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Glauber evolution with Kac potentials: III. Spinodal decomposition

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Abstract. This is the last of a series of papers on the Glauber dynamics of spin systems in $\mathbb{Z}^d$ with Kac potentials. It deals with phase separation, studying the evolution of an initial state which is a Bernoulli measure with zero average while the temperature of the Glauber dynamics is below the critical value. The state with 0 magnetization is then thermodynamically unstable and we prove that it is so also dynamically. In fact the stable phases, that have magnetization $\pm m_\beta$, develop into non-trivial patterns after times proportional to $\log \gamma^{-1}$, γ^{-1} the range of the Kac interaction. We characterize the typical spin configurations, both during the separation and when this is completed. In particular, we study the magnetization pattern at the boundaries of the clusters and the development of the interfaces.

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1. Introduction

This is the third and last paper of a series devoted to the Glauber dynamics of spin systems in $\mathbb{Z}^d$ interacting with Kac potentials. Here we study phase separation by considering the system initially in an equilibrium state at infinite temperature with 0 magnetization density, i.e. a Bernoulli measure with zero averages. We then let the system evolve with the Glauber dynamics at a temperature which is below the critical one (of the Lebowitz–Penrose theory, a statistical mechanics model for the van der Waals phase diagram, see [8]). The set-up is meant to model a quenching experiment where the system is rapidly cooled down to a temperature below the critical one, which is then kept fixed. At this temperature the phase with 0 magnetization is thermodynamically unstable, but stationary for the 'mesoscopic dynamics', i.e. the limit evolution when $\gamma \to 0$, recall that the interaction range of the Kac potential is γ^{-1} and that in the mesoscopic limit times are not scaled with γ, see again [8] for a discussion on the physical meaning of this and the other possible scaling limits.

The problem of phase separation is to determine whether and in the affirmative when and how, for each $\gamma > 0$, the true, stable phases develop. We give here a first, rough answer and in section 2 more precise and detailed statements which are proved afterwards.

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(i) *There is a sharp and non-random phase separation time (even though the phenomenon is due to a random fluctuation!)*. More precisely there are two times, t_c and $t^\star$, both dependent on γ, such that in $[0, t_c]$ the magnetization is still infinitesimal as $\gamma \to 0$, while at time $t^\star$ the phases are fully developed. t_c is proportional to $\log \gamma^{-1}$ by a constant fixed by the temperature and the interaction. $t^\star$ is 'very close' to t_c, in the sense that $(t^\star - t_c)/\log \gamma^{-1}$ vanishes as $\gamma \to 0$. Thus in the macro-time-scale $\log \gamma^{-1}$, t_c and $t^\star$ cannot be distinguished (as $\gamma \to 0$) hence the time when the phases separate is deterministic and sharp.

(ii) *A non-trivial spatial structure*. At time $t^\star$ the space is divided into 'large clusters' where the magnetization is alternatively equal to $\pm m_\beta$, the equilibrium magnetization at the given inverse temperature β (of the dynamics) in the Lebowitz–Penrose theory. The typical diameter of a cluster is $\gamma^{-1}[\log \gamma^{-1}]^{1/2}$, in lattice units.

(iii) *Existence of an interface*. There is a universal magnetization pattern $\bar{m}(s)$ at the boundaries between adjacent clusters of different phases, s a length parameter on a line normal to the boundary measured in units of the interaction length. $\bar{m}(s)$ solves the non-local mean-field equation

$$\bar{m}(s) = \tanh\{\beta \bar{J} \star \bar{m}(s)\} \qquad \lim_{s\to\pm\infty} \bar{m}(s) = \pm m_\beta \tag{1.1}$$

where $\bar{J}$ is related to the spin–spin interaction and is defined in the next section. Existence and uniqueness (modulo translations) for (1.1) are proven in [12]. Thus the magnetization pattern along the normal to the boundary between phases is the same for all clusters.

(iv) *Random geometry of the clusters*. The boundaries of the clusters at time $t^\star$ are the 0's of a distinguished Gaussian process. Its qualitative features, like the percolation probabilities, are to some extent known [19].

