integral operators are obtained by linearizing a non local evolution equation for a non conserved order parameter describing the phases of a fluid. We prove a Perron-Frobenius theorem showing that there is an isolated, simple minimal eigenvalue strictly positive for L finite, going to zero exponentially fast in L. We lower bound, uniformly on L, the spectral gap by applying a generalization of the Cheeger's inequality. These results are useful for deriving spectral properties for non local Cahn-Hilliard type of equations in problems of interface dynamics, see [16].

1. Introduction

We study the spectrum of an integral operator acting on L^2 functions defined in intervals $[-L, L] \subset \mathbb{R}$, for value of L large. This problem arises when analyzing layered equilibria and front dynamics for the conservative, nonlocal, quasilinear evolution equation typified by

$$\partial_t m(t,x) = \nabla \cdot \left\{ \nabla m(t,x) - \beta(1 - m(t,x)^2)(J \star \nabla m)(t,x) \right\}, \tag{1.1}$$

where $\beta > 1$,

$$(J \star m)(x) = \int_{\mathbb{R}} J(x,y) m(y) dy$$

and $J(\cdot,\cdot)$ is a regular, symmetric, translational invariant, non negative function with compact support and integral equal to one. This equation (1.1) first appeared in the literature in a paper [14] on the dynamics of Ising systems with a long–range interaction and so–called "Kawasaki" or "exchange" dynamics and later it was rigorously derived in [11]. In this physical context, $m(x,t) \in [-1,1]$ is the spin magnetization density. It has been formally shown by Giacomin and Lebowitz [12], that in the sharp interface limit, i.e the limit in which the phase domain is very large with respect to the size of the interfacial region and time is suitable rescaled, the limit motion is given by Mullins Sekerka motion, a quasi-static free boundary problem in which the mean curvature of the interface plays a fundamental role. Equation (1.1) could be considered as a non local type of Cahn-Hilliard equation. Our intention is to provide basic spectral estimates useful for deriving higher dimensions spectral results in order to establish rigorously the relation between (1.1) and the singular limit motion described by the Mullins Sekerka equations, see [16]. We recall some previous results useful to better contextualize the problem. When $\beta > 1$ there is a phase transition in the underlying spin system, [15]. The pure phases correspond to the stationary spatially homogeneous solutions of (1.1) satisfying

$$m = \tanh \beta m.$$

For $\beta > 1$ there are three and only three different roots denoted

$$\pm m_\beta, 0.$$

Date: November 19, 2014.
1991 *Mathematics Subject Classification.* Primary 60J25; secondary 82A05.
Key words and phrases. Integral operators, Cheeger's constant, interfaces .
E.O. supported by ROMA TRE University.

The two phases $\pm m_\beta$ are thermodynamically stable while $m = 0$ is unstable. These statements, established in the context of the theory of Equilibrium Statistical Mechanics, see [15], are reflected by the corresponding stability properties of the space homogeneous solution of (1.1), see [12]. Equations (1.1) has also stationary solutions connecting the two coexisting phases: they are all identical modulo translations and reflection, see [12], to the "istanton" $\bar{m}(\cdot)$ which is $C^\infty(\mathbb{R})$, strictly increasing, antisymmetric function which identically verifies

$$\bar{m}(x) = \tanh \beta (J \star \bar{m})(x), \qquad x \in \mathbb{R}. \tag{1.2}$$

$\bar{m}(\cdot)$ is the stationary pattern that connects the minus and the plus phases as

$$\lim_{x \to \pm\infty} \bar{m}(x) = \pm m_\beta, \tag{1.3}$$

and it can be interpreted as a diffuse interface. The first results on these stationary patterns were obtained when analyzing the non conservative equation

$$\partial_t m(t,x) = -m(t,x) + \tanh \beta (J \star m)(t,x). \tag{1.4}$$

Equation (1.4) has been derived from the Glauber (non conservative) dynamic of an Ising spin system interacting via a Kac potential, see [6]. Since both the equations (1.1) and (1.4) have been derived from the same Ising spin systems, the first by a conservative dynamic the latter by a non conservative one, both have as equilibrium solutions the homogeneous solution $\pm m_\beta$ and the stationary patterns connecting the two homogeneous phases. Stability properties of $\bar{m}$ has been derived either for the conservative evolution (1.1), see [2], [3] and [4] either for the nonconservative evolution (1.4), see [8]. We recall only previous results which are used in this paper. As proved in [8] the interface described by the istanton is "stable" for equation (1.4) and any initial datum "close to the instanton" is attracted and eventually converges exponentially to some translate of the instanton. Linearizing the evolution equation (1.4) at $\bar{m}$ one obtains the integral operator

$$\mathcal{L}v = v - \beta(1 - \bar{m}^2) J \star v \tag{1.5}$$

which is selfadjoint when $v \in L^2(\mathbb{R}, \frac{1}{\beta(1-\bar{m}^2)}dx)$. The spectrum of this operator has been studied in [7]. It has been proved that the spectrum of $\mathcal{L}$ is positive, the lower bound of the spectrum is 0 which is an eigenvalue of multiplicity one and the corresponding eigenvalue is $\bar{m}'(\cdot)$, i.e

$$\mathcal{L}\bar{m}' = 0. \tag{1.6}$$

The remaining part of the spectrum is strictly bigger then some positive number. In this paper we consider operators of the type of the operator $\mathcal{L}$ defined in (1.5) but acting over functions in bounded intervals $[-L, L]$, L large. It might be helpful to compare heuristically what we are doing with similar problems analyzed previously in the context of reaction diffusion equations and Cahn-Hilliard equations. Dividing by $\beta(1 - \bar{m}^2)$ the operator $\mathcal{L}$ we can define a new operator

$$\mathcal{G}v = \frac{v}{\beta(1-\bar{m}^2)} - J \star v = -\left[J \star v - v\right] + f''(\bar{m})v$$

where

$$f''(\bar{m}) = -1 + \frac{1}{\beta(1-\bar{m}^2)}$$

and

$$f(m) = -\frac{1}{2}m^2 + \frac{1}{\beta}\left[\left(\frac{1+m}{2}\right)\ln\left(\frac{1+m}{2}\right) + \left(\frac{1-m}{2}\right)\ln\left(\frac{1-m}{2}\right)\right]$$

is a double equal well potential. The operator $\mathcal{G}$ on $L^2(\mathbb{R}, dx)$ and the operator $\mathcal{L}$ on $L^2(\mathbb{R}, \frac{1}{\beta(1-\bar{m}^2)}dx)$ have the same spectrum. Assume that v is smooth, taking into account that J is symmetric and therefore the first moment is null, we have that $J \star v - v \simeq \Delta v$. Heuristically $-\left[J \star v - v\right] + f''(\bar{m})v$ is equal to $-\Delta v + f''(\bar{m})v$. So the problem we are dealing with is in the same spirit of the problem dealt by De Mottoni and Schatzman, see [10, subsection 5.4]. They studied the spectrum of $-\Delta v + W(\bar{\theta})v$ in the finite interval $[-L, L]$ with Neuman boundary conditions. We denoted by $W(\bar{\theta})$ the corresponding of $f''(\bar{m})$ in [10]. This was a basic result to obtain higher dimension spectral results for the Cahn-Hilliard equations, see for example [5] and [1]. In this paper we establish results for the spectrum of one dimensional integral operator in the finite interval $[-L, L]$. The main difficulty is to show that the spectral gap of our integral operator is bounded uniformly on L. This is achieved by applying a generalization of Cheeger's inequality, proven in [13] and lower bounding in our context the Cheeger's constant.

2. Notations and Results

Let $T_L = [-L, L]$ be a real interval, $L \geq 1$. We are actually interested when L is large.

