

SPECTRAL PROPERTIES OF INTEGRAL OPERATORS IN PROBLEMS OF INTERFACE DYNAMICS

ENZA ORLANDI

ABSTRACT. We consider a family of integral operators which appears when analyzing layered equilibria and front dynamics of a phase kinetics equation with a conservation law. We study the spectra of these operators in L^2 and derive a lower bound for the associated quadratic forms in terms of the H^{-1} norm.

1. INTRODUCTION

The purpose of this paper is to derive spectral estimates for a family of integral operators which appear when analyzing layered equilibria and front dynamics of a phase kinetics equation with a conservation law. We start by recalling some background.

Consider in the torus \mathbb{T}^d the nonlocal and nonlinear evolution equation

$$\frac{\partial}{\partial t} m(x, t) = \nabla \cdot (\nabla m(x, t) - \beta(1 - m(x, t)^2)(J \star \nabla m)(x, t)) \quad (1.1)$$

where $\beta > 1$, \star denotes convolution and J is a smooth, spherically symmetric probability density with compact support. This equation first appeared in the literature in a paper [13] on the dynamics of Ising systems with a long-range interaction and so-called ‘‘Kawasaki’’ or ‘‘exchange’’ dynamics and later it was rigorously derived in [10]. In this physical context, $m(x, t) \in [-1, 1]$ is the magnetization density at x at time t , viewed on the length scale of the interaction, and β is the inverse temperature. This introduction is not the place to fully explain the physical origins of the equation (1.1), and familiarity with them is not needed to understand our results or their proofs. We refer to the previous quoted paper for more physical insight. The equation (1.1) can be written in a gradient flow form. To do this, we introduce the free energy functional $\mathcal{F}(m)$:

$$\mathcal{F}(m) = \int_{\mathbb{T}^d} [V(m(x)) - V(m_\beta)] dx + \frac{1}{4} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} J(x - y) [m(x) - m(y)]^2 dx dy, \quad (1.2)$$

where $V(m)$ is

$$V(m) = -\frac{1}{2}m^2 + \frac{1}{\beta} \left[\left(\frac{1+m}{2} \right) \ln \left(\frac{1+m}{2} \right) + \left(\frac{1-m}{2} \right) \ln \left(\frac{1-m}{2} \right) \right]. \quad (1.3)$$

For $\beta > 1$, this potential function V is a symmetric double well potential on $[-1, 1]$. We denote the positive minimizer of V on $[-1, 1]$ by m_β . It is easy to see that m_β is the positive solution of the equation

$$m_\beta = \tanh(\beta m_\beta). \quad (1.4)$$

Then equation (1.1) can be written as

$$\frac{\partial}{\partial t} m = \nabla \cdot \left(\sigma(m) \nabla \left(\frac{\delta \mathcal{F}}{\delta m} \right) \right) \quad (1.5)$$

Date: November 25, 2014.

1991 Mathematics Subject Classification. Primary 60J25; secondary 82A05.

Key words and phrases. Integral operators, spectrum, Cahn-Hilliard, interfaces.

E.O. supported by Università di ROMA TRE.

where the *mobility* $\sigma(m)$ is given by

$$\sigma(m) = \beta(1 - m^2). \quad (1.6)$$

Formally one derives

$$\frac{d}{dt} \mathcal{F}(m(t)) = - \int \left| \nabla \left(\frac{\delta \mathcal{F}}{\delta m} \right) \right|^2 \sigma(m(t)) dx \quad (1.7)$$

thus \mathcal{F} is a Lyapunov function for (1.1). This suggests that the free energy should want to tend locally to one of the two minimizing values, $\pm m_\beta$, and that the interface between a region of $+m_\beta$ magnetization and a region of $-m_\beta$ magnetization should have a “profile” – in the direction orthogonal to the interface – that makes the transition from one local equilibrium to the other in a way that minimizes the free energy. This is indeed the case, as it has been shown in [3], [4] in dimension $d = 1$ and in [6] in dimension $d \leq 3$. Also clearly, the minimizers of the free energy $\pm m_\beta$ represent the “pure phases” of the system. However, unless the initial data m_0 happens to satisfy $\int_{\mathbb{T}} m_0(x) dx = \pm m_\beta |\mathbb{T}|$, these “pure phases” cannot be reached because of the conservation law. Instead, what will eventually be produced is a region in which $m(x) \approx +m_\beta$, with $m(x) \approx -m_\beta$ in its complement, and with a smooth transition across its boundary. This is referred to a *phase segregation*, and the boundary is the *interface* between the two phases. If we “stand far enough back” from \mathbb{T} , all we see is the interface, and we do not see any structure across the interface – the structure now being on an invisibly small scale. The evolution of m under the (1.1), or another evolution equation of this type, drives an evolution of the interface. To see any evolution of the interface, one must wait a long time. More specifically, let λ be a small parameter, and introduce new variables τ and ξ through

$$\tau = \lambda^3 t \quad \text{and} \quad \xi = \lambda x .$$

Then of course

$$\frac{\partial}{\partial t} = \lambda^3 \frac{\partial}{\partial \tau} \quad \text{and} \quad \frac{\partial}{\partial x} = \lambda \frac{\partial}{\partial \xi} .$$

Hence if $m(x, t)$ is a solution of (1.1), and we define $m^\lambda(\xi, \tau)$ by $m^\lambda(\xi, \tau) = m(x(\xi), t(\tau))$, we obtain

$$\frac{\partial}{\partial \tau} m^\lambda(\xi, \tau) = \frac{1}{\lambda} \nabla \cdot \left(\sigma(m^\lambda) \nabla \left[\frac{1}{\beta} \operatorname{arctanh} m^\lambda - (J^\lambda \star m^\lambda) \right] \right) (\xi, \tau) \quad (1.8)$$

where we denoted $J^\lambda(\xi) = \lambda^{-d} J(\lambda^{-1} \xi)$. One should just bear in mind that now we are looking at the evolution over a *very* long time scale when λ is small.

One might hope that for small values of λ , *all information about the evolution on m^λ is contained in the evolution of the interface Γ_τ* . This is indeed the case. The sharp interface limit of equations tipified by (1.8) has been investigated by Giacomin and Lebowitz, see [11], where it is heuristically proven that the limit motion is driven by the Mullins–Sekerka flow. They applied asymptotic analysis in the same spirit of the heuristic work of Pego [15] who derived the sharp interfaces limit for the Cahn-Hilliard equation.

The heuristic analysis of Pego has been rigorously proven for the Cahn-Hilliard equation by [1] in the 1994 and later on, applying a different method, in [5]. In both the papers the proof is based on two steps. The first step is to construct approximate solutions to the Cahn-Hilliard equation, which are, in the limit, close to the the Mullins–Sekerka flow. The second step is to show that the family of approximate solutions is indeed suitably close to the solution of the Cahn-Hilliard equation. To show this second step spectral estimates are needed. These were proven in [2], in dimensions $d = 2$, and [7] in any dimensions.

In [1] the family of the approximate solutions to Cahn-Hilliard equations are constructed by matched asymptotic expansions. In [5], the approximate solutions are constructed by an alternative method. The method based on the Hilbert expansion used in kinetic theory besides its relative simplicity, it leads to calculable higher order corrections to the interface motion. More important in this context is that the above approach allows to construct approximate solutions to the non local evolution equation (1.8).

Hence, to prove rigorously the heuristic analysis done by Giacomini and Lebowitz in [11], one might first construct approximate solutions and this can be done applying the same method as in [5]. Then one needs to derive spectral estimates to show that the constructed approximate solutions are close in some convenient norm to the solution of (1.8). In this paper we prove such spectral estimates. We set the problem in a bounded domain $\Omega \subset \mathbb{R}^2$. The restriction at dimension $d = 2$ is purely technical. In $d = 2$ we still can use global set of coordinates to represent the operator when close to the interface, Γ . Namely in such a case any simple, smooth, closed, one dimensional curve can be mapped into a one dimensional circle. This allows to use Fourier series by going to the universal cover. In dimension $d \geq 3$ this would not be possible and one needs to deal with local coordinates and, possibly, Fourier transforms. Indeed, we believe that the result holds in any dimension and we leave this problem to further investigation.

We consider a family m_A^λ of smooth functions which as $\lambda \rightarrow 0$ approach a step function with values $\pm m_\beta$ which is discontinuous along a smooth curve $\Gamma \subset \Omega$. This family consists of approximate solutions to (1.8), which can be constructed as in [5]. The functions m_A^λ have a very specific behaviour in the direction orthogonal to Γ , as $\lambda \rightarrow 0$. The specific form of m_A^λ is given in (2.9). To simplify notations from now on we drop the index λ . The linear equation obtaining linearising (1.8) at m_A , is ¹

$$\frac{\partial}{\partial \tau} v(\xi, \tau) = \frac{1}{\lambda} \nabla \cdot \left(\sigma(m_A) \nabla \cdot \left[\frac{v}{\sigma(m_A)} - (J^\lambda \star_\Omega v) \right] \right) (\xi, \tau), \quad \xi \in \Omega, \quad (1.9)$$

where $(J^\lambda \star_\Omega v)(\xi) = \int_\Omega J^\lambda(\xi - \xi') v(\xi') d\xi'$, $v = m^\lambda - m_A$ and we assume that $v(\xi, 0) = 0$, $\xi \in \Omega$. By the conservation law, $\int_\Omega d\xi v(\xi, \tau) = 0$ for all $\tau \geq 0$. To simplify the explanation let us first pretend that $\sigma(m_A) = 1$ in $\nabla \cdot \sigma(m_A) \nabla$. Let us then denote

$$\frac{1}{\lambda} (A_{m_A}^\lambda v)(\xi) = \frac{1}{\lambda} \left\{ \frac{v(\xi)}{\sigma(m_A(\xi))} - (J^\lambda \star_\Omega v)(\xi) \right\}, \quad \xi \in \Omega. \quad (1.10)$$

Accordingly, to lower bound the spectrum of the linear operator on the right hand side of (1.9) in $H^{-1}(\Omega)$ it suffices to show that

$$\frac{1}{\lambda} \langle\langle v, A_{m_A}^\lambda v \rangle\rangle \geq -C \|v\|_{H^{-1}(\Omega)}^2, \quad (1.11)$$

where $\langle\langle v, g \rangle\rangle = \int_\Omega v(\xi) g(\xi) d\xi$ and $C > 0$ does not depend on λ . In the general case, when $\sigma(m_A) \neq 1$, one can argue in the same manner by using a weighted H^{-1} norm, the weigh being $\sigma(m_A)$. Because $0 < a \leq \sigma(m_A) \leq \beta$ the weighted H^{-1} norm is equivalent to the H^{-1} . So we will prove (1.11), with $v \in H^{-1}(\Omega)$.

As in Alikakos and Fusco, [2], or in X. Chen, [7], we first prove a lower bound of the spectrum of the operator in (1.10) in the L^2 norm, see Theorem 2.4, i.e

$$\langle\langle v, A_{m_A}^\lambda v \rangle\rangle \geq -C \lambda^2 \|v\|_{L^2(\Omega)}^2. \quad (1.12)$$

Then, when

$$\langle\langle v, A_{m_A}^\lambda v \rangle\rangle \leq 0 \quad (1.13)$$

we show that

$$\lambda \|v\|_{L^2(\Omega)}^2 \leq \|v\|_{H^{-1}(\Omega)}^2 \quad (1.14)$$

see Theorem 2.5. In this way we prove the lower bound (1.11).

¹We are now considering (1.8) in $\Omega \Subset \mathbb{R}^2$ with non flux boundary conditions and with the convolution operator “restricted” in Ω .

1.1. Sketch of the proof. The proof of (1.11) presents some similarities with the one given in [2] and in [7] for the Cahn-Hilliard (C-H) case. Nevertheless, the implementation of each single step requires different technique due to the non locality of the operator. In particular, we cannot follow the method in [2] and in [7], as we cannot split the integral kernel in (1.10) into a tangential and in normal part, whereas expressions like $\|\nabla v\|^2$ split in such a way naturally.

Now we summarize how we proceed in proving (1.11). We consider a neighborhood of Γ , $\mathcal{N}(\Gamma)$. We map the curve Γ in a circle T having perimeter equal to the length of the curve L and each point $\xi \in \mathcal{N}(\Gamma)$ is mapped to $(s, r) \in \mathcal{T} = T \times [-d_0, d_0]$ in a diffeomorphic way. We denote by $\alpha(s, r)$ the Jacobian of the map. We take a subset $\mathcal{N}_1(\Gamma) \subset \mathcal{N}(\Gamma)$ and we assume that

$$\inf_{\xi \in \Omega \setminus \mathcal{N}_1(\Gamma)} \frac{1}{\sigma(m_A(\xi))} > C^* > 1. \quad (1.15)$$

To take advantage of (1.15) we split the quadratic form (1.12) in two integrals: one over $\mathcal{N}_1(\Gamma)$ the other in $\Omega \setminus \mathcal{N}_1(\Gamma)$. To this end we introduce the indicator function of the set $\mathcal{N}_1(\Gamma)$, $\eta_1(\xi) = 1$ when $\xi \in \mathcal{N}_1(\Gamma)$, $\eta_1(\xi) = 0$ in $\Omega \setminus \mathcal{N}_1(\Gamma)$ and $\eta_2(\xi) = 1 - \eta_1(\xi)$. Because of the non locality of the operator, taking into account that $\eta_2(\xi)\eta_1(\xi) = 0$ for $\xi \in \Omega$ and the symmetry of $J^\lambda(\cdot)$ we obtain that

$$\begin{aligned} \int_{\Omega} (A_{m_A}^\lambda v)(\xi)v(\xi)d\xi &= \int_{\Omega} (A_{m_A}^\lambda \eta_1 v)(\xi)\eta_1(\xi)v(\xi)d\xi \\ &+ \int_{\Omega} (A_{m_A}^\lambda \eta_2 v)(\xi)\eta_2(\xi)v(\xi)d\xi \\ &- 2 \int_{\Omega} d\xi \eta_1(\xi)v(\xi)(J^\lambda \star \eta_2 v)(\xi). \end{aligned} \quad (1.16)$$

Condition (1.15) implies

$$\int_{\Omega} (A_{m_A}^\lambda \eta_2 v)(\xi)\eta_2(\xi)v(\xi)d\xi \geq (C^* - 1)\|\eta_2 v\|_{L^2(\Omega)}^2 > 0.$$

Even if we show that

$$\int_{\Omega} (A_{m_A}^\lambda \eta_1 v)(\xi)\eta_1 v(\xi)d\xi \geq -C\lambda^2 \int_{\Omega} v^2(\xi)d\xi, \quad (1.17)$$

the last term of (1.16) might create problems for getting estimate (1.12). The first task in proving estimate (1.12) is therefore to show that there exists $\mathcal{N}_1(\Gamma)$ so that

$$\left| \int_{\Omega} d\xi \eta_1(\xi)v(\xi)(J^\lambda \star \eta_2 v)(\xi) \right| \leq C\lambda^2 \|v\|_{L^2(\Omega)}^2. \quad (1.18)$$

This is proven at the beginning of the proof of Theorem 2.4. The following step is to show (1.17). We write the quadratic form on the left hand side of (1.17) in local variables. Notice that the integrals are on the set $\mathcal{N}(\Gamma)$. In these local variables the convolution operator $(J^\lambda \star \eta_1 v)(\xi)$ becomes equal, up to order λ^2 , to an operator which is not a convolution anymore but it is still self-adjoint with respect to a weighed Lebesgue measure on \mathcal{T} , see Lemma 4.1. In this way we show, see Lemma 4.2, that setting

$$\begin{aligned} \hat{v}(s, r) &= \sqrt{\alpha(s, r)}v(s, r) \\ \int_{\Omega} (A_{m_A}^\lambda \eta_1 v)(\xi)\eta_1 v(\xi)d\xi &\geq \int_{\mathcal{T}} ds dr (L^\lambda \hat{v})(s, r)\hat{v}(s, r) - C\lambda^2 \|v\|_{L^2(\Omega)}^2. \end{aligned} \quad (1.19)$$

It turns out that the operator L^λ is conjugate to the operator $\mathbb{I} - \mathcal{P}^\lambda$, where \mathcal{P}^λ is an integral operator, positivity improving. Therefore L^λ and $\mathbb{I} - \mathcal{P}^\lambda$ have the same spectrum. By the Perron-Frobenius Theorem we know that the principal eigenfunction is point wise positive. To estimate the principal eigenvalue we still need other ingredients. We show that when the operator L^λ acts on functions depending only on the signed distance r from Γ , becomes equal, up to order λ^2 , to a one dimensional operator $L_1^{\lambda, s}$, $s \in T$. The knowledge of the spectrum of $L_1^{\lambda, s}$ (obtained using the results of [14]) together with the properties of the eigenfunctions associated to the principal eigenvalues of $L_1^{\lambda, s}$ and \mathcal{P}^λ allows to

upper and lower bound the principal eigenvalue μ_0 of L^λ . In this way we show that $-C\lambda^2 \leq \mu_0 \leq C\lambda^2$. Hence (1.17) follows. To prove the $H^{-1}(\Omega)$ lower bound, see (1.11), we cannot proceed as in [2] and in [7]. For the non local operator (1.10) a bound on its quadratic form does not imply any bound on the gradient of the function. Nevertheless, we are still able to show that when

$$\int_{\Omega} (A_{m_A}^\lambda \eta_1 v)(\xi) \eta_1 v(\xi) d\xi \leq \lambda^2 \|v\|_{L^2(\Omega)}^2 \quad (1.20)$$

then v in local variables enjoys the following decomposition, see Theorem 6.1,

$$v(s, r) = Z(s) \frac{1}{\sqrt{\lambda}} \psi_0^0\left(\frac{r}{\lambda}\right) + v^R(s, r) \quad (1.21)$$

where $\|Z\|_{L^2(\mathcal{T})} \simeq 1$, $\|\nabla Z\|_{L^2(\mathcal{T})} \leq C$, $\|v^R\|_{L^2(\mathcal{N}(\Gamma))}^2 \leq C\lambda^2$ and $\psi_0^0(\cdot)$ is a strictly positive, even, smooth function, exponential decreasing. In such a way we obtain a decomposition similar to the one in [2] and [7] and we can proceed as in [7] to prove (1.14). To prove the decomposition (1.21) a more complete analysis of the spectrum of the operator L^λ is needed. Let $\{\mu_k\}_{k \in \mathbb{N}}$ be the eigenvalues of L^λ in $L^2(\mathcal{T})$. Essentially we prove that $-C\lambda^2 + c_0 k^2 \lambda^2 \leq \mu_k \leq C\lambda^2 + c_1 k^2 \lambda^2$, for $k \leq \frac{h_0}{\lambda}$, and, for $k > \frac{h_0}{\lambda}$, $\mu_k \geq \nu > 0$, where C , c_0 , c_1 , h_0 and ν are positive, real numbers independent on λ . In addition, when $k \leq \frac{h_0}{\lambda}$, we also need, and establish, a precise knowledge of the shape of the associated eigenfunctions. To obtain such informations on the spectrum of the operator L^λ we make a judicious use of Fourier analysis.

Acknowledgements I am indebted to Sao Carvalho and Eric Carlen for discussions on this and related problems, discussions which started many years ago when visiting Georgia Tech and never stopped. I also benefited from suggestions by Carlangelo Liverani.

2. NOTATIONS AND RESULTS

Let Ω be a bounded domain in \mathbb{R}^2 with sufficiently smooth boundary and let $\Gamma \subset \Omega$ be a simple, smooth, closed curve, boundary of an open set Ω^- . We denote by $\Omega^+ = \Omega \setminus (\Omega^- \cup \Gamma)$. We denote by C a constant which might change from one occurrence to the other, independent on λ .

2.1. The interaction J^λ . Let J be a smooth spherically symmetric, translation invariant, probability density on \mathbb{R}^2 with compact support, $\{\xi \in \mathbb{R}^2 : |\xi| \leq 1\}$. We assume $J \in C^1(\mathbb{R}^2)$. We say that $\xi \in \mathbb{R}^2$ and $\xi' \in \mathbb{R}^2$ interact each other if $J(\xi - \xi') > 0$. For $\xi = (\xi_1, \xi_2)$ we denote

$$\bar{J}(\xi_1) = \int_{\mathbb{R}} J(\xi) d\xi_2, \quad (2.1)$$

and for $\lambda \in (0, 1]$

$$J^\lambda(\xi) = \frac{1}{\lambda^2} J\left(\frac{\xi}{\lambda}\right), \quad \bar{J}^\lambda(\xi_1) = \frac{1}{\lambda} \bar{J}\left(\frac{\xi_1}{\lambda}\right).$$

The scaling is such that for all $\lambda \in (0, 1]$

$$\int_{\mathbb{R}^2} J^\lambda(\xi) d\xi = 1, \quad \int_{\mathbb{R}} \bar{J}^\lambda(\xi_1) d\xi_1 = 1.$$

2.2. Interaction in bounded domain. In a bounded domain Q we require the interaction to act only when ξ and ξ' are in Q . Let v be a function having support Q . We denote for $\xi \in Q$

$$(J \star v)(\xi) = (J \star_Q v)(\xi) = \int_Q J(\xi - \xi') v(\xi') d\xi'. \quad (2.2)$$

We do not add the suffix Q , unless confusion arises. Same notation when J is replaced by J^λ .

2.3. Local variables in a neighborhood of Γ . We parametrize Γ by arc length. Let L be the length of Γ and let T be a circle of length L . Let $\gamma : T \rightarrow \Gamma$ be the map parametrizing Γ so that for $s \in T$, $|\gamma'(s)| = 1$. We assume $\gamma \in C^5(T)$. We denote by $\nu(s)$ a smoothly varying unit vector normal to Γ at the point $\gamma(s)$. We denote by $k(s)$ the signed curvature defined by $\nu'(s) = -k(s)\gamma'(s)$. We therefore have

$$|\nu(s)| = 1, \quad |\gamma'(s)| = 1, \quad \nu'(s) = -k(s)\gamma'(s), \quad \gamma''(s) = k(s)\nu(s), \quad \gamma'(s) \cdot \nu(s) = 0, \quad (2.3)$$

where for two vectors w_1 and w_2 in \mathbb{R}^2 , $w_1 \cdot w_2$ is the scalar product. Let $d(\xi, \Gamma)$ be the euclidean distance of the point ξ from Γ and $r = r(\xi, \Gamma)$ be the signed distance of a point ξ from Γ with the convention that $r < 0$ when $\xi \in \Omega^-$ and $r > 0$ when $\xi \in \Omega^+$. Let $d_0 > 0$ be so that

$$\mathcal{N}(d_0) = \{\xi \in \mathbb{R}^2 : d(\xi, \Gamma) \leq d_0\} \subset \Omega.$$

We require d_0 to be small enough so that the map $\rho : T \times [-d_0, d_0] \rightarrow \mathcal{N}(d_0)$,

$$\xi = \gamma(s) + \nu(s)r = \rho(s, r), \quad (2.4)$$

is a diffeomorphism. We denote $I = [-d_0, d_0]$, $\mathcal{T} = T \times I$ and $\alpha(s, r)$ for $(s, r) \in \mathcal{T}$ the Jacobian of the local change of variables

$$\alpha(s, r) = \det \left(\frac{\partial \rho(s, r)}{\partial (s, r)} \right) = 1 - rk(s). \quad (2.5)$$

We further require d_0 to be small enough so that

$$\sup_{s \in T} |k(s)|d_0 \leq \frac{1}{2}. \quad (2.6)$$

This implies $\frac{3}{2} \geq \alpha(s, r) \geq \frac{1}{2}$ for all $(s, r) \in \mathcal{T}$.

A function $u(\xi)$ for $\xi \in \mathcal{N}(d_0)$ becomes by the change of coordinates $v(s, r) = u(\rho(s, r))$, $(s, r) \in \mathcal{T}$. In the sequel we identify functions of variable ξ and functions of variable (s, r) in the domain $\mathcal{N}(d_0)$. We often lift the function v on the universal cover of \mathcal{T} without mentioning. We write

$$\int_{\mathcal{N}(d_0)} u(\xi) d\xi = \int_{\mathcal{T}} u(s, r) \alpha(s, r) ds dr \quad (2.7)$$

and for v and w in $L^2(\mathcal{T})$

$$\langle v, w \rangle = \int_{\mathcal{T}} v(s, r) w(s, r) ds dr. \quad (2.8)$$

2.4. The approximate solution $m_A(\cdot)$. Let $\beta > 1$, m_β be the strictly positive solution of $m = \tanh \beta m$ and \bar{m} be the unique antisymmetric solution of

$$m(z) = \tanh(\beta \bar{J} \star m)(z) \quad \text{in } \mathbb{R}, \quad m(0) = 0, \quad \lim_{z \rightarrow \pm\infty} m(z) = \pm m_\beta. \quad (2.9)$$

Notice that $m_\beta < 1$ and it can be proven that $\bar{m} \in C^\infty(\mathbb{R})$, it is strictly increasing, and there exist $a > 0$, $\alpha > \alpha_0 > 0$ and $c > 0$ so that

$$\begin{aligned} 0 < m_\beta^2 - \bar{m}^2(z) &\leq ce^{-\alpha|z|}, \\ |\bar{m}'(z) - a\alpha e^{-\alpha|z|}| &\leq ce^{-\alpha_0|z|}. \end{aligned} \quad (2.10)$$

A proof of these estimates and related results can be found in [16, Chapter 8, Section 8.2]. We assume that $m_A(\cdot)$ in $\mathcal{N}(d_0)$ has the expansion

$$m_A(\xi) = \bar{m}\left(\frac{r(\xi, \Gamma)}{\lambda}\right) + \lambda[h_1\left(\frac{r(\xi, \Gamma)}{\lambda}\right)g(s(\xi)) + \phi(\xi)] + \lambda^2 q^\lambda(\xi) \quad \xi \in \mathcal{N}(d_0). \quad (2.11)$$

We assume that $h_1 \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $\lim_{|z| \rightarrow \infty} h_1(z) = 0$ exponentially fast and that h_1 is an even function. Hence if v is an odd function

$$\int_{\mathbb{R}} h_1(z) v(z) dz = 0. \quad (2.12)$$

In particular, since \bar{m} is an odd function,

$$\int_{\mathbb{R}} \frac{\bar{m}(z)}{\sigma^2(\bar{m}(z))} h_1(z) (\bar{m}'(z))^2 dz = 0 \quad (2.13)$$

where $\sigma(m) = \beta(1 - m^2)$, see Remark 2.2. We assume that g and ϕ are smooth functions, at least C^3 ,

$$\sup_{\lambda \in (0,1]} \sup_{\xi \in \mathcal{N}(d_0)} (g(s(\xi)) + \phi(\xi) + q^\lambda(\xi)) \leq C, \quad (2.14)$$

$$|\nabla^\Gamma m_A(\xi)| \leq C\lambda, \quad \xi \in \mathcal{N}(d_0), \quad (2.15)$$

where ∇^Γ is the tangential derivative to Γ . The function $\phi(\cdot)$ has a Lipschitz norm bounded uniformly with respect to λ :

$$\|\phi\|_{\text{Lip}(\mathcal{N}(d_0))} \leq C. \quad (2.16)$$

We assume that away of the interface,

$$m_A(\xi) \simeq \pm m_\beta \quad \xi \in \Omega^\pm \cap [\Omega \setminus \mathcal{N}(d_0)]$$

where \simeq means up to correction of order λ . We require that there exist $C^* > 1$, $a > 0$ and $\lambda_0 > 0$ so that for $\lambda \leq \lambda_0$

$$\inf_{\xi \in \Omega \setminus \mathcal{N}(\frac{d_0}{2})} \frac{1}{\sigma(m_A(\xi))} > C^*, \quad (2.17)$$

and

$$a \leq \sigma(m_A(\xi)) \leq \beta, \quad \xi \in \Omega. \quad (2.18)$$

Remark 2.1. *The assumptions regarding m_A are suggested by the preliminary results obtained by applying the method of [5] to the construction of approximate solutions of (1.8). The condition (2.15) is stronger than the corresponding requirement used in proving the spectral estimates in the Cahn-Hilliard case. In [7], it is required $|\nabla^\Gamma m_A(\xi)| \leq C$, for $\xi \in \mathcal{N}(d_0)$, although the function approximating the solution of the Cahn-Hilliard case constructed in [1] satisfies condition (2.15), see [1, formula (5.2)].*

Remark 2.2. *Condition (2.13) corresponds to the condition required by Alikakos, Fusco and Chen in the Cahn-Hilliard case, see for example [7, formula (1.12)]. In the Chen notation one needs*

$$\int_{\mathbb{R}} f''(\bar{m}(z)) h_1(z) (\bar{m}'(z))^2 dz = 0, \quad (2.19)$$

where $f''(\cdot)$ is the third derivative of a double well potential. Let $V(m)$ be the double well potential defined in (1.3). We have $f(m) = V'(m) = -m + \frac{1}{2\beta} \ln \frac{1+m}{1-m}$, $f'(m) = -1 + \frac{1}{\beta} \frac{1}{(1-m^2)}$, $f''(m) = 2 \frac{m}{\beta(1-m^2)^2}$. Inserting this into (2.19) gives condition (2.13).