(v) *Predictability of the spatial patterns*. The actual positions of the clusters at time $t^\star$ are completely determined by the spin configuration at any earlier time, except time 0, when times are measured in the macroscopic scale $\log \gamma^{-1}$. (The statement becomes true with probability one in the limit as $\gamma \to 0$).

After time $t^\star$ the clusters should move by mean curvature (MBC), in the time-scale $\log \gamma^{-1}$ and when the space dimension is $d \geqslant 2$. The conditional tense is for the sake of precision, as there is a proof of such a statement for the motion of a single cluster, [8, 22], and the presence of the others should not change its behaviour, but there is no explicit proof in the literature. The MBC is consistent with theory and experimental observations [16], according to which the typical cluster size should grow like $t^{1/2}$ and this follows from the fact that the MBC is invariant under the diffusive scaling. At very long times, however, when the clusters are very large and the interface correspondingly flat, this moves very slowly by mean curvature so that the random fluctuations due to the intrinsic randomness of the microscopic evolution become competitive; at even longer times tunnelling effects with the appearence of new clusters of the opposite phase will no longer be negligible.

In $d = 1$ there is no MBC and we expect no significant change in the time-scale $\log \gamma^{-1}$. At times $\exp\{b[\log \gamma^{-1}]^{1/2}\}$ for some $b > 0$, the boundaries of the clusters will move significantly because of the mechanism described by [4, 15], for the Ginzburg–Landau equation that should apply to our case as well, (the velocity of the boundary depends on the exponential lengths of the neighbouring clusters which, according to (ii) above, are proportional to $[\log \gamma^{-1}]^{1/2}$, in units of interaction lengths). Thus shorter clusters disappear first and after due time the clusters which have survived are so long and the above mechanism so slow that fluctuations become competitive. Fluctuations are relevant at times of the order of γ^{-1}, transposing to this context results for the Ginzburg–Landau equation with noise [1, 2, 18]. These questions are currently under investigation as well as a characterization of the distribution of the clusters' lengths at times of the order γ^{-1}, when fluctuations take over. The true asymptotic limit, $t \to \infty$ with γ fixed (and small), is described by the Gibbs distribution (which is the only invariant measure for the Glauber dynamics). Typical Gibbs spin configurations have clusters of length $e^{c\gamma^{-1}}$, $c > 0$, thus a lot of phenomena have to occur after $t^\star$ before reaching the true Gibbsian equilibrium. The mentioned properties of the Gibbs measure and their other features may be found in [5].

The spinodal decomposition has already been studied for the Glauber+Kawasaki process which has been introduced in [6] to model reaction–diffusion equations: macroscopically finite volumes have been considered in [13], macroscopically infinite volumes in [14] (for $d = 1$) and [20] ($d = 2, 3$). Our results are both in qualitative and quantitative agreement with these papers, provided the parameters which determine the models are properly related.

In section 2 we recall the definition of the model, state the main results and outline the proofs. In section 3 we study the process in the time interval $[0, t_c]$ when the magnetization density, while growing, remains an infinitesimally small function of γ, proving the theorems stated in section 2.3. In section 4 we study the geometry of the interfaces (proving the theorems stated in section 2.4) and we report the proofs of some local central limit estimates used throughout the paper. In section 5 we study the development of the interfaces for the discretized ions non-local evolution equation and we describe the phase separation for the spin dynamics, thus proving the theorems stated in section 2.5.

2. Definitions and results

This section is divided into five subsections. In the first two we recall and adapt to the present case definitions from [8]. In the third one we analyse the system in the time interval $[0, t_c]$, in the fourth one the structure of the interfaces and in the fifth one the full development of the phases in the time interval $[t_c, t^\star]$.

2.1. Basic definitions

We start with the definition of the Glauber dynamics with Kac potential.