2.1. **The interaction.** Let $J(x)$, $|x| \leq 1$ be a symmetric, translational invariant probability kernel, i.e $\int J(x)dx = 1$. We assume that $J \in C^1(\mathbb{R})$, i.e it is continuous differentiable. To define the interaction between x and y in $\mathbb{R}$ we set, by an abuse of notation, $J(x, y) = J(y - x)$. For a function v defined on T_L we set

$$(J \star_b v)(x) = \int_{T_L} J(x - y)v(y)dy. \tag{2.1}$$

The suffix b is to reminds the reader that the integral is on the bounded interval T_L. Notice $\int_{T_L} J(x, y)dy = b(x)$ with $b(x) \in [\frac{1}{2}, 1]$ for $x \in T_L$. There are other ways to derive from J an integral kernel acting only on functions on the bounded interval T_L. One is the following

$$J^{neum}(x, y) = J(x, y) + J(x, 2L - y) + J(x, -2L - y), \tag{2.2}$$

where $2L - y$ is the image of y under reflection on the right boundary $\{L\}$ and $-2L - y$ is the image of y under reflection on the left boundary $\{-L\}$. By the assumption on J, $J^{neum}(x, y) = J^{neum}(y, x)$ and $\int J^{neum}(x, y)dy = 1$ for all $x \in T_L$. The choice to define by boundary reflections the interaction (2.2) has the advantage to keep J^{neum} a symmetric probability kernel. This definition first appeared in the paper [9, Section 2] and it was called there "Neuman" interaction. In [9] the authors studied spectral properties of operators closely related to the operator $\mathcal{L}$, see (1.5), defined on the space of the continuous symmetric functions on $\mathbb{R}$, $C^{\text{sym}}(\mathbb{R})$.

We will consider in this paper operators with the integral kernel (2.1) acting on Hilbert spaces. We could denote (2.1) the Dirichelet interaction kernel. Our results can be, with minor modifications, immediately extended to the case when the integral kernel is J^{neum}.

2.2. **The istanton.** We call istanton the antisymmetric solution $\bar{m}$ of (1.2) with conditions at infinity given in (1.3). The function $\bar{m} \in C^\infty(\mathbb{R})$, it is strictly increasing, and there exist $a > 0$, $\alpha_0 > \alpha > 0$ and $c > 0$ so that

$$\begin{aligned} 0 < m_\beta^2 - \bar{m}^2(x) \leq ce^{-\alpha|x|}\ , \\ |\bar{m}'(x) - a\alpha e^{-\alpha|x|}| \leq ce^{-\alpha_0|x|}. \end{aligned} \tag{2.3}$$

A proof of these estimates and related results can be found in Chapter 8, Section 8.2 of the book [17].

2.3. **The Operator.** For $\beta > 1$ set $p(x) = \beta(1 - \bar{m}^2(x))$ where $\bar{m}$ is the istanton. By the properties of $\bar{m}$ we have that

$$\lim_{|x|\to\infty} p(x) = \beta(1 - m_\beta^2) < 1, \tag{2.4}$$

and

$$\beta \geq p(x) \geq \beta(1 - m_\beta^2) > 0, \qquad x \in \mathbb{R}. \tag{2.5}$$

Denote

$$\mathcal{H} = L^2(T_L, \frac{1}{p(x)}dx),$$

and for $v \in \mathcal{H}$ and $w \in \mathcal{H}$

$$\langle v, w\rangle = \int_{T_L} v(x) w(x) \frac{1}{p(x)} dx,$$

$$\|v\|^2 = \int_{T_L} v^2(x) \frac{1}{p(x)} dx. \tag{2.6}$$

To stress the dependence of $\mathcal{H}$ on L we will add, when needed, a suffix L, writing $\mathcal{H}_L$. We denote by

$$\|v\|_2, \qquad \|v\|_\infty,$$

respectively the $L^2(T_L, dx)$ and the $L^\infty(T_L, dx)$ norm of a function v. Let $\mathcal{L}^0$ be the operator acting on $\mathcal{H}$ as

$$(\mathcal{L}^0 g)(x) = g(x) - p(x)(J \star_b g)(x). \tag{2.7}$$

2.4. **Results.** The following results for the operator $\mathcal{L}^0$ hold for any fixed value of L large enough.

Theorem 2.1. *For any $\beta > 1$ there exists $L_1(\beta)$ so that for $L \geq L_1(\beta)$ the following holds.*

(0)The operator $\mathcal{L}^0$ is a bounded, quasi compact, selfadjoint operator on $\mathcal{H}$.

(1) There exist $\mu_1^0 \in \mathbb{R}$ and $\psi_1^0 \in \mathcal{H}$, ψ_1^0 strictly positive in T_L so that

$$\mathcal{L}^0 \psi_1^0 = \mu_1^0 \psi_1^0.$$

The eigenvalue μ_1^0 has multiplicity one and any other point of the spectrum is strictly bigger than μ_1^0. There exist $c > 0$ independent on L so that

$$0 \leq \mu_1^0 \leq c e^{-2\alpha L}, \tag{2.8}$$

where $\alpha > 0$ is given in (2.3)*. Further $\psi_1^0 \in C^\infty(T_L)$, $\psi_1^0(z) = \psi_1^0(-z)$ for $z \in T_L$.*

(2) Let μ_2^0 be the second eigenvalue of $\mathcal{L}^0$. We have that

$$\mu_2^0 = \inf_{\langle \psi, \psi_1^0\rangle = 0; \|\psi\| = 1} \langle \psi, \mathcal{L}^0 \psi\rangle \geq D, \tag{2.9}$$

where $D > 0$ independent on L is given in (3.63)*.*

(3) Let ψ_1^0 be the normalized eigenfunction corresponding to μ_1^0 we have

$$\|\psi_1^0 - \frac{\bar{m}'}{\|\bar{m}'\|}\| \leq C e^{-2\alpha L}, \tag{2.10}$$

where $C > 0$ is a constant independent on L.

3. Proof of the results

To prove Theorem 2.1 we introduce the following auxiliary operators. Denote by $\mathcal{A}$ the linear integral operator acting on functions $g \in \mathcal{H}$

$$\mathcal{A} g(x) = p(x)(J \star_b g)(x). \tag{3.1}$$

We denote by $\mathcal{B}$ the operator acting on $L^2(\mathbb{R}, \frac{1}{p(x)} dx)$:

$$\mathcal{B} g(x) = p(x)(J \star g)(x). \tag{3.2}$$

The operator $\mathcal{B}$ has been studied in [7] and we will use that, recall (1.6),

$$\bar{m}'(x) = (\mathcal{B} \bar{m}')(x). \tag{3.3}$$

We have the following result.

Theorem 3.1. *Take $L \geq 1$. The operator $\mathcal{A}$ is a compact, selfadjoint operator on $\mathcal{H}$, positivity improving. Further, there exist $\nu_0 > 0$ and $v_0 \in \mathcal{H}$, v_0 strictly positive even function, so that*

$$\mathcal{A}v_0(x) = \nu_0 v_0(x) \qquad x \in T_L. \tag{3.4}$$

The eigenvalue ν_0 has multiplicity one and any other point of the spectrum is strictly inside the ball of radius ν_0. The eigenfunction v_0 is in $C^\infty(T_L)$.

Proof. It is immediate to see that

$$\langle \mathcal{A}v, w\rangle = \langle v, \mathcal{A}w\rangle.$$

The compactness can be shown by proving that any bounded set of $\mathcal{H}$ is mapped by $\mathcal{A}$ in a relatively compact set. Namely since $J(\cdot,\cdot)$ is continuous in $T_L \times T_L$ and T_L is compact, then $J(\cdot,\cdot)$ is uniformly continuous. Thus given $\epsilon > 0$, we can find $\delta > 0$ so that $|x-y| \leq \delta$ implies $|J(x,z) - J(y,z)| \leq \epsilon$ for all $z \in T_L$. The same holds for $p(\cdot)J(\cdot,z)$. Let $B_M = \{v \in \mathcal{H} : \|v\|^2 \leq M\}$. If $v \in B_M$ and $|x-y| \leq \delta$ we have

$$|(\mathcal{A}v)(x) - (\mathcal{A}v)(y)| \leq \epsilon c(\beta, J)\|v\| \leq \epsilon c(\beta, J)M,$$

where $c(\beta, J) > 0$ depends only on β and J. Therefore the functions $\mathcal{A}[B_M] = \{w \in \mathcal{H} : w = \mathcal{A}v,\ v \in B_M\}$ are equicontinuous. Since they are also uniformly bounded by $c(\beta)\|J\|_2 M$, where $c(\beta) > 0$, we can use the Ascoli theorem to conclude that for every sequence $\{v_n\} \in B_M$, the sequence $\{\mathcal{A}v_n\}$ has a convergent subsequence (the limit might not be in $\mathcal{A}[B_M]$) in $C[T_L]$ and therefore in $\mathcal{H}$. To show the positivity improving we take $v(z) \geq 0$, for $z \in T_L$, $v \neq 0$, and show that $(J \star_b v)(z) > 0$. Namely, assume that there exists $z^* \in T_L$ so that $(J \star_b v)(z^*) = 0$ then since $v(z) \geq 0$ for $z \in T_L$ and $J \geq 0$ we have that $v(z) = 0$ for $z \in (z^* - 1, z^* + 1)$. Repeating the same argument for $z \in (z^* - 1, z^* + 1)$ we obtain that $v(z) = 0$ for $z \in T_L$. In this way we obtain a contradiction. Therefore the positivity improving property is proven. From the hypothesis on J, it is easy to verify that for any given $L \geq 1$ there exists an integer n_L such that for $n \geq n_L$, there is $\zeta > 0$ so that for any x and y in T_L

$$\int_{T_L \times_n T_L} \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_n J(x, x_1)J(x_1, x_2)\ldots J(x_n, y) > \zeta. \tag{3.5}$$