2.5. Main Results. Denote by $A_{m_A}^\lambda$ the operator acting on functions $v \in L^2(\Omega)$:

$$(A_{m_A}^\lambda v)(\xi) = \frac{v(\xi)}{\sigma(m_A(\xi))} - (J^\lambda \star_\Omega v)(\xi) \quad \xi \in \Omega. \quad (2.20)$$

For $\lambda \leq \lambda_0$, $\sigma(m_A(\xi))$ is strictly positive for $\xi \in \Omega$, see (2.18), therefore the operator (2.20) is well defined. Denote

$$\mathcal{X} = \left\{ v \in H^1(\Omega) : \Delta w = v, \int v = 0 \right\}.$$

We have the following main result.

Theorem 2.3. *Set $\beta > 1$. There exists $\lambda_1 > 0$ such that for $\lambda \in (0, \lambda_1]$*

$$\inf_{\{v \in \mathcal{X}\}} \frac{1}{\lambda} \frac{\int_\Omega (A_{m_A}^\lambda v(\xi)) v(\xi) d\xi}{\int_\Omega |\nabla w(\xi)|^2 d\xi} \geq -C. \quad (2.21)$$

The proof of Theorem 2.3 is based on two important intermediate results.

Theorem 2.4. *Set $\beta > 1$. There exists λ_2 such that for $\lambda \in (0, \lambda_2]$*

$$\int_{\Omega} A_{m_A}^{\lambda} v(\xi) v(\xi) d\xi \geq -C\lambda^2 \int_{\Omega} v^2(\xi) d\xi.$$

Theorem 2.5. *Set $\beta > 1$. There exists λ_4 such that if for $\lambda \leq \lambda_4$, $v \in \mathcal{X}$*

$$\int_{\Omega} A_{m_A}^{\lambda} v(\xi) v(\xi) d\xi \leq C\lambda^2 \int_{\Omega} v^2(\xi) d\xi \quad (2.22)$$

then

$$\|\nabla w\|_{L^2(\Omega)}^2 \geq C\lambda \|v\|_{L^2(\Omega)}^2. \quad (2.23)$$

where w solves $\Delta w = v$ in Ω with Neumann boundary conditions on $\partial\Omega$.

Theorem 2.4 and Theorem 2.5 imply the thesis of Theorem 2.3. Preliminary to our analysis is the study of the spectra of one dimensional linear operators. This is done in Section 3. In Section 4 we prove Theorem 2.4. In Section 5 we study the spectrum of a two dimensional convolution operator. The knowledge of it allows to prove in Section 6 the representation theorem for function v so that (1.21) holds. This is the main ingredient to show Theorem 2.5 which is proven in Section 7. In the Appendix, Section 8, we collect some estimates needed to prove the results.

3. ONE DIMENSIONAL CONVOLUTION OPERATORS IN ENLARGED INTERVALS.

Denote by $z = \frac{r}{\lambda}$ the stretched variable and by $I_{\lambda} = [-\frac{d_0}{\lambda}, \frac{d_0}{\lambda}]$ the stretched interval. Also we denote by v the generic function of (s, r) and by V the generic function of (s, z) . Define for $V \in L^2(I_{\lambda})$ the following operator

$$(\mathcal{L}^0 V)(z) = \frac{V(z)}{\sigma(\bar{m}(z))} - (\bar{J} \star_{I_{\lambda}} V)(z), \quad z \in I_{\lambda}. \quad (3.1)$$

Preliminary to the analysis of the spectrum of \mathcal{L}^0 is the knowledge of the spectrum of the following operator \mathcal{L} defined on the space $L^2(\mathbb{R})$.

$$(\mathcal{L}V)(z) = \frac{V(z)}{\sigma(\bar{m}(z))} - (\bar{J} \star V)(z), \quad z \in \mathbb{R}. \quad (3.2)$$

Spectral properties of \mathcal{L} are given in [8]. The spectrum of \mathcal{L} is positive, the lower bound of the spectrum is 0 which is an eigenvalue of multiplicity one and the corresponding eigenvalue is $\bar{m}'(\cdot)$, i.e

$$\mathcal{L}\bar{m}' = 0. \quad (3.3)$$

The remaining part of the spectrum is strictly bigger then some positive number. The operator \mathcal{L}^0 is the “restriction” of the operator \mathcal{L} in the bounded interval I_{λ} . The spectrum of \mathcal{L}^0 is studied in [14]. We collect in Theorem 3.1 stated below the main results. Denote

$$(V, W) = \int_{I_{\lambda}} V(z)W(z)dz, \quad \|V\|^2 = \int_{I_{\lambda}} V(z)^2 dz.$$

Theorem 3.1 ([14]). *For any $\beta > 1$ there exists $\lambda_0(\beta)$ so that for $\lambda \leq \lambda_0(\beta)$ the following holds.*

- (0) *The operator \mathcal{L}^0 is a bounded, self adjoint operator on $L^2(I_{\lambda})$.*
- (1) *There exist $\mu_0^0 \in \mathbb{R}$ and $\psi_0^0 \in L^2(I_{\lambda})$, ψ_0^0 strictly positive in I_{λ} so that*

$$\mathcal{L}^0 \psi_0^0 = \mu_0^0 \psi_0^0. \quad (3.4)$$

The eigenvalue μ_0^0 has multiplicity one.

$$0 \leq \mu_0^0 \leq C e^{-\frac{2\alpha}{\lambda}}, \quad (3.5)$$

where $\alpha > 0$ is given in (2.10). Further $\psi_0^0 \in C^\infty(I_\lambda)$, $\psi_0^0(z) = \psi_0^0(-z)$ for $z \in I_\lambda$. The spectrum of \mathcal{L}^0 is discrete and any other eigenvalue is strictly bigger than μ_0^0 .

(2) Let μ_2^0 be the second eigenvalue of \mathcal{L}^0 and $D > 0$ independent on λ . We have that

$$\mu_2^0 = \inf_{(V, \psi_0^0)=0; \|V\|=1} (V, \mathcal{L}^0 V) \geq D. \quad (3.6)$$

(3) Let ψ_0^0 be the normalized eigenfunction corresponding to μ_0^0 we have

$$\|\psi_0^0 - \frac{\bar{m}'}{\|\bar{m}'\|}\| \leq C e^{-\frac{2\alpha}{\lambda}}. \quad (3.7)$$

The point (0) it is easy to prove. To show point (1) one first notes that the Perron-Frobenius Theorem holds for the operator $(\mathcal{P}^0 g)(z) = p(z)(\bar{J} \star_{I_\lambda} g)(z)$, $g \in L^2(I_\lambda)$, since \bar{J} is a positivity improving integral kernel. The operator \mathcal{L}^0 is conjugate to the operator $\mathbb{I} - \mathcal{P}^0$. This implies immediately the result stated in point (1). Estimate (3.6) is obtained by applying the operator to a convenient trial function and using (2.13). The most difficult part is to show point (2). This has been obtained by applying a generalization of Cheeger's inequality. For more details see [14]. Next we introduce a family of one dimensional operators. For any $s \in T$ and for m_A given in (2.11), let

$$(\mathcal{L}^s V)(s, z) = \frac{V(s, z)}{\sigma(m_A(s, \lambda z))} - (\bar{J} \star_{I_\lambda, z} V)(s, z), \quad z \in I_\lambda \quad (3.8)$$

be the operator acting on $L^2(I_\lambda)$ where

$$(\bar{J} \star_{I_\lambda, z} V)(s, z) = \int_{I_\lambda} \bar{J}(z - z') V(s, z') dz'. \quad (3.9)$$

We stress that \mathcal{L}^s acts for any fixed s only on the z variable of V . We denote

$$\langle V, W \rangle_s = \int_{I_\lambda} V(s, z) W(s, z) dz, \quad \|V\|_s^2 = \int_{I_\lambda} V(s, z)^2 dz.$$

By the definition of m_A given in (2.11) we have

$$\frac{1}{\beta(1 - m_A^2(s, \lambda z))} = \frac{1}{\beta(1 - \bar{m}^2(z))} \left[1 + \lambda \frac{2\bar{m}(z)}{(1 - \bar{m}^2(z))} [h_1(z)g(s) + \phi(s, \lambda z)] \right] + q^\lambda(s, \lambda z)\lambda^2. \quad (3.10)$$

By the point wise bound (2.14) we have that

$$\left| \frac{1}{\beta(1 - m_A^2(s, \lambda z))} - \frac{1}{\beta(1 - \bar{m}^2(z))} \right| \leq C\lambda.$$

Therefore the operator \mathcal{L}^s is for each $s \in T$, a λ -perturbation of the operator \mathcal{L}^0 , i.e.

$$\sup_{\{\|V\|_s=1\}} |\langle (\mathcal{L}^s - \mathcal{L}^0)V, V \rangle_s| \leq C\lambda.$$

Nevertheless, by (2.13), it is possible to show that the perturbation on the principal eigenvalue of \mathcal{L}^0 is of order λ^2 . We have the following result.

Theorem 3.2. *For any $\beta > 1$ there exists $\lambda_0 = \lambda_0(\beta)$ so that for $\lambda \leq \lambda_0$ the following holds.*

(1) *For all $s \in T$, the operator \mathcal{L}^s is a bounded, selfadjoint operator on $L^2(I_\lambda)$. There exist $\mu_1(s) \in \mathbb{R}$ and $\Psi_1(s, \cdot) \in L^2(I_\lambda)$, $\Psi_1(s, \cdot)$ strictly positive in I_λ so that*

$$\mathcal{L}^s \Psi_1(s, \cdot) = \mu_1(s) \Psi_1(s, \cdot). \quad (3.11)$$

The eigenvalue $\mu_1(s)$ has multiplicity one and any other point of the spectrum is strictly bigger than $\mu_1(s)$.

(2) We have that for all $s \in T$

$$C\lambda^2 \geq \mu_1(s) = \inf_{\|V\|_s=1} \langle \mathcal{L}^s V, V \rangle_s \geq -C\lambda^2, \quad (3.12)$$

$$\Psi_1(s, \cdot) = \frac{1}{\|\bar{m}'\|} \bar{m}'(\cdot) + \Psi_1^R(s, \cdot) \quad (3.13)$$

where

$$\sup_{s \in T} \|\Psi_1^R\|_s \leq C\lambda. \quad (3.14)$$

Moreover, there exist $z_1 > 0$ and $\zeta_1 > 0$ independent on λ so that

$$\Psi_1(s, z) \geq \zeta_1, \quad |z| \leq z_1, \quad s \in T. \quad (3.15)$$

(3) There exists $\gamma > 0$ such for every $\lambda \in (0, \lambda_0]$ and $s \in T$

$$\mu_2(s) = \inf_{\langle \Psi, \Psi_1 \rangle_s = 0; \|\Psi\|_s = 1} \langle \Psi, \mathcal{L}^s \Psi \rangle_s \geq \gamma. \quad (3.16)$$

(4)

$$\sup_{s \in T} \|\nabla_s \Psi_1\|_s \leq C \|\nabla_s m_A\|_{L^\infty(\mathcal{N}(d_0))}. \quad (3.17)$$

Proof. By the symmetry of \bar{J} and (2.18), immediately one gets that \mathcal{L}^s is a bounded, self-adjoint operator on $L^2(I_\lambda)$, for $s \in T$. Point (1) of the theorem follows by the positivity improving property of the integral kernel \bar{J} , similarly as done in proving point (1) of Theorem 3.1. As a consequence we have that the principal eigenvalue of the spectrum of \mathcal{L}^s , $\mu_1(s)$, has multiplicity one and any other point of the spectrum of \mathcal{L}^s is strictly bigger than $\mu_1(s)$. Further the associated eigenfunction $\Psi_1(s, \cdot)$ does not change sign and we assume that it is positive.

(2) We show (3.12). Taking into account (3.10), we split the operator \mathcal{L}^s as following

$$\langle \mathcal{L}^s V, V \rangle_s = \langle \mathcal{L}^0 V, V \rangle_s + \langle [\mathcal{L}^s - \mathcal{L}^0] V, V \rangle_s, \quad (3.18)$$

where \mathcal{L}^0 is the operator defined in (3.1). We write

$$\langle [\mathcal{L}^s - \mathcal{L}^0] V, V \rangle_s = I_{2,s}(V) + I_{3,s}(V) + I_{4,s}(V) \quad (3.19)$$

where

$$I_{2,s}(V) = \lambda 2g(s)\beta \int_{I_\lambda} dz \frac{1}{\sigma^2(\bar{m}(z))} \bar{m}(z) h_1(z) V(s, z)^2, \quad (3.20)$$

$$I_{3,s}(V) = \lambda 2\beta \int_{I_\lambda} dz \frac{1}{\sigma^2(\bar{m}(z))} \bar{m}(z) \phi(s, \lambda z) V(s, z)^2, \quad (3.21)$$

$$I_{4,s}(V) = \lambda^2 \beta \int_{I_\lambda} dz \frac{1}{\sigma^2(\bar{m}(z))} \bar{m}(z) q^\lambda(s, \lambda z) V(s, z)^2. \quad (3.22)$$

Take as trial function

$$\bar{V}(s, z) = \frac{\bar{m}'(z)}{\|\bar{m}'\|_{L^2(I_\lambda)}} \quad s \in T, \quad z \in I_\lambda. \quad (3.23)$$

By the variational form for the eigenvalues

$$\mu_1(s) \leq \langle \bar{V}, \mathcal{L}^s \bar{V} \rangle_s = \langle \mathcal{L}^0 \bar{V}, \bar{V} \rangle_s + \langle [\mathcal{L}^s - \mathcal{L}^0] \bar{V}, \bar{V} \rangle_s. \quad (3.24)$$

Next we compute the right hand side of (3.24). We have

$$\begin{aligned}
\langle \bar{V}, \mathcal{L}^0 \bar{V} \rangle_s &= \frac{1}{\|\bar{m}'\|_{L^2(I_\lambda)}^2} \int_{I_\lambda} dz \left\{ \frac{\bar{m}'(z)}{\sigma(\bar{m})} - (\bar{J} \star_{I_\lambda} \bar{m}')(z) \right\} \bar{m}'(z) \\
&= \frac{1}{\|\bar{m}'\|_{L^2(I_\lambda)}^2} \int_{I_\lambda} dz \left\{ \frac{\bar{m}'(z)}{\sigma(\bar{m})} - (\bar{J} \star \bar{m}')(z) \right\} \bar{m}'(z) - \frac{1}{\|\bar{m}'\|_{L^2(I_\lambda)}^2} \int_{I_\lambda} dz \bar{m}'(z) \int_{I_\lambda^c} \bar{J}(z-z') \bar{m}'(z') dz' \\
&= -\frac{1}{\|\bar{m}'\|_{L^2(I_\lambda)}^2} \int_{I_\lambda} dz \bar{m}'(z) \int_{I_\lambda^c} \bar{J}(z-z') \bar{m}'(z') dz',
\end{aligned} \tag{3.25}$$

since, see (3.3), $(\mathcal{L}\bar{m}')(z) = \frac{\bar{m}'(z)}{\sigma(\bar{m}(z))} - (\bar{J} \star \bar{m}')(z) = 0$. By (2.10) we have that

$$|\langle \bar{V}, \mathcal{L}^0 \bar{V} \rangle_s| \leq \frac{1}{\|\bar{m}'\|_{L^2(I_\lambda)}^2} \int_{I_\lambda} dz \bar{m}'(z) \int_{I_\lambda^c} \bar{J}(z-z') \bar{m}'(z') dz' \leq C e^{-2\alpha \frac{d_0}{\lambda}}. \tag{3.26}$$

We split, as in (3.19), the second term in (3.24)

$$\langle [\mathcal{L}^s - \mathcal{L}^0] \bar{V}, \bar{V} \rangle_s = I_{2,s}(\bar{V}) + I_{3,s}(\bar{V}) + I_{4,s}(\bar{V}). \tag{3.27}$$

By condition (2.13)

$$I_{2,s}(\bar{V}) = 0.$$

Applying again (2.13)

$$I_{3,s}(\bar{V}) = \lambda 2\beta \frac{1}{\|\bar{m}'\|_{L^2(I_\lambda)}^2} \int_{I_\lambda} dz \frac{1}{\sigma^2(\bar{m}(z))} \bar{m}(z) [\phi(s, \lambda z) - \phi(s, 0)] (\bar{m}'(z))^2. \tag{3.28}$$

By (2.16) and the exponentially decreasing of \bar{m}' , see (2.10), we have

$$|I_{3,s}(\bar{V})| \leq \lambda 2\beta \frac{1}{\|\bar{m}'\|_{L^2(I_\lambda)}^2} \left| \int_{I_\lambda} dz \frac{1}{\sigma^2(\bar{m}(z))} \bar{m}(z) |\phi(s, \lambda z) - \phi(s, 0)| (\bar{m}'(z))^2 \right| \leq \lambda^2 C.$$

By (2.14) and (2.10), we have that

$$|I_{4,s}(\bar{V})| \leq C \lambda^2.$$

Therefore, recalling (3.26) we get

$$\mu_1(s) \leq \langle \bar{V}, \mathcal{L}^s \bar{V} \rangle_s \leq C \lambda^2. \tag{3.29}$$

We need to show $\mu_1(s) \geq -C \lambda^2$. Let $\Psi_1(s, \cdot)$ be the normalized positive eigenfunction associated to $\mu_1(s)$. Since $\mu_1(s) \leq C \lambda^2$, by Lemma 8.1 stated in the Appendix, for λ small enough, $\Psi_1(s, \cdot)$ as function of z decays to zero exponentially fast for z large enough. Set

$$\Psi_1(s, \cdot) = a(s) \psi_0^0(\cdot) + (\psi_0^0)^\perp(s, \cdot), \tag{3.30}$$

where $\psi_0^0(\cdot)$ is the normalized eigenfunction associated to the principal eigenvalue of \mathcal{L}^0 , see (3.7),

$$a(s) = \int_{I_\lambda} \Psi_1(s, z) \psi_0^0(z) dz, \quad \int_{I_\lambda} \psi_0^0(z) (\psi_0^0)^\perp(s, z) dz = 0 \quad s \in T.$$

By (3.18) we have that

$$\mu_1(s) = \langle \Psi_1, \mathcal{L}^s \Psi_1 \rangle_s = \langle \Psi_1, \mathcal{L}^0 \Psi_1 \rangle_s + \langle [\mathcal{L}^s - \mathcal{L}^0] \Psi_1, \Psi_1 \rangle_s. \tag{3.31}$$

By (3.30) we obtain

$$\begin{aligned}
\langle \Psi_1, \mathcal{L}^0 \Psi_1 \rangle_s &= a^2(s) \langle \psi_0^0, \mathcal{L}^0 \psi_0^0 \rangle_s + \langle (\psi_0^0)^\perp, \mathcal{L}^0 (\psi_0^0)^\perp \rangle_s \\
&\geq a^2(s) \mu_0^0 + \mu_2^0 \|(\psi_0^0)^\perp\|_s^2,
\end{aligned} \tag{3.32}$$

where μ_0^0 and μ_2^0 are, respectively, the first and second eigenvalue of \mathcal{L}^0 . We split the second term on the right hand side of (3.31) as in (3.27) obtaining

$$\langle [\mathcal{L}^s - \mathcal{L}^0] \Psi_1, \Psi_1 \rangle_s = I_{2,s}(\Psi_1) + I_{3,s}(\Psi_1) + I_{4,s}(\Psi_1). \quad (3.33)$$

By the L^∞ bounds on σ , q^λ and \bar{m} we have

$$|I_{4,s}(\Psi_1)| \leq C\lambda^2. \quad (3.34)$$

Taking into account (3.30) we get

$$\begin{aligned} I_{2,s}(\Psi_1) &= \lambda g(s) \beta a(s) \int_{I_\lambda} dz \frac{1}{\sigma^2(\bar{m}(z))} \bar{m}(z) h_1(z) \psi_0^0(z)^2 \\ &+ \lambda g(s) \beta \int_{I_\lambda} dz \frac{1}{\sigma^2(\bar{m}(z))} \bar{m}(z) h_1(z) [(\psi_0^0)^\perp(s, z)^2 + 2a(s) \psi_0^0(z) (\psi_0^0)^\perp(s, z)]. \end{aligned} \quad (3.35)$$

By (3.7) and (2.13) we have

$$\left| \int_{I_\lambda} dz \frac{1}{\sigma^2(\bar{m}(z))} \bar{m}(z) h_1(z) (\psi_0^0(z))^2 \right| \leq C e^{-\frac{s}{\lambda}}. \quad (3.36)$$

By the L^∞ bounds on σ , h_1 and \bar{m} we have

$$\left| \int_{I_\lambda} dz \frac{1}{\sigma^2(\bar{m}(z))} \bar{m}(z) h_1(z) [(\psi_0^0)^\perp(s, z)^2 + 2a(s) \psi_0^0(z) (\psi_0^0)^\perp(s, z)] \right| \leq C \|(\psi_0^0)^\perp\|_s. \quad (3.37)$$

Therefore, by (3.36) and (3.37),

$$|I_{2,s}(\Psi_1)| \leq \lambda C \|(\psi_0^0)^\perp\|_s. \quad (3.38)$$

For the third term in the right hand side of (3.31), we have

$$\begin{aligned} I_{3,s}(\Psi_1) &= \lambda \beta \int_{I_\lambda} dz \frac{1}{\sigma^2(\bar{m}(z))} \bar{m}(z) \phi_1(0, s) (\Psi_1(s, z))^2 \\ &+ \lambda \int_{I_\lambda} dz \frac{1}{\sigma(\bar{m}(z))} \bar{m}(z) [\phi_1(\lambda z, s) - \phi_1(0, s)] (\Psi_1(s, z))^2 \\ &\leq \lambda C \|(\psi_0^0)^\perp\|_s + C\lambda^2. \end{aligned} \quad (3.39)$$

We bound the first term on the right hand side of (3.39), as in (3.35), by splitting Ψ_1 , see (3.30), taking into account that (2.13) hold. The bound for the second term on the right hand side of (3.39) is a consequence of the Lipschitz bound (2.16) for ϕ and the exponentially decay to zero of $\Psi_1(s, \cdot)$. Therefore, see (3.31), (3.32), (3.35) and (3.39) and (3.34) we obtain

$$\langle \Psi_1, \mathcal{L}^s \Psi_1 \rangle_s \geq a^2(s) \mu_0^0 + \mu_2^0 \|(\psi_0^0)^\perp\|_s^2 - \lambda \|(\psi_0^0)^\perp\|_s - C\lambda^2. \quad (3.40)$$

By Theorem 3.1 we have that $\mu_0^0 \geq 0$, $\mu_2^0 \geq D$. This, together with the upper bound (3.29), implies

$$\|(\psi_0^0)^\perp\|_s \leq C\lambda. \quad (3.41)$$

Further, writing Ψ_1 as in (3.30) we have

$$1 = \|\Psi_1\|_s^2 = a^2(s) \|\psi_0^0\|_s^2 + \|(\psi_0^0)^\perp\|_s^2 = a^2(s) + \|(\psi_0^0)^\perp\|_s^2.$$

Then

$$a^2(s) = 1 - \|(\psi_0^0)^\perp\|_s^2 \geq 1 - C\lambda^2, \quad s \in T. \quad (3.42)$$

From (3.40) and (3.41) we get therefore

$$\langle \Psi_1, \mathcal{L}^s \Psi_1 \rangle_s \geq -C\lambda^2. \quad (3.43)$$

Next we show (3.13). By (3.7) there exists a function $R(z)$, $z \in I_\lambda$, $\|R\| \leq Ce^{-\frac{\alpha}{\lambda}}$ so that $\psi_0^0(\cdot) = \frac{\bar{m}'(\cdot)}{\|\bar{m}'\|} + R(\cdot)$. Therefore by (3.30) we have

$$\Psi_1(s, \cdot) = a(s)\psi_0^0(\cdot) + (\psi_0^0)^\perp(s, \cdot) = a(s) \left[\frac{\bar{m}'(\cdot)}{\|\bar{m}'\|} + R(\cdot) \right] + (\psi_0^0)^\perp(s, \cdot).$$

Denote by

$$\Psi_1^R(s, \cdot) = a(s)R(\cdot) + (\psi_0^0)^\perp(s, \cdot).$$

By (3.41)

$$\|\Psi_1^R\|_s \leq \|(\psi_0^0)^\perp\|_s + Ce^{-\frac{\alpha}{\lambda}} \leq C\lambda \quad s \in T$$

and (3.13) is proven. The (3.15) is a consequence of exponential decay of $\Psi_1(s, \cdot)$, see Lemma 8.1. Further

$$\int_{I_\lambda} \Psi_1(s, z)^2 dz = 1, \quad \forall s \in T.$$

Therefore we must have that there exists $z_1 > 0$ independent on λ so that

$$\int_{-z_1}^{z_1} \Psi_1(s, z)^2 dz \geq \frac{1}{2}. \quad (3.44)$$

This implies that there exists $\zeta_1 > 0$, independent on λ so that

$$\Psi_1(s, z) \geq \zeta_1, \quad |z| \leq z_1, \forall s \in T.$$

Namely, if this is false, there exists $\bar{z} \in [-z_1, z_1]$ so that $\Psi_1(s, \bar{z}) = 0$. Since \bar{J} is positivity improving and Ψ_1 is an eigenfunction one can easily show that $\Psi_1(s, z) = 0$ for $z \in [-z_1, z_1]$. This is impossible since (3.44) holds.