Definition 2.1.1a. *The Glauber dynamics.*
For any $\gamma \in (0, 1]$ the Glauber dynamics is the Markov process with state space $\{-1, 1\}^{\mathbb{Z}^d}$ and generator L_γ which acts on the cylinder functions f as

$$L_\gamma f(\sigma) = \sum_{x \in \mathbb{Z}^d} c(x, \sigma)[f(\sigma^x) - f(\sigma)] \qquad \sigma \in \mathbb{Z}^d \tag{2.1}$$

σ^x is obtained from σ by flipping the spin at x, i.e.

$$\sigma^x(y) = \begin{cases} \sigma(y) & \text{if} \quad y \neq x \\ -\sigma(x) & \text{if} \quad y = x \end{cases} \tag{2.2}$$

$$c_\gamma(x, \sigma) = \frac{e^{-\beta\sigma(x)h_\gamma(x,\sigma)}}{e^{-\beta h_\gamma(x,\sigma)} + e^{\beta h_\gamma(x,\sigma)}} = \frac{1}{2}\big[1 - \sigma(x)\tanh \beta h_\gamma(x, \sigma)\big] \tag{2.3}$$

$$h_\gamma(x, \sigma) = (J_\gamma \circ \sigma)(x) = \sum_{y \in \mathbb{Z}^d : y \neq x} J_\gamma(x, y)\sigma(y) \tag{2.4}$$

where $J_\gamma(x, y)$ is the Kac potential:

$$J_\gamma(x, y) = \gamma^d J(\gamma|x - y|)\,. \tag{2.5}$$

We suppose $J \geqslant 0$, $J \in C^\infty$, $J(|r|) = 0$ *for all* $r \geqslant 1$ *and, denoting by* β *the inverse temperature, we assume that*

$$\beta > \beta_{\rm crit} = 1 \qquad \int_{\mathbb{R}^d} {\rm d}r\, J(|r|) = 1\,. \tag{2.6}$$

We also assume that $J(|r|)$ *is a non-increasing function (this assumption is essential only in the proof of proposition 5.1.4).*

We finally fix the initial measure μ_0 *as the Bernoulli measure on* $\{-1,1\}^{\mathbb{Z}^d}$ *with 0 averages, i.e.* $\mathbb{E}_{\mu_0}(\sigma(x)) = 0$ *for all* $x \in \mathbb{Z}^d$*, and we denote by* μ_t^γ *the law of the process at time* t *starting from* μ_0.

Recall that the Lebowitz–Penrose inverse critical temperature $\beta_{\rm crit}$ is equal to 1 in our system. The relevant quantity in what follows is the magnetization density. After definition 2.1.5 of [8] we set

Definition 2.1.1b. *Block spins.*
For any function f *on* $\mathbb{Z}^d$ *we let*

$$\mathcal{A}_{\gamma,x,b_0}(f) = \frac{1}{|B_{\gamma,x,b_0}|} \sum_{y \in B_{\gamma,x,b_0}} f(y) \tag{2.7}$$

$$B_{\gamma,x,b_0} = \{y : |y-x| \leqslant \gamma^{-b_0}\} \qquad 0 < b_0 < 1\,. \tag{2.8}$$

The block spin magnetization at time $t \geqslant 0$ *is the expression in (2.7) with* $f = \sigma(\cdot,t)$*, the spin configuration at time* t.

As explained in [8], there are different time and space scales each one relevant for its corresponding phenomena.

Definition 2.1.2. *Space and time-scales.*
The microscopic scale is (x,t), $x \in \mathbb{Z}^d$ *and* $t \geqslant 0$*, i.e. spaces measured in lattice units and times proportionally to the spin flip unit.*

The mesoscopic scale is (r,t) *with* $t \geqslant 0$ *and*

$$r = \gamma x \qquad r \in \mathbb{R}^d\,.$$

Time is unchanged while space is measured in interaction range units, thus it is shrunk by γ *with respect to the micro-scale.*

The macroscopic scale is (ξ,τ) *with*

$$\xi = \lambda r \qquad \tau = \lambda^2 t \tag{2.9}$$

$$\lambda = (\log\gamma^{-1})^{-1/2}\,. \tag{2.10}$$

It is determined by the time when the phase separation occurs and by the size of the clusters of each phase.