The proof of (3.5) follows immediately from [9, Lemma 3.3]. So given $L \geq 1$ and $n \geq n_L$, where n_L is chosen so that (3.5) holds, denote for x and y in T_L

$$K(x, y) = p(x) \int \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_n p(x_1)J(x, x_1)p(x_2)J(x_1, x_2)\ldots p(x_n)J(x_n, y). \tag{3.6}$$

Then one can apply the classical Perron Frobenius Theorem to the kernel $K(\cdot,\cdot)$. As a consequence we have that the maximum eigenvalue of the spectrum of $\mathcal{A}$, which we denote ν_0, has multiplicity one and any other point of the spectrum of $\mathcal{A}$ is strictly smaller than ν_0. Further the eigenfunction associated to ν_0 does not change sign. So we assume that it is positive and we denote it v_0. Next we show that v_0 is even. Denote by $w(x) = v_0(-x)$. Since J and $p(\cdot)$ are even functions we have that

$$(\mathcal{A}w)(x) = (\mathcal{A}v_0)(-x) = \nu_0 v_0(-x) = \nu_0 w(x).$$

We then deduce that the function w is an eigenfunction associated to ν_0. Since ν_0 ha multiplicity one we must have that $w(x) = v_0(x)$. Therefore v_0 is even. Next we show that $v_0 \in C^\infty(T_L)$. We start proving that it is $C^1(T_L)$. Since $p(\cdot)$ is $C^\infty(\mathbb{R})$ and $J \in C^1(\mathbb{R})$ differentiating we obtain

$$\nu_0 v_0'(z) = p'(z)(J \star_b v_0)(z) + p(z)(J \star_b v_0)'(z). \tag{3.7}$$

Therefore $v_0 \in C^1(T_L)$. Since $(J \star_b v_0)'(z) = (J \star_b v_0')(z)$, differentiating again (3.7) we can show that $v_0 \in C^2(T_L)$. Repeating the argument we obtain $v_0 \in C^\infty(T_L)$. □

Lemma 3.2. (Lower bound on ν_0) *There exists positive constant $c>0$ independent on L so that for any $L \ge 1$*

$$\nu_0 \ge 1 - ce^{-2\alpha L}, \tag{3.8}$$

where $\alpha > 0$ is the constant in (2.3).

Proof. Consider the following trial function

$$h(x) = \frac{\bar{m}'(x)}{\|\bar{m}'\|}, \qquad -L \le x \le L. \tag{3.9}$$

By the variational formula for eigenvalue we have that

$$\nu_0 \ge \langle \mathcal{A}h, h\rangle.$$

$$\begin{aligned}\langle \mathcal{A}h, h\rangle &= \int \mathrm{d}x \frac{1}{p(x)} h(x)p(x)(J \star_b h)(x) \\ &= \frac{1}{\|\bar{m}'\|^2}\int \mathrm{d}x \frac{1}{p(x)}\bar{m}'(x)(\mathcal{B}\bar{m}')(x) + \int \mathrm{d}x \frac{1}{p(x)} h(x)p(x)\left[(J\star_b h)(x) - \frac{1}{\|\bar{m}'\|}(J\star \bar{m}')(x)\right],\end{aligned} \tag{3.10}$$

where $\mathcal{B}$ is the operator defined in (3.2). By (3.3) and (2.6) we have that

$$\frac{1}{\|\bar{m}'\|^2}\int \mathrm{d}x \frac{1}{p(x)}\bar{m}'(x)(\mathcal{B}\bar{m}')(x) = 1.$$

Further, since $\bar{m}'$ is even a we have

$$\int \mathrm{d}x h(x)\left[(J\star_b h)(x) - \frac{1}{\|\bar{m}'\|}(J\star\bar{m}')(x)\right] = 2\int_{L-1}^{L} \mathrm{d}x h(x)\left[(J\star_b h)(x) - \frac{1}{\|\bar{m}'\|}(J\star\bar{m}')(x)\right]. \tag{3.11}$$

For $x \in [L-1, L]$ we have

$$(J\star_b h)(x) - \frac{1}{\|\bar{m}'\|}(J\star\bar{m}')(x) = \frac{1}{\|\bar{m}'\|}\left[\int_{x-1}^{L} \mathrm{d}y J(x,y)\bar{m}'(y) - \int_{x-1}^{x+1}\mathrm{d}y J(x,y)\bar{m}'(y)\right].$$

We obtain

$$\int_{x-1}^{L} \mathrm{d}y J(x,y)\bar{m}'(y) - \int_{x-1}^{x+1}\mathrm{d}y J(x,y)\bar{m}'(y) = -\int_{L}^{x+1}\mathrm{d}y J(x,y)\bar{m}'(y) \ge -ce^{-\alpha L}, \tag{3.12}$$

since $\bar{m}'(\cdot)$ is strictly positive and exponentially decreasing, see (2.3), where we denoted by c a positive constant independent on L. Inserting (3.12) in (3.11) we obtain the lower bound (3.8). □

Remark 3.3. *It is easy to verify that if $\mathcal{A}$ is defined replacing $J\star_b$ with the integral kernel J^{neum} we will have $\nu_0 \ge 1 + ce^{-2\alpha L}$.*

Remark 3.4. *Since $\mathcal{A}$ is a bounded operator one can easily upper bound the eigenvalue ν_0. Taking into account that $\|v_0\| = 1$ and $\|\mathcal{A}v_0\|^2 \le \beta^2$ we have that*

$$\nu_0 = \langle \mathcal{A}v_0, v_0\rangle \le \|\mathcal{A}v_0\|\|v_0\| = \|\mathcal{A}v_0\| \le \beta.$$

In Lemma 3.7, given below, we will prove a more accurate upper bound, i.e $\nu_0 < 1$.

Next we show that the eigenfunctions associated to the principal eigenvalue ν_0 and to certain eigenvalues ν of $\mathcal{A}$ close to ν_0 decay exponentially fast when x large enough. The proof of this result is based on proving that there exists a point $r_0 \in (0, L)$ so that $\frac{p(r_0)}{\nu} < 1$. When $\nu \ge 1$ it is enough to find r_0 so that $p(r_0) < 1$. By (2.4), for L large enough, such r_0 always exists. But, the lower bound on ν_0 proven in Lemma 3.2 tell us that ν_0 might be smaller than one, although exponentially close to 1. To threat this case we introduce a cut -off ϵ_0, which depends on β. We will consider only $\nu \ge 1 - \frac{\epsilon_0}{2}$, for a suitable choice of ϵ_0. This allows to find r_0 depending on ϵ_0 so that $\frac{p(r_0)}{\nu} < 1$. Obviously the rate of decay of the eigenfunctions depends on the chosen cut- off.

Lemma 3.5. *For any $\epsilon_0 \in (0, \frac{(1-\sigma(m_\beta))}{2})$ there exists $r_0 = r_0(\epsilon_0)$ and $L_0 = L_0(\epsilon_0) > 0$ so that for $L \geq L_0$ the following holds. Let $\nu > 1 - \frac{\epsilon_0}{2}$ be an eigenvalue of the operator $\mathcal{A}$ on $\mathcal{H}$ and ψ be any of the corresponding normalized eigenfunctions. We have*

$$|\psi(x)| \leq Ce^{-\alpha(\epsilon_0)|x|} \qquad |x| \geq r_0, \tag{3.13}$$

where $\alpha(\epsilon_0)$ is given in (3.18).