(3) Let $\Psi_2(s, \cdot)$ be one of the normalized eigenfunctions corresponding to the second eigenvalue $\mu_2(s)$ of \mathcal{L}^s , then

$$\int_{I_\lambda} \Psi_1(s, z)\Psi_2(s, z)dz = 0, \quad \forall s \in T.$$

Split $\Psi_1(s, \cdot)$ as in (3.30). We obtain

$$0 = \langle \Psi_1, \Psi_2 \rangle_s = a(s)\langle \psi_0^0, \Psi_2 \rangle_s + \langle (\psi_0^0)^\perp, \Psi_2 \rangle_s.$$

Therefore, taking into account (3.42),

$$\langle \psi_0^0, \Psi_2 \rangle_s = -\frac{1}{a(s)}\langle (\psi_0^0)^\perp, \Psi_2 \rangle_s.$$

Hence, by (3.41),

$$|\langle \psi_0^0, \Psi_2 \rangle_s| \leq \frac{1}{|a(s)|}\|(\psi_0^0)^\perp\|_s \leq C\lambda. \quad (3.45)$$

Further we assume that $\mu_2(s)$ satisfies the hypothesis of Lemma 8.1. In fact either it is small and therefore satisfies the hypothesis of Lemma 8.1, either it is large, than there is nothing to prove. Then we can argue as in Lemma 8.1 and show that $\Psi_2(s, \cdot)$, as function of z decays exponentially fast to zero. So we can decompose $\Psi_2(s, \cdot)$ as

$$\Psi_2(s, \cdot) = \langle \Psi_2, \psi_0^0 \rangle_s \psi_0^0(\cdot) + [\Psi_2(s, \cdot) - \langle \Psi_2, \psi_0^0 \rangle_s \psi_0^0(\cdot)]. \quad (3.46)$$

We therefore obtain

$$\mu_2(s) = \langle \mathcal{L}^s \Psi_2, \Psi_2 \rangle_s = \langle \mathcal{L}^0 \Psi_2, \Psi_2 \rangle_s + \langle [\mathcal{L}^s - \mathcal{L}^0] \Psi_2, \Psi_2 \rangle_s.$$

By (3.19)

$$|\langle [\mathcal{L}^s - \mathcal{L}^0] \Psi_2, \Psi_2 \rangle_s| \leq C\lambda.$$

Then inserting (3.46) we have that

$$\begin{aligned}\mu_2(s) &\geq \mu_0^0 \langle \Psi_2, \psi_0^0 \rangle_s^2 + \mu_2^0 [1 - \langle \Psi_2, \psi_0^0 \rangle_s^2] - C\lambda \\ &\geq D[1 - C\lambda^2] - C\lambda\end{aligned}\tag{3.47}$$

where $D > 0$ is the lower bound in (3.6) and we applied estimate (3.45). For λ small enough, there exists $\gamma > 0$ so that (3.16) holds.

(4) To prove (3.17) we differentiate the eigenvalue equation for Ψ_1 with respect to s . We obtain

$$[\mathcal{L}^s - \mu_1(s)]\Psi_{1s} = \mu_{1s}(s)\Psi_1 - \frac{d}{ds} \left(\frac{1}{\sigma(m_A)} \right) \Psi_1.\tag{3.48}$$

Since

$$\int \Psi_{1s}(z)\Psi_1(z)dz = 0\tag{3.49}$$

$$\langle [\mathcal{L}^s - \mu_1(s)]\Psi_{1s}, \Psi_{1s} \rangle_s = -\left\langle \frac{d}{ds} \frac{1}{\sigma(m_A)} \Psi_1, \Psi_{1s} \right\rangle_s.\tag{3.50}$$

Therefore,

$$|\langle [\mathcal{L}^s - \mu_1(s)]\Psi_{1s}, \Psi_{1s} \rangle_s| \leq C \|\Psi_{1s}\|_s \|\nabla_s m_A\|_{L^\infty(\mathcal{N}(d_0))}.$$

On the other hand, by (3.16)

$$\langle [\mathcal{L}^s - \mu_1(s)]\Psi_{1s}, \Psi_{1s} \rangle_s \geq (\gamma - \mu_1(s)) \|\Psi_{1s}\|_s^2.$$

Then

$$\|\Psi_{1s}\|_s \leq \frac{C}{(\gamma - \mu_1(s))} \|\nabla_s m_A\|_{L^\infty(\mathcal{N}(d_0))} \leq C \|\nabla_s m_A\|_{L^\infty(\mathcal{N}(d_0))}.\tag{3.51}$$

This completes the proof of the theorem. \square

In the following we deal with functions defined in $I = [-d_0, d_0]$. To this end, for $u(s, \cdot) \in L^2(I)$, $s \in T$, denote

$$(L_1^{\lambda, s} u)(s, r) = \frac{u(s, r)}{\sigma(m_A(s, r))} - (\bar{J}^\lambda \star_{I, r} u)(s, r),\tag{3.52}$$

where

$$(\bar{J}^\lambda \star_{I, r} u)(s, r) = \int_I \bar{J}^\lambda(r - r') u(s, r') dr'.\tag{3.53}$$

The subscript 1 in (3.52) is to remind the reader that $L_1^{\lambda, s}$ acts only on functions of the r -variable. We immediately have the following.

Proposition 3.3. *The spectrum of $L_1^{\lambda, s}$ on $L^2(I)$ is equal to the spectrum of \mathcal{L}^s on $L^2(I_\lambda)$. In particular the principal eigenvalue of $L_1^{\lambda, s}$ is*

$$\psi_1(s, r) = \frac{1}{\sqrt{\lambda}} \Psi_1\left(s, \frac{r}{\lambda}\right),\tag{3.54}$$

where $\Psi_1(s, \cdot)$ is the principal eigenvalue of \mathcal{L}^s .

Proof. The operator $\mathcal{L}^s : L^2(I_\lambda) \rightarrow L^2(I_\lambda)$ and the operator $L_1^s : L^2(I) \rightarrow L^2(I)$ are conjugate. Namely, let $T : L^2(I_\lambda) \rightarrow L^2(I)$ be the map so that $(TU)(s, r) = \frac{1}{\sqrt{\lambda}} U\left(s, \frac{r}{\lambda}\right)$. The map T is an isometry:

$$\int_I ((TU)(s, r))^2 dr = \int_I \frac{1}{\lambda} U^2\left(s, \frac{r}{\lambda}\right) dr = \int_{I_\lambda} U^2(s, z) dz$$

and $L_1^{\lambda, s} = T\mathcal{L}^s T^{-1}$. Therefore the spectrum of L_1^s is equal to the spectrum of \mathcal{L}^s and (3.54) holds \square

4. L^2 ESTIMATES

This section is devoted to the proof of Theorem 2.4. Let $\eta(\cdot)$ be the indicator function of the set $\mathcal{N}(d_0)$, $\eta(\xi) = 1$ when $\xi \in \mathcal{N}(d_0)$, $\eta(\xi) = 0$ when $\xi \notin \mathcal{N}(d_0)$. In Subsection 4.1 we bound from below $\int_{\mathcal{N}(d_0)} (A_{m_A}^\lambda \eta u)(\xi) \eta(\xi) u(\xi) d\xi$ in term of the quadratic form of the local operator L^λ defined in (4.6). Then, in Subsection 4.2 we bound from below the quadratic form of the local operator L^λ . Finally in Subsection 4.3 we show Theorem 2.4.

4.1. Bound from below of $\int_{\mathcal{N}(d_0)} (A_{m_A}^\lambda \eta u)(\xi) \eta(\xi) u(\xi) d\xi$. We start writing in local coordinates when $\xi \in \mathcal{N}(d_0)$ the integral $\int_{\mathcal{N}(d_0)} J^\lambda(\xi - \xi') u(\xi') d\xi'$.

Lemma 4.1. *Let $\xi \in \mathcal{N}(d_0)$, $\xi = \rho(s, r)$, $(s, r) \in \mathcal{T}$ be the change of variables defined in (2.4), $u(s, r) = u(\rho(s, r))$ and $\alpha(s, r)$ as in (2.5). We have that*

$$\begin{aligned} \int_{\mathcal{N}(d_0)} J^\lambda(\xi - \xi') u(\xi') d\xi' &= \int_{\mathcal{T}} J^\lambda(s, s', r, r') u(s', r') \alpha(s', r') ds' dr' \\ &+ \int_{\mathcal{T}} R_1^\lambda(s, s', r, r') u(s', r') \alpha(s', r') ds' dr' \\ &+ \int_{\mathcal{T}} R_2^\lambda(s, s', r, r') u(s', r') \alpha(s', r') ds' dr', \end{aligned} \quad (4.1)$$

where, for $s^* = \frac{s+s'}{2}$, $r^* = \frac{r+r'}{2}$ and $\alpha(s^*, r^*) = 1 - k(s^*)r^*$

$$J^\lambda(s, s', r, r') = J^\lambda((s - s')\alpha(s^*, r^*), (r - r')), \quad (4.2)$$

R_1^λ is defined in (8.17) and R_2^λ in (8.18). Further we have that

$$\left| \int_{\mathcal{T}} R_1^\lambda(s, s', r, r') \alpha(s', r') ds' dr' \right| \leq C\lambda^2, \quad (4.3)$$

$$\left| \int_{\mathcal{T}} R_2^\lambda(s, s', r, r') \alpha(s', r') ds' dr' \right| \leq C\lambda^4. \quad (4.4)$$

For the proof of the lemma it is enough that Γ is a C^3 curve. The proof is simple although lengthy and it is reported in the appendix. The decomposition obtained in Lemma 4.1 turns out to be very useful. Set for $u \in L^2(\mathcal{T})$

$$(B^\lambda u)(s, r) = \int_{\mathcal{T}} J^\lambda(s, s', r, r') \alpha(s^*, r^*) u(s', r') ds' dr', \quad (4.5)$$

and

$$(L^\lambda u)(s, r) = \frac{u(s, r)}{\sigma(m_A(s, r))} - (B^\lambda u)(s, r). \quad (4.6)$$

The operator L^λ is selfadjoint in $L^2(\mathcal{T})$. We have the following result.

Lemma 4.2. *Let $\eta(\cdot)$ be the indicator function of the set $\mathcal{N}(d_0)$. Set*

$$\hat{u}(s, r) = \sqrt{\alpha(s, r)} u(s, r). \quad (4.7)$$

We get that

$$\int_{\mathcal{N}(d_0)} (A_{m_A}^\lambda \eta u)(\xi) \eta(\xi) u(\xi) d\xi \geq \langle \hat{u}, L^\lambda \hat{u} \rangle - \lambda^2 C \|u\|_{\mathcal{N}(d_0)}^2. \quad (4.8)$$

Proof. By changing variables we have that

$$\int_{\mathcal{N}(d_0)} (A_{m_A}^\lambda \eta u)(\xi) \eta(\xi) u(\xi) d\xi = \int_{\mathcal{T}} ds dr \alpha(s, r) u(s, r) \left\{ \frac{u(s, r)}{\sigma(m_A(s, r))} - (J^\lambda \star \eta u)(s, r) \right\}.$$

Taking into account Lemma 4.1 we have

$$\begin{aligned} & \int_{\mathcal{T}} ds dr \alpha(s, r) u(s, r) \left\{ \frac{u(s, r)}{\sigma(m_A(s, r))} - (J^\lambda \star \eta u)(s, r) \right\} \\ & \geq \int_{\mathcal{T}} ds dr \alpha(s, r) u(s, r) \left\{ \frac{u(s, r)}{\sigma(m_A(s, r))} - \int_{\mathcal{T}} J^\lambda(s, s', r, r') u(s', r') \alpha(s', r') ds' dr' \right\} - \lambda^2 C \|u\|_{L^2(\mathcal{N}(d_0))}^2. \end{aligned} \quad (4.9)$$

By (4.3) and (4.4) one obtains

$$\begin{aligned} & \left| \int_{\mathcal{T}} ds dr \alpha(s, r) u(s, r) \int_{\mathcal{T}} ds' dr' R_1^\lambda(s, s', r, r') u(s', r') \alpha(s', r') \right| \leq C \lambda^2 \|u\|_{L^2(\mathcal{T}, \alpha(s, r) ds dr)}^2 = C \lambda^2 \|u\|_{L^2(\mathcal{N}(d_0))}^2, \\ & \left| \int_{\mathcal{T}} ds dr \alpha(s, r) u(s, r) \int_{\mathcal{T}} ds' dr' R_2^\lambda(s, s', r, r') u(s', r') \alpha(s', r') \right| \leq C \lambda^4 \|u\|_{L^2(\mathcal{T}, \alpha(s, r) ds dr)}^2 = C \lambda^4 \|u\|_{L^2(\mathcal{N}(d_0))}^2. \end{aligned}$$

Set \hat{u} as in (4.7) and

$$(\hat{\mathcal{D}}^\lambda \hat{u})(s, r) = \sqrt{\alpha(s, r)} \int_{\mathcal{T}} J^\lambda(s, s', r, r') \sqrt{\alpha(s', r')} \hat{u}(s', r') ds' dr'. \quad (4.10)$$

We have immediately that

$$\begin{aligned} & \int_{\mathcal{T}} ds dr \alpha(s, r) u(s, r) \left\{ \frac{u(s, r)}{\sigma(m_A(s, r))} - \int_{\mathcal{T}} J^\lambda(s, s', r, r') u(s', r') \alpha(s', r') ds' dr' \right\} \\ & = \int_{\mathcal{T}} ds dr \hat{u}(s, r) \left\{ \frac{\hat{u}(s, r)}{\sigma(m_A(s, r))} - (\hat{\mathcal{D}}^\lambda \hat{u})(s, r) \right\} \\ & = \int_{\mathcal{T}} ds dr \left\{ \frac{\hat{u}(s, r)}{\sigma(m_A(s, r))} - (B^\lambda \hat{u})(s, r) \right\} \\ & + \int_{\mathcal{T}} ds dr \hat{u}(s, r) (C^\lambda \hat{u})(s, r) \end{aligned} \quad (4.11)$$

where

$$(C^\lambda \hat{u})(s, r) = \int_{\mathcal{T}} J^\lambda(s, s', r, r') \left[\sqrt{\alpha(s, r) \alpha(s', r')} - \alpha(s^*, r^*) \right] \hat{u}(s', r') ds' dr'. \quad (4.12)$$

By Lemma 4.3 given below we have that

$$|\langle \hat{u}, C^\lambda \hat{u} \rangle| \leq C \lambda^2 \|\hat{u}\|_{L^2(\mathcal{T})}^2. \quad (4.13)$$

Taking into account definition (4.6), from (4.11) we get (4.8). \square

Lemma 4.3. *Take $|s - s'| \leq \lambda$ and $|r - r'| \leq \lambda$. We have*

$$\sqrt{\alpha(s, r) \alpha(s', r')} = \alpha(s^*, r^*) \sqrt{1 + b(s, s', r, r')} \quad (4.14)$$

where $b(s, s', r, r')$ is defined in (4.17) and

$$|b(s, s', r, r')| \leq C \lambda^2. \quad (4.15)$$

Proof. Adding and subtracting $k(s^*)r^*$ we have

$$\begin{aligned} \alpha(s, r) \alpha(s', r') & = \alpha(s^*, r^*)^2 + \alpha(s^*, r^*) [2k(s^*)r^* - k(s')r' - k(s)r] \\ & + [k(s^*)r^* - k(s')r'] [k(s^*)r^* - k(s)r] \\ & = \alpha(s^*, r^*)^2 [1 + b(s, s', r, r')] \end{aligned} \quad (4.16)$$

where

$$b(s, s', r, r') = \frac{[2k(s^*)r^* - k(s')r' - k(s)r]}{\alpha(s^*, r^*)} + \frac{[k(s^*)r^* - k(s')r'] [k(s^*)r^* - k(s)r]}{\alpha(s^*, r^*)^2}. \quad (4.17)$$

Next we show (4.15). We have

$$[2k(s^*)r^* - k(s')r' - k(s)r] = [k(s^*) - k(s)]r + [k(s^*) - k(s')]r'. \quad (4.18)$$

Taylor expanding $k(\cdot)$ we have

$$k(s') = k(s^*) + k'(s^*)[s' - s^*] + \frac{1}{2}k''(s^*)[s' - s^*]^2 + \frac{1}{6}k'''(\tilde{s})[s' - s^*]^3, \quad (4.19)$$

where $\tilde{s} \in (s, s')$. Similarly we proceed for $k(s)$. By (4.18), taking into account (4.19) and that $s' - s^* = \frac{[s' - s]}{2}$ and $s - s^* = \frac{[s - s']}{2}$ we obtain

$$\begin{aligned} & |[k(s^*) - k(s')]r' + [k(s^*) - k(s)]r| \\ & \leq \left| k'(s^*) \frac{[s' - s]}{2} [r - r'] + \frac{1}{4}k''(s^*)[s' - s]^2 r^* + \frac{1}{48}k'''(\tilde{s})[s' - s]^3 [r - r'] \right| \leq C\lambda^2. \end{aligned} \quad (4.20)$$

By similar computations

$$|[k(s^*)r^* - k(s')r'] [k(s^*)r^* - k(s)r]| \leq C\lambda^2.$$

□

4.2. Properties of L^λ . The operator L^λ acts on function of (s, r) . In Corollary 4.5, stated below, we show that when the operator L^λ acts on functions depending only on the variable r , it is, in the L^2 norm, λ^2 close to the one dimensional operator $L_1^{\lambda, s}$, defined in (3.52). Preliminarily we need to see how the operator B^λ , defined in (4.5), acts on functions depending only on r . This is the content of the next lemma. The symmetry of \bar{J} is essential to obtain the estimate (4.22).

Lemma 4.4. *Let B^λ be the operator defined in (4.5) and v a function depending only on the r variable. For any $s \in T$ we have*

$$(B^\lambda v)(r) = (\bar{J}^\lambda \star_I v)(r) + (\Gamma^{\lambda, s} v)(r), \quad r \in I \quad (4.21)$$

where $\Gamma^{\lambda, s}$ is the operator defined in (4.26),

$$|\Gamma^{\lambda, s} v(r)| \leq C\lambda^2 (\bar{J}^\lambda \star_I |v|)(r), \quad r \in I. \quad (4.22)$$

When $v \in L^2(I)$,

$$\|\Gamma^{\lambda, s} v\|_{L^2(I)} \leq C\lambda^2 \|v\|_{L^2(I)}. \quad (4.23)$$

Proof. Taking into account that the support of J^λ , see the definition of the operator B^λ , is the ball of radius λ centred in $(s, r) \in \mathcal{T}$ we make the following local change of variable: For each $(s, r) \in \mathcal{T}$ and $r' \in I$,

$$w = f(s') = -(s - s')\alpha(s^*, r^*), \quad |s - s'| \leq \lambda. \quad (4.24)$$

Notice that we are not explicitly writing the dependence on (s, r, r') . Denoting by f' the derivative with respect to s' of f , we have

$$f'(s') = \alpha(s^*, r^*) + \frac{1}{2}(s - s')k'(s^*)r^*.$$

By (2.6) and for λ small enough, $f'(s') > 0$ when $|s - s'| \leq \lambda$. By the inverse function theorem we have

$$ds' = \frac{1}{f'(s'(w))} dw, \quad f'(s'(w)) = \alpha(s^*(w), r^*) + \frac{1}{2} \frac{w}{\alpha(s^*(w), r^*)} k'(s^*(w))r^*,$$

where by an abuse of notation we set $s^*(w) = \frac{1}{2}s + \frac{1}{2}f^{-1}(w)$. We have

$$\begin{aligned} (B^\lambda v)(r) &= \int_{\mathcal{T}} J^\lambda((s-s')\alpha(s^*, r^*), (r-r'))\alpha(s^*, r^*)v(r')ds'dr' \\ &= \int_I dr' \int_{f^{-1}(|s-s'| \leq \lambda)} J^\lambda(w, r-r')\alpha(s^*(w), r^*)\frac{1}{f'(s'(w))}v(r')dw \\ &= \int_I dr' \int J^\lambda(w, r-r')v(r')dw + (\Gamma^{\lambda, s}v)(r) \end{aligned} \quad (4.25)$$

where

$$(\Gamma^{\lambda, s}v)(r) = \int_{\mathcal{T}} dr'dw J^\lambda(w, r-r') \left[\frac{1}{1 + \frac{wk'(s^*(w))r^*}{\alpha(s^*(w), r^*)^2}} - 1 \right] v(r'). \quad (4.26)$$

For r and r' in I , the support of $J^\lambda(w, r-r')$ is the ball of radius λ centered in the point $(0, r) \in \mathcal{T}$. Hence we have for $r \in I$ and $r' \in I$

$$\int J^\lambda(w, r-r')dw = \bar{J}^\lambda(r-r'), \quad (4.27)$$

where \bar{J}^λ is defined in Subsection 2.1. Therefore from (4.25), (4.26) and (4.27) we have (4.21). Denote shortly by $a = \frac{wk'(s^*(w))r^*}{\alpha(s^*(w), r^*)^2}$. Since $|w| \leq \lambda$ we have that $|a| \leq C\lambda$. Writing $\frac{1}{1+a} = \sum_{k=0}^{\infty} (-a)^k$ we have that

$$\left[\frac{1}{1+a} - 1 \right] = -a + \sum_{k=2}^{\infty} (-a)^k.$$

Therefore from (4.25) we obtain

$$\begin{aligned} (\Gamma^{\lambda, s}v)(r) &= - \int_I dr' \int J^\lambda(w, r-r') \frac{wk'(s^*(w))r^*}{\alpha(s^*(w), r^*)^2} v(r')dw \\ &\quad + \int_I dr' \int J^\lambda(w, r-r') \sum_{k=2}^{\infty} (-a)^k v(r')dw. \end{aligned} \quad (4.28)$$

For the second term we have

$$\begin{aligned} &\left| \int_I dr' \int J^\lambda(w, r-r') \sum_{k=2}^{\infty} (-a)^k v(r')dw \right| \\ &\leq C\lambda^2 \int_I dr' \int J^\lambda(w, r-r') |v(r')|dw = C\lambda^2 \int_I dr' \bar{J}^\lambda(r-r') |v(r')|. \end{aligned} \quad (4.29)$$

For the first term of (4.28) develop in Taylor expansion around $w = 0$

$$\frac{k'(s^*(w))r^*}{\alpha(s^*(w), r^*)^2} = \frac{k'(s^*(0))r^*}{\alpha(s^*(0), r^*)^2} + g(\tilde{w})w$$

where we denote by g the derivative of $\frac{k'(s^*(w))r^*}{\alpha(s^*(w), r^*)^2}$ with respect to w . We have that

$$\begin{aligned} &\int_{[-d_0, d_0]} dr' \int J^\lambda(w, r-r') \frac{wk'(s^*(w))r^*}{\alpha(s^*(w), r^*)^2} v(r')dw \\ &= \int_{[-d_0, d_0]} dr' \frac{k'(s^*(0))r^*}{\alpha(s^*(0), r^*)^2} v(r') \int J^\lambda(w, r-r') w dw \\ &\quad + \int_{[-d_0, d_0]} dr' \int J^\lambda(w, r-r') g(\tilde{w}) w^2 v(r') dw. \end{aligned} \quad (4.30)$$

By the symmetry of J^λ

$$\int J^\lambda(w, r - r') w dw = 0.$$

Since $|g(\tilde{w})|w^2 \leq C\lambda^2$, from (4.30), (4.29) and (4.28) we obtain

$$|(\Gamma^{\lambda,s}v)(r)| \leq C\lambda^2(\bar{J}^\lambda \star |v|)(r). \quad (4.31)$$

□

Corollary 4.5. *Let $v \in L^2(I)$ we have that*

$$(L^\lambda v)(r) = (L_1^{\lambda,s}v)(r) + (\Gamma^{\lambda,s}v)(r), \quad (4.32)$$

where $\Gamma^{\lambda,s}$ is the operator defined in (4.26) and

$$|\Gamma^{\lambda,s}v(r)| \leq C\lambda^2(\bar{J}^\lambda \star |v|)(r). \quad (4.33)$$

The proof is immediate by recalling the definition of L^λ given in (4.6) and $L_1^{\lambda,s}$ given in (3.52).

Corollary 4.6.

$$\int_{\mathcal{T}} J^\lambda(s, s', r, r') \alpha(s^*, r^*) ds' dr' = 1 + (\Gamma^{\lambda,s})(r) \quad (4.34)$$

where we denote by $(\Gamma^{\lambda,s})(r)$ the quantity defined in (4.26) when applied to the function $v(r) = 1$ for all $r \in I$ and

$$|(\Gamma^{\lambda,s})(r)| \leq C\lambda^2.$$

Further

$$\int_{\mathcal{T}} J^\lambda(s, s', r, r') \alpha(s^*, r^*) ds' = J^\lambda(r - r') + (\Gamma_1^{\lambda,s})(r, r'), \quad (4.35)$$

where $\Gamma_1^{\lambda,s}$ is defined in (4.38),

$$|(\Gamma_1^{\lambda,s})(r, r')| \leq C\lambda^2 J^\lambda(r - r'). \quad (4.36)$$

Proof. The proof of (4.34) is straightforward. Take $v(r) = 1$ in Lemma 4.4 and the thesis follows. To show (4.35), we proceed similarly as in Lemma 4.4. For r and r' in I , we have as in (4.25)

$$\begin{aligned} & \int_{\mathcal{T}} J^\lambda((s - s')\alpha(s^*, r^*), r - r') \alpha(s^*, r^*) ds' \\ &= \int J^\lambda(w, r - r') \alpha(s^*(w), r^*) \frac{1}{f'(s'(w))} dw \\ &= \int J^\lambda(w, r - r') dw + (\Gamma_1^{\lambda,s})(r) = \bar{J}^\lambda(r - r') + (\Gamma_1^{\lambda,s})(r, r') \end{aligned} \quad (4.37)$$

where

$$(\Gamma_1^{\lambda,s})(r, r') = \int J^\lambda(w, r - r') \left[\frac{1}{1 + \frac{wk'(s^*(w))r^*}{\alpha(s^*(w), r^*)^2}} - 1 \right] dw. \quad (4.38)$$

Similarly as done in (4.28) and (4.29), (4.30) and (4.31) we have

$$|(\Gamma_1^{\lambda,s})(r, r')| \leq C\lambda^2 \bar{J}^\lambda(r - r'). \quad (4.39)$$

□

4.3. Two dimensional integral operators in enlarged bounded domains. Let \mathcal{T}_λ be the enlarged cylinder

$$\mathcal{T}_\lambda = T \times I_\lambda, \quad I_\lambda = \left[-\frac{d_0}{\lambda}, \frac{d_0}{\lambda}\right]. \quad (4.40)$$

Notice that the circle T is kept unchanged. Denote $z^* = \frac{z+z'}{2}$, $s^* = \frac{s+s'}{2}$ and, see (2.5),

$$\alpha(s^*, z^*) = 1 - \lambda k(s^*)z^*, \quad (4.41)$$

$$J^c(s, s', z, z') = \frac{1}{\lambda} J \left(\frac{(s-s')}{\lambda} \alpha(s^*, z^*), (z-z) \right) \alpha(s^*, z^*), \quad (4.42)$$

and

$$(\mathcal{B}V)(s, z) = \int_{\mathcal{T}_\lambda} J^c(s, s', z, z') V(s', z') ds' dz'. \quad (4.43)$$

The J^c is obtained by rescaling only the r -variable of the integral kernel defining the operator B^λ , see (4.5). Denote for $V \in L^2(\mathcal{T}_\lambda)$

$$(\mathcal{A}V)(s, z) = \frac{1}{\sigma(m_A(s, z))} V(s, z) - \int_{\mathcal{T}_\lambda} J^c(s, s', z, z') V(s', z') ds' dz', \quad (4.44)$$

where J^c is defined in (4.42). We avoid to write explicitly the dependence on λ on the previous notations, but the reader should bear in mind that if not explicitly written, there is almost always an hidden dependence on λ . In this section we study the spectrum of the integral operator $\mathcal{A} : L^2(\mathcal{T}_\lambda) \rightarrow L^2(\mathcal{T}_\lambda)$, defined in (4.44). The operator \mathcal{A} is conjugate to $L^\lambda : L^2(\mathcal{T}) \rightarrow L^2(\mathcal{T})$, hence the spectrum of \mathcal{A} and L^λ are the same, see Theorem 4.8 We have the following result.