We finally define the 'critical time' t_c *in meso and* τ_c *in macro units as*

$$\tau_c = \frac{d}{2\alpha} \qquad \alpha = \beta - 1 > 0 \qquad t_c = \tau_c \lambda^{-2} \tag{2.11}$$

and

$$t^\star = t_c + (\log\log\gamma^{-1})^2 \equiv \tau_c\lambda^{-2} + (\log\lambda^{-2})^2\,. \tag{2.12}$$

2.2. The mesoscopic regime

The mesoscopic behaviour of the model is described by a deterministic non-local equation. In [8] it has been proven that the block spin magnetization $\mathcal{A}_{\gamma,x}\big(\sigma(\cdot,t)\big)$, equations (2.7), (2.8), with $x = [\gamma^{-1}r]$, converges (in probability) as $\gamma \to 0$, to $m(r,t)$, where

$$\frac{\partial m}{\partial t}(r,t) = -m(r,t) + \tanh\big\{\beta(J \star m)(r,t)\big\} \tag{2.13}$$

$$(J \star m)(r,t) = \int \mathrm{d}r'\, J(|r-r'|)m(r',t) \tag{2.14}$$

under suitable assumptions on the initial datum. In our case $m(r,0) \equiv 0$ hence, in the mesoscopic regime, $m(r,t) \equiv 0$. To observe the phase separation we then need a more accurate analysis which takes into account deviations from the limit behaviour (2.13).

Also at the level of the mesoscopic equation, however, phase separation may still be observed, such as when we replace the initial limit magnetization $m(r,0) \equiv 0$ by the actual spin configurations, $m(r,0) \to \sigma([\gamma^{-1}r])$. The result yields a qualitatively but not quantitatively correct picture, as we shall see. The analysis at this point takes advantage of several similarities with the reaction–diffusion (Allen–Cahn) equation. To see the relation with the Allen–Cahn equation, we add and subtract to the right-hand side of (2.13) the term $\tanh \beta m$, giving

$$\frac{\partial m}{\partial t} = \mathcal{R}(m) + \mathcal{D}(m) \tag{2.15}$$

where

$$\mathcal{R}(m) = -m + \tanh \beta m \tag{2.16}$$

$$\mathcal{D}(m) = \tanh \beta(J \star m) - \tanh \beta m\,. \tag{2.17}$$

The equation (2.15) has to be compared to the Allen–Cahn equation

$$\frac{\partial m(r,t)}{\partial t} = \frac{\beta D}{2}\Delta m(r,t) - V'\big(m(r,t)\big) \qquad D = \int \mathrm{d}r\, J(|r|)r^2 \tag{2.18}$$

with $\mathcal{D}(m)$ playing the role of the diffusive term in (2.18) and $-V'(m) = \mathcal{R}(m)$. $V(m)$ is then a symmetric double-well potential whose minima are $\pm m_\beta$,

$$m_\beta = \tanh\{\beta m_\beta\} \tag{2.19}$$

namely the equilibrium magnetizations of the Lebowitz–Penrose theory. Notice that $m \equiv 0$ is also stationary, the stability properties are determined by

$$\mathcal{R}'(0) < 0 \qquad \text{if} \quad \beta < 1 \qquad \mathcal{R}'(0) = 0 \qquad \text{if} \quad \beta = 1 \tag{2.20}$$

$$\mathcal{R}'(0) > 0 \qquad \mathcal{R}'(\pm m_\beta) < 0 \qquad \beta > 1\,. \tag{2.21}$$

It is convenient to proceed with the lattice analogue of (2.13):

Definition 2.2.1. *The discretized equation.*
For any $\gamma > 0$ we denote by $m_\gamma(x,t)$, $x \in \mathbb{Z}^d$, $t \geqslant 0$, the solution of

$$\frac{\mathrm{d}m_\gamma(x,t)}{\mathrm{d}t} = -m_\gamma(x,t) + \tanh\{\beta(J_\gamma \circ m_\gamma)(x,t)\} \tag{2.22}$$

$$(J_\gamma \circ m_\gamma)(x,t) = \sum_y J_\gamma(x,y)m_\gamma(y,t)\,. \tag{2.23}$$