Proof. Choose $r_0 = r_0(\epsilon_0) > 0$ such that

$$p(x) < 1 - \epsilon_0, \qquad |x| \geq r_0. \tag{3.14}$$

This is possible since $1 - \epsilon_0 > \sigma(m_\beta)$ and (2.4) holds. Choose L_0 large enough so that for $L \geq L_0$, $r_0 \leq \frac{L}{2}$. By assumption we have

$$\psi(x) = \frac{1}{\nu} p(x)(J \star_b \psi)(x), \qquad x \in T_L. \tag{3.15}$$

Note that for $|x| \geq r_0$, by (3.14) and since $\nu > 1 - \frac{\epsilon_0}{2}$

$$\frac{1}{\nu} p(x) < 1.$$

Take $x = r_0 + n$ where n is any integer so that $r_0 + 2n \leq L$. We have that

$$\psi(r_0 + n) = \frac{1}{\nu} p(r_0 + n)(J \star_b \psi)(r_0 + n). \tag{3.16}$$

Iterate n times (3.16). The support of the n fold convolution $(J)^n$ is the interval $[r_0, r_0 + 2n]$. Since $p(x)$ is decreasing we obtain that

$$\begin{aligned} |\psi(r_0 + n)| &\leq \left(\frac{1}{\nu} p(r_0)\right)^n |(J)^n \star_b \psi)(r_0 + n)| \leq \left(\frac{1}{1 - \frac{\epsilon_0}{2}} p(r_0)\right)^n \|J\|_2 \|\psi\|_2 \\ &\leq \beta \|J\|_2 e^{-n\alpha(\epsilon_0)} \|\psi\| = \beta \|J\|_2 e^{-n\alpha(\epsilon_0)} \end{aligned} \tag{3.17}$$

where

$$\alpha(\epsilon_0) = \ln\left(\frac{1 - \frac{\epsilon_0}{2}}{p(r_0)}\right) > 0. \tag{3.18}$$

□

Lemma 3.6. (Properties of v_0) *For any $\epsilon_0 \in (0, \frac{(1-\sigma(m_\beta))}{2})$ there exists $r_0 = r_0(\epsilon_0)$ and $L_1 = L_1(\epsilon_0) > 0$ so that the following holds. Take $L \geq L_1$ and let v_0 be the strictly positive normalized eigenfunction of $\mathcal{A}$ on $\mathcal{H}_L$ corresponding to ν_0, see Theorem 3.1. We have that*

$$v_0(x) \leq e^{-\alpha(\epsilon_0)|x|} \qquad |x| \geq r_0 \tag{3.19}$$

where $\alpha(\epsilon_0)$ is given in (3.18),

$$v_0'(x) < 0 \quad \text{for} \quad x \geq r_0, \qquad v_0'(x) > 0 \quad \text{for} \quad x \leq -r_0, \tag{3.20}$$

$$\gamma \geq \frac{v_0(x)}{v_0(y)} \geq \frac{1}{\gamma} \qquad |x - y| \leq 1, \quad \text{for any} \quad x, y \in T_L \tag{3.21}$$

where $\gamma = \gamma(\epsilon_0) > 1$ is defined in (3.28). *There exists $r_1 > 0$ and $\zeta_1 > 0$ independent of L so that*

$$v_0(x) \geq \zeta_1 \qquad \text{for} \quad |x| \leq r_1. \tag{3.22}$$

Proof. Fix $\epsilon_0 \in (0, \frac{(1-\sigma(m_\beta))}{2})$ and, as in Lemma 3.5, let $r_0 = r_0(\epsilon_0)$ be so that

$$p(x) < 1 - \epsilon_0, \qquad |x| \geq r_0. \tag{3.23}$$

Take $L_0(\epsilon_0)$ so that for $L \geq L_0(\epsilon_0)$, $r_0 \leq \frac{L}{2}$. By Lemma 3.2 for any $L \geq 1$, $\nu_0 \geq 1 - ce^{-2\alpha L}$, where $\alpha > 0$ is the constant in (2.3). Take $L_2(\epsilon_0)$ so large so that for $L \geq L_2(\epsilon_0)$,

$$\nu_0 \geq 1 - ce^{-2\alpha L} > 1 - \frac{\epsilon_0}{2}. \tag{3.24}$$

Set $L_1(\epsilon_0) = \max\{L_0(\epsilon_0), L_2(\epsilon_0)\}$. Then for $L \geq L_1(\epsilon_0)$, ν_0 satisfies the requirement of Lemma 3.5 and therefore v_0 decays exponentially, proving (3.19). Next we prove (3.20). Take $x \geq r_0$. Since v_0 satisfies the eigenvalue equation, see (3.4), we have that

$$\begin{aligned} v_0(x) &= \frac{1}{\nu_0} p(x)(J \star_b v_0)(x) \leq \frac{1}{\nu_0} p(x) \sup_{x-1 \leq y \leq \min\{x+1, L\}} v_0(y) \int_{x-1}^{x+1} J(x,y) \mathrm{d}y \\ &\leq \frac{1}{\nu_0} p(x) \sup_{x-1 \leq y \leq \min\{x+1, L\}} v_0(y) \leq \frac{1}{\nu_0} p(r_0) \sup_{x-1 \leq y \leq \min\{x+1, L\}} v_0(y) < \sup_{x-1 \leq y \leq \min\{x+1, L\}} v_0(y) \end{aligned} \tag{3.25}$$

since $\frac{1}{\nu_0} p(r_0) < 1$, see (3.23) and (3.24).The (3.25) says that v_0 cannot achieve a local maximum, when $x \geq r_0$. This implies that v_0 is strictly decreasing when $x \geq r_0$. Suppose, by contradiction, that v_0 increases when $x \geq r_0$. By (3.19), v_0 must decreases at some $x_0 \geq r_0$. Then v_0 would have a local maximum at x_0. By (3.25) this is impossible. Since v_0 is a symmetric function, it is strictly increasing for $x < -r_0$. Next we show (3.21). By (3.5) there are n and $\zeta > 0$ so that for any x, y, x' in T_L, such that $|x - y| \leq 1$ and $|x' - y| \leq 1$, $(J)^n(x, x') \geq \zeta$. Then, taking into account that $\nu_0 \leq \beta$, see Remark 3.4, we have

$$\begin{aligned} v_0(x) &= \frac{1}{\nu_0} p(x)(J \star_b v_0)(x) = \frac{1}{\nu_0^n} p(x) \int dx_1 \ldots \int dx_n p(x_1) \ldots p(x_{n-1}) J(x, x_1) \ldots J(x_{n-1}, x_n) v_0(x_n) \\ &\geq \frac{1}{\beta^n} \beta^n (1 - m_\beta^2)^n \zeta \int_{\max\{y-1, -L\}}^{\min\{y+1, L\}} dx' v_0(x'). \end{aligned} \tag{3.26}$$

On the other hand

$$v_0(y) = \frac{1}{\nu_0} p(y)(J \star_b v_0)(y) \leq \frac{\beta}{\nu_0} \int_{T_L} dx' J(y, x') v(x') \leq \frac{\beta}{\nu_0} \|J\|_\infty \int_{\max\{y-1, -L\}}^{\min\{y+1, L\}} dx' v_0(x'). \tag{3.27}$$

Therefore, by (3.24), for n large enough there exists $\gamma = \gamma(\epsilon_0) > 1$ so that

$$\frac{v_0(x)}{v_0(y)} \geq \frac{(1 - m_\beta^2)^n \zeta \nu_0}{\beta \|J\|_\infty} \geq \frac{(1 - m_\beta^2)^n \zeta (1 - \frac{\epsilon_0}{2})}{\beta \|J\|_\infty} \equiv \frac{1}{\gamma}. \tag{3.28}$$

We show (3.22). Since $v_0(x) > 0$ certainly $v_0(x) \geq c_L > 0$ for $x \in T_L$. We would like to show that there exists an interval independent on L so that for x in such an interval , $v_0(x) \geq \zeta > 0$ with $\zeta > 0$ independent on L. This is shown exploiting that v_0 is exponentially decreasing for $|x| \geq r_0$, see (3.19). Since

$$\|v_0\|^2 = 1$$

we must have that there exists $r_1 > 0$, independent on L, so that

$$\int_{-r_1}^{r_1} \frac{1}{p(x)} (v_0(x))^2 dx \geq \frac{1}{2}. \tag{3.29}$$

Then there exists $\zeta_1 > 0$ independent on L so that $v_0(x) \geq \zeta_1 > 0$ for $x \in [-r_1, r_1]$. Namely, suppose that this is false. Then there will be $x_0 \in [-r_1, r_1]$ such that $v_0(x_0) = 0$. Since $\mathcal{A}$ is positivity improving $v_0(x) = 0$ for all $x \in [-r_1, r_1]$. This is impossible since (3.29). □