Theorem 4.7. (0) *The operator \mathcal{A} is a bounded, self adjoint operator on $L^2(\mathcal{T}_\lambda)$ with discrete spectrum.*

(1) *There exist $\mu_0 \in \mathbb{R}$ and $\Phi_0 \in L^2(\mathcal{T}_\lambda)$, Φ_0 strictly positive in \mathcal{T}_λ so that*

$$\mathcal{A}\Phi_0 = \mu_0\Phi_0. \quad (4.45)$$

The eigenvalue μ_0 has multiplicity one and any other point of the spectrum is strictly bigger than μ_0 .

(2) *There exists $\zeta_1 > 0$ and $z_1 > 0$ independent on λ so that*

$$\Phi_0(s, z) \geq \zeta_1 \quad |z| \geq z_1. \quad (4.46)$$

(3) *There exists $C > 0$ independent on λ so that*

$$-C\lambda^2 \leq \mu_0 \leq C\lambda^2. \quad (4.47)$$

The proof is given at the end of the subsection. This result immediately implies the following.

Theorem 4.8. *Let L^λ be the operator defined in (4.6). The spectrum of L^λ is equal to spectrum of \mathcal{A} . In particular we have*

$$\langle u, L^\lambda u \rangle \geq \mu_0 \|u\|_{L^2(\mathcal{T})}^2 \geq -\lambda^2 C \|u\|_{L^2(\mathcal{T})}^2.$$

The thesis immediately follows by Theorem 4.7 and noticing that L^λ and \mathcal{A} are conjugate. The proof is similar to the one given in Proposition 3.3. The following results are straightforward consequences of Lemma 4.4 and Corollary 4.6.

Proposition 4.9. *Let $V \in L^2(I_\lambda)$. For any $s \in T$ we have*

$$(\mathcal{B}V)(z) = (\bar{J} \star_{I_\lambda} V)(z) + (\Gamma^s V)(z), \quad (4.48)$$

$$\int_{\mathcal{T}_\lambda} ds' dz' J^c(s, s', z, z') = 1 + (\Gamma^s)(z), \quad (4.49)$$

$$\int_{\mathcal{T}_\lambda} ds' J^c(s, s', z, z') = \bar{J}(z - z') + (\Gamma_1^s)(z, z'), \quad (4.50)$$

$$|(\Gamma^s V)(z)| \leq C\lambda^2 (\bar{J} \star_{I_\lambda} |V|)(z), \quad |\Gamma^s(z)| \leq C\lambda^2, \quad |\Gamma_1^s(z, z')| \leq C\lambda^2 \bar{J}(z - z'). \quad (4.51)$$

Proof. Taking into account the definition of J^c , see (4.42), applying similar argument as in Lemma 4.4 and Corollary 4.6 one gets the statements. \square

Denote

$$\begin{aligned} p(s, z) &= \sigma(m_A(s, z)), \quad (s, z) \in \mathcal{T}_\lambda, \\ \mathcal{H} &= \left\{ u : \int_{\mathcal{T}_\lambda} u(s, z)^2 \frac{1}{p(s, z)} dz ds < \infty \right\}, \\ \langle\langle V, U \rangle\rangle &= \int_{\mathcal{T}_\lambda} V(s, z) U(s, z) \frac{1}{p(s, z)} dz ds, \\ \|V\|_{\mathcal{H}}^2 &= \int_{\mathcal{T}_\lambda} V(s, z)^2 \frac{1}{p(s, z)} dz ds, \end{aligned} \quad (4.52)$$

and \mathcal{P} the linear integral operator acting on functions $V \in \mathcal{H}$

$$(\mathcal{P}V)(s, z) = p(s, z) \int_{\mathcal{T}_\lambda} J^c(s, s', z, z') V(s', z') ds' dz'. \quad (4.53)$$

By the property of m_A stated in Subsection 2.4, $\beta \geq p(s, z) > a > 0$ and $p \in C^1(\mathcal{T}_\lambda)$.

Theorem 4.10. *The operator \mathcal{P} is a compact, self-adjoint operator on \mathcal{H} , positivity improving. Further, there exist $\nu_0 > 0$ and $V_0 \in \mathcal{H}$, V_0 strictly positive function, so that*

$$\mathcal{P}V_0 = \nu_0 V_0. \quad (4.54)$$

The eigenvalue ν_0 has multiplicity one and any other point of the spectrum is strictly inside the ball of radius ν_0 . The eigenfunction $V_0 \in C^1(\mathcal{T}_\lambda)$. Further there exists $z_1 > 0$ and $\zeta > 0$ independent on λ so that

$$V_0(s, z) \geq \zeta, \quad |z| \leq z_1, \forall s \in T. \quad (4.55)$$

Proof. It is immediate to see that

$$\langle\langle \mathcal{P}V, W \rangle\rangle = \langle\langle V, \mathcal{P}W \rangle\rangle.$$

The compactness can be shown by proving that any bounded set of \mathcal{H} is mapped by \mathcal{P} in a relatively compact set. To show the positivity improving we take $V \geq 0$ in \mathcal{T} , $V \neq 0$, and show that for all $(s, z) \in \mathcal{T}_\lambda$,

$$(\mathcal{P}V)(s, z) > 0. \quad (4.56)$$

Namely, assume that there exists $(\bar{s}, \bar{z}) \in \mathcal{T}_\lambda$ so that $(\mathcal{P}V)(\bar{s}, \bar{z}) = 0$. Since $J^c \geq 0$ we have that $V(s, z) = 0$ for all $(s, z) \in Q_\lambda(\bar{s}, \bar{z}) = \{(s, z) \in \mathcal{T}_\lambda : |s - \bar{s}| \leq \lambda, |z - \bar{z}| \leq 1\}$. Repeating the same argument for points (s, z) in $Q_\lambda(\bar{s}, \bar{z})$ we end up that $V(s, z) = 0$ for $(s, z) \in \mathcal{T}_\lambda$, obtaining a contradiction. Therefore the positivity improving property is proven. From the positivity hypothesis on J^c , there exists an integer n_λ such that for $n \geq n_\lambda$, there is $\zeta > 0$ so that for any $\underline{x} = (s, z)$ and $\bar{x} = (\bar{s}, \bar{z})$ in \mathcal{T}_λ

$$\int d\underline{x}_1 d\underline{x}_2 \dots d\underline{x}_n J^c(\underline{x}, \underline{x}_1) J^c(\underline{x}_1, \underline{x}_2) \dots J^c(\underline{x}_n, \bar{x}) > \zeta. \quad (4.57)$$

Denote for $\underline{x} \in \mathcal{T}$ and $\bar{x} \in \mathcal{T}$

$$K(\underline{x}, \bar{x}) = p(\underline{x}) \int d\underline{x}_1 d\underline{x}_2 \dots d\underline{x}_n p(\underline{x}_1) J^c(\underline{x}, \underline{x}_1) p(\underline{x}_2) J^c(\underline{x}_1, \underline{x}_2) \dots p(\underline{x}_n) J^c(\underline{x}_n, \bar{x}). \quad (4.58)$$

Then one can apply the classical Perron Frobenius Theorem to the kernel $K(\cdot, \cdot)$. As a consequence we have that the maximum eigenvalue of the spectrum of \mathcal{P} , which we denote ν_0 , has multiplicity one and any other point of the spectrum of \mathcal{P} is strictly smaller than ν_0 . Let V_0 be the eigenfunction associated to ν_0 . Then it does not change sign and we assume that it is positive. Differentiating the eigenvalue equation, taking into account that $p \in C^1(\mathcal{T}_\lambda)$ we get that $V_0 \in C^1(\mathcal{T}_\lambda)$. Next we show (4.55). From

(4.61) of Lemma 4.11, stated below, we have that $\int_T V_0(s, z)^2 ds$ is exponentially decreasing for $|z| \geq z_0$. Further

$$\int_{\mathcal{T}_\lambda} \frac{1}{p(s, z)} V_0(s, z)^2 ds dz = 1.$$

Therefore we must have that there exists $z_1 > 0$ independent on λ so that

$$\int_{-z_1}^{z_1} \frac{1}{p(s, z)} \int_T V_0(s, z)^2 ds dz \geq \frac{1}{2}. \quad (4.59)$$

This implies that there exists $\zeta > 0$, independent on λ so that

$$V_0(s, z) \geq \zeta, \quad |z| \leq z_1, \forall s \in T.$$

Namely, if this is false, there exists $(\bar{s}, \bar{z}) \in T \times [-z_1, z_1]$ so that $V_0(\bar{s}, \bar{z}) = 0$. Since \mathcal{P} is positivity improving and V_0 is an eigenfunction, repeating the argument done after formula (4.56), we get $V_0(s, z) = 0$ in $T \times [-z_1, z_1]$. This is impossible since (4.59) holds. \square

Lemma 4.11. *For any $\epsilon_0 \in (0, \frac{1-\sigma(m_\beta)}{2})$ there exists $\lambda_0 = \lambda_0(\epsilon_0) > 0$ so that for $\lambda \leq \lambda_0$ the following holds. Let $\nu > 1 - \epsilon_0$ be an eigenvalue of the operator \mathcal{P} on \mathcal{H} and Ψ be any of the corresponding normalized eigenfunctions. There exists $z_0 = z_0(\epsilon_0) \in I_\lambda$ independent on λ so that*

$$|\Psi(s, z)| \leq \frac{C}{\sqrt{\lambda}} e^{-\alpha(\epsilon_0)|z|} \|\Psi\|_{\mathcal{H}} \quad |z| \geq z_0, \quad \forall s \in T, \quad (4.60)$$

where $\alpha(\epsilon_0)$ is given in (4.67). Further

$$\int_T |\Psi(s, z)|^2 ds \leq C e^{-\alpha(\epsilon_0)|z|} \|\Psi\|_{\mathcal{H}}^2, \quad |z| \geq z_0. \quad (4.61)$$

Proof. For any ϵ_0 , choose $\lambda_0 = \lambda_0(\epsilon_0) > 0$, $z_0 = z_0(\epsilon_0) > 0$, such that for for $\lambda \leq \lambda_0$

$$p(s, z) < 1 - 2\epsilon_0, \quad |z| \geq z_0, \quad (s, z) \in \mathcal{T}_\lambda. \quad (4.62)$$

This is possible since $|p(s, z) - \sigma(\bar{m}(z))| \leq C\lambda$, $\lim_{|z| \rightarrow \infty} \sigma(\bar{m}(z)) = \sigma(m_\beta)$ and $1 - 2\epsilon_0 < \sigma(m_\beta)$. Further $\nu > 1 - \epsilon_0$, hence, by (4.62), we have for $|z| \geq z_0$,

$$\frac{1}{\nu} p(s, z) < \frac{1 - 2\epsilon_0}{1 - \epsilon_0}. \quad (4.63)$$

Take $\lambda_0 = \lambda_0(\epsilon_0)$ small enough so that $2z_0 \in I_\lambda$. Take $z = z_0 + n$ where n is any integer so that $z_0 + 2n \in \mathcal{T}_\lambda$. By the eigenvalue equation we have

$$\Psi(s, z_0 + n) = \frac{1}{\nu} (\mathcal{P}\Psi)(s, z_0 + n). \quad (4.64)$$

Iterating n times (4.64) and by (4.63) we obtain

$$|\Psi(s, z_0 + n)| \leq \left(\frac{1 - 2\epsilon_0}{1 - \epsilon_0} \right)^n |(J^c)^n \Psi(s, z_0 + n)| \leq \left(\frac{1 - 2\epsilon_0}{1 - \epsilon_0} \right)^n \|(J^c)^n \Psi\|_\infty. \quad (4.65)$$

By Proposition 4.9 we have that

$$\|J^c\|_1 = \int_T ds' dz' J^c(s, s', z, z') = 1 + \Gamma^s(z),$$

where $|\Gamma^s(z)| \leq C\lambda^2$. Hence, by Jensen inequality we get

$$\|(J^c)^n \Psi\|_\infty \leq \|J^c\|_2 \|(J^c)^{n-1} \Psi\|_2 \leq \frac{1}{\sqrt{\lambda}} C \|J\|_2 (\|J^c\|_1)^{n-1} \|\Psi\|_{\mathcal{H}} \leq \frac{1}{\sqrt{\lambda}} C (1 + C\lambda^2)^{n-1} \|J\|_2 \|\Psi\|_{\mathcal{H}},$$

where we estimated $\|J^c\|_2 \leq \frac{C}{\sqrt{\lambda}}\|J\|_2$. Take λ small enough so that $\frac{1}{2} \ln \left(\frac{1-\epsilon_0}{1-2\epsilon_0} \right) \geq \ln(1 + C\lambda^2)$. Hence, by (4.65), we have

$$|\Psi(s, z_0 + n)| \leq \frac{1}{\sqrt{\lambda}} C e^{-n\alpha(\epsilon_0)} \|J\|_2 \|\Psi\|_{\mathcal{H}}, \quad (4.66)$$

where

$$\alpha(\epsilon_0) = \frac{1}{2} \ln \left(\frac{1 - \epsilon_0}{1 - 2\epsilon_0} \right) > 0. \quad (4.67)$$

To get (4.61) we proceed as above obtaining

$$\int_T ds |\Psi(s, z_0 + n)|^2 \leq \left(\frac{1 - 2\epsilon_0}{1 - \epsilon_0} \right)^{2n} \int_T ds |(J^c)^n \Psi(s, z_0 + n)|^2. \quad (4.68)$$

By Proposition 4.9 we have that

$$\int_T ds' J^c(s, s', z, z') = \bar{J}(z - z') + \Gamma_1^s(z, z')$$

where $|\Gamma_1^s(z, z')| \leq C\lambda^2 \bar{J}(z - z')$. Set $\bar{z} = z_0 + n$, by Jensen inequality we have

$$\begin{aligned} \int_T ds |(J^c)^n \Psi(s, z_0 + n)|^2 &= \int_T ds \left(\int_{\mathcal{T}_\lambda} ds' dz' J^c(s, s', \bar{z}, z') |(J^c)^{n-1} \Psi(s', z')| \right)^2 \\ &\leq (1 + C\lambda^2) \int_{\mathcal{T}_\lambda} ds' dz' \left(\int_T ds J^c(s, s', \bar{z}, z') \right) |(J^c)^{n-1} \Psi(s', z')|^2 \\ &\leq (1 + C\lambda^2) \int_{\mathcal{T}_\lambda} ds' dz' \bar{J}(\bar{z} - z') |(J^c)^{n-1} \Psi(s', z')|^2 + C\lambda^2 \int_{\mathcal{T}_\lambda} ds' dz' \bar{J}(\bar{z} - z') |(J^c)^{n-1} \Psi(s', z')|^2 \\ &= (1 + 2C\lambda^2) \int_{\mathcal{T}_\lambda} ds' dz' \bar{J}(\bar{z} - z') |(J^c)^{n-1} \Psi(s', z')|^2 \leq (1 + 2C\lambda^2) \sup_z \bar{J}(z) \int_{\mathcal{T}_\lambda} ds' dz' |(J^c)^{n-1} \Psi(s', z')|^2 \\ &\leq (1 + 2C\lambda^2)^n \sup_z \bar{J}(z) \|\Psi\|_{\mathcal{H}}^2 \leq C(1 + 2C\lambda^2)^n \|\Psi\|_{\mathcal{H}}^2. \end{aligned} \quad (4.69)$$

Therefore by (4.68), (4.69) and (4.67)

$$\int_T ds |\Psi(s, z_0 + n)|^2 \leq C \left(\frac{1 - 2\epsilon_0}{1 - \epsilon_0} \right)^{2n} (1 + 2C\lambda^2)^n \|\Psi\|_{\mathcal{H}}^2 \leq C e^{-4n\alpha(\epsilon_0)} e^{n \ln(1 + 2C\lambda^2)} \|\Psi\|_{\mathcal{H}}^2. \quad (4.70)$$

Take λ small enough so that $3\alpha(\epsilon_0) \geq \ln(1 + 2C\lambda^2)$, we then obtain (4.61). \square

Proof of Theorem 4.7 The points (0), (1) and (2) are an immediate consequence of Theorem 4.10. Namely, let $T : \mathcal{H} \rightarrow L^2(\mathcal{T}_\lambda)$ so that $TV = \frac{V}{\sqrt{p}}$. The map T is an isometry and the operator $\mathbb{I} - \mathcal{P} : \mathcal{H} \rightarrow \mathcal{H}$ is therefore conjugate to the operator $\mathcal{A} : L^2(\mathcal{T}_\lambda) \rightarrow L^2(\mathcal{T}_\lambda)$. This means that the spectrum of the two operators are equal, moreover if V is an eigenfunction of \mathcal{P} , then TV is an eigenfunction of \mathcal{A} . Next we show the upper bound in (4.47). By the variational form for the eigenvalues

$$\mu_0 = \inf_{V \in L^2(\mathcal{T}_\lambda), \|V\|=1} \langle \mathcal{A}V, V \rangle \leq \langle \mathcal{A}\bar{V}, \bar{V} \rangle,$$

where $\bar{V} = \frac{\bar{m}'(z)}{\|\bar{m}'\|_{L^2(\mathcal{T}_\lambda)}}$. Since \bar{V} is a function of only the z -variable, by Proposition 4.9,

$$\mathcal{A}\bar{V} = \mathcal{L}^s \bar{V} + \Gamma^s \bar{V}.$$

Hence

$$\mu_0 \leq \langle \mathcal{A}\bar{V}, \bar{V} \rangle = \int_T ds \langle \bar{V}(s, \cdot), \mathcal{L}^s \bar{V}(s, \cdot) \rangle_s + \int_T ds \langle \bar{V}(s, \cdot), \Gamma^s \bar{V}(s, \cdot) \rangle_s.$$

By (3.29)

$$\int_T ds \langle \bar{V}(s, \cdot), \mathcal{L}^s \bar{V}(s, \cdot) \rangle_s \leq C\lambda^2,$$

and by (4.51)

$$\int_T ds \langle \bar{V}(s, \cdot), \Gamma^s \bar{V}(s, \cdot) \rangle_s \leq C\lambda^2.$$

Therefore

$$\mu_0 \leq C\lambda^2. \quad (4.71)$$

Let Φ_0 be the normalized positive eigenfunction associated to μ_0 . Multiply the eigenvalue equation by Ψ_1 , see (3.11) the principal eigenvalue of \mathcal{L}^s , defined in (3.8). We have, since \mathcal{A} is self-adjoint

$$\mu_0 \langle \Phi_0, \Psi_1 \rangle = \langle \mathcal{A}\Phi_0, \Psi_1 \rangle = \langle \Phi_0, \mathcal{A}\Psi_1 \rangle.$$

By Taylor formula $\Psi_1(s', z) = \Psi_1(s, z) + (s - s')\nabla_s \Psi_1(\tilde{s}, z)$, where $\tilde{s} \in (s, s')$. By Proposition 4.9

$$\begin{aligned} \mathcal{A}\Psi_1 &= \frac{\Psi_1}{\sigma(m_A)} - \int_{\mathcal{T}_\lambda} J^c(s, s', z, z') \Psi_1(s', z') ds' dz' \\ &= \frac{\Psi_1}{\sigma(m_A)} - \int_{\mathcal{T}_\lambda} J^c(s, s', z, z') \Psi_1(s, z') ds' dz' - \int_{\mathcal{T}_\lambda} J^c(s, s', z, z') (s - s') \nabla_s \Psi_1(\tilde{s}, z') ds' dz' \\ &= (\mathcal{L}^s \Psi_1)(s, z) + \Gamma^s \Psi_1 - \int_{\mathcal{T}_\lambda} J^c(s, s', z, z') (s - s') \nabla_s \Psi_1(\tilde{s}, z') ds' dz'. \end{aligned} \quad (4.72)$$

Therefore

$$\begin{aligned} \mu_0 \langle \Phi_0, \Psi_1 \rangle &= \langle \Phi_0, \mathcal{L}^s \Psi_1 \rangle + \langle \Phi_0, \Gamma^s \Psi_1 \rangle \\ &+ \int_{\mathcal{T}_\lambda} ds dz \Phi_0(s, z) \int_{\mathcal{T}_\lambda} J^c(s, s', z, z') (s - s') \nabla_s \Psi_1(\tilde{s}, z') ds' dz'. \end{aligned} \quad (4.73)$$

By (3.17) and (2.15) we estimate

$$\left| \int_{\mathcal{T}_\lambda} ds dz \Phi_0(s, z) \int_{\mathcal{T}_\lambda} J^c(s, s', z, z') (s - s') \nabla_s \Psi_1(\tilde{s}, z') ds' dz' \right| \leq C\lambda \|\Phi_0\| \|\nabla_s \Psi_1\| \leq C\lambda \|\nabla_s m_A\| \leq C\lambda^2,$$

$$\langle \Phi_0, \Gamma^s \Psi_1 \rangle \leq C\lambda^2.$$

Therefore, we have

$$\mu_0 \langle \Phi_0, \Psi_1 \rangle \geq \inf_s \mu_1^s \langle \Phi_0, \Psi_1 \rangle - C\lambda^2.$$

We need to show that

$$\langle \Phi_0, \Psi_1 \rangle \geq C. \quad (4.74)$$

By (3.15) and (4.46) there exist $z_1 > 0$ and $\zeta_1 > 0$ independent on λ so that $\Phi_0(s, z)\Psi_1(s, z) \geq \zeta_1$ for $|z| \leq z_1$ and for all $s \in T$. Therefore (4.74) holds and we obtain

$$\mu_0 \geq -C\lambda^2.$$

□

4.4. Proof of Theorem 2.4. As explained in the introduction, we would like to take advantage of (2.17), splitting the quadratic form associated to the operator $A_{m_A}^\lambda$ in two integrals. One integral is over the region $\Omega \setminus \mathcal{N}(\frac{d_0}{2})$ and because of (2.17) is positive. The second integral is over the region $\mathcal{N}(\frac{d_0}{2})$, i.e near the surface Γ and we can estimate it from below by applying Lemma 4.2 and Lemma 4.8. But because of the non locality of the operator this argument does not work. Namely when splitting the integral of the quadratic form there is an extra term which might spoil the estimate. Here, we show that it is always possible to find a way to split the integral of the quadratic form associated to the operator $A_{m_A}^\lambda$ to obtain the desired estimate.

Define for any integer $k \in \{0, \dots, N\}$, for $N = [\frac{1}{\lambda}]$ where for $x \in \mathbb{R}$, $[x]$ is the integer part of x , a sequences of cut off functions

$$\eta_1^k(\xi) = \begin{cases} 1 & \text{when } \xi \in \mathcal{N}(\frac{d_0}{2}(1 + \lambda k)) \\ 0 & \text{otherwise,} \end{cases} \quad (4.75)$$

$\eta_2^k(\xi) = 1 - \eta_1^k(\xi)$ and set

$$s_k = 2 \int_{\Omega} d\xi \eta_1^k(\xi) v(\xi) (J^\lambda \star \eta_2^k v)(\xi).$$

Let $k = 0$. We have, taking into account that $\eta_2^0(\xi) \eta_1^0(\xi) = 0$ for $\xi \in \Omega$ and the symmetry of $J^\lambda(\cdot)$

$$\begin{aligned} \int_{\Omega} (A_{m_A}^\lambda v)(\xi) v(\xi) d\xi &= \int_{\Omega} (A_{m_A}^\lambda \eta_1^0 v)(\xi) \eta_1^0(\xi) v(\xi) d\xi \\ &+ \int_{\Omega} (A_{m_A}^\lambda \eta_2^0 v)(\xi) \eta_2^0(\xi) v(\xi) d\xi - s_0 \end{aligned} \quad (4.76)$$

Because of (2.17)

$$\int_{\Omega} (A_{m_A}^\lambda \eta_2^0 v)(\xi) \eta_2^0(\xi) v(\xi) d\xi \geq (C^* - 1) \|\eta_2^0 v\|_{L^2(\Omega)}^2 > 0.$$

By Lemma 4.2 and Lemma 4.8 we have that

$$\int_{\Omega} (A_{m_A}^\lambda \eta_1^0 v)(\xi) \eta_1^0 v(\xi) d\xi \geq -C\lambda^2 \|v\|_{L^2(\Omega)}^2.$$

But the last term of (4.76) might create problems. Obviously if

$$s_0 < 0 \quad (4.77)$$

or if

$$s_0 \leq \delta^* \|\eta_2^0 v\|_{L^2(\Omega)}^2 \quad (4.78)$$

for $\delta^* > 0$ so that $(C^* - 1 - \delta^*) > 0$ or if

$$s_0 \leq C\lambda^2 \|v\|_{L^2(\Omega)}^2 \quad (4.79)$$

then the theorem would be proven. But this might not be the case. We proceed recursively as following. Assume that no one of the three conditions (4.77), (4.78), (4.79) hold. Take $\delta > 0$ so that $2\frac{\delta}{1-\delta} \leq \delta^*$ Notice that since we are assuming that (4.77) does not hold, $s_0 > 0$. We must have

$$s_0 > \delta \left[s_0 + 2\|\eta_2^0 v\|_{L^2(\Omega)}^2 \right]. \quad (4.80)$$

Namely if the reverse inequality holds in (4.80) then

$$s_0 \leq 2\frac{\delta}{1-\delta} \|\eta_2^0 v\|_{L^2(\Omega)}^2 \leq \delta^* \|\eta_2^0 v\|_{L^2(\Omega)}^2, \quad (4.81)$$

which is (4.78). But we are assuming that (4.78) does not hold. Notice that the integral defining s_0 has support in a stripe of width 2λ around $\Gamma_{\pm \frac{d_0}{2}} = \{\xi \in \Omega : r(\xi, \Gamma) = \pm \frac{d_0}{2}\}$. So we might try to split

the integral of the quadratic form moving by λ from $\Gamma_{\pm \frac{d_0}{2}}$, i.e applying the cut off function η_1^1 . If one of the following conditions holds

$$\begin{cases} s_1 \leq 0, \\ s_1 \leq \delta^* \|\eta_2^1 v\|_{L^2(\Omega)}^2, \\ s_1 \leq \lambda^2 \|v\|_{L^2(\Omega)}^2, \end{cases} \quad (4.82)$$

we can conclude the proof of the theorem. If not then

$$s_1 > \delta \left[s_1 + 2\|\eta_2^1 v\|_{L^2(\Omega)}^2 \right]. \quad (4.83)$$

By (4.80),

$$s_1 = \sum_{i=0}^1 s_i - s_0 \leq \sum_{i=0}^1 s_i - \delta \left[s_0 + 2\|\eta_2^0 v\|_{L^2(\Omega)}^2 \right]. \quad (4.84)$$

Notice that

$$s_1 = 2 \int_{\Omega} d\xi \eta_1^1(\xi) v(\xi) (J^\lambda \star \eta_2^1 v)(\xi) \leq 2 \int_{\Omega} d\xi \eta_2^0(\xi) |v(\xi)| |(J^\lambda \star \eta_2^0 v)(\xi)| \leq 2\|\eta_2^0 v\|_{L^2(\Omega)}^2. \quad (4.85)$$