We denote by $m_\gamma(x,t|\sigma)$ the (unique) solution of (2.22) with

$$m_\gamma(x,0|\sigma) = \sigma(x)\,. \tag{2.24}$$

For any given $t_0 > 0$, *we also define* $m_{\gamma,t_0}(x,t|\sigma)$ *as the solution of (2.22) for* $t \geqslant t_0$ *with*

$$m_{\gamma,t_0}(x,t_0|\sigma) = \sigma(x)\,. \tag{2.25}$$

In the next subsection we study the statistical solutions of (2.22), namely the random variables $\{m_{\gamma,t_0}(\cdot,t|\sigma)\}$ solutions of (2.22) with the random initial datum σ having distribution μ_0. Loosely speaking therefore, each initial datum has a weight, given by μ_0, that it carries unchanged at all the later times, when it evolves according to (2.22). Collecting all such weights, we reconstruct, at any time t, a probability measure, which is the distribution of the variables $\{m_{\gamma,t_0}(\cdot,t|\sigma)\}$ and it is called a 'statistical solution of (2.22)'.

2.3. Evolution in the time interval $[0, t_c]$

We start with a heuristic analysis of the statistical solutions of (2.22). The initial datum, σ in (2.24), is obviously not close to $m \equiv 0$ in a L_∞ norm. But one may argue from (2.22) that the relevant quantity in the evolution is $(J_\gamma \circ m_\gamma)$ rather than m_γ. We then introduce the variable $u_\gamma(x,t) = J_\gamma \circ m_\gamma(x,t)$, $u_\gamma(\cdot,0) = (J_\gamma \circ \sigma)$. By the independence of the spins and a central limit theorem estimate, one finds out that the typical values of $(J_\gamma \circ \sigma)$ are of the order of $\gamma^{d/2}$. We thus expect that a good approximation to the true solution is obtained by solving the linearized equation around $u \equiv 0$, namely,

$$\frac{\mathrm{d}u_\gamma(x,t)}{\mathrm{d}t} = -u_\gamma(x,t) + \beta(J_\gamma \circ u_\gamma)(x,t) = \alpha_\gamma u_\gamma(x,t) + \beta(J_\gamma \circ u_\gamma)(x,t) - \beta \hat{J}_{\gamma,0} u_\gamma(x,t) \tag{2.26}$$

where

$$\alpha_\gamma = \beta \hat{J}_{\gamma,0} - 1 \quad \text{and } \hat{J}_{\gamma,0} = \sum_y J_\gamma(0,y)\,. \tag{2.27}$$

To write the solution to (2.26) explicitly, we set

Definition 2.3.1. *Let*

$$p_t^\gamma(x,y) = p_t^\gamma(0,x-y) \tag{2.28}$$

$$p_t^\gamma(0,x) = \mathrm{e}^{-c^\star t} \sum_{n=0}^{\infty} \frac{(\beta t)^n}{n!} \sum_{x_1,\ldots,x_{n-1}} J_\gamma(0,x_1)\cdots J_\gamma(x_{n-1},x) \qquad c^\star = \beta \hat{J}_{\gamma,0}\,. \tag{2.29}$$

We also set

$$q_t(r,r') = q_t(0,r-r') =: q_t(r-r') \tag{2.30}$$

$$q_t(r) = \mathrm{e}^{-\beta t} \sum_{n=1}^{\infty} \frac{(\beta t)^n}{n!} \int \mathrm{d}r_1 \ldots \mathrm{d}r_{n-1} \prod_{i=1}^{n} J(|r_i - r_{i-1}|) \qquad r_n = r, \quad r_0 = 0\,. \tag{2.31}$$

Observe that because $n \geqslant 1$, $q_t(r)$ is a smooth C^∞ function. For t large the contribution of $n = 0$ is negligible so that in this case and for $\gamma \to 0$, q_t approximates p_t^γ and this will be proven in section 4.3.