Lemma 3.7. *We have that for any L large enough*

$$\nu_0 < 1. \tag{3.30}$$

Proof. Multiply (3.4) in $L^2(T_L, \frac{1}{p(x)}dx)$ with the trial function h introduced in (3.9) we have

$$\nu_0\langle v_0, h\rangle = \langle \mathcal{A}v_0, h\rangle = \langle v_0, \mathcal{A}h\rangle.$$

Write

$$\mathcal{A}h = \mathcal{B}\frac{\bar{m}'}{\|\bar{m}'\|} + p[J\star_b h - J\star\frac{\bar{m}'}{\|\bar{m}'\|}].$$

By (3.3) we have $\langle v_0, \mathcal{B}\frac{\bar{m}'}{\|\bar{m}'\|}]\rangle = \langle v_0, h\rangle$ and

$$\begin{aligned}\langle v_0, p[J\star_b h - J\star\frac{\bar{m}'}{\|\bar{m}'\|}]\rangle &= \int v_0(x)\frac{1}{\|\bar{m}'\|}[(J\star_b \bar{m}')(x) - (J\star\bar{m})(x)]dx\\ &= \frac{2}{\|\bar{m}'\|}\int_{L-1}^{L} \mathrm{d}x v_0(x)\left[(J\star_b \bar{m}')(x) - (J\star\bar{m}')(x)\right]\\ &= -\frac{2}{\|\bar{m}'\|}\int_{L-1}^{L}\mathrm{d}x v_0(x)\int_{L}^{x+1}\mathrm{d}y J(x,y)\bar{m}'(y) < 0\end{aligned} \tag{3.31}$$

since v_0 is an even positive function. Since (3.22) and $\bar{m}'$ is strictly positive and exponential decaying

$$\langle v_0, h\rangle \geq C$$

where C is a positive constant independent on L. We then have

$$\nu_0\langle v_0, h\rangle < \langle v_0, h\rangle$$

dividing by $\langle v_0, h\rangle$ we get (3.30). □

Theorem 3.1 shows that for any fixed L the operator $\mathcal{A}$ has a spectral gap, which might depend on L. We want to prove that the spectral gap can be upper bounded uniformly with respect to L. We achieve this following close the paper of Gregory Lawler and Alan Sokal, [13]. We apply a generalization of the Cheeger's inequality for positive recurrent continuous time jump processes and estimated the Cheeger's constant in our context. Denote by

$$Q(x,y) = \frac{p(x)}{\nu_0}J(x,y)\frac{v_0(y)}{v_0(x)} \qquad x, y \in T_L \tag{3.32}$$

and consider the operator

$$(Qf)(x) = \int Q(x,y)f(y)\mathrm{d}y \tag{3.33}$$

for $f \in L^2(T_L, \pi(x)dx)$ where

$$\pi(x) = \frac{v_0^2(x)}{p(x)}. \tag{3.34}$$

The operator Q is selfadjoint in $L^2(T_L, \pi(x)dx)$, it is a positivity-preserving linear contraction on $L^2(T_L, \pi(x)dx)$ [and in fact on all the space $L^p(T_L, \pi(x)dx)$]. The constant function 1 is an eigenfunction of Q with eigenvalue 1. Denote by T the map from $L^2(T_L, \frac{1}{p(x)}dx)$ to $L^2(T_L, \pi(x)dx)$, so that

$$Tf = \frac{f}{v_0}$$

The map T is an isometry, $\|f\|_{L^2(T_L, \frac{1}{p(x)}dx)} = \|Tf\|_{L^2(T_L, \pi(x)dx)}$, and

$$Qf = T\mathcal{A}T^{-1}f.$$

Therefore the spectrum of $\mathcal{A}$ is equal to the spectrum of Q.

Denote by $B = I - Q$ where I is the identity operator on $L^2(T_L, \pi(x)dx)$. We have the following obvious result.

Lemma 3.8. *The spectrum of B is equal to the spectrum of $I - \mathcal{A}$, where I is the identity operator on $L^2(T_L, \frac{1}{p(x)}dx)$.*

Next we show that the spectrum of B restricted to functions orthogonal in $L^2(T_L, \pi(x)dx)$ to the constant functions, so that $\int f(x)\pi(x)dx = 0$, is strictly positive. The gap is bounded by a constant independent on L. To short notation we denote for functions f and g in $L^2(T_L, \pi(x)dx)$

$$\int_{T_L} f(x)g(x)\pi(x)\mathrm{d}x = (f, g). \tag{3.35}$$

Denote by

$$\nu_1 = \inf_{\{f:(f,1)=0\}} \frac{(f, Bf)}{(f, f)}.$$

We will show that there exists a constant D independent on L so that $\nu_1 \geq D > 0$. We obtain this by applying [13, Theorem 2.1] and estimating the Cheeger's constant. First notice that the linear bounded operator B defined on $L^2(T_L, \pi(x)dx)$ can be written as

$$(Bg)(x) \equiv \int Q(x, y)[g(x) - g(y)]\mathrm{dy}, \tag{3.36}$$

i.e as the generator of a continuous time markovian jump process with transition rate kernel $Q(\cdot, \cdot)$ and invariant probability π. Define, see [13], the Cheeger's constant as

$$k \equiv \inf_{A \subset \mathcal{S}, 0 < \pi(A) < 1} k(A) \tag{3.37}$$

where $\mathcal{S}$ denotes the $\pi-$ measurable sets of T_L and

$$k(A) \equiv \frac{\int \pi(x)\mathrm{dx}\mathbb{1}_{\mathrm{A}}(\mathrm{x}) \left(\int \mathrm{Q}(\mathrm{x}, \mathrm{y})\mathbb{1}_{\mathrm{A^c}}(\mathrm{y})\mathrm{dy}\right)}{\pi(A)\pi(A^c)} = \frac{(\mathbb{1}_A, Q\mathbb{1}_{A^c})}{\pi(A)\pi(A^c)}. \tag{3.38}$$

Taking into account that $(\mathbb{1}_A, \mathbb{1}_{A^c}) = 0$ we can write (3.38) as the following:

$$k(A) = \frac{(\mathbb{1}_A, Q\mathbb{1}_{A^c})}{\pi(A)\pi(A^c)} = -\frac{(\mathbb{1}_A, B\mathbb{1}_{A^c})}{\pi(A)\pi(A^c)} = \frac{(\mathbb{1}_A, B\mathbb{1}_A)}{\pi(A)\pi(A^c)}. \tag{3.39}$$

Since

$$\int \pi(x)Q(x, y)\mathrm{dx} = \frac{1}{\nu_0}\mathrm{v_0(y)(J \star_b v_0)(y)} = \pi(\mathrm{y}) \tag{3.40}$$

$$\int \pi(x)Q(x, y)\mathrm{dy} = \frac{1}{\nu_0}\mathrm{v_0(x)(J \star_b v_0)(x)} = \pi(\mathrm{x}), \tag{3.41}$$

the constant M appearing in [13] in our case is simply $M = 1$. Next, we recall [13, Theorem 2.1], which in the present context reads:

Theorem 3.9. *([13]) Let B be a bounded selfadjoint operator on $L^2(T_L, \pi(x)dx)$ whose marginals can be estimated in term of the invariant measure, see (3.40) and (3.41). Then*

$$\kappa\frac{k^2}{8} \leq \nu_1 \leq k \tag{3.42}$$

where k is defined in (3.37), (3.38) and κ is a positive constant

$$\kappa \equiv \inf_{\mathcal{D}} \sup_c \frac{\left(\mathbb{E}|(X + c)^2 - (Y + c)^2|\right)^2}{\mathbb{E}|(X + c)^2|} \tag{3.43}$$

where the infimum is taken over all distributions $\mathcal{D}$ of i.i.d. real-valued random variable (X, Y) with variance 1.

It can be proved that $\kappa \geq 1$, see [13, Proposition 2.2]. The interesting and deeper part of the previous theorem is the lower bound of ν_1. It states that if there does not exists a set A for which the flow from A to A^c is unduly small then the Markov chain must have rapid convergence to equilibrium, or more precisely that B restricted to function orthogonal to the constant must have spectrum strictly positive. To estimate from below the Cheeger's constant it is convenient to make the following definitions.