Therefore

$$s_1 \leq (1 - \delta) \sum_{i=0}^1 s_i. \quad (4.86)$$

Next we show that there exists $\bar{k} \in \{0, \dots, N\}$ so that if for $j \in \{0, \dots, \bar{k} - 1\}$ no one of the following conditions is satisfied

$$\begin{cases} s_j \leq 0 \\ s_j \leq \delta^* \|\eta_2^j v\|_{L^2(\Omega)}^2 \\ s_j \leq \lambda^2 \|v\|_{L^2(\Omega)}^2 \end{cases} \quad (4.87)$$

then

$$s_{\bar{k}} \leq \lambda^2 \|v\|_{L^2(\Omega)}^2.$$

Namely, reiterating the argument done in the case of s_1 we have that for $k \in \{0, \dots, N\}$

$$s_k \leq (1 - \delta)^k \sum_{i=0}^k s_i.$$

Denote $\delta_0 = -\log(1 - \delta) > 0$. Take λ small enough so that $\frac{1}{\delta_0} \log \frac{1}{\lambda^2} < [\frac{1}{\lambda}] = N$ and set $\bar{k} = [\frac{1}{\delta_0} \log \frac{1}{\lambda^2}]$. With such a choice $\bar{k} < N$. We have

$$\begin{aligned} s_{\bar{k}} &\leq e^{-\delta_0 \bar{k}} \sum_{i=0}^{\bar{k}} s_i \\ &\leq \lambda^2 \sum_{i=0}^{\bar{k}} s_i \leq C \lambda^2 \|v\|_{L^2(\Omega)}^2. \end{aligned} \quad (4.88)$$

We then split

$$\begin{aligned} \int_{\Omega} (A_{m_A}^\lambda v)(\xi) v(\xi) d\xi &= \int_{\Omega} (A_{m_A}^\lambda \eta_1^{\bar{k}} v)(\xi) \eta_1^{\bar{k}}(\xi) v(\xi) d\xi \\ &\quad + \int_{\Omega} (A_{m_A}^\lambda \eta_2^{\bar{k}} v)(\xi) \eta_2^{\bar{k}}(\xi) v(\xi) d\xi - s_{\bar{k}}. \end{aligned} \quad (4.89)$$

Because of (2.17)

$$\int_{\Omega} (A_{m_A}^\lambda \eta_2^{\bar{k}} v)(\xi) \eta_2^{\bar{k}}(\xi) v(\xi) d\xi \geq (C^* - 1) \|\eta_2^{\bar{k}} v\|_{L^2(\Omega)} > 0.$$

By Lemma 4.2 and Lemma 4.8 we have that

$$\int_{\Omega} (A_{m_A}^{\lambda} \eta_1^{\bar{k}} v)(\xi) \eta_1^{\bar{k}} v(\xi) d\xi \geq -C\lambda^2 \|v\|_{L^2(\Omega)}^2,$$

and

$$s_{\bar{k}} \leq C\lambda^2 \|v\|_{L^2(\Omega)}^2.$$

Theorem is proved. \square

5. TWO DIMENSIONAL CONVOLUTION OPERATORS IN ENLARGED BOUNDED DOMAINS.

Essential ingredient to show the H^{-1} - estimate, stated in Theorem 2.5, is the knowledge of the spectrum of the operator defined below, see (5.1). For $V \in L^2(\mathcal{T}_{\lambda})$, where \mathcal{T}_{λ} is the enlarged cylinder defined in (4.40), denote

$$(\mathcal{G}^{\lambda} V)(s, z) = \frac{V(s, z)}{\sigma(\bar{m}(z))} - \int_{\mathcal{T}_{\lambda}} \frac{1}{\lambda} J\left(\frac{s-s'}{\lambda}, z-z'\right) V(s', z') dz' ds', \quad (5.1)$$

where J is the symmetric probability density on \mathbb{R}^2 defined in Subsection 2.1. We study the spectrum of the operator \mathcal{G}^{λ} on $L^2(\mathcal{T}_{\lambda})$ by Fourier analysis. For $h \in \mathbb{R}$, let $J^h(\cdot)$ be the h component of the Fourier transform of $J(s, \cdot)$, $s \in \mathbb{R}$:

$$J^h(z) = \int_{\mathbb{R}} J(s, z) e^{ihs} ds = \int_{\mathbb{R}} J(s, z) \cos(hs) ds. \quad (5.2)$$

The last identity holds because J is an even function of s . This implies that $J^h = J^{-h}$. Since $J \in C^1(\mathbb{R}^2)$ we have that

$$|J^h(z)| \leq \frac{C(z)}{(1+|h|)}, \quad (5.3)$$

where $C(z) = 0$ when $|z| > 1$ and

$$C(z) = \int ds \left| \frac{d}{ds} J(s, z) \right|, \quad \text{for } |z| \leq 1. \quad (5.4)$$

Further

$$J^0(z) = \int J(s, z) ds = \bar{J}(z), \quad (5.5)$$

see Subsection 2.1. For $w \in L^2(I_{\lambda})$ denote

$$(\mathcal{L}^h w)(z) = \left[\frac{w(z)}{\sigma(\bar{m}(z))} - (J^h \star_{I_{\lambda}} w)(z) \right]. \quad (5.6)$$

Proposition 5.1. *The operator \mathcal{L}^h , $h \in \mathbb{R}$, is a bounded, self-adjoint operator on $L^2(I_{\lambda})$. The spectrum of \mathcal{L}^h is discrete.*

Proof. The proof is straightforward. One needs to exploit that $\mathcal{P}^h w = \sigma(\bar{m}) J^h \star w$ is a bounded integral operator in $L^2(I_{\lambda})$ and that \mathcal{L}^h is conjugate to $\mathbb{I} - \mathcal{P}^h$. \square

By general arguments one can deduce informations about the spectrum of \mathcal{G}^{λ} by the knowledge of the spectrum of \mathcal{L}^h . Any $V \in L^2(\mathcal{T}_{\lambda})$ can be expanded in Fourier complex series as the following

$$V(s, z) = \sum_{k \in \mathbb{Z}_L} e^{iks} u_k(z) \quad (5.7)$$

where $\mathbb{Z}_L = \frac{2\pi}{L}\mathbb{Z}$ and $u_k(z) = \frac{1}{L} \int_T V(s, z) e^{-iks} ds$. Let $\mathcal{F} : L^2(\mathcal{T}_{\lambda}) \rightarrow \bigoplus_{k \in \mathbb{Z}_L} H_k$, $H_k = L^2(I_{\lambda})$, be the isometry induced by the Fourier expansion. We denote by $W = (u_k)_{k \in \mathbb{Z}_L}$ an element of $\bigoplus_{k \in \mathbb{Z}_L} H_k$.

Theorem 5.2. *Let $\sigma(\mathcal{G}^\lambda)$ be the spectrum of \mathcal{G}^λ in $L^2(\mathcal{T}_\lambda)$ and $\sigma(\mathcal{L}^h)$ the spectrum of \mathcal{L}^h in $L^2(I_\lambda)$. Then*

$$\sigma(\mathcal{G}^\lambda) = \bigcup_{\{k \in \mathbb{Z}_L\}} \sigma(\mathcal{L}^{\lambda k}).$$

Proof. Let $\tilde{\mathcal{G}}^\lambda = \bigoplus_{k \in \mathbb{Z}_L} \mathcal{L}^{\lambda k}$ be the operator defined on $\bigoplus_{k \in \mathbb{Z}_L} H_k$ so that $\tilde{\mathcal{G}}^\lambda W = (\mathcal{L}^{\lambda k} u_k)_{k \in \mathbb{Z}_L}$. We have that

$$\mathcal{G}^\lambda = \mathcal{F}^{-1} \tilde{\mathcal{G}}^\lambda \mathcal{F}.$$

Being conjugate $\tilde{\mathcal{G}}^\lambda$ and \mathcal{G}^λ have the same spectrum. By [9], $\sigma(\tilde{\mathcal{G}}^\lambda) = \bigcup_{\{k \in \mathbb{Z}_L\}} \sigma(\mathcal{L}^{\lambda k})$. The thesis of theorem follows. \square

The aim is then to study the spectrum of \mathcal{L}^h , $h \in R$. It turns out that when $|h| > h_0$, where h_0 is a positive real number, conveniently chosen, the spectrum of \mathcal{L}^h is strictly positive and can be lower bounded by a positive constant depending on h_0 but not on h . For $|h| \leq h_0$ the spectrum of \mathcal{L}^h is still positive but the lower bound does depend on h . In this case, we are able to give upper and lower bound of the principal eigenvalue of \mathcal{L}^h which turns out to be very useful. We analyse these type of behaviour in Proposition 5.6 and Proposition 5.7. Next, we show that eigenfunctions associated to small eigenvalues decay exponentially for $|z|$ large enough. This result is valid for all $\{\mathcal{L}^h\}_h$.

Proposition 5.3. *For any $\epsilon_0 \in (0, \frac{(1-\sigma(m_\beta))}{2})$, there exists $z_0 > 0$, $\lambda_0 = \lambda_0(\epsilon_0)$ and $\alpha(\epsilon_0) > 0$ so that for $\lambda \leq \lambda_0$ the following holds. Let $\mu \leq \frac{\epsilon_0}{2}$ be an eigenvalue of \mathcal{L}^h and ψ be any of the corresponding eigenfunctions. There is $\alpha(\epsilon_0) > 0$ and $z_0 = z_0(\epsilon_0)$ independent on h and λ so that*

$$|\psi(z)| \leq C e^{-\alpha(\epsilon_0)|z|} \quad |z| \geq z_0. \quad (5.8)$$

The proof is similar to the one given in [14][Lemma 3.5] and it is therefore omitted.

Remark 5.4. *Notice that z_0 and $\alpha(\epsilon_0)$ depend only on ϵ_0 and not on h . Applying the argument as in [14][Lemma 3.5] one ends up with $|\psi(z)| \leq \beta \|J^h\|_2 e^{-\alpha(\epsilon_0)|z|}$. But it is immediate to see that $\|J^h\|_2 \leq \|J^0\|_2$.*

The spectrum of the operator \mathcal{L}^0 , i.e. when $h = 0$ has been studied in [14]. When $|h| \leq \frac{\pi}{2}$ the integral kernel J^h is positivity improving, i.e if $v(z) \geq 0$ and $v(z) \neq 0$ for $z \in I_\lambda$, then $\int_{I_\lambda} dz' J^h(z-z')v(z') > 0$. Hence we could apply the same type of arguments used in [14] to study the spectrum of the operator \mathcal{L}^h when $|h| \leq \frac{\pi}{2}$. In Proposition 5.6 we summarise the results for the spectrum of \mathcal{L}^h , $|h| \leq h_0$, where $h_0 \leq \frac{\pi}{2}$ is suitable chosen. To prove a uniform (in $|h| \leq h_0$ and $\lambda > 0$) lower bound for the gap of \mathcal{L}^h we apply perturbation theory.

Proposition 5.5. *Let J^h , $|h| \leq \frac{\pi}{2}$, be the integral kernel defined in (5.2). We have that there exists $h_0 > 0$ so that for $|h| \leq h_0$,*

$$\bar{J}(z) - \frac{1}{2}h^2 \bar{J}_{tan}(z) \leq J^h(z) \leq \bar{J}(z) - \frac{1}{4}h^2 \bar{J}_{tan}^\lambda(z), \quad (5.9)$$

where $\bar{J}_{tan}(\cdot)$ is defined in (5.10),

$$0 \leq \bar{J}_{tan}(z) < \bar{J}(z).$$

Proof. By Taylor expanding $\cos h\xi$ we have

$$1 - \frac{1}{2}h^2\xi^2 + \frac{1}{4!}h^4 \geq \cos h\xi \geq 1 - \frac{1}{2}h^2\xi^2.$$

Denote

$$\bar{J}_{tan}(z) = \int J(\xi, z)\xi^2 d\xi. \quad (5.10)$$

We have

$$\bar{J}(z) - \frac{1}{2}h^2 \bar{J}_{tan}(z) + \frac{1}{4!}h^4 \int J(\xi, z)\xi^4 d\xi \geq J^h(z) \geq \bar{J}(z) - \frac{1}{2}h^2 \bar{J}_{tan}(z).$$

Since

$$\int J(\xi, z)\xi^4 d\xi \leq \bar{J}_{tan}(z)$$

taking $h_0 > 0$ so that when $|h| \leq h_0$, $\frac{1}{4}h^2 \leq \frac{1}{4}$ we obtain

$$\bar{J}(z) - \frac{1}{2}h^2\bar{J}_{tan}(z) \leq J^h(z) \leq \bar{J}(z) - \frac{1}{4}h^2\bar{J}_{tan}^\lambda(z).$$

Therefore we get (5.9). □

Proposition 5.6. *There exists $h_0 \in (0, \frac{\pi}{2})$ so that for $|h| \leq h_0$, the following holds for \mathcal{L}^h defined in (5.6) on $L^2(I_\lambda)$.*

(1) *There exists $\mu_0^h \in \mathbb{R}$ and ψ_0^h strictly positive in I_λ , ψ_0^h even function, so that*

$$\mathcal{L}^h \psi_0^h = \mu_0^h \psi_0^h.$$

The eigenvalue μ_0^h has multiplicity one and any other eigenvalue is strictly bigger than μ_0^h .

(2) *Let μ_0^0 be the principal eigenvalue of the operator \mathcal{L}^0 . We have that*

$$\mu_0^0 < \mu_0^h \leq \mu_0^0 + \frac{1}{2}h^2.$$

(3) *There exists $D > 0$ independent on λ and h so that*

$$\inf_{\|\psi\|=1, \langle \psi, \psi_0^h \rangle = 0} \langle \mathcal{L}^h \psi, \psi \rangle \geq D, \quad \forall h : |h| \leq h_0. \quad (5.11)$$

(4) *The principal eigenvector ψ_0^h is such that*

$$\|\psi_0^h - \psi_0^0\|^2 \leq Ch^2. \quad (5.12)$$

(5) *There exists $z_0 > 0$ and $\zeta_0 > 0$ independent on h and λ so that*

$$\psi_0^h(z) \geq \zeta_0, \quad |z| \leq z_0.$$

(6) *There exists $C_0 > 0$, independent on λ and h , so that*

$$\mu_0^h \geq \mu_0^0 + C_0 h^2. \quad (5.13)$$

Proof. For $|h| \leq \frac{\pi}{2}$, for $|s| \leq 1$, the integral kernel J^h , in the definition of \mathcal{L}^h , is non negative for $z \in I_\lambda$. Applying the Perron Frobenius Theorem to the operator $(\mathcal{A}^h g)(z) = \sigma(\bar{m})(J^h \star g)(z)$, $z \in I_\lambda$ and proceeding as in [14] [Theorem 2.1] we prove point (1). To show (2) and (3) we apply standard perturbation theory for bounded selfadjoint operators, see [12]. Define the following family of operators indexed by ν :

$$A_\nu = \mathcal{L}^0 + \nu B, \quad \nu \in [0, 1]$$

where

$$B = \mathcal{L}^h - \mathcal{L}^0.$$

The family A_ν connects in a smooth way the unperturbed operator \mathcal{L}^0 to \mathcal{L}^h . We have that

$$Bw(z) = [\mathcal{L}^h - \mathcal{L}^0]w(z) = \int J(\xi, z - z')[1 - \cos(h\xi)]d\xi w(z')dz'.$$

Notice that for $|h| \leq h_0$, B leaves invariant the cone of the positive functions. Further, by (5.9),

$$\|B\| = \sup_{\{\|w\|=1\}} \langle w, [\mathcal{L}^h - \mathcal{L}^0]w \rangle \leq \frac{1}{2}h^2. \quad (5.14)$$

Since \mathcal{L}^0 has an isolated simple eigenvalue and a spectral gap D , independent on λ , see Theorem 3.1, the \mathcal{L}^h for all

$$|h| \leq \sqrt{\frac{D}{3}} \equiv h_0$$

will have an isolated simple eigenvalue and a spectral gap bigger or equal of $D/4$. Moreover the principal eigenvalue μ_0^ν and eigenvector ψ_0^ν of A_ν are analytic in ν . By Perron Frobenius Theorem $\psi_0^\nu > 0$ and we assume $\langle \psi_0^\nu, \psi_0^\nu \rangle = 1$. Next we would like to show that that $\mu_0^\nu > \mu_0^0$. We derive with respect to ν the eigenvalue equation

$$A_\nu \psi_0^\nu = \mu_0^\nu \psi_0^\nu.$$

We have

$$\begin{aligned} B\psi_0^\nu + A_\nu \partial_\nu(\psi_0^\nu) &= \partial_\nu(\mu_0^\nu) \psi_0^\nu + \mu_0^\nu \partial_\nu(\psi_0^\nu), \\ \langle \psi_0^\nu, [B\psi_0^\nu + A_\nu \partial_\nu(\psi_0^\nu)] \rangle &= \langle \psi_0^\nu, [\partial_\nu(\mu_0^\nu) \psi_0^\nu + \mu_0^\nu \partial_\nu(\psi_0^\nu)] \rangle. \end{aligned}$$

Therefore

$$\langle \psi_0^\nu, B\psi_0^\nu \rangle + \mu_0^\nu \langle \psi_0^\nu, \partial_\nu(\psi_0^\nu) \rangle = \partial_\nu(\mu_0^\nu) \langle \psi_0^\nu, \psi_0^\nu \rangle + \mu_0^\nu \langle \psi_0^\nu, \partial_\nu(\psi_0^\nu) \rangle.$$

We have

$$\langle \psi_0^\nu, B\psi_0^\nu \rangle = \partial_\nu(\mu_0^\nu).$$

Hence

$$\mu_0^\nu = \mu_0^0 + \int_0^\nu \langle \psi_0^{\nu'}, B\psi_0^{\nu'} \rangle d\nu'. \quad (5.15)$$

Since $\psi_0^\nu > 0$ and B is a positive operator $\mu_0^\nu > \mu_0^0$. By (5.14),

$$\mu_0^\nu \leq \mu_0^0 + \nu \frac{1}{2} h^2.$$

When $\nu = 1$, $\mathcal{A}_{\{\nu=1\}} = \mathcal{L}^h$, $\mu_0^1 = \mu_0^h$ and we have

$$\mu_0^h \leq \mu_0^0 + \frac{1}{2} h^2.$$

Next we show (5.12). For $h \neq 0$, split

$$\psi_0^0 = a\psi_0^h + (\psi_0^h)^\perp. \quad (5.16)$$

Then

$$a^2 + \|(\psi_0^h)^\perp\|^2 = 1 \quad (5.17)$$

$$\langle \mathcal{L}^h \psi_0^0, \psi_0^0 \rangle = a^2 \mu_0^h + \langle \mathcal{L}^h (\psi_0^h)^\perp, (\psi_0^h)^\perp \rangle \geq a^2 \mu_0^h + \frac{D}{4} \|(\psi_0^h)^\perp\|^2. \quad (5.18)$$

Further

$$\begin{aligned} \langle \mathcal{L}^h \psi_0^0, \psi_0^0 \rangle &= \langle \mathcal{L}^0 \psi_0^0, \psi_0^0 \rangle + \langle (\mathcal{L}^h - \mathcal{L}^0) \psi_0^0, \psi_0^0 \rangle \\ &\leq \mu_0^0 + \frac{1}{2} h^2. \end{aligned} \quad (5.19)$$

By (5.17), (5.18) and $\mu_0^h > 0$ we have that

$$\mu_0^0 + \frac{1}{2} h^2 \geq a^2 \mu_0^h + \frac{D}{4} \|(\psi_0^h)^\perp\|^2 \geq \frac{D}{4} \|(\psi_0^h)^\perp\|^2.$$

By [14] [Theorem 2.2, formula (2.8)], $0 \leq \mu_0^0 \leq C e^{-2\alpha \frac{1}{\lambda}}$. It follows that there exists $C > 0$ independent on λ and h so that

$$\|(\psi_0^h)^\perp\|^2 \leq Ch^2. \quad (5.20)$$

By (5.17) $a^2 = 1 - \|(\psi_0^h)^\perp\|^2$. This, together with decomposition (5.16) and (5.20) implies (5.12).

The proof of the point (5) can be done as in [14] [Lemma 3.6, formula (3.22)]. The proof is similar to the one given in Theorem 3.2 when proving (3.15).

To show (6) we need to lower bound, see (5.15), $\int_0^1 \langle \psi_{\nu'}^0, B\psi_{\nu'}^0 \rangle d\nu'$. First of all we note that the same type of argument of point (5) applies to ψ_ν^0 the principal eigenvalue of \mathcal{A}_ν . Namely $\psi_\nu^0(z)$, $z \in I_\lambda$, is positive and exponentially decaying when $|z|$ large enough. Hence, as in point (5), there exists $z_0 > 0$ and $\zeta_0 > 0$ so that

$$\psi_\nu^0(z) \geq \zeta_0, \quad |z| \leq z_0, \quad \forall \nu \in [0, 1], \quad (5.21)$$

where $\zeta_0 > 0$ and $z_0 > 0$ are independent on λ . Hence, by Proposition 5.5 we have that

$$\begin{aligned} \langle \psi_{\nu'}^0, B\psi_{\nu'}^0 \rangle &= \int_{I_\lambda} dz \psi_{\nu'}^0(z) \int_{I_\lambda} [J(z-z') - J^h(z-z')] \psi_{\nu'}^0(z') dz' \\ &\geq 2\zeta_0^2 z_0 \frac{1}{4} h^2 \int \bar{J}_{tan}(z') dz' \equiv C_0 h^2. \end{aligned} \quad (5.22)$$

The statement follows. \square

Proposition 5.7. *Let \mathcal{L}^h be the operator defined in (5.6) on $L^2(I_\lambda)$. Let h_0 be as in Proposition 5.6. There exists $\nu = \nu(h_0) > 0$ independent on h and λ so that for $|h| > h_0$*

$$\langle w, \mathcal{L}^h w \rangle \geq \nu \|w\|^2. \quad (5.23)$$

The proof of Proposition 5.7 follows from Proposition 5.8 and Proposition 5.9.

Proposition 5.8. *Let \mathcal{L}^h be the operator defined in (5.6) on $L^2(I_\lambda)$. There exists $h_1 = h_1(\beta, J) > 0$ independent on λ so that for $|h| > h_1$*

$$\langle w, \mathcal{L}^h w \rangle \geq \frac{1}{2} \beta^{-1} \|w\|^2. \quad (5.24)$$

Proof. By (5.3)

$$|\langle w, J^h w \rangle| \leq \frac{1}{(1+|h|)} \int_{I_\lambda} dz |w(z)| \int dz' C(z-z') |w(z')| \leq \frac{C_2}{(1+|h|)} \|w\|^2$$

where

$$C_2 = \int_{I_\lambda} C(z) dz = \int_{I_\lambda} dz \int_T ds \left| \frac{d}{ds} J(s, z) \right|. \quad (5.25)$$

Hence, by definition of \mathcal{L}^h and (5.4), we get

$$\sup_{\{w: \|w\|=1\}} \langle w, \mathcal{L}^h w \rangle \geq \sup_{\{w: \|w\|=1\}} \left(\frac{1}{\beta} - \frac{C_2}{(1+|h|)} \right) \|w\|^2. \quad (5.26)$$

Choosing $h_1 = h_1(\beta, J) > 0$, so that $\frac{1}{2} \beta^{-1} \geq \frac{C_2}{(1+|h|)}$ we get (5.27). \square

Proposition 5.9. *Let \mathcal{L}^h be the operator defined in (5.6) over functions $L^2(I_\lambda)$. For any given $h_0 > 0$ and $h_1 > 0$, there exists $\nu = \nu(h_0, h_1) > 0$ so that for $h_0 \leq |h| \leq h_1$*

$$\langle w, \mathcal{L}^h w \rangle \geq \nu \|w\|^2. \quad (5.27)$$

Proof. We first show that there exists $c_1 = c_1(h_0, h_1) > 0$ so that

$$|J^h(z)| \leq \bar{J}(z)(1 - c_1), \quad h_0 \leq |h| \leq h_1, \quad z \in (-1, 1). \quad (5.28)$$

To show this we argue by contradiction. Assume that there exists $\bar{z} \in (-1, 1)$ and \bar{h} so that $|J^{\bar{h}}(\bar{z})| = \bar{J}(\bar{z})$. Then $J^{\bar{h}}(\bar{z}) = e^{i\theta} \bar{J}(\bar{z})$ for some $\theta \in \mathbb{R}$, and therefore

$$\bar{J}(\bar{z}) = e^{-i\theta} J^{\bar{h}}(\bar{z})$$

which means

$$\int_{\mathbb{R}} J(\xi, \bar{z}) \left[1 - e^{-i\theta} e^{i\bar{h}\xi} \right] d\xi = 0. \quad (5.29)$$

This implies that the real part of (5.29), i.e

$$\int_{\mathbb{R}} J(\xi, \bar{z}) \left[1 - \cos(\theta - \bar{h}\xi) \right] d\xi = 0.$$

This forces $J(\xi, \bar{z}) = 0$ for all $\xi \in \mathbb{R}$, which is a contradiction. Since for any given z the set $\{J^h(z), h_0 \leq |h| \leq h_1\}$ is a compact subset of \mathbb{R} , and $|J^h(z)| < \bar{J}(z)$, then (5.28) follows. Hence, by definition of \mathcal{L}^h and (5.28), we get, for $1 > a > 0$,

$$\begin{aligned} \langle w, \mathcal{L}^h w \rangle &= (1-a) \int dz \frac{w^2(z)}{\sigma(\bar{m}(z))} + a \left[\int dz \frac{w^2(z)}{\sigma(\bar{m}(z))} - \langle |w|(\bar{J} \star |w|) \rangle \right] \\ &\quad + a \langle |w|(\bar{J} \star |w|) \rangle - \langle w, J^h \star w \rangle. \end{aligned} \quad (5.30)$$

By (5.28),

$$|\langle w, J^h \star w \rangle| \leq \langle |w|, |J^h| \star |w| \rangle \leq (1-c_1) \langle |w|, \bar{J} \star |w| \rangle.$$

Then, taking $a = 1 - c_1$ we have that

$$a \langle |w|, \bar{J} \star |w| \rangle - \langle w, J^h \star w \rangle \geq 0.$$

Further, since

$$\inf_{\{w: \|w\|=1\}} \langle |w|, \mathcal{L}^0 |w| \rangle \geq \inf_{\{w: \|w\|=1\}} \langle w, \mathcal{L}^0 w \rangle \geq 0, \quad (5.31)$$

and $\sigma(\bar{m}(z)) \leq \beta$

$$\inf_{\{w: \|w\|=1\}} \langle w, \mathcal{L}^h w \rangle \geq \frac{c_1}{\beta} \equiv \nu. \quad (5.32)$$

□

6. REPRESENTATION FORMULA FOR FUNCTIONS WITH SMALL ENERGY.

In this section, we show a representation theorem for functions having small energy, see (6.1) below. The representation stated in Theorem 6.1 is reminiscent of the one obtained by X.Chen, see [7, Lemma 2.4] in the C-H case. The proof, as explained in the introduction, is different.

Theorem 6.1. *Take $f \in H^1(\mathcal{N}(d_0))$ such that $\|f\|_{L^2(\mathcal{N}(d_0))} = 1$, and*

$$\int_{\mathcal{N}(d_0)} (A_{m_A}^\lambda f(\xi)) f(\xi) d\xi \leq C\lambda^2. \quad (6.1)$$

Then, there exists $\lambda_0 > 0$, so that for any $\lambda \in (0, \lambda_0)$ we can construct $Z(\cdot) \in H^1(T)$, $f^R(\cdot, \cdot) \in L^2(\mathcal{N}(d_0))$ such that

$$f(s, r) = Z(s) \frac{1}{\sqrt{\alpha(s, r)}} \frac{1}{\sqrt{\lambda}} \psi_0^0\left(\frac{r}{\lambda}\right) + f^R(s, r), \quad (6.2)$$

where $\psi_0^0(\cdot)$ is the first eigenvalue of \mathcal{L}^0 , see Theorem 3.1,

$$1 - C\lambda^2 \leq \|Z\|_{L^2(T)}^2 \leq 1, \quad \|\nabla Z\|_{L^2(T)} \leq C \quad (6.3)$$

$$\|f^R\|_{L^2(\mathcal{N}(d_0))}^2 \leq C\lambda^2. \quad (6.4)$$

Set $\hat{f}(s, r) = \sqrt{\alpha(s, r)} f(s, r)$. By Lemma 4.2, (6.1) implies

$$C\lambda^2 \geq \int_{\mathcal{N}(d_0)} A_{m_A}^\lambda f(\xi) f(\xi) d\xi \geq \langle \hat{f}, L^\lambda \hat{f} \rangle - \lambda^2 C \|f\|_{L^2(\mathcal{N}(d_0))}^2. \quad (6.5)$$

Since by assumption $\|f\|_{L^2(\mathcal{N}(d_0))}^2 = 1$ we have

$$\lambda^2 C \geq \langle \hat{f}, L^\lambda \hat{f} \rangle. \quad (6.6)$$

Further setting

$$\hat{f}(s, r) = \frac{1}{\sqrt{\lambda}} V(s, \frac{r}{\lambda})$$

we have that (6.6) is equivalent to

$$\lambda^2 C \geq \langle V, \mathcal{A}V \rangle, \quad (6.7)$$

where \mathcal{A} is the operator defined in (4.44). Therefore Theorem 6.1 follows once we show the following theorem.