Going back to (2.26), we have

$$u_\gamma(x,t) = \mathrm{e}^{\alpha_\gamma t}(p_t^\gamma \circ u_\gamma)(x,0) \qquad (p_t^\gamma \circ u_\gamma)(x,0) = \sum_y p_t^\gamma(x,y) u_\gamma(y,0)\,. \tag{2.32}$$

If $u_\gamma(y,0) = \gamma^{d/2}$ identically, then $u_\gamma(\cdot,t)$ would be finite at t_c, the critical time of definition 2.1.2. The action of $p_t^\gamma(x,\cdot)$ in (2.32) on the true $u_\gamma(y,0)$ is essentially to average it over regions whose diameter is of the order of $\gamma^{-1}\sqrt{t}$. In fact, each jump covers a distance of the order γ^{-1}, the mean number of jumps goes like t and the jumps have an independent identical symmetric distribution. The average of $[\gamma^{-1}\sqrt{t}]^d$ independent spins goes typically like $[\gamma^{-1}\sqrt{t}]^{-d/2}$, hence, at t_c, u_γ is still infinitesimal, i.e. is of the order $t^{-d/4} = \lambda^{d/2}$ and $u_\gamma(x,t)$ varies significantly only when x varies over distances of the order $\gamma^{-1}\sqrt{t}$, i.e. the macro-scale $(\lambda\gamma)^{-1}$.

The approximation provided by the statistical solution gives the correct result, namely that the magnetization is infinitesimal until t_c:

Theorem 2.3.2. *For any $\xi \in \mathbb{R}^d$, and any $\tau \in [0, \tau_c]$ let*

$$\mathcal{M}_{\gamma,b_0}(\xi,\tau) = \mathcal{A}_{\gamma,x,b_0}(\sigma(,\cdot,\lambda^{-2}\tau)) \qquad x = [\lambda^{-1}\gamma^{-1}\xi]\,. \tag{2.33}$$

Then for any $b_0 < 1$, sufficiently close to 1, for any $\delta > 0$ and any $R > 0$

$$\lim_{\gamma\to 0} P^\gamma_{\mu_0}\Big(\sup_{\tau\leqslant\tau_c}\ \sup_{|\xi|\leqslant R} |\mathcal{M}_{\gamma,b_0}(\xi,\tau)| > \delta\Big) = 0\,. \tag{2.34}$$

Furthermore, for any integer $n \geqslant 1$ any $R > 0$, setting $R_\gamma = R(\lambda\gamma)^{-1}$,

$$\lim_{\gamma\to 0}\ \sup_{\substack{x_1\neq\ldots\neq x_n\\ |x_i|\leqslant R_\gamma}} \left|\mathbb{E}^\gamma_{\mu_0}\left(\prod_{i=1}^n \sigma(x_i,\lambda^{-2}\tau)\right)\right| = 0 \qquad \textit{for all} \quad 0\leqslant\tau\leqslant\tau_c\,. \tag{2.35}$$

The statistical solution of (2.22) provide a more accurate description of the system except for an initial time layer when they have the same order as the fluctuations.

In section 3 we will prove the following theorem:

Theororem 2.3.3. *There is $\delta > 0$ and given any $b_0 < 1$, sufficiently close to 1, any $\epsilon > 0$, $R > 0$ and any $\tau_0 \in (0,\tau_c)$, there are c and γ_0 so that the following holds. For any $\gamma \leqslant \gamma_0$ there is $\mathcal{G}^{(0)}_\gamma \subset \{-1,1\}^{\mathbb{Z}^d}$ so that*

$$\mathbb{P}^\gamma_{\mu_0}\big(\sigma(\cdot,\lambda^{-2}\tau_0) \in \mathcal{G}^{(0)}_\gamma\big) > 1-\epsilon \tag{2.36}$$

and for any $\sigma \in \mathcal{G}^{(0)}_\gamma$ and any $\tau_0 < \tau \leqslant \tau_c$

$$\sup_{\lambda\gamma|x|\leqslant R} |m_{\gamma,\lambda^{-2}\tau_0}(x,\lambda^{-2}\tau|\sigma)| \leqslant c\big[\lambda^{d/2}\gamma^{-\alpha\tau+d/2} + \mathrm{e}^{-\lambda^{-2}(\tau-\tau_0)}\big] \tag{2.37}$$