Definition 1. *A family $\mathcal{S}_0 \subset \mathcal{S}$ is said to be dense if for all $A \in \mathcal{S}$ and all $\epsilon > 0$ there exists $A_0 \in \mathcal{S}_0$ such that $\pi(A \Delta A_0) \leq \epsilon$* [1].

Definition 2. *We say that the sets A and A_0 are separated (for the operator B) if*

$$\pi(A \cap A_0) = (1\!\!1_A, B 1\!\!1_{A_0}) = (1\!\!1_{A_0}, B 1\!\!1_A) = 0.$$

Theorem 3.10. *For $\beta > 1$ there exists $L_1(\beta)$ so that for $L \geq L_1(\beta)$ the Cheeger's constant associated to the operator B on $L^2(T_L, \pi(x)dx)$, defined in* (3.37) *and* (3.38) *is bounded below by a positive constant D, given in* (3.63)*, depending on β and on the interaction J, but independent on L so that*

$$k \geq D. \tag{3.44}$$

Proof. For $\beta > 1$, fix any $\epsilon_0 = \epsilon_0(\beta)$, $\epsilon_0 \in (0, \frac{(1-\sigma(m_\beta))}{2})$ and take $L_1(\beta)$ so that Lemma 3.6 holds. For any $L \geq L_1(\beta)$ we estimate the Cheeger's constant see (3.37) for the operator B. We need to evaluate the minimum over all measurable sets A of T_L. This would be rather difficult. By [13, Lemma 4.1] we can restrict to consider only subsets $A \subset \mathcal{S}_0$, $\mathcal{S}_0$ being a family of measurable sets dense in $\mathcal{S}$. Therefore we can restrict to consider countable unions of intervals. From the definition (3.39) it is enough to consider countable union of intervals $A = \cup_{i \in I} A_i$, pairwise separated of non zero $\pi-$ measure, such that A^c also has non zero $\pi-$ measure. Namely, by (3.39),

$$k(\cup_{i \in I} A_i) \geq \frac{\sum_i (1\!\!1_{A_i}, B 1\!\!1_{A_i})}{\pi(A)\pi(A^c)}.$$

The inequality is strict when $\{A_i, i \in I\}$ are not pairwise separated, see Definition 2. Further if $\{A_i, i \in I\}$ are pairwise separated of non zero $\pi-$ measure then, if $A = \cup_{i \in I} A_i$ and $\pi(A^c) \neq 0$ from point (b) of Lemma 4.3 of [13] we have that there exists $i_0 \in I$ so that

$$k(A_{i_0}) < k(A).$$

So it is enough to consider single intervals A, such that $\pi(A) < \frac{1}{2}$. Since the support of the interaction kernel $J(\cdot)$ is one, it is convenient distinguish two cases: $|A| \leq \frac{1}{2}$, $|A| > \frac{1}{2}$, where $|A|$ is the Lebesgue measure of A.

- Assume $|A| \leq \frac{1}{2}$. Take $A = [a, b]$. We have, see (3.38),

$$\begin{aligned} k(A) &\equiv \frac{\int \pi(x) \mathrm{dx} 1\!\!1_{\mathrm{A}}(\mathrm{x}) \left(\int_{\mathrm{T_L}} \mathrm{p(x) J(x,y)} \frac{\mathrm{v_0(y)}}{\mathrm{v_0(x)}} 1\!\!1_{\mathrm{A^c}}(\mathrm{y}) \mathrm{dy} \right)}{\pi(A)\pi(A^c)} \\ &\geq \frac{1}{\gamma} \beta (1 - m_\beta^2) \frac{\int \pi(x) \mathrm{dx} 1\!\!1_{\mathrm{A}}(\mathrm{x}) \left(\int_{\mathrm{b}}^{\mathrm{(x+1)\wedge L}} \mathrm{J(x,y) dy} + \int_{\mathrm{(x-1)\vee(-L)}}^{\mathrm{a}} \mathrm{J(x,y) dy} \right)}{\pi(A)\pi(A^c)} \\ &\geq \frac{1}{\gamma} \beta (1 - m_\beta^2) \inf_{\{x \in A\}} \left(\int_b^{(x+1)\wedge L} J(x,y) \mathrm{dy} + \int_{\mathrm{(x-1)\vee(-L)}}^{\mathrm{a}} \mathrm{J(x,y) dy} \right) \end{aligned} \tag{3.45}$$

[1] $A \Delta A_0 = (A \setminus A_0) \cup (A_0 \setminus A)$

where we used (2.5), (3.21) and $\pi(A^c) \le 1$. When $[a,b] \subset [-L+1, L-1]$ we have that

$$\inf_{\{x\in A\}} \left(\int_b^{(x+1)\wedge L} J(x,y)\mathrm{dy} + \int_{(\mathrm{x}-1)\vee(-\mathrm{L})}^{\mathrm{a}} \mathrm{J(x,y)dy} \right) = \inf_{\{x\in A\}} \left(\int_b^{(x+1)} J(x,y)\mathrm{dy} + \int_{(\mathrm{x}-1)}^{\mathrm{a}} \mathrm{J(x,y)dy} \right)$$
$$\ge 2\int_{\frac{1}{2}}^1 J(0,z)\mathrm{dz}, \tag{3.46}$$

by the translational invariance of J. Namely $J(x,y) = J(0, y-x)$ and translating the interval $[a,b]$ so that its middle point $\frac{b-a}{2}$ is at 0 the interval A is translated in an interval A' symmetric with respect to the origin. Set $A' = [-\frac{c}{2}, \frac{c}{2}]$, with $c = b - a \le \frac{1}{2}$, we have

$$\begin{aligned}
&\inf_{x\in A} \left(\int_b^{(x+1)} J(x,y)\mathrm{dy} + \int_{(\mathrm{x}-1)}^{\mathrm{a}} \mathrm{J(x,y)dy} \right) = \inf_{x\in A'} \left(\int_{\frac{c}{2}}^{(x+1)} J(x,y)\mathrm{dy} + \int_{(\mathrm{x}-1)}^{-\frac{c}{2}} \mathrm{J(x,y)dy} \right) \\
&= \left(\int_{\frac{c}{2}-x}^1 J(0,z)\mathrm{dz} + \int_{-1}^{-\frac{c}{2}-\mathrm{x}} \mathrm{J(0,z)dz} \right).
\end{aligned} \tag{3.47}$$

By the symmetry of J the infimum for $x \in [-\frac{c}{2}, \frac{c}{2}]$ of (3.47) is reached at $x = 0$. Since $c \le \frac{1}{2}$ we get (3.46). Assume now that $[a,b] \cap [L-1, L] \ne \emptyset$. We have for $x \in [a,b]$

$$\begin{aligned}
&\left(\int_b^{(x+1)\wedge L} J(x,y)\mathrm{dy} + \int_{(\mathrm{x}-1)\vee(-\mathrm{L})}^{\mathrm{a}} \mathrm{J(x,y)dy} \right) = \left(\int_b^{(x+1)\wedge L} J(x,y)\mathrm{dy} + \int_{(\mathrm{x}-1)}^{\mathrm{a}} \mathrm{J(x,y)dy} \right) \\
&\ge \int_{(x-1)}^a J(x,y)\mathrm{dy} \ge \int_{\mathrm{a}-\frac{1}{2}}^{\mathrm{a}} \mathrm{J(x,y)dy} = \int_{\frac{1}{2}}^1 \mathrm{J(0,y)dy}.
\end{aligned} \tag{3.48}$$

Similarly one proceeds when $[a,b] \cap [-L, -L+1] \ne \emptyset$. Therefore one concludes when $|A| \le \frac{1}{2}$

$$k(A) \ge \frac{1}{\gamma}\beta(1-m_\beta^2)\int_{\frac{1}{2}}^1 J(0,y) \equiv D_1. \tag{3.49}$$