Theorem 6.2. *Take $V \in H^1(\mathcal{T}_\lambda)$, $\|V\| = 1$,*

$$\langle V, \mathcal{A}V \rangle \leq C\lambda^2, \quad (6.8)$$

where \mathcal{A} is the operator defined in (4.44). Let $\psi_0^0(\cdot)$ be the first eigenvalue of \mathcal{L}^0 , see Theorem 3.1, we have

$$V(s, z) = Z(s)\psi_0^0(z) + V^R(s, z), \quad (6.9)$$

$$1 - \lambda^2 \leq \|Z\|_{L^2(T)}^2 \leq 1, \quad \|\nabla Z\| \leq C, \quad \|V^R\|^2 \leq C\lambda^2. \quad (6.10)$$

The proof of Theorem 6.2 is based on a deeper knowledge of the spectrum of the operator \mathcal{A} in $L^2(\mathcal{T}_\lambda)$. We explain in Subsection 6.1 how to prove (6.9) when the operator \mathcal{A} is replaced by the operator \mathcal{G} defined in (5.1). Then in Subsection 6.2 we write the operator \mathcal{A} in term of \mathcal{G} plus extra terms. In Subsection 6.3 we show Theorem 6.2.

6.1. Toy model. We explain the method for proving Theorem 6.2 in a simpler context. Replace the operator \mathcal{A} with the operator \mathcal{G}^λ defined in (5.1). Notice that \mathcal{G}^λ defined in (5.1) is different from the \mathcal{A} defined in (4.44). Namely \bar{m} replaces m_A and the convolution term replaces J^c . Assume

$$\langle V, \mathcal{G}^\lambda V \rangle \leq C\lambda^2. \quad (6.11)$$

We would like to prove that (6.9) and (6.10) follow. Write $V(s, z) = \sum_{k \in \mathbb{Z}_L} e^{i2\pi \frac{k}{L}s} u_k(z)$, see (5.7). By (6.11) and simple computations

$$C\lambda^2 \geq \langle V, \mathcal{G}^\lambda V \rangle = \sum_{k \in \mathbb{Z}_L} \langle \mathcal{L}^{k\lambda} u_k, u_k \rangle. \quad (6.12)$$

We then apply the spectral results for $\{\mathcal{L}^h\}_h$ obtained above. Let $h_0 > 0$ be as in Proposition 5.6. When $|k| \leq \frac{h_0}{\lambda}$ split

$$u_k = \alpha_k \psi_0^{k\lambda} + u_k^\perp$$

where $\psi_0^{k\lambda}$ is the principal eigenvalue of $\mathcal{L}^{k\lambda}$ and

$$\int dz \psi_0^{k\lambda}(z) u_k^\perp(z) = 0.$$

By Proposition 5.6 and Proposition 5.7 we have

$$\begin{aligned} \sum_{k \in \mathbb{Z}_L} \langle \mathcal{L}^{k\lambda} u_k, u_k \rangle &\geq \sum_{|k| \leq \frac{h_0}{\lambda}} \langle \mathcal{L}^{k\lambda} u_k, u_k \rangle + \nu \sum_{|k| > \frac{h_0}{\lambda}} \|u_k\|^2 \\ &= \sum_{|k| \leq \frac{h_0}{\lambda}} [\mu_0^{k\lambda} \alpha_k^2 + \langle u_k^\perp, \mathcal{L}^{k\lambda} u_k^\perp \rangle] + \nu \sum_{|k| > \frac{h_0}{\lambda}} \|u_k\|^2 \\ &\geq \sum_{|k| \leq \frac{h_0}{\lambda}} [\mu_0^{k\lambda} \alpha_k^2 + D \|u_k^\perp\|^2] + \nu \sum_{|k| > \frac{h_0}{\lambda}} \|u_k\|^2. \end{aligned} \quad (6.13)$$

Therefore, see (6.12),

$$C\lambda^2 \geq \sum_{|k| \leq \frac{h_0}{\lambda}} [\mu_0^{k\lambda} \alpha_k^2 + D \|u_k^\perp\|^2] + \nu \sum_{|k| > \frac{h_0}{\lambda}} \|u_k\|^2. \quad (6.14)$$

Since $\mu_0^{k\lambda} \geq 0$, see (5.13), the three terms on the right hand side of (6.14) are positive, hence

$$\sum_{|k| \leq \frac{h_0}{\lambda}} \mu_0^{k\lambda} \alpha_k^2 \leq C\lambda^2, \quad (6.15)$$

$$\sum_{|k| \leq \frac{h_0}{\lambda}} \|u_k^\perp\|^2 \leq \frac{C}{D} \lambda^2, \quad (6.16)$$

$$\sum_{|k| > \frac{h_0}{\lambda}} \|u_k\|^2 \leq \frac{C}{\nu} \lambda^2. \quad (6.17)$$

By Proposition 5.6, since $\mu_0^0 \geq 0$,

$$\begin{aligned} \sum_{|k| \leq \frac{h_0}{\lambda}} \mu_0^{k\lambda} \alpha_k^2 &= \sum_{|k| \leq \frac{h_0}{\lambda}} [\mu_0^{k\lambda} - \mu_0^0 + \mu_0^0] \alpha_k^2 \geq \sum_{|k| \leq \frac{h_0}{\lambda}} [\mu_0^{k\lambda} - \mu_0^0] \alpha_k^2 \\ &> C_0 \sum_{|k| \leq \frac{h_0}{\lambda}} [(k\lambda)^2] \alpha_k^2. \end{aligned} \quad (6.18)$$

Hence, by (6.15)

$$\sum_{|k| \leq \frac{h_0}{\lambda}} k^2 \alpha_k^2 \leq \frac{C}{C_0}. \quad (6.19)$$

Define

$$Z(s) = \sum_{|k| \leq \frac{h_0}{\lambda}} e^{iks} \alpha_k, \quad (6.20)$$

$$V^R(s, z) = V_2(s, z) + V_3(s, z), \quad (6.21)$$

where

$$V_2(s, z) = \sum_{|k| \leq \frac{h_0}{\lambda}} e^{iks} u_k^\perp + \sum_{|k| > \frac{h_0}{\lambda}} e^{iks} u_k(z), \quad (6.22)$$

$$V_3(s, z) = \sum_{|k| \leq \frac{h_0}{\lambda}} e^{iks} \alpha_k [\psi_0^{k\lambda} - \psi_0^0]. \quad (6.23)$$

Then

$$V(s, z) = Z(s) \psi_0^0(z) + V^R(s, z). \quad (6.24)$$

By (6.16) and (6.17)

$$\|V_2\|^2 \leq C \lambda^2.$$

By (5.12) and (6.19)

$$\|V_3\|^2 = \sum_{|k| \leq \frac{h_0}{\lambda}} \alpha_k^2 \|\psi_0^{k\lambda} - \psi_0^0\|^2 \leq \lambda^2 \sum_{|k| \leq \frac{h_0}{\lambda}} \alpha_k^2 k^2 \leq \lambda^2 C.$$

Hence $\|V^R\|^2 \leq C \lambda^2$, $1 - C \lambda^2 \leq \|Z\|^2 \leq 1$,

$$\|\nabla_s Z\|^2 = \sum_{|k| \leq \frac{h_0}{\lambda}} k^2 \alpha_k^2 \leq C. \quad (6.25)$$

In this way we get the decomposition (6.9), (6.10) in the toy model. The proof of Theorem 6.2 is more complicated because terms outside diagonal are present.

6.2. Expansion of \mathcal{A} in term of \mathcal{G} . In this subsection we decompose $\mathcal{A} = \mathcal{G} + \mathcal{R}$, but the L^2 norm of \mathcal{R} is not of order λ^2 . Nevertheless we show that when V satisfies (6.8) then $\langle \mathcal{A}V, V \rangle \simeq \langle \mathcal{G}V, V \rangle + C\lambda^2\|V\|^2$.

We start writing the operator \mathcal{B} defined in (4.43) in term of a convolution operator plus an operator involving the first derivative with respect to the s - variable of V , plus a remainder. The remainder for general $L^2(\mathcal{T}_\lambda)$ functions is of order 1 in $L^2(\mathcal{T}_\lambda)$, so it is not small. To get the remainder small we need to require decay properties of V . They hold when V satisfies (6.8).

Lemma 6.3. *Let $V \in H^1(\mathcal{T}_\lambda)$ and \mathcal{B} be the operator defined in (4.43). We have*

$$\begin{aligned} (\mathcal{B}V)(s, z) &= \int_{\mathcal{T}_\lambda} \frac{1}{\lambda} J\left(\frac{(s-s')}{\lambda}, (z-z')\right) V(s', z') ds' dz' \\ &\quad - \lambda^2 \int_{\mathcal{T}_\lambda} \frac{1}{\lambda} J\left(\frac{(s-s')}{\lambda}, (z-z')\right) \left[\frac{(s-s')}{\lambda} k(s^*) z^* \alpha(s^*, z^*) D_{s'} V(s', z') \right] ds' dz' \\ &\quad + (R^\lambda V)(s, z) \end{aligned} \quad (6.26)$$

where R^λ is defined in (6.33).

Proof. Set $x_0 = \frac{(s-s')}{\lambda}$ and $x = \frac{(s-s')}{\lambda} \alpha(s^*, z^*)$. We expand J in Taylor formula, up to second order, at the point x_0 and for any given $z \in I_\lambda$. We obtain

$$\begin{aligned} J(x, z) &= J(x_0, z) + D_1 J(x_0, z) (x - x_0) \\ &\quad + \frac{1}{2} \int_{x_0}^x D_{11} J(x', z) (x_0 - x') dx'. \end{aligned} \quad (6.27)$$

We denoted by $D_1 J$ and by $D_{11} J$ the first derivative and respectively the second derivative of J with respect to the first argument. Recalling that the support of J is the ball of radius 1 we have that $|s-s'| \leq \lambda$, $|z-z'| \leq 1$ and $(x-x_0) = -\frac{1}{\lambda}(s-s')\lambda k(s^*)z^* \simeq \lambda k(s^*)z^*$. We set

$$M(s, s', z, z') = \frac{1}{2} \int_{x_0}^x D_{11} J(x', z) (x_0 - x') dx'. \quad (6.28)$$

By the properties of J and the boundness of the curvature

$$|M(s, s', z, z')| \leq \lambda^2 C(z^*)^2 \mathbb{1}_{|s-s'| \leq \lambda} \mathbb{1}_{|z-z'| \leq 1}. \quad (6.29)$$

By (6.27), taking into account that $\alpha(s^*, z^*) = 1 - \lambda k(s^*)z^*$, we have

$$\begin{aligned} (\mathcal{B}V)(s, z) &= \int_{\mathcal{T}_\lambda} \frac{1}{\lambda} J\left(\frac{(s-s')}{\lambda} \alpha(s^*, z^*), (z-z')\right) \alpha(s^*, z^*) V(s', z') ds' dz' \\ &= \int_{\mathcal{T}_\lambda} \frac{1}{\lambda} J\left(\frac{(s-s')}{\lambda}, (z-z')\right) V(s', z') ds' dz' \\ &\quad - \lambda \int_{\mathcal{T}_\lambda} \frac{1}{\lambda} J\left(\frac{(s-s')}{\lambda}, (z-z')\right) k(s^*) z^* V(s', z') ds' dz' \\ &\quad - \lambda \int_{\mathcal{T}_\lambda} \frac{1}{\lambda} (D_1 J)\left(\frac{(s-s')}{\lambda}, (z-z')\right) \frac{1}{\lambda} (s-s') k(s^*) z^* \alpha(s^*, z^*) V(s', z') ds' dz' \\ &\quad + \frac{1}{\lambda} \int_{\mathcal{T}_\lambda} M(s, s', z, z') \alpha(s^*, z^*) V(s', z') ds' dz'. \end{aligned} \quad (6.30)$$

Since

$$(D_1 J)\left(\frac{(s-s')}{\lambda}, (z-z')\right) = \lambda (D_s J)\left(\frac{(s-s')}{\lambda}, (z-z')\right) = -\lambda (D_{s'} J)\left(\frac{(s-s')}{\lambda}, (z-z')\right)$$

we obtain

$$\begin{aligned}
& \int_{\mathcal{T}_\lambda} (D_1 J) \left(\frac{(s-s')}{\lambda}, (z-z') \right) \frac{1}{\lambda} (s-s') k(s^*) z^* \alpha(s^*, z^*) V(s', z') ds' dz' \\
&= - \int_{\mathcal{T}_\lambda} (D_{s'} J) \left(\frac{(s-s')}{\lambda}, (z-z') \right) (s-s') k(s^*) z^* \alpha(s^*, z^*) V(s', z') ds' dz' \\
&= \int_{\mathcal{T}_\lambda} J \left(\frac{(s-s')}{\lambda}, (z-z') \right) D_{s'} [(s-s') k(s^*) z^* \alpha(s^*, z^*) V(s', z')] ds' dz' \\
&= - \int_{\mathcal{T}_\lambda} J \left(\frac{(s-s')}{\lambda}, (z-z') \right) k(s^*) z^* [1 - \lambda k(s^*) z^*] V(s', z') ds' dz' \tag{6.31} \\
&+ \int_{\mathcal{T}_\lambda} J \left(\frac{(s-s')}{\lambda}, (z-z') \right) (s-s') k_{s'}(s^*) z^* \alpha(s^*, z^*) V(s', z') ds' dz' \\
&+ \int_{\mathcal{T}_\lambda} J \left(\frac{(s-s')}{\lambda}, (z-z') \right) (s-s') k(s^*) z^* \alpha(s^*, z^*) D_{s'} V(s', z') ds' dz' \\
&+ \int_{\mathcal{T}_\lambda} J \left(\frac{(s-s')}{\lambda}, (z-z') \right) (s-s') k(s^*) z^* D_{s'} [\alpha(s^*, z^*)] V(s', z') ds' dz'.
\end{aligned}$$

We replaced $\alpha(s^*, z^*) = 1 - \lambda k(s^*) z^*$ in the first term of the last equality of (6.31). We insert (6.31) into the third term of the last equality in (6.30). Notice that we must multiply (6.31) by $-\lambda \frac{1}{\lambda}$. We obtain

$$\begin{aligned}
& \int_{\mathcal{T}_\lambda} J^c(s, s', z, z') V(s', z') ds' dz' = \int_{\mathcal{T}_\lambda} \frac{1}{\lambda} J \left(\frac{(s-s')}{\lambda}, (z-z) \right) V(s', z') ds' dz' \\
&- \lambda \int_{\mathcal{T}_\lambda} \frac{1}{\lambda} J \left(\frac{(s-s')}{\lambda}, (z-z') \right) [(s-s') k(s^*) z^* \alpha(s^*, z^*) D_{s'} V(s', z')] ds' dz' \tag{6.32} \\
&+ (R^\lambda V)(s, z),
\end{aligned}$$

where

$$\begin{aligned}
(R^\lambda V)(s, z) &= -\lambda^2 \int_{\mathcal{T}_\lambda} \frac{1}{\lambda} J \left(\frac{(s-s')}{\lambda}, (z-z') \right) [(k(s^*) z^*)^2 V(s', z')] ds' dz' \\
&- \lambda^2 \int_{\mathcal{T}_\lambda} \frac{1}{\lambda} J \left(\frac{(s-s')}{\lambda}, (z-z') \right) \left[\frac{(s-s')}{\lambda} \frac{1}{2} k_{s'}(s^*) z^* \alpha(s^*, z^*) V(s', z') \right] ds' dz' \\
&- \lambda^3 \int_{\mathcal{T}_\lambda} \frac{1}{\lambda} J \left(\frac{(s-s')}{\lambda}, (z-z') \right) \left[\frac{(s-s')}{\lambda} k(s^*) k_{s'}(s^*) (z^*)^2 V(s', z') \right] ds' dz' \tag{6.33} \\
&+ \int_{\mathcal{T}_\lambda} \frac{1}{\lambda} M(s, s', z, z') \alpha(s^*, z^*) V(s', z') ds' dz',
\end{aligned}$$

with M defined in (6.28). The term

$$-\lambda \int_{\mathcal{T}_\lambda} \frac{1}{\lambda} J \left(\frac{(s-s')}{\lambda}, (z-z) \right) k(s^*) z^* V(s', z') ds' dz'$$

appearing in the third lines of (6.30) cancels with the first addend of the last equality of (6.31) when multiplied by $-\lambda \frac{1}{\lambda}$. \square

We have the following

Lemma 6.4. *Let $V \in L^2(\mathcal{T}_\lambda)$ with the property that there exists z_0 and $a > 0$ so that*

$$\int_T ds V(s, z)^2 \leq e^{-a|z|} \|V\|^2 \quad \text{for } |z| \geq z_0. \tag{6.34}$$

Let R^λ be the quantity defined in (6.33), there exists $C = C(z_0) > 0$ so that

$$\|R^\lambda V\| \leq \lambda^2 C \|V\|.$$

Proof. Denote

$$(R_1^\lambda V)(s, z) = -\lambda^2 \int_{\mathcal{T}_\lambda} \frac{1}{\lambda} J\left(\frac{(s-s')}{\lambda}, (z-z')\right) [k(s^*)z^*]^2 V(s', z') ds' dz',$$

$$(R_2^\lambda V)(s, z) = -\lambda^2 \int_{\mathcal{T}_\lambda} \frac{1}{\lambda} J\left(\frac{(s-s')}{\lambda}, (z-z')\right) \left[\frac{(s-s')}{\lambda} \frac{1}{2} k_{s'}(s^*) z^* \alpha(s^*, z^*) V(s', z') \right] ds' dz',$$

$$(R_3^\lambda V)(s, z) = -\lambda^3 \int_{\mathcal{T}_\lambda} \frac{1}{\lambda} J\left(\frac{(s-s')}{\lambda}, (z-z')\right) \left[\frac{(s-s')}{\lambda} k(s^*) k_{s'}(s^*) (z^*)^2 V(s', z') \right] ds' dz'.$$

By Jensen inequality, (6.34) and taking into account that $|z-z'| \leq 1$, we have

$$\begin{aligned} \|R_1^\lambda V\|^2 &= \lambda^4 \int_{\mathcal{T}_\lambda} dsdz \left\{ \int_{\mathcal{T}_\lambda} \frac{1}{\lambda} J\left(\frac{(s-s')}{\lambda}, (z-z')\right) [k(s^*)z^*]^2 V(s', z') ds' dz' \right\}^2 \\ &\leq \lambda^4 C \int_{\mathcal{T}_\lambda} dsdz \int_{\mathcal{T}_\lambda} \frac{1}{\lambda} J\left(\frac{(s-s')}{\lambda}, (z-z')\right) [(z'+1)^2 V(s', z')]^2 ds' dz' \\ &\leq \lambda^4 C(z_0) \|V\|^2. \end{aligned} \quad (6.35)$$

We estimate similarly the term $R_2^\lambda V$ and $R_3^\lambda V$. By (6.29) and (6.34) we have

$$\begin{aligned} \|R_4^\lambda V\|^2 &\equiv \int_{\mathcal{T}_\lambda} dsdz \left\{ \frac{1}{\lambda} \int_{\mathcal{T}_\lambda} M(s, s', z, z') \alpha(s^*, z^*) V(s', z') ds' dz' \right\}^2 \leq \\ &\leq C \lambda^4 \int_{\mathcal{T}_\lambda} dsdz \left\{ \int_{\mathcal{T}_\lambda} (z^*)^2 \frac{1}{\lambda} \mathbb{I}_{|s-s'| \leq \lambda} \mathbb{I}_{|z-z'| \leq 1} |V(s', z')| ds' dz' \right\}^2 \\ &\leq C \lambda^4 \int_{\mathcal{T}_\lambda} dsdz \int_{\mathcal{T}_\lambda} \frac{1}{\lambda} \mathbb{I}_{|s-s'| \leq \lambda} \mathbb{I}_{|z-z'| \leq 1} (z^*)^4 V(s', z')^2 ds' dz' \\ &\leq C(z_0) \lambda^4 \|V\|^2. \end{aligned} \quad (6.36)$$

Since $R^\lambda V = \sum_{i=1}^4 R_i^\lambda V$, the thesis follows. \square

Lemma 6.5. *Set $V(s, z) = \sum_{h \in \mathbb{Z}_L} e^{ihs} u_h(z)$, see (5.7). We have*

$$\begin{aligned} &\int_{\mathcal{T}_\lambda} dsdz V(s, z) (\mathcal{B}V)(s, z) \\ &= \sum_{h \in \mathbb{Z}_L, k \in \mathbb{Z}_L} \left\{ \delta_{h,k} \int_{I_\lambda} dz u_h(z) \int_{I_\lambda} J^{\lambda k}(z-z') u_k(z') dz' + F_1^\lambda(u_h, u_k) \right\} + \int_{\mathcal{T}_\lambda} dsdz V(s, z) (R^\lambda V)(s, z) \end{aligned} \quad (6.37)$$

where $J^{\lambda k}$ is defined in (5.2), $F_1^\lambda(u_h, u_k)$ in (6.41).

Proof. From (6.26) we have

$$\begin{aligned}
& \int_{\mathcal{T}_\lambda} dsdz V(s, z) (\mathcal{B}V)(s, z) \\
&= \sum_{h, k} \int_{\mathcal{T}_\lambda} dsdz e^{ihs} u_h(z) \int_{\mathcal{T}_\lambda} \frac{1}{\lambda} J\left(\frac{(s-s')}{\lambda}, (z-z')\right) e^{iks'} u_k(z') ds' dz' \\
& - \lambda^2 \sum_{h, k} \int_{\mathcal{T}_\lambda} dsdz e^{-ihs} u_h(z) \int_{\mathcal{T}_\lambda} \frac{1}{\lambda} J\left(\frac{(s-s')}{\lambda}, (z-z')\right) \left[\frac{(s-s')}{\lambda} k(s^*) z^* i k e^{iks'} u_k(z') \right] ds' dz' \\
& + \int_{\mathcal{T}_\lambda} dsdz V(s, z) (R^\lambda V)(s, z).
\end{aligned} \tag{6.38}$$

We have that

$$\begin{aligned}
& \int_{\mathcal{T}_\lambda} dsdz e^{-ihs} u_h(z) \int_{\mathcal{T}_\lambda} \frac{1}{\lambda} J\left(\frac{(s-s')}{\lambda}, (z-z')\right) e^{iks'} u_k(z') ds' dz' \\
&= \int_{\mathcal{T}_\lambda} dsdz e^{-ihs} u_h(z) e^{iks} J^{\lambda k}(z-z') u_k(z') = \delta_{h, k} \int_{I_\lambda} dz u_h(z) \int_{I_\lambda} J^{\lambda k}(z-z') u_k(z') dz'
\end{aligned} \tag{6.39}$$

where, see (5.2),

$$J^{\lambda k}(z-z') = \int_T \frac{1}{\lambda} J\left(\frac{(s-s')}{\lambda}, (z-z')\right) e^{ik\lambda \frac{(s'-s)}{\lambda}} ds' = \int J(w, (z-z')) e^{-ik\lambda w} dw. \tag{6.40}$$

Set

$$\begin{aligned}
F_1^\lambda(u_h, u_k) &= -ik\lambda^2 \int_{\mathcal{T}_\lambda} dsdz e^{-ihs} u_h(z) e^{iks} \int_{\mathcal{T}_\lambda} \frac{1}{\lambda} J\left(\frac{(s-s')}{\lambda}, (z-z')\right) \frac{(s-s')}{\lambda} k(s^*) e^{ik\lambda \frac{(s-s')}{\lambda}} z^* u_k(z') ds' dz' \\
&= -ik\lambda^2 \int_{I_\lambda} dz u_h(z) \int_{I_\lambda} z^* u_k(z') dz' \int J(w, (z-z')) w e^{-ik\lambda w} \int_T ds e^{i(k-h)s} k(s + \frac{1}{2}\lambda w) dw \\
&= -ik\lambda^2 \int_{I_\lambda} dz u_h(z) \int_{I_\lambda} z^* u_k(z') dz' \int J(w, (z-z')) w e^{-i\frac{1}{2}(3k-h)\lambda w} \int_T ds' e^{i(k-h)s'} k(s') \\
&= -k\lambda^2 \int_{I_\lambda} dz u_h(z) \int_{I_\lambda} z^* u_k(z') dz' \int J(w, (z-z')) w \sin\left(\frac{1}{2}(3k-h)\lambda w\right) \int_T ds' e^{i(k-h)s'} k(s').
\end{aligned} \tag{6.41}$$

We use that $s^* = \frac{s+s'}{2} = s + \frac{s'-s}{2}$ and therefore $k(s^*(w)) = k(s + \frac{1}{2}\lambda w)$. \square

Remark 6.6. Notice that when v and w are even function

$$F_1^\lambda(v, w) = 0.$$

Lemma 6.7. Take $\|V\| = 1$, $V(s, z) = \sum_{k \in \mathbb{Z}_L} e^{iks} u_k(z)$, see (5.7),

$$\langle V, \mathcal{A}V \rangle \leq C\lambda^2, \tag{6.42}$$

where the operator \mathcal{A} is defined in (4.44). Then

$$\langle V, \mathcal{A}V \rangle - \langle V, R_0^\lambda V \rangle = \sum_{k \in \mathbb{Z}_L, h \in \mathbb{Z}_L} \{ \delta_{h, k} \langle u_h, \mathcal{L}^{k\lambda} u_k \rangle + F_1^\lambda(u_h, u_k) + F_2^\lambda(u_h, u_k) + F_3^\lambda(u_h, u_k) \}, \tag{6.43}$$

$$\langle V, R_0^\lambda V \rangle \leq C\lambda^2, \tag{6.44}$$

see (6.60), where F_1^λ is given in (6.41), F_2^λ in (6.56) and F_3^λ in (6.57). Further we have for u and v even function

$$F_1^\lambda(u, v) = F_2^\lambda(u, v) = F_3^\lambda(u, v) = 0, \tag{6.45}$$

$$|F_1^\lambda(u_h, u_k)| \leq \lambda^2 |k| C \frac{1}{(1+|k-h|)^3} \frac{1}{(1+|\frac{1}{2}(3k-h)|\lambda)} \|u_h\| \|u_k\|, \tag{6.46}$$

$$|F_2^\lambda(u_h, u_k) + F_3^\lambda(u_h, u_k)| \leq C\lambda \|u_h\| \|u_k\| C \frac{1}{(1 + |k - h|)^3}. \quad (6.47)$$

Proof. From (6.42) it follows that V satisfies (6.34), i.e there exists z_0 and $a > 0$ so that

$$\int_T ds V(s, z)^2 \leq e^{-a|z|} \|V\|^2 \quad \text{for } |z| \geq z_0. \quad (6.48)$$