where $m_{\gamma,\lambda^{-2}\tau_0}$ is defined in definition 2.1.1. Moreover,

$$\mathbb{P}^\gamma_{\sigma,\lambda^{-2}\tau_0}\Big(\sup_{\lambda\gamma|x|\leqslant R} |\mathcal{A}_{\gamma,x,b_0}(\sigma(\cdot,\lambda^{-2}\tau) - m_{\gamma,\lambda^{-2}\tau_0}(\cdot,\lambda^{-2}\tau|\sigma)| \geqslant \gamma^{\delta-\alpha\tau+d/2}\Big) \leqslant \epsilon \tag{2.38}$$

where $\mathbb{P}^\gamma_{\sigma,t}$ is the law of the process which starts from σ at time t.

By equation (2.37), the order of magnitude of m_γ is the same as predicted by the linear theory. By equation (2.38) m_γ is a good approximation to the block spin averages since the error is much smaller than the magnitude of m_γ itself. The above arguments cannot be used past t_c as the solution will become finite so that the linear evolution is no longer a good approximation. However, the spatial pattern of the decomposition are already encoded in the typical configurations at any time $t = \lambda^{-2}\tau$, $\tau \leqslant \tau_c$, $\tau \neq 0$ as we shall see.

2.4. Interfaces

We begin by describing the interfaces in macroscopic variables.

Definition 2.4.1. *The set $\mathcal{U}$.*
The set $\mathcal{U}$ consists of all the functions $u(\xi)$, $\xi \in \mathbb{R}^d$, with values $\pm m_\beta$, such that the discontinuity set Σ of u intersects any spherical region into finitely many connected regular surfaces of codimension 1 and at finite distance from each other.

An interface is a connected component of Σ. If it is a closed surface, the region in its interior is called a cluster of the phase $+$, $[-]$, if in that region the value of $u(\xi)$ is $+m_\beta$, (respectively $-m_\beta$).

We will obtain elements u of $\mathcal{U}$ starting from the functions $m_\gamma(x, t|\sigma)$, as we are going to see.

Definition 2.4.2. *The function ℓ_γ.*
Given τ_0 and τ, $0 < \tau_0 < \tau < \tau_c$, for any $\sigma \in \{-1, 1\}^{\mathbb{Z}^d}$, any $r \in \mathbb{R}^d$ and any $\gamma > 0$, we define

$$\ell_\gamma(r|\sigma) = \lambda^{-d/2}\gamma^{-(\tau_c-\tau)\alpha} \int \mathrm{d}r'\, q_{\lambda^{-2}(\tau_c-\tau)}(r - r')m_{\gamma,\lambda^{-2}\tau_0}([r'\gamma^{-1}], \lambda^{-2}\tau|\sigma) \tag{2.39}$$

where $q_t(r)$ is defined in (2.30), (2.31) and m_{γ,τ_0} in (2.25).

Recall that $q_t(r)$ is a smooth function so that $\ell_\gamma \in C^\infty$. Actually it is convenient to have ℓ_γ expressed in macro-variables:

$$\hat{\ell}_\gamma(\xi|\sigma) = \ell_\gamma\big(\xi\lambda^{-1}|\sigma\big)\,. \tag{2.40}$$

Let Λ be an open bounded set in $\mathbb{R}^d$ and $C^n(\Lambda)$ be the space of functions which have n bounded derivatives in Λ, equipped with the sup norm for the function and its first n derivatives. We then define for any n the probability $\mathcal{P}_{\gamma,\tau_0}$ on $C^n(\Lambda)$ which is the image via the map (2.39) of $\mu_{\lambda^{-2}\tau_0}$. Observe that $\mathcal{P}_{\gamma,\tau_0}$ is supported on the intersection over n of all $C^n(\Lambda)$.