- When $|A| > \frac{1}{2}$ we have

$$\begin{aligned}
k(A) &\equiv \frac{\int \pi(x)\mathrm{dx}\mathbb{1}_{\mathrm{A}}(\mathrm{x}) \left(\int \mathrm{Q(x,y)}\mathbb{1}_{\mathrm{A^c}}(\mathrm{y})\mathrm{dy} \right)}{\pi(A)\pi(A^c)} \\
&= \frac{\left(\int_a^b \pi(x)\mathrm{dx} \int_{\mathrm{b}}^{(\mathrm{x}+1)\wedge\mathrm{L}} \mathrm{Q(x,y)dy} + \int_{\mathrm{a}}^{\mathrm{b}} \pi(\mathrm{x})\mathrm{dx} \int_{(\mathrm{x}-1)\vee(-\mathrm{L})}^{\mathrm{a}} \mathrm{Q(x,y)dy} \right)}{\pi(A)\pi(A^c)} \\
&\ge \frac{1}{\gamma} \frac{\left(\int_{b-\frac{1}{2}}^b v_0^2(x)\mathrm{dx} \int_{\mathrm{b}}^{(\mathrm{x}+1)\wedge\mathrm{L}} \mathrm{J(x,y)dy} + \int_{\mathrm{a}}^{\mathrm{a}+\frac{1}{2}} \mathrm{v}_0^2(\mathrm{x})\mathrm{dx} \int_{(\mathrm{x}-1)\vee(-\mathrm{L})}^{\mathrm{a}} \mathrm{J(x,y)dy} \right)}{\pi(A)\pi(A^c)}
\end{aligned} \tag{3.50}$$

where we used (3.21). To lower bound the last term in (3.50) we distinguish two cases. The first when the interval $A \subset [-r_0, r_0]$, where r_0 is the positive real number (independent on L) introduced in Lemma 3.5. The second case when A is not a subset of $[-r_0, r_0]$. In the first case we have, see (3.50), and assuming without loss of generality that $r_0+1 \le L$, that

$$\begin{aligned}&\left(\int_{b-\frac{1}{2}}^{b} v_0^2(x)\mathrm{dx}\int_{\mathrm{b}}^{(\mathrm{x}+1)\wedge\mathrm{L}}\mathrm{J(x,y)dy}+\int_{\mathrm{a}}^{\mathrm{a}+\frac{1}{2}}\mathrm{v}_0^2(\mathrm{x})\mathrm{dx}\int_{(\mathrm{x}-1)\vee(-\mathrm{L})}^{\mathrm{a}}\mathrm{J(x,y)dy}\right)\\&=\left(\int_{b-\frac{1}{2}}^{b} v_0^2(x)\mathrm{dx}\int_{\mathrm{b}}^{(\mathrm{x}+1)}\mathrm{J(x,y)dy}+\int_{\mathrm{a}}^{\mathrm{a}+\frac{1}{2}}\mathrm{v}_0^2(\mathrm{x})\mathrm{dx}\int_{(\mathrm{x}-1)}^{\mathrm{a}}\mathrm{J(x,y)dy}\right).\end{aligned} \tag{3.51}$$

From (3.50) and (3.51) we get that

$$\begin{aligned}k(A)&\ge\frac{1}{\gamma}\left(\int_0^{\frac{1}{2}}J(z)dz\right)\frac{\left(\int_{b-\frac{1}{2}}^{b}v_0^2(x)\mathrm{dx}+\int_{\mathrm{a}}^{\mathrm{a}+\frac{1}{2}}\mathrm{v}_0^2(\mathrm{x})\mathrm{dx}\right)}{\pi(A)\pi(A^c)}\\&\ge\frac{4}{\gamma}\zeta_1^2\left(\int_0^{\frac{1}{2}}J(z)dz\right)\end{aligned} \tag{3.52}$$

since (3.22) and $\pi(A)\pi(A^c) \le \frac{1}{4}$. When A is not a subset of $[-r_0, r_0]$, but $[b-\frac{1}{2}, b] \subset [-r_0, r_0]$ or $[a, a+\frac{1}{2}] \subset [-r_0, r_0]$ then we proceed in a way similarly to the previous case. Assume that $[b-\frac{1}{2}, b] \subset [-r_0, r_0]$, then

$$k(A)\ge\frac{1}{\gamma\pi(A)\pi(A^c)}\int_{b-\frac{1}{2}}^{b}v_0^2(x)\mathrm{dx}\int_{\mathrm{b}}^{(\mathrm{x}+1)}\mathrm{J(x,y)dy}\ge\frac{4}{\gamma}\zeta_1^2\int_0^{\frac{1}{2}}\mathrm{J(0,y)dy} \tag{3.53}$$

since (3.22) and $\pi(A)\pi(A^c) \le \frac{1}{4}$. Similarly one proceeds when $[a, a+\frac{1}{2}] \subset [-r_0, r_0]$.

If neither of the two intervals $[b-\frac{1}{2}, b]$ and $[a, a+\frac{1}{2}]$ is a subset of $[-r_0, r_0]$ then the interval $[a, b]$ is on the left or on the right of $[-r_0, r_0]$ or either $[-r_0, r_0] \subset [b+\frac{1}{2}, a-\frac{1}{2}]$. The last case is incompatible with the fact that $\pi(A) < \frac{1}{2}$ and the fact that $v_0(\cdot)$ is exponentially decaying for $|x| \ge r_0$.

Assume that A is at the right of $[-r_0, r_0]$. We then have, see (3.50),

$$\begin{aligned}&\left(\int_{b-\frac{1}{2}}^{b} v_0^2(x)\mathrm{dx}\int_{\mathrm{b}}^{(\mathrm{x}+1)\wedge\mathrm{L}}\mathrm{J(x,y)dy}+\int_{\mathrm{a}}^{\mathrm{a}+\frac{1}{2}}\mathrm{v}_0^2(\mathrm{x})\mathrm{dx}\int_{(\mathrm{x}-1)\vee(-\mathrm{L})}^{\mathrm{a}}\mathrm{J(x,y)dy}\right)\\&\ge\int_a^{a+\frac{1}{2}}v_0^2(x)\mathrm{dx}\int_{(\mathrm{x}-1)}^{\mathrm{a}}\mathrm{J(x,y)dy}\ge\int_0^{\frac{1}{2}}\mathrm{J(0,y)dy}\int_{\mathrm{a}}^{\mathrm{a}+\frac{1}{2}}\mathrm{v}_0^2(\mathrm{x})\mathrm{dx}.\end{aligned} \tag{3.54}$$

The quantity $\int_a^{a+\frac{1}{2}} v_0^2(x)\mathrm{dx}$ can be very small since the exponentially decreasing of v_0. To obtain a lower bound we estimate from above $\pi(A)$ in term of $\int_a^{a+\frac{1}{2}} v_0^2(x)\mathrm{dx}$. We have

$$\begin{aligned}\pi(A)&=\int_A\frac{v_0^2(x)}{p(x)}\mathrm{dx}\le\frac{1}{\beta(1-\mathrm{m}_\beta^2)}\int_{\mathrm{a}}^{\mathrm{b}}\mathrm{v}_0^2(\mathrm{x})\mathrm{dx}\le\frac{1}{\beta(1-\mathrm{m}_\beta^2)}\int_{\mathrm{a}}^{\mathrm{L}}\mathrm{v}_0^2(\mathrm{x})\mathrm{dx}\\&=\frac{1}{\beta(1-m_\beta^2)}\sum_{k=0}^{2(L-a)-1}\int_{a+\frac{k}{2}}^{a+\frac{1+k}{2}}v_0^2(x)\mathrm{dx},\end{aligned} \tag{3.55}$$

and

$$\int_{a+\frac{k}{2}}^{a+\frac{1+k}{2}}v_0^2(x)\mathrm{dx}=\int_{\mathrm{a}+\frac{\mathrm{k}}{2}-1}^{\mathrm{a}+\frac{1+\mathrm{k}}{2}-1}\mathrm{v}_0^2(\mathrm{x}+1)\mathrm{dx}=\int_{\mathrm{a}+\frac{\mathrm{k}}{2}-1}^{\mathrm{a}+\frac{1+\mathrm{k}}{2}-1}\mathrm{v}_0^2(\mathrm{x})\frac{\mathrm{v}_0^2(\mathrm{x}+1)}{\mathrm{v}_0^2(\mathrm{x})}\mathrm{dx}\le\mathrm{d}_1^2\int_{\mathrm{a}+\frac{\mathrm{k}}{2}-1}^{\mathrm{a}+\frac{1+\mathrm{k}}{2}-1}\mathrm{v}_0^2(\mathrm{x})\mathrm{dx} \tag{3.56}$$

where $0 < d_1 < 1$ is a constant independent on L obtained as following.