Namely, by Lemma 4.11, one deduces that the eigenfunctions $\{\Psi_i\}_i$ associated to eigenvalues of \mathcal{A} smaller than some ϵ_0 decay as (6.48). Therefore we can write for some positive integer N , $V = \sum_i^N a_i \Psi_i(s, z)$ where $a_i = \int_{\mathcal{T}_\lambda} ds dz \Psi_i(s, z) V(s, z)$. We get

$$\int_T ds V(s, z)^2 = \sum_i^N a_i^2 \int_T ds \Psi_i^2(s, z) \leq e^{-a|z|} C \sum_i^N a_i^2 \|\Psi_i\|^2 = C e^{-a|z|} \|V\|^2 \quad \text{for } |z| \geq z_0. \quad (6.49)$$

By (3.10) and definitions (3.20), (3.21) and (3.22) we obtain

$$\begin{aligned} \int_{\mathcal{T}_\lambda} ds dz \frac{V(s, z)^2}{\sigma(m_A(s, \lambda z))} &= \int_{\mathcal{T}_\lambda} ds dz \frac{V(s, z)^2}{\beta(1 - \bar{m}^2(z))} \\ &+ \int_T ds [I_{2,s}(V) + I_{3,s}(V) + I_{4,s}(V)]. \end{aligned} \quad (6.50)$$

We write

$$\begin{aligned} I_{3,s}(V) &= \lambda \int_{\mathcal{T}_\lambda} ds dz \frac{(V(s, z))^2}{\beta(1 - \bar{m}^2(z))^2} \bar{m}(z) \phi(s, 0) \\ &+ \lambda^2 \int_{\mathcal{T}_\lambda} ds dz \frac{(V(s, z))^2}{\beta(1 - \bar{m}^2(z))^2} \bar{m}(z) [\phi(s, z) - \phi(s, 0)]. \end{aligned} \quad (6.51)$$

By (2.16) and (6.48) we get

$$\left| \int_{\mathcal{T}_\lambda} ds dz \frac{(V(s, z))^2}{\beta(1 - \bar{m}^2(z))^2} \bar{m}(z) [\phi(s, z) - \phi(s, 0)] \right| \leq C \|V\|^2. \quad (6.52)$$

Hence by (6.50) and (6.51)

$$\begin{aligned} \int_{\mathcal{T}_\lambda} ds dz \frac{(V(s, z))^2}{\sigma(m_A(s, \lambda z))} &= \int_{\mathcal{T}_\lambda} ds dz \frac{(V(s, z))^2}{\beta(1 - \bar{m}^2(z))} \\ &+ 2\lambda \int_{\mathcal{T}_\lambda} ds dz \frac{(V(s, z))^2}{\beta(1 - \bar{m}^2(z))^2} \bar{m}(z) [h_1(z)g(s) + \phi(s, 0)] + \lambda^2 \int_{\mathcal{T}_\lambda} ds dz R_1(s, z) (V(s, z))^2, \end{aligned} \quad (6.53)$$

where we denoted by

$$R_1(s, z) = \frac{1}{\beta(1 - \bar{m}^2(z))^2} \bar{m}(z) [\phi(s, z) - \phi(s, 0)] + q^\lambda(s, \lambda z).$$

By (6.52) and (2.14) we have that

$$\langle V, R_1 V \rangle \leq C \|V\|^2. \quad (6.54)$$

Set $V(s, z) = \sum_k e^{iks} u_k(z)$. Taking into account (6.53) we have

$$\begin{aligned} \int_{\mathcal{T}_\lambda} ds dz \frac{(V(s, z))^2}{\sigma(m_A(s, \lambda z))} &= \sum_k \int_{I_\lambda} dz \frac{(u_k(z))^2}{\sigma(\bar{m}(z))} + \sum_k \sum_h [F_2^\lambda(u_h, u_k) + F_3^\lambda(u_h, u_k)] \\ &+ \lambda^2 \int_{\mathcal{T}_\lambda} ds dz R_1(s, z) (V(s, z))^2, \end{aligned} \quad (6.55)$$

where

$$F_2^\lambda(u_h, u_k) = 2\lambda \int_{I_\lambda} dz \frac{u_k(z) u_h(z)}{\beta(1 - \bar{m}^2(z))^2} \bar{m}(z) h_1(z) \int_T ds e^{i(k-h)s} g(s) \quad (6.56)$$

$$F_3^\lambda(u_h, u_k) = 2\lambda \int_{I_\lambda} dz \frac{u_k(z)u_h(z)}{\beta(1-\bar{m}^2(z))^2} \bar{m}(z) \int_T ds e^{i(k-h)s} \phi(s, 0). \quad (6.57)$$

Then by Lemma 6.5

$$\begin{aligned} & \langle V, L^\lambda V \rangle - \langle V, R^\lambda V \rangle + \lambda^2 \langle V, R_1 V \rangle \\ &= \sum_{k,h} \{ \delta_{h,k} \langle u_h, \mathcal{L}^{k\lambda} u_k \rangle + F_1^\lambda(u_h, u_k) + F_2^\lambda(u_h, u_k) + F_3^\lambda(u_h, u_k) \}. \end{aligned} \quad (6.58)$$

Since V satisfies the decay property (4.61) we can apply Lemma 6.4 obtaining

$$\langle V, R^\lambda V \rangle \leq C\lambda^2 \|V\|^2. \quad (6.59)$$

Denote

$$\langle V, R_0^\lambda V \rangle = \lambda^2 \langle V, R_1 V \rangle + \langle V, R^\lambda V \rangle. \quad (6.60)$$

By (6.54) and (6.59) we get (6.44). By inspection one realises that when u and v are even (6.45) holds. Further, since

$$u_h(z) = \int V(s, z) e^{ihs} ds$$

then, see (4.61),

$$|u_h(z)| \leq \int |V(s, z)| ds \leq C \left(\int |V(s, z)|^2 ds \right)^{\frac{1}{2}} \leq C e^{-a|z|} \|V\|, \quad |z| \geq z_0, \quad (6.61)$$

where $a > 0$ and $z_0 > 0$ do not depend on λ . To estimate F_i , $i = 1, 2, 3$ we use the smoothness of J , Γ and m_A . We therefore use estimate (5.3),

$$\begin{aligned} \left| \int_T ds e^{i(k-h)s} k(s) \right| &\leq \frac{C}{(1+|k-h|)^3}, \\ \left| \int_T ds e^{i(k-h)s} g(s) \right| &\leq C \frac{1}{(1+|k-h|)^3}, \\ \left| \int_T ds e^{i(k-h)s} \Phi(s, 0) \right| &\leq C \frac{1}{(1+|k-h|)^3}. \end{aligned} \quad (6.62)$$

To estimate F_1^λ we bound $|z^*| \leq |z+1|$ since J has support in the ball of radius 1. The exponential decay of the u_h , see (6.61), is essential to control the growing of $|z+1|$. We get

$$\begin{aligned} |F_1^\lambda(u_h, u_k)| &\leq \lambda^2 |k| C \frac{1}{(1+|k-h|)^3} \frac{1}{(1+|\frac{1}{2}(3k-h)|\lambda)} \int_{I_\lambda} dz |u_h(z)| |z+1| \int_{I_\lambda} C(z-z') |u_k(z')| dz' \\ &\leq \lambda^2 |k| C \frac{1}{(1+|k-h|)^3} \frac{1}{(1+|\frac{1}{2}(3k-h)|\lambda)} \|u_h\| \|u_k\|, \end{aligned} \quad (6.63)$$

$$|F_2^\lambda(u_h, u_k) + F_3^\lambda(u_h, u_k)| \leq C\lambda \|u_h\| \|u_k\| \frac{1}{(1+|k-h|)^3}. \quad (6.64)$$

□

Proof of Theorem 6.2 By assumption, $\langle V, \mathcal{A}V \rangle \leq C\lambda^2$, see (6.8), hence Lemma (6.7) holds. Take $V(s, z)$ as in (5.7) and consider the representation of $\langle V, \mathcal{A}V \rangle$ given in (6.43). We start considering the diagonal term, i.e when $h = k$. We need to lower bound the following quantity.

$$\sum_k \{ \langle u_k, \mathcal{L}^{k\lambda} u_k \rangle + F_1^\lambda(u_k, u_k) + F_2^\lambda(u_k, u_k) + F_3^\lambda(u_k, u_k) \}. \quad (6.65)$$

Split $\sum_k = \sum_{|\lambda k| \leq h_0} + \sum_{|\lambda k| > h_0}$ where h_0 is as in Proposition 5.6. When $|\lambda k| \leq h_0$ split

$$u_k = \alpha_k \psi_0^{k\lambda} + u_k^\perp \quad (6.66)$$

where $\psi_0^{k\lambda}$ is the principal eigenvalue of $\mathcal{L}^{k\lambda}$ and

$$\int dz \psi_0^{k\lambda}(z) u_k^\perp(z) = 0.$$

By (5.11)

$$\langle u_k, \mathcal{L}^{k\lambda} u_k \rangle = \mu_0^{k\lambda} \alpha_k^2 + \langle u_k^\perp, \mathcal{L}^{k\lambda} u_k^\perp \rangle \geq \mu_0^{k\lambda} \alpha_k^2 + D \|u_k^\perp\|^2.$$

By Proposition 5.6 the principal eigenvalue of the operator $\mathcal{L}^{k\lambda}$ is even, hence for $i = \{1, 2, 3\}$

$$F_i^\lambda(\psi_0^{h\lambda}, \psi_0^{k\lambda}) = 0, \quad |\lambda h| \leq h_0, \quad |\lambda k| \leq h_0, \quad (6.67)$$

and

$$F_i^\lambda(u_k, u_h) = F_i^\lambda(\alpha_k \psi_0^{k\lambda}, u_h^\perp) + F_i^\lambda(u_k^\perp, \alpha_h \psi_0^{h\lambda}) + F_i^\lambda(u_k^\perp, u_h^\perp). \quad (6.68)$$

Property (6.67) is essential to obtain the final estimate. Taking into account (6.46), (6.47) and (6.67) we have

$$|F_1^\lambda(u_k, u_k)| \leq \lambda^2 |k| C \frac{1}{(1 + |k|\lambda)} [2|\alpha_k| \|u_k^\perp\| + \|u_k^\perp\|^2] \leq \lambda C [|\alpha_k| \|u_k^\perp\| + \|u_k^\perp\|^2], \quad (6.69)$$

$$|F_2^\lambda(u_k, u_k) + F_3^\lambda(u_k, u_k)| \leq \lambda C [2|\alpha_k| \|u_k^\perp\| + \|u_k^\perp\|^2]. \quad (6.70)$$

Therefore

$$\begin{aligned} & \sum_{|k| \leq \frac{h_0}{\lambda}} \{ \langle u_k, \mathcal{L}^{k\lambda} u_k \rangle + F_1^\lambda(u_k, u_k) + F_2^\lambda(u_k, u_k) + F_3^\lambda(u_k, u_k) \} \\ & \geq \sum_{|k| \leq \frac{h_0}{\lambda}} \{ \mu_0^{k\lambda} \alpha_k^2 + D \|u_k^\perp\|^2 - \lambda C [\|u_k^\perp\| + \|u_k^\perp\|^2] \} \\ & \geq \sum_{|k| \leq \frac{h_0}{\lambda}} \{ \mu_0^{k\lambda} \alpha_k^2 + [D - C\lambda] \|u_k^\perp\|^2 - \lambda C |\alpha_k| \|u_k^\perp\| \}. \end{aligned} \quad (6.71)$$

When $|k| > \frac{h_0}{\lambda}$, taking advantage that $\lambda^2 \frac{|k|}{(1 + |k|\lambda)} \leq \lambda$, and by (6.46), (6.47) we get

$$|\sum_i F_i^\lambda(u_k, u_k)| \leq \lambda C \|u_k\|^2.$$

Hence

$$\begin{aligned} & \sum_{|k| > \frac{h_0}{\lambda}} \{ \langle u_k, \mathcal{L}^{k\lambda} u_k \rangle + F_1^\lambda(u_k, u_k) + F_2^\lambda(u_k, u_k) + F_3^\lambda(u_k, u_k) \} \\ & \geq \sum_{|k| > \frac{h_0}{\lambda}} [\nu - \lambda C] \|u_k\|^2. \end{aligned} \quad (6.72)$$

Next we estimate the terms outside diagonal, i.e $h \neq k$:

$$\begin{aligned} & \sum_{k, h; k \neq h} [F_1^\lambda(u_h, u_k) + F_2^\lambda(u_h, u_k) + F_3^\lambda(u_h, u_k)] \\ & = \left[\sum_{|k\lambda| \leq h_0} \sum_{|h\lambda| \leq h_0, h \neq k} + \sum_{|k\lambda| \leq h_0} \sum_{|h\lambda| > h_0} + \sum_{|k\lambda| > h_0} \sum_{|h\lambda| \leq h_0} + \sum_{|k\lambda| > h_0} \sum_{|h\lambda| > h_0, h \neq k} \right] \sum_i F_i^\lambda(u_h, u_k). \end{aligned} \quad (6.73)$$

We estimate each term of (6.73). We control the double sums using that

$$\forall k \quad \sum_h \frac{1}{(1 + |k - h|)^3} \leq C, \quad \sum_h \frac{1}{(1 + |k - h|)^2} \leq C.$$

When $|k\lambda| \leq h_0$ we decompose u_k as in (6.66). When $|k\lambda| \leq h_0$ and $|h\lambda| \leq h_0$ we take advantage of (6.67) and (6.68). By (6.46) we obtain

$$\begin{aligned}
& \sum_{|k\lambda| \leq h_0} \sum_{|h\lambda| \leq h_0, h \neq k} |F_1^\lambda(u_h, u_k)| \\
& \leq \lambda^2 \sum_{|k\lambda| \leq h_0} \sum_{|h\lambda| \leq h_0, h \neq k} \left\{ |k| \frac{1}{(1+|k-h|)^3} \frac{1}{(1+\frac{1}{2}(3k-h)|\lambda|)} [|\alpha_k| \|u_h^\perp\| + \|u_h^\perp\| \|u_k^\perp\|] \right\} \\
& \leq \lambda h_0 \sum_{|k\lambda| \leq h_0} \sum_{|h\lambda| \leq h_0, h \neq k} \frac{1}{(1+|k-h|)^3} \left[|\alpha_k| \|u_h^\perp\| + \frac{1}{2} \|u_h^\perp\|^2 + \frac{1}{2} \|u_k^\perp\|^2 \right] \\
& \leq \lambda h_0 C \left\{ \sum_{|h\lambda| \leq h_0} \|u_h^\perp\|^2 + \sum_{|k\lambda| \leq h_0} \sum_{|h\lambda| \leq h_0, h \neq k} \frac{1}{(1+|k-h|)^3} |\alpha_k| \|u_h^\perp\| \right\}.
\end{aligned} \tag{6.74}$$

Taking into account (6.67) and (6.68), applying estimate (6.47) and proceeding as above we get

$$\begin{aligned}
& \sum_{|k\lambda| \leq h_0} \sum_{|h\lambda| \leq h_0, h \neq k} |F_2^\lambda(u_h, u_k) + F_3^\lambda(u_h, u_k)| \\
& \leq \lambda C \left\{ \sum_{|h\lambda| \leq h_0} \|u_h^\perp\|^2 + \sum_{|k\lambda| \leq h_0} \sum_{|h\lambda| \leq h_0, h \neq k} \frac{1}{(1+|k-h|)^3} |\alpha_k| \|u_h^\perp\| \right\}.
\end{aligned} \tag{6.75}$$

Next we consider the case when $|k\lambda| \leq h_0$ and $|h\lambda| > h_0$. By (6.46), we get

$$\begin{aligned}
|F_1^\lambda(u_h, u_k)| & \leq \lambda^2 |k| \frac{1}{(1+|k-h|)^3} \frac{1}{(1+\frac{1}{2}(3k-h)|\lambda|)} \|u_h\| [\|u_k^\perp\| + |\alpha_k|] \\
& \leq \lambda h_0 \frac{1}{(1+|k-h|)^3} \left[\frac{1}{2} \|u_h\|^2 + \frac{1}{2} \|u_k^\perp\|^2 + |\alpha_k| \|u_h\| \right].
\end{aligned} \tag{6.76}$$

By (6.47), proceeding as above,

$$|F_2^\lambda(u_h, u_k) + F_3^\lambda(u_h, u_k)| \leq \lambda \frac{1}{(1+|k-h|)^3} \left[\frac{1}{2} \|u_h\|^2 + \frac{1}{2} \|u_k^\perp\|^2 + |\alpha_k| \|u_h\| \right]. \tag{6.77}$$

Therefore

$$\begin{aligned}
& \sum_{|k\lambda| \leq h_0} \sum_{|h\lambda| > h_0} \sum_i |F_i^\lambda(u_h, u_k)| \\
& \leq \lambda C \sum_{|k\lambda| \leq h_0} \sum_{|h\lambda| > h_0} \frac{1}{(1+|k-h|)^3} \left[\frac{1}{2} \|u_h\|^2 + \frac{1}{2} \|u_k^\perp\|^2 + |\alpha_k| \|u_h\| \right] \\
& \leq \lambda C \sum_{|k\lambda| \leq h_0} \|u_k^\perp\|^2 + \lambda C \sum_{|h\lambda| > h_0} \|u_h\|^2 + \lambda C \sum_{|k\lambda| \leq h_0} \sum_{|h\lambda| > h_0} \frac{1}{(1+|k-h|)^3} |\alpha_k| \|u_h\|.
\end{aligned} \tag{6.78}$$

When $|k\lambda| > h_0$ and $|h\lambda| < h_0$ the estimate of $|F_2^\lambda(u_h, u_k) + F_3^\lambda(u_h, u_k)|$ gives similar terms as in (6.77), with h replacing k . To estimate F_1^λ we note that $|k| \leq |k-h| + |h|$. Therefore by (6.46)

$$\begin{aligned}
& \sum_{|k\lambda| > h_0} \sum_{|h\lambda| \leq h_0} |F_1^\lambda(u_h, u_k)| \\
& \leq \lambda^2 \sum_{|k\lambda| > h_0} \sum_{|k\lambda| \leq h_0} |k| \frac{1}{(1+|k-h|)^3} \frac{1}{(1+|\frac{1}{2}(3k-h)|\lambda)} \|u_h\| \|u_k\| \\
& \leq \lambda^2 \sum_{|k\lambda| > h_0} \sum_{|h\lambda| \leq h_0} |h| \frac{1}{(1+|k-h|)^3} \|u_h\| \|u_k\| \\
& + \lambda^2 \sum_{|k\lambda| > h_0} \sum_{|h\lambda| \leq h_0} |k-h| \frac{1}{(1+|k-h|)^3} \|u_h\| \|u_k\| \\
& \leq \lambda h_0 \sum_{|k\lambda| > h_0} \sum_{|h\lambda| \leq h_0} \frac{1}{(1+|k-h|)^3} \|u_h\| \|u_k\| + \lambda^2 \sum_{|k\lambda| > h_0} \sum_{|h\lambda| \leq h_0} \frac{1}{(1+|k-h|)^2} \|u_h\| \|u_k\|.
\end{aligned} \tag{6.79}$$

Since $|h\lambda| \leq h_0$, we split u_h as in (6.66). Insert this decomposition into (6.79) we get

$$\begin{aligned}
& \sum_{|k\lambda| > h_0} \sum_{|h\lambda| \leq h_0} |F_1^\lambda(u_h, u_k)| \\
& \leq \lambda \sum_{|k\lambda| > h_0} \|u_k\|^2 + \lambda \sum_{|h\lambda| < h_0} \|u_h^\perp\|^2 + \lambda \sum_{|k\lambda| > h_0} \sum_{|h\lambda| \leq h_0} \frac{1}{(1+|k-h|)^3} |\alpha_h| \|u_k\|.
\end{aligned} \tag{6.80}$$

Next we consider the case when $|k\lambda| > h_0$ and $|h\lambda| > h_0, h \neq k$. The main point is to control the $|k|$ in the estimate of F_1^λ . When k and h have opposite sign, then

$$\frac{|k|}{1+|k-h|} \leq 1. \tag{6.81}$$

When k and h have the same sign it might be that $\frac{|k|}{1+|k-h|} > 1$. But if this is case, then for these values of h and k

$$\frac{k\lambda}{(1+\frac{1}{2}(3k-h)\lambda)} \leq 2. \tag{6.82}$$

This statement can be easily verified. Take $k > 0$ the case with $k < 0$ can be treated in the same way. Assume that (6.81) does not hold. This means that, for given k , $|k-h| < k-1$. This inequality is satisfied for $h \in \{1, \dots, 2k-1\}$. For such value of h we have that $3k-h \geq k$ and therefore (6.82) holds. Hence

$$\sum_{|k\lambda| > h_0} \sum_{|h\lambda| > h_0, h \neq k} F_1^\lambda(u_h, u_k) = \sum'_{|k\lambda| > h_0} \sum'_{|h\lambda| > h_0, h \neq k} F_1^\lambda(u_h, u_k) + \sum''_{|k\lambda| > h_0} \sum''_{|h\lambda| > h_0, h \neq k} F_1^\lambda(u_h, u_k), \tag{6.83}$$

where \sum' is a sum restricted to the h and k so that (6.81) holds, and \sum'' is a sum restricted to the h and k so that (6.81) does not hold. By (6.46) and (6.81) we get

$$\begin{aligned}
& \sum'_{|k\lambda| > h_0} \sum'_{|h\lambda| > h_0, h \neq k} F_1^\lambda(u_h, u_k) \\
& \leq \lambda^2 C \sum'_{|k\lambda| > h_0} \sum'_{|h\lambda| > h_0, h \neq k} |k| \frac{1}{(1+|k-h|)^3} \frac{k\lambda}{(1+\frac{1}{2}(3k-h)\lambda)} \|u_h\| \|u_k\| \leq \lambda^2 C \sum_{|k\lambda| > h_0} \sum_{|h\lambda| > h_0, h \neq k} \frac{1}{(1+|k-h|)^2} \|u_h\| \|u_k\| \\
& \leq \lambda^2 C \sum_{|k\lambda| > h_0} \sum_{|h\lambda| > h_0, h \neq k} \frac{1}{(1+|k-h|)^2} \left[\frac{1}{2} \|u_h\|^2 + \frac{1}{2} \|u_k\|^2 \right] \leq \lambda^2 C \sum_{|k\lambda| > h_0} \|u_k\|^2.
\end{aligned} \tag{6.84}$$

By (6.46) and (6.82) we get

$$\begin{aligned}
& \sum_{|k\lambda|>h_0}'' \sum_{|h\lambda|>h_0, h\neq k}'' F_1^\lambda(u_h, u_k) \\
& \leq \lambda^2 C \sum_{|k\lambda|>h_0}'' \sum_{|h\lambda|>h_0, h\neq k}'' |k| \frac{1}{(1+|k-h|)^3} \frac{k\lambda}{(1+\frac{1}{2}(3k-h)\lambda)} \|u_h\| \|u_k\| \leq \lambda C \sum_{|k\lambda|>h_0} \sum_{|h\lambda|>h_0, h\neq k} \frac{1}{(1+|k-h|)^3} \|u_h\| \|u_k\| \\
& \leq \lambda C \sum_{|k\lambda|>h_0} \sum_{|h\lambda|>h_0, h\neq k} \frac{1}{(1+|k-h|)^3} \left[\frac{1}{2} \|u_h\|^2 + \frac{1}{2} \|u_k\|^2 \right] \leq C\lambda \sum_{|k\lambda|>h_0} \|u_k\|^2.
\end{aligned} \tag{6.85}$$

By the previous estimates we have

$$\begin{aligned}
& \sum_{k, h; k\neq h} [F_1^\lambda(u_h, u_k) + F_2^\lambda(u_h, u_k) + F_3^\lambda(u_h, u_k)] \\
& \leq \lambda C \left(\sum_{|k\lambda|\leq h_0} \|u_k^\perp\|^2 + \sum_{|k\lambda|>h_0} \|u_k\|^2 \right) \\
& + \lambda C \sum_{|k\lambda|\leq h_0} \sum_{|h\lambda|\leq h_0, h\neq k} \frac{1}{(1+|k-h|)^3} |\alpha_k| \|u_h^\perp\| \\
& + \lambda C \sum_{|k\lambda|\leq h_0} \sum_{|h\lambda|>h_0} \frac{1}{(1+|k-h|)^3} |\alpha_k| \|u_h\| \\
& + \lambda C \sum_{|k\lambda|>h_0} \sum_{|h\lambda|\leq h_0} \frac{1}{(1+|k-h|)^3} |\alpha_h| \|u_k\|.
\end{aligned} \tag{6.86}$$

Define

$$b_h = \begin{cases} \|u_h^\perp\|, & |h\lambda| \leq h_0 \\ \|u_h\|, & |h\lambda| > h_0. \end{cases}$$

Adding to the (6.86) the terms on the diagonal, i.e $F_i^\lambda(u_k, u_k)$, for $i = 1, 2, 3$, we get

$$\begin{aligned}
& \left| \sum_{k, h} [F_1^\lambda(u_h, u_k) + F_2^\lambda(u_h, u_k) + F_3^\lambda(u_h, u_k)] \right| \\
& \leq \lambda C \left(\sum_{|k\lambda|\leq h_0} \|u_k^\perp\|^2 + \sum_{|k\lambda|>h_0} \|u_k\|^2 \right) \\
& + \lambda C \sum_{|k\lambda|\leq h_0} \sum_h \frac{1}{(1+|k-h|)^3} |\alpha_k| b_h.
\end{aligned} \tag{6.87}$$

By Schwartz

$$\begin{aligned}
& \sum_{|k\lambda|\leq h_0} \sum_h \frac{1}{(1+|k-h|)^3} |\alpha_k| b_h \leq \sqrt{\sum_{|k\lambda|\leq h_0} \sum_h \frac{1}{(1+|k-h|)^3} |\alpha_k|^2} \sqrt{\sum_{|k\lambda|\leq h_0} \sum_h \frac{1}{(1+|k-h|)^3} b_h^2} \\
& \leq C \sqrt{\sum_{|k\lambda|\leq h_0} |\alpha_k|^2} \sqrt{\sum_h b_h^2} \leq C \sqrt{\sum_h b_h^2}.
\end{aligned} \tag{6.88}$$

Taking into account (6.43) and (6.44) and estimates (6.71), (6.72) and (6.86) we have

$$\begin{aligned} \langle V, \mathcal{A}V \rangle - \langle V, R_0^\lambda V \rangle &= \sum_{k,h} \{ \delta_{h,k} \langle u_h, \mathcal{L}^{k\lambda} u_k \rangle + F_1^\lambda(u_h, u_k) + F_2^\lambda(u_h, u_k) + F_3^\lambda(u_h, u_k) \} \\ &\geq \sum_{|k| \leq \frac{h_0}{\lambda}} \mu_0^{k\lambda} \alpha_k^2 + [D - C\lambda] \sum_{|k| \leq \frac{h_0}{\lambda}} \|u_k^\perp\|^2 + [\nu - \lambda C] \sum_{|k| > \frac{h_0}{\lambda}} \|u_k\|^2 - \lambda \sqrt{\sum_h b_h^2} \end{aligned} \quad (6.89)$$

Since $\langle V, \mathcal{A}V \rangle \leq C\lambda^2$, see (6.8), and $|\langle V, R_0^\lambda V \rangle| \leq C\lambda^2$, see (6.44), taking $C \leq \min\{[D - C\lambda], [\nu - \lambda C]\}$ we get

$$\begin{aligned} C\lambda^2 &\geq \sum_{|k| \leq \frac{h_0}{\lambda}} \mu_0^{k\lambda} \alpha_k^2 + C \sum_k b_k^2 - \lambda \sqrt{\sum_k b_k^2} \\ &\geq C \sum_k b_k^2 - \lambda \sqrt{\sum_k b_k^2}. \end{aligned} \quad (6.90)$$