From (3.20) and equation (3.4), taking into account that v_0 is strictly decreasing for $|x| \geq r_0$ we obtain that

$$\frac{v_0(x+1)}{v_0(x)} \leq \frac{1}{\nu_0}\beta(1-\bar{m}^2(x+1)) < 1, \qquad |x| \geq r_0, \tag{3.57}$$

see (3.23) and (3.24). Denote

$$d_1 = \sup_{|x|\geq r_0} \frac{1}{\nu_0}\beta(1-\bar{m}^2(x+1)) = \frac{1}{\nu_0}\beta(1-\bar{m}^2(r_0)) < 1. \tag{3.58}$$

Reiterating (3.56) we get that

$$\int_{a+\frac{k}{2}}^{a+\frac{k+1}{2}} v_0^2(x)\mathrm{d}x \leq (\mathrm{d}_1^2)^{\mathrm{k}} \int_{\mathrm{a}}^{\mathrm{a}+\frac{1}{2}} \mathrm{v}_0^2(\mathrm{x})\mathrm{d}\mathrm{x}. \tag{3.59}$$

Therefore from (3.55) we obtain

$$\pi(A) \leq \left(\int_a^{a+\frac{1}{2}} v_0^2(x)\mathrm{d}x\right) \sum_{k=0}^{\infty} d_1^{2k} = \frac{1}{1-d_1^2}\int_a^{a+\frac{1}{2}} v_0^2(x)\mathrm{d}x. \tag{3.60}$$

From (3.53), when A is on the right of $[-r_0, r_0]$ we obtain

$$k(A) \geq 2\frac{\beta(1-m_\beta^2)}{\gamma}\left(\int_0^{\frac{1}{2}} J(z)dz\right)\frac{1}{1-d_1^2}. \tag{3.61}$$

Similar argument works when A is on the left of $[-r_0, r_0]$. From (3.53) and (3.61), when $|A| \geq \frac{1}{2}$ we obtain that

$$k(A) \geq \min\left\{2\frac{\beta(1-m_\beta^2)}{\gamma}\left(\int_0^{\frac{1}{2}} J(0,z)dz\right)\frac{1}{1-d_1^2}, \frac{2}{\gamma}\zeta_1^2\left(\int_0^{\frac{1}{2}} J(0,z)dz\right)\right\} \equiv D_2. \tag{3.62}$$

Denote by D

$$D = \min\{D_1, D_2\}. \tag{3.63}$$

The thesis follows. □

Proof of Theorem 2.1 For $\beta > 1$, fix any $\epsilon_0 = \epsilon_0(\beta)$, $\epsilon_0 \in (0, \frac{(1-\sigma(m_\beta))}{2})$ and take $L_1(\beta)$ so that Lemma 3.6 and Theorem 3.10 hold. Recall that $\mathcal{L}^0 = I - \mathcal{A}$, where I is the identity operator and $\mathcal{A}$ is the operator defined in (3.1). By Theorem 3.1 we have immediately that $\mathcal{L}^0$ is a bounded, selfadjoint, quasi compact operator. The smallest eigenvalue of $\mathcal{L}^0$ is $\mu_1^0 = 1-\nu_0$ where ν_0 is the maximum eigenvalue of $\mathcal{A}$ and $\psi_1^0 = v_0$ is the corresponding eigenfunction. The (2.8) follows from Lemma 3.2 and Lemma 3.7. Point (2) is a direct consequence of Lemma 3.8, (3.42) and Theorem 3.10. Next we show point (3). Split

$$\frac{\bar{m}'}{\|\bar{m}'\|} = a\psi_1^0 + (\psi_1^0)^{ort}. \tag{3.64}$$

Then

$$a^2 + \|(\psi_1^0)^{ort}\|^2 = 1 \tag{3.65}$$

$$\frac{1}{\|\bar{m}'\|^2}\langle \mathcal{L}^0\bar{m}', \bar{m}'\rangle = a^2\mu_1^0 + \langle \mathcal{L}^0(\psi_1^0)^{ort}, (\psi_1^0)^{ort}\rangle \geq a^2\mu_1^0 + \mu_2^0\|(\psi_1^0)^{ort}\|^2. \tag{3.66}$$

By Lemma 3.2

$$\frac{1}{\|\bar{m}'\|^2}\langle \mathcal{L}^0\bar{m}', \bar{m}'\rangle \leq ce^{-2\alpha L}$$

hence from (3.65) and (3.66)

$$ce^{-2\alpha L} \geq (1-\|(\psi_1^0)^{ort}\|^2)\mu_1^0 + \mu_2^0\|(\psi_1^0)^{ort}\|^2.$$

By (2.8) and (2.9), that there exists a $C > 0$ independent on L so that

$$\|(\psi_1^0)^{ort}\|^2 \le Ce^{-2\alpha L}. \tag{3.67}$$

The (2.10) follows by (3.64), (3.65) and (3.67). □

References

[1] N. Alikakos, G. Fusco *The spectrum of the Cahn-Hilliard operator for generic interface in higher space dimensions.* Indiana University Math. J. **42**, No2, 637–674 (1993).

[2] E. A. Carlen, M. C. Carvalho, E. Orlandi, *Algebraic rate of decay for the excess free energy and stability of fronts for a non–local phase kinetics equation with a conservation law I* J. Stat. Phy. **95** N 5/6, 1069-1117 (1999)

[3] E. A. Carlen, M. C. Carvalho, E. Orlandi *Algebraic rate of decay for the excess free energy and stability of fronts for a no–local phase kinetics equation with a conservation law, II* Comm. Par. Diff. Eq. **25** N 5/6, 847-886 (2000)

[4] E. A. Carlen, E. Orlandi *Stability of planar fronts for a non-local phase kinetics equation with a conservation law in* $d \ge 3$ Reviews in Mathematical Physics Vol. 24, **4** (2012)

[5] Xinfu Chen *Spectrum for the Allen- Cahn, Cahn-Hilliard, and phase-field equations for generic interfaces.* Commun in Partial Differential Equations **19**, No7-8, 1371–1395 (1994).

[6] A. De Masi, E. Orlandi, E. Presutti, L. Triolo, *Glauber evolution with Kac potentials I. Mesoscopic and macroscopic limits, interface dynamics* Nonlinearity **7** 633-696 (1994).

[7] A. De Masi, E. Orlandi, E. Presutti, L. Triolo, *Stability of the interface in a model of phase separation* Proc.Royal Soc. Edinburgh **124A** 1013-1022 (1994).

[8] A. De Masi, E. Orlandi, E. Presutti, L. Triolo, *Uniqueness of the instanton profile and global stability in non local evolution equations* Rendiconti di Matematica.Serie Vll **14**, (1994).

[9] A. De Masi, E.Olivieri, E. Presutti *Spectral properties of integral operators in problems of interface dynamics and metastability.*, Markov Process Related Fields 4**1**, 27-112(1998).

[10] P. De Mottoni , M Schatzman *Geometrical Evolution of Developed Interfaces* Trans. Amer. Math. Soc, Vol. 347, No. 5, 1533-1589 (1995)

[11] G. Giacomin, J. Lebowitz, *Phase segregation dynamics in particle systems with long range interactions I: macroscopic limits* J. Stat. Phys. **87**, 37–61 (1997).

[12] G. Giacomin, J. Lebowitz, *Phase segregation dynamics in particle systems with long range interactions II: interface motions* SIAM J. Appl. Math.**58**, No 6, 1707–1729 (1998).

[13] G. F. Lawler and A. Sokal *Bounds on the* L^2 *spectrum for Markov Chains and Markov processes: a generalization of Cheeger's inequality.*, Transactions of the AMS **309**, No 2, 557-580 (1988).

[14] J. Lebowitz , E.Orlandi and E. Presutti. *A particle model for spinodal decomposition* J. Stat. Phys. **63**, 933-974, (1991).

[15] J. Lebowitz, O. Penrose, *Rigorous treatment of metastable states in the Van der waals Maxwell theory* J. Stat. Phys **3**, 211-236 (1971).

[16] E.Orlandi , *Spectral properties of integral operators in problems of interface dynamics* preprint.

[17] E. Presutti, *Scaling Limits In Statistical Mechanics and Microstructures in Continuum Mechanics* Springer (2009)

Enza Orlandi, Dipartimento di Matematica e Fisica, Università di Roma Tre, L.go S.Murialdo 1, 00156 Roma, Italy.
E-mail address: orlandi@mat.uniroma3.it