In the last inequality we use that $\mu_0^{k\lambda} > \mu_0^0 > 0$, see Proposition 5.6 and Theorem 3.1. Therefore

$$C\lambda^2 + \lambda \sqrt{\sum_k b_k^2} \geq C \sum_k b_k^2. \quad (6.91)$$

This inequality immediately implies

$$\sqrt{\sum_k b_k^2} \leq C\lambda. \quad (6.92)$$

Lower bounding (6.89) by (6.92) we get

$$C\lambda^2 \geq \sum_{|k| \leq \frac{h_0}{\lambda}} \mu_0^{k\lambda} \alpha_k^2 + [D - C\lambda] \sum_{|k| \leq \frac{h_0}{\lambda}} \|u_k^\perp\|^2 + [\nu - \lambda C] \sum_{|k| > \frac{h_0}{\lambda}} \|u_k\|^2. \quad (6.93)$$

We get similar estimates as in Subsection 6.1 (Toy model). This implies

$$\sum_{|k| \leq \frac{h_0}{\lambda}} \mu_0^{k\lambda} \alpha_k^2 \leq C\lambda^2, \quad (6.94)$$

$$\sum_{|k\lambda| \leq h_0} \|u_k^\perp\|^2 \leq C\lambda^2, \quad (6.95)$$

$$\sum_{|k\lambda| > h_0} \|u_k\|^2 \leq C\lambda^2. \quad (6.96)$$

We therefore define

$$Z(s) = \sum_{|k| \leq \frac{h_0}{\lambda}} e^{iks} \alpha_k \quad (6.97)$$

$$V^R(s, z) = \sum_{|k| \leq \frac{h_0}{\lambda}} e^{iks} u_k^\perp(z) + \sum_{|k| > \frac{h_0}{\lambda}} e^{iks} u_k(z) + \sum_{|k| \leq \frac{h_0}{\lambda}} e^{iks} \alpha_k [\psi_0^{k\lambda}(z) - \psi_0^0(z)]. \quad (6.98)$$

Then

$$V(s, z) = Z(s) \psi_0^0(z) + V^R(s, z) \quad (6.99)$$

and, proceeding as in Subsection 6.1, the requirements (6.10) hold. \square

7. PROOF OF THEOREM 2.5

Proof. If $v = 0$ then the assertion of the theorem holds. So let v be non identically equal to zero and assume that $\int_{\Omega} v^2(\xi) d\xi = 1$. Proceeding as in the proof of Theorem 2.4 there exists $\bar{k} \in \{0, \dots, N\}$, with $N = \lceil \frac{1}{\lambda} \rceil$, and cut off functions $\eta_1 = \eta_1^{\bar{k}}$ and $\eta_2 = 1 - \eta_1^{\bar{k}}$ so that

$$\begin{aligned} \int_{\Omega} (A_{m_A}^{\lambda} v)(\xi) v(\xi) d\xi &= \int_{\Omega} (A_{m_A}^{\lambda} \eta_1 v)(\xi) \eta_1(\xi) v(\xi) d\xi \\ &+ \int_{\Omega} (A_{m_A}^{\lambda} \eta_2 v)(\xi) \eta_2(\xi) v(\xi) d\xi \\ &- 2 \int_{\Omega} d\xi \eta_1(\xi) v(\xi) (J^{\lambda} \star \eta_2 v)(\xi), \end{aligned} \quad (7.1)$$

$$\int_{\Omega} (A_{m_A}^{\lambda} \eta_2 v)(\xi) \eta_2(\xi) v(\xi) d\xi \geq (C^* - 1) \|\eta_2 v\|_{L^2(\Omega)} > 0, \quad (7.2)$$

and

$$\left| 2 \int_{\Omega} d\xi \eta_1(\xi) v(\xi) (J^{\lambda} \star \eta_2 v)(\xi) \right| \leq \lambda^2 C \|v\|_{L^2(\Omega)}^2. \quad (7.3)$$

Without loss of generality we keep on denoting by $\mathcal{N}(d_0)$ the set where $\eta_1(\xi) = 1$. The (2.22), (7.2) and (7.3) imply that

$$\int_{\mathcal{N}(d_0)} (A_{m_A}^{\lambda} \eta_1 v)(\xi) \eta_1(\xi) v(\xi) d\xi \leq C \lambda^2 - \int_{\Omega \setminus \mathcal{N}(d_0)} (A_{m_A}^{\lambda} \eta_2 v)(\xi) \eta_2 v(\xi) d\xi \leq C \lambda^2. \quad (7.4)$$

By (2.22) and (7.3)

$$\int_{\Omega \setminus \mathcal{N}(d_0)} (A_{m_A}^{\lambda} \eta_2 v)(\xi) v(\xi) d\xi \leq C \lambda^2 - \int_{\mathcal{N}(d_0)} (A_{m_A}^{\lambda} \eta_1^N v)(\xi) \eta_1(\xi) v(\xi) d\xi. \quad (7.5)$$

By Lemma 4.2 and Lemma 4.8 we have that

$$\int_{\mathcal{N}(d_0)} (A_{m_A}^{\lambda} \eta_1 v)(\xi) \eta_1 v(\xi) d\xi \geq -C \lambda^2 \int_{\Omega} v^2(\xi) d\xi = -C \lambda^2. \quad (7.6)$$

Then, from (7.5), taking into account (7.6) we obtain

$$\int_{\Omega \setminus \mathcal{N}(d_0)} (A_{m_A}^{\lambda} \eta_1 v)(\xi) v(\xi) d\xi \leq C \lambda^2. \quad (7.7)$$

This together with (7.2) implies

$$\int_{\Omega \setminus \mathcal{N}(d_0)} v^2(\xi) d\xi \leq C \lambda^2, \quad (7.8)$$

hence

$$\int_{\mathcal{N}(d_0)} v^2(\xi) d\xi \geq 1 - C \lambda^2. \quad (7.9)$$

Therefore, see (7.4) and (7.9), we can apply Theorem 6.1 decomposing v as

$$v(r, s) = Z(s) \frac{1}{\sqrt{\alpha(s, r)}} \frac{1}{\sqrt{\lambda}} \psi_0^0\left(\frac{r}{\lambda}\right) + v^R(s, r), \quad (7.10)$$

where $\psi_0^0(\cdot)$ is the first eigenvalue of \mathcal{L}^0 , see Theorem 3.1, with

$$\|v^R\|_{L^2(\mathcal{N}(d_0))}^2 \leq \lambda^2 C, \quad (7.11)$$

and

$$1 - C \lambda^2 \leq \|Z\|_{L^2(T)}^2 \leq 1, \quad \|\nabla Z\|_{L^2(T)} \leq C. \quad (7.12)$$

Set

$$\Delta w = v. \quad (7.13)$$

Denote

$$\hat{w} = \frac{1}{\sqrt{\lambda}} w.$$

Then, from (7.10)

$$\Delta \hat{w} = \frac{1}{\sqrt{\lambda}} \left[Z(s) \frac{1}{\sqrt{\alpha(s,r)}} \frac{1}{\sqrt{\lambda}} \psi_0^0\left(\frac{r}{\lambda}\right) + v^R(s,r) \right].$$

To show (2.23) it is enough to prove that

$$\|\nabla \hat{w}\|_{L^2(\Omega)} \geq C \quad (7.14)$$

for some positive C independent on λ . Let $\delta \in (0, \frac{1}{2}]$ a small constant to be determined and let $\chi \in C_0^\infty(\mathbb{R})$ be a cut-off function, such that

$$\chi(x) = 1 \quad \text{if} \quad |x| \leq \frac{1}{2}, \quad \chi(x) = 0 \quad \text{if} \quad |x| > 1, \quad x\chi'(x) \leq 0 \quad \text{in} \quad \mathbb{R}. \quad (7.15)$$

Set $\chi^\delta(x) = \chi(\frac{x}{\delta})$, with χ as in (7.15). We have

$$\begin{aligned} & \int_{\mathcal{N}(d_0)} Z(s(\xi)) \chi^\delta(r(\xi, \Gamma)) \Delta \hat{w}(\xi) d\xi \\ &= \int_{T \times [-\delta, \delta]} Z(s) \chi^\delta(r) \left\{ Z(s) \frac{1}{\sqrt{\alpha(s,r)}} \frac{1}{\lambda} \psi_0^0\left(\frac{r}{\lambda}\right) + \frac{1}{\sqrt{\lambda}} v^R(s,r) \right\} \alpha(s,r) dr ds. \end{aligned} \quad (7.16)$$

By (3.7), for $\delta > \lambda$, $\int_{[-\delta, \delta]} \frac{1}{\lambda} \psi_0^0\left(\frac{r}{\lambda}\right) dr \geq 2\bar{m}(\frac{\delta}{\lambda}) + C e^{-\alpha \frac{\delta}{\lambda}} \geq C$. This, together with (7.12), allows to lower bound

$$\begin{aligned} & \int_{T \times [-\delta, \delta]} \sqrt{\alpha(r,s)} Z^2(s) \chi^\delta(r) \frac{1}{\lambda} \psi_0^0\left(\frac{r}{\lambda}\right) dr ds \\ & \geq C \inf_{\{(r,s) \in T \times [-\delta, \delta]\}} \sqrt{\alpha(s,r)} \|Z^2\|_{L^2(T)}^2 \int_{[-\delta, \delta]} \frac{1}{\lambda} \psi_0^0\left(\frac{r}{\lambda}\right) dr \geq C \|Z^2\|_{L^2(T)}^2 \geq C(1 - \lambda^2). \end{aligned} \quad (7.17)$$

By Schwartz inequality

$$\begin{aligned} & \left| \frac{1}{\sqrt{\lambda}} \int_{T \times [-\delta, \delta]} Z(s) \chi^\delta(r) v^R(r,s) \alpha(r,s) dr ds \right| \\ & \leq \frac{1}{\sqrt{\lambda}} \|v^R\|_{L^2(\mathcal{N}(\delta))} \left(\int_{T \times [-\delta, \delta]} \alpha(r,s) Z^2(s) [\chi^\delta(r)]^2 dr ds \right)^{\frac{1}{2}} \\ & \leq \frac{1}{\sqrt{\lambda}} \|v^R\|_{L^2(\mathcal{N}(\delta))} \left(\sup_{\{(r,s) \in T \times [-\delta, \delta]\}} |\alpha(r,s)| \right)^{\frac{1}{2}} \|Z\|_{L^2(T)} \delta^{\frac{1}{2}} \leq C \delta^{\frac{1}{2}} \lambda^{\frac{1}{2}}. \end{aligned} \quad (7.18)$$

The last inequality is obtained applying (7.11). Then from (7.16), (7.17) and (7.18) we obtain

$$\int_{\mathcal{N}(d_0)} Z(s(\xi)) \chi^\delta(r(\xi, \Gamma)) \Delta \hat{w}(\xi) d\xi \geq C(1 - \lambda^2) - C \delta^{\frac{1}{2}} \lambda^{\frac{1}{2}}. \quad (7.19)$$

On the other hand we can calculate

$$\begin{aligned}
& \int_{\mathcal{N}(d_0)} Z(s(\xi)) \chi^\delta(r(\xi, \Gamma)) \Delta \hat{w}(\xi) d\xi \\
&= - \int_{\mathcal{N}(d_0)} \nabla \{Z(s(\xi)) \chi^\delta(r(\xi, \Gamma))\} \cdot \{\nabla \hat{w}(\xi)\} d\xi \\
&= - \int_{\mathcal{N}(d_0)} \{\nabla Z(s(\xi)) \chi^\delta(r(\xi, \Gamma)) + Z(s(\xi)) \nabla \chi^\delta(r(\xi, \Gamma))\} \cdot \nabla \hat{w}(\xi) d\xi \\
&\leq \|\nabla \hat{w}\|_{L^2(\mathcal{N}(\delta))} \left\{ \|\nabla Z\|_{L^2(T)} \delta^{\frac{1}{2}} + \sup_{r \in [-\delta, \delta]} |\nabla \chi^\delta(r)| \delta^{\frac{1}{2}} \|Z\|_{L^2(T)} \right\} \\
&\leq \|\nabla \hat{w}\|_{L^2(\mathcal{N}(\delta))} \left[\delta^{\frac{1}{2}} + \delta^{-\frac{1}{2}} \right] C,
\end{aligned} \tag{7.20}$$

where we estimated $\|Z\|_{L^2(T)}$ and $\|\nabla Z\|_{L^2(T)}$, as in (7.12) and

$$\left(\int_{\mathcal{N}(d_0)} (\nabla Z(s(\xi)))^2 \chi^\delta(r(\xi, \Gamma)) d\xi \right)^{\frac{1}{2}} \leq C \|\nabla Z\|_{L^2(T)} \delta^{\frac{1}{2}}.$$

Combining (7.20) with the estimates (7.19) we obtain for δ small enough

$$\|\nabla \hat{w}\|_{L^2(\mathcal{N}(\delta))} \geq \frac{C(1 - \lambda^2) - \delta^{\frac{1}{2}} \lambda^{\frac{1}{2}}}{\left[\delta^{\frac{1}{2}} + \delta^{-\frac{1}{2}} \right]} \geq C > 0,$$

hence (7.14). The theorem is proved. \square

8. APPENDIX

Lemma 8.1. *Let $\mu(s)$ be any eigenvalue of \mathcal{L}^s such that*

$$\epsilon_0 = \frac{1}{\sigma(m_\beta)} - 1 - \sup_{s \in T} \mu(s) > 0. \tag{8.1}$$

Let $\psi(s, \cdot)$ be one of the normalized eigenfunctions corresponding to $\mu(s)$. There exists $z_0 = z_0(\epsilon_0) > 0$ and $\lambda_0 \equiv \lambda_0(\epsilon_0) > 0$ such that for $\lambda \in (0, \lambda_0]$, we have that for $|z| \geq z_0$, for all $s \in T$

$$|\psi(s, z)| \leq e^{-\alpha(\epsilon_0)(|z| - z_0)} \|\bar{J}\|_2, \tag{8.2}$$

where $\alpha(\epsilon_0)$ is given in (8.13).

Proof. Let $\mu(s)$ and $\psi(s, \cdot)$ be as in the hypothesis, then

$$\frac{\psi(s, z)}{\sigma(m_A(s, \lambda z))} - (\bar{J} \star_z \psi)(s, z) = \mu(s) \psi(s, z) \quad z \in I_\lambda. \tag{8.3}$$

By the definition of m_A , see (2.11) and (2.14), there exists $C > 0$ so that

$$\sup_{s \in T} |\sigma(m_A(s, \lambda z)) - \sigma(\bar{m}(z))| \leq \lambda C. \tag{8.4}$$

Since $\lim_{|z| \rightarrow \infty} \sigma(\bar{m}(z)) = \sigma(m_\beta)$, there exists $z_0 = z_0(\epsilon_0) > 0$ so that for $|z| \geq z_0$

$$\frac{1}{\sigma(\bar{m}(z))} - 1 > \sup_{s \in T} \mu(s) + \frac{\epsilon_0}{2}. \tag{8.5}$$

Set for $|z| \geq z_0$

$$A^0(s, z) = \frac{1}{\frac{1}{\sigma(\bar{m}(z))} - \mu(s)} = \frac{\sigma(\bar{m}(z))}{1 - \mu(s) \sigma(\bar{m}(z))}.$$

By (8.5) there exists $\epsilon_1 = \epsilon_1(\epsilon_0)$ so that for all $s \in T$

$$0 < A^0(s, z) < 1 - \epsilon_1, \quad |z| \geq z_0. \quad (8.6)$$

Set for $|z| \geq z_0$

$$A^\lambda(s, z) = \frac{1}{\frac{1}{\sigma(m_A(s, \lambda z))} - \mu(s)} = \frac{\sigma(m_A(s, \lambda z))}{1 - \mu(s)\sigma(m_A(s, \lambda z))}.$$

By (8.4) we have that

$$|A^\lambda(s, z) - A^0(s, z)| \leq C\lambda, \quad \forall s \in T. \quad (8.7)$$

Choose then $\lambda_0 = \lambda_0(\epsilon_0)$ small enough so that that for $\lambda \leq \lambda_0$, $z_0 < \frac{1}{2\lambda}$ and

$$|A^\lambda(s, z) - A^0(s, z)| \leq \frac{\epsilon_1}{2} \quad \forall s \in T. \quad (8.8)$$

By (8.6) and (8.8) we have that

$$A^\lambda(s, z) \leq A^0(s, z) + \frac{\epsilon_1}{2} < 1 - \frac{\epsilon_1}{2}, \quad \forall s \in T, \quad \forall |z| \geq z_0. \quad (8.9)$$

From (8.3)

$$\psi(s, z) = A^\lambda(s, z)(\bar{J} \star_z \psi)(s, z). \quad (8.10)$$

Suppose $z = z_0 + n$ where n is any integer such that $z_0 + 2n \leq \frac{1}{\lambda}$. Same can be done when $z < 0$ and by simple interpolation argument for any $z \in [z_0 + n, z_0 + n + 1)$. We have, see (8.10),

$$|\psi(s, z_0 + n)| \leq A^\lambda(s, z_0 + n)|(\bar{J} \star_z \psi)(s, z_0 + n)|. \quad (8.11)$$

We iterate n times (8.11). The support of n fold convolution is the interval $[z_0, z_0 + 2n]$. By (8.9) we obtain the following estimate

$$\begin{aligned} |\psi(s, z_0 + n)| &\leq [1 - \frac{\epsilon_1}{2}]^n |(\bar{J})^n \star_z \psi)(s, z_0 + n)| \\ &\leq [1 - \frac{\epsilon_1}{2}]^n \|(\bar{J})^n\|_2 \|\psi(s, \cdot)\|_2 = [1 - \frac{\epsilon_1}{2}]^n \|\bar{J}\|_2 = e^{-n\alpha(\epsilon_0)} \|\bar{J}\|_2 \end{aligned} \quad (8.12)$$

where, since $\epsilon_1 = \epsilon_1(\epsilon_0)$,

$$\alpha(\epsilon_0) = \log \frac{1}{[1 - \frac{\epsilon_1}{2}]} \quad (8.13)$$

The thesis follows. \square

Proof of Lemma 4.1 Take ξ and ξ' in $\mathcal{N}(d_0)$. Write in local variables $\xi = \gamma(s) + \nu(s)r$ and $\xi' = \gamma(s') + \nu(s')r'$. It is convenient to express the difference

$$\xi - \xi' = \gamma(s) + \nu(s)r - [\gamma(s') + \nu(s')r']$$

in term of $s^* = \frac{s+s'}{2}$ and $r^* = \frac{r+r'}{2}$, the middle points between s and s' and r and r' respectively. This allows to get some cancellations. We have

$$\begin{aligned} \gamma(s) &= \gamma(s^*) + \gamma'(s^*)(s - s^*) + \frac{1}{2}\gamma''(s^*)(s - s^*)^2 + \frac{1}{6}\gamma'''(\tilde{s})(s - s^*)^3, \\ \nu(s) &= \nu(s^*) + \nu'(s^*)(s - s^*) + \frac{1}{2}\nu''(s^*)(s - s^*)^2 + \frac{1}{6}\nu'''(\tilde{s})(s - s^*)^3 \end{aligned}$$

where $\tilde{s} \in (s, s')$ and it might change from one occurrence to the other. Similarly expressions hold for $\gamma(s')$ and $\nu(s')$. Since $s - s^* = \frac{s-s'}{2}$ and $s' - s^* = \frac{s'-s}{2}$ we have

$$\gamma(s) - \gamma(s') = \gamma'(s^*)(s - s') + \frac{1}{24}\gamma'''(\tilde{s})(s - s')^3.$$

Note that the term

$$\frac{1}{2}\gamma''(s^*)(s-s^*)^2 - \frac{1}{2}\gamma''(s^*)(s'-s^*)^2 = 0.$$

Further

$$\begin{aligned} \nu(s)r - \nu(s')r' &= \nu(s^*)[r - r'] + \nu'(s^*)(s - s')r^* \\ &\quad + \frac{1}{8}\nu''(s^*)(s - s')^2(r - r') + \frac{1}{24}\nu'''(\tilde{s})(s - s')^3r^*. \end{aligned} \quad (8.14)$$

Taking into account that $\nu'(s) = -k(s)\gamma'(s)$, we have

$$\begin{aligned} \xi - \xi' &= \gamma'(s^*)(s - s')[1 - k(s^*)r^*] + \nu(s^*)(r - r') \\ &\quad + \frac{1}{8}\nu''(s^*)(s - s')^2(r - r') \\ &\quad + \frac{1}{24}\nu'''(\tilde{s})(s - s')^3r^* + \frac{1}{24}\gamma'''(\tilde{s})(s - s')^3. \end{aligned} \quad (8.15)$$

Denote by a and b the following vectors

$$\begin{aligned} a &= \gamma'(s^*)(s - s')[1 - k(s^*)r^*] + \nu(s^*)(r - r'), \\ b &= \frac{1}{8}\nu''(s^*)(s - s')^2(r - r') + \frac{1}{24}[\nu'''(\tilde{s})(s - s')^3r^* + \gamma'''(\tilde{s})(s - s')^3]. \end{aligned}$$

It is important to notice that for $|s - s'| \leq \lambda$ and $|r - r'| \leq \lambda$ we have $|b| \leq C\lambda^3$. By Taylor expansion up to the second order of $J^\lambda(\cdot)$ we get that there exists $c^* \in \mathbb{R}^2$ so that

$$J^\lambda(\xi - \xi') = J^\lambda(a) + (\nabla J^\lambda)(a) \cdot b + \frac{1}{2}b \cdot (D^2 J^\lambda)(c^*) \cdot b, \quad (8.16)$$

where we denote by $(\nabla J^\lambda)(a)$ the gradient of J^λ computed in a and by $D^2 J^\lambda(c^*)$ the matrix of the second derivatives of $J^\lambda(\cdot)$ computed at c^* . Notice that $\|\nabla J^\lambda\|_\infty \leq C\lambda^{-d-1}$, since λ^{-d} comes from the normalization and λ^{-1} by differentiating one time, $\|D^2 J^\lambda\|_\infty \leq C\lambda^{-2-d}$, since λ^{-d} comes from the normalization and λ^{-2} by differentiating twice. When $|s - s'| \leq \lambda$ and $|r - r'| \leq \lambda$ we then obtain

$$\begin{aligned} |(\nabla J^\lambda)(a) \cdot b| &\leq C\lambda^{2-d} \\ |b \cdot (D^2 J^\lambda)(c^*) \cdot b| &\leq C\lambda^{4-d}. \end{aligned}$$

Define

$$\begin{aligned} J^\lambda(s, s', r, r') &= J^\lambda(a) = J^\lambda((s - s')[1 - k(s^*)r^*], (r - r')) \\ R_1^\lambda(s, s', r, r') &= (\nabla J^\lambda)(a) \cdot b \end{aligned} \quad (8.17)$$

$$R_2^\lambda(s, s', r, r') = \frac{1}{2}b \cdot (D^2 J^\lambda)(c^*) \cdot b. \quad (8.18)$$

Therefore, see (2.7), we obtain

$$\begin{aligned} \int_{\mathcal{N}(d_0)} J^\lambda(\xi - \xi')u(\xi')d\xi' &= \int_{\mathcal{T}} J^\lambda(s, s', r, r')u(s', r')\alpha(s', r')ds'dr' \\ &\quad + \int_{\mathcal{T}} R_1^\lambda(s, s', r, r')u(s', r')\alpha(s', r')ds'dr' \\ &\quad + \int_{\mathcal{T}} R_2^\lambda(s, s', r, r')u(s', r')\alpha(s', r')ds'dr', \end{aligned} \quad (8.19)$$

where

$$\begin{aligned} \left| \int_{\mathcal{T}} R_1^\lambda(s, s', r, r')\alpha(s', r')ds'dr' \right| &\leq C\lambda^2, \\ \left| \int_{\mathcal{T}} R_2^\lambda(s, s', r, r')\alpha(s', r')ds'dr' \right| &\leq C\lambda^4. \end{aligned}$$

□

REFERENCES

- [1] N. Alikakos, P.W. Bates, X Chen, *Convergence of the Cahn-Hilliard equation to the Hele-Shaw model*. Arch. Rat. Mech. Anal. **128**, 165–205 (1994).
- [2] N. Alikakos, G. Fusco *The spectrum of the Cahn-Hilliard operator for generic interface in higher space dimensions*. Indiana University Math. J. **42**, No2, 637–674 (1993).
- [3] E. A. Carlen, M. C. Carvalho, E. Orlandi, *Algebraic rate of decay for the excess free energy and stability of fronts for a non-local phase kinetics equation with a conservation law I* J. Stat. Phys. **95** N 5/6, 1069-1117 (1999)
- [4] E. A. Carlen, M. C. Carvalho, E. Orlandi *Algebraic rate of decay for the excess free energy and stability of fronts for a non-local phase kinetics equation with a conservation law, II* Comm. Par. Diff. Eq. **25** N 5/6, 847-886 (2000)
- [5] E. A. Carlen, M. C. Carvalho, E. Orlandi, *Approximate Solution of the Cahn-Hilliard Equation via Corrections to the Mullins-Sekerka Motion*, Arch. Rat. Mech. Anal., **178**, 1–55 (2005).
- [6] E. A. Carlen, E. Orlandi *Stability of planar fronts for a non-local phase kinetics equation with a conservation law in $d \geq 3$* Reviews in Mathematical Physics Vol. **24**, **4** (2012)
- [7] Xinfu Chen *Spectrum for the Allen- Cahn, Cahn-Hilliard, and phase-field equations for generic interfaces*. Comm. Partial Differential Equations **19**, No7-8, 1371–1395 (1994).
- [8] A. De Masi, E. Orlandi, E. Presutti, L. Triolo, *Stability of the interface in a model of phase separation* Proc.Royal Soc. Edinburgh **124A** 1013-1022 (1994).
- [9] O. Cevik, Z. Ismailov, *Spectrum of the direct sum of operators* Electron. J. Differential Equations, No. 210, 8 pp. (2012)
- [10] G. Giacomin, J. Lebowitz, *Phase segregation dynamics in particle systems with long range interactions I: macroscopic limits* J. Stat. Phys. **87**, 37–61 (1997).
- [11] G. Giacomin, J. Lebowitz, *Phase segregation dynamics in particle systems with long range interactions II: interface motions* SIAM J. Appl. Math. **58**, No 6, 1707–1729 (1998).
- [12] Tosio Kato *Perturbation Theory for Linear Operators* Springer-Verlag, Berlin Heidelberg New York 1980
- [13] J. Lebowitz , E.Orlandi and E. Presutti. *A particle model for spinodal decomposition* J. Stat. Phys. **63**, 933-974, (1991).
- [14] E. Orlandi , *Spectral properties of integral operators in bounded intervals* <http://arxiv.org/abs/1411.5221>
- [15] R.L. Pego, *Front migration in the nonlinear Cahn-Hilliard equation*. Proc. R. Soc. Lond.A **422**, 261–278 (1989).
- [16] E. Presutti, *Scaling Limits In Statistical Mechanics and Microstructures in Continuum Mechanics* Springer (2009)

ENZA ORLANDI, DIPARTIMENTO DI MATEMATICA E FISICA, UNIVERSITÀ DI ROMA TRE, L.GO S.MURIALDO 1, 00156 ROMA, ITALY.

E-mail address: orlandi@mat.uniroma3.it