Theorem 2.4.3. *For any $\epsilon > 0$, there is γ_0 such that the following holds. For all $\gamma < \gamma_0$ there is a set $G_\gamma^{(1)} \subset \{-1, 1\}^{\mathbb{Z}^d}$ such that $\mu_{\lambda^{-2}\tau_0}(G_\gamma^{(1)}) > 1 - \epsilon$ and for $\sigma \in G_\gamma^{(1)}$ the function $u(\xi) := m_\beta$ sign $\{\hat{\ell}_\gamma(\xi|\sigma)\}$ (set equal to m_β when $\hat{\ell}_\gamma = 0$) belongs to the set $\mathcal{U}$. Furthermore, for any n and any bounded regular set Λ, the probabilities $(C^n(\Lambda), \mathcal{P}_{\gamma,\tau_0})$ converge to $(C^n(\Lambda), \mathcal{P})$ where $\mathcal{P}$ is the law of the Gaussian process with 0 average and covariance*

$$\mathcal{E}\big(X(\xi)X(\xi')\big) = \left(1 + \frac{1}{\alpha}\right) \frac{\alpha^{d/2}}{(\pi d\beta D)^{d/2}}\, \mathrm{e}^{-\alpha(\xi-\xi')^2/(d\beta D)} \qquad \textit{for any} \quad \xi, \xi' \in \mathbb{R}^d \tag{2.41}$$

$$D = \int \mathrm{d}r\, J(|r|)r^2\,. \tag{2.42}$$

The relation between the interface described by the function $u(\xi)$ of theorem 2.4.3 and the actual spin configuration is established in the next subsection.

2.5. Evolution in the time interval $[t_c, t^\star]$

We will prove that for any $\tau_0 < \tau_c$ there is a set of good configurations which has large probability and such that if we start from a good configuration at time $\lambda^{-2}\tau_0$, then the block spin magnetization at any later times $t \in [\tau_0\lambda^{-2}, t^\star]$ is close to m_{γ,τ_0} defined in definition 2.2.1.

Theorem 2.5.1. *For any $\epsilon > 0$, $b_0 < 1$, sufficiently close to 1, $R > 0$, $0 < \tau_0 < \tau_c$ there is γ_0 such that the following holds. For all $\gamma < \gamma_0$ there is a set $\mathcal{G}_\gamma^{(2)} \subset \{-1,1\}^{\mathbb{Z}^d}$ such that*

$$\mathbb{P}^\gamma_{\mu_0}\big(\sigma(\cdot,\lambda^{-2}\tau_0) \in \mathcal{G}_\gamma^{(2)}\big) > 1 - \epsilon$$

and if $\sigma \in \mathcal{G}_\gamma^{(2)}$ then there is $\delta > 0$ such that

$$\sup_{\tau_c\lambda^{-2}\leqslant t\leqslant t^\star} \mathbb{P}^\gamma_{\sigma,\lambda^{-2}\tau_0}\left(\left\{\sup_{|x|\leqslant R(\lambda\gamma)^{-1}} \mathcal{A}_{\gamma,x,b_0}\big(\sigma(\cdot,t) - m_{\gamma,\lambda^{-2}\tau_0}(\cdot,t|\sigma)\big) \geqslant \gamma^\delta\right\}\right) < \epsilon\,. \tag{2.43}$$

We are then left with the study of the solutions of (2.22) with a random value σ at time $\tau_0\lambda^{-2}$. Theorem 2.5.3 below states that $m_{\gamma,\lambda^{-2}\tau_0}(y,t^\star|\sigma)$ is close to $m_\beta\,\mathrm{sign}\hat{\ell}(\gamma\lambda x|\sigma)$ if x is sufficiently far from the interface (see definition 2.4.1). The values of m_{γ,τ_0} close to the interfaces are also known. We describe them by means of the following definition.

Definition 2.5.2. *The instanton.*
The instanton is an antisymmetric and non-identically zero solution of the one-dimensional equation (1.1) with

$$\bar{J}(s) = \int_{\mathbb{R}^{d-1}} \mathrm{d}r\; J\big((s^2 + r^2)^{1/2}\big)\,. \tag{2.44}$$

It is proven in [12] that the instanton exists (provided $\beta > 1$) and that it is unique in the class of functions that are asymptotically strictly positive, (negative), as $s \to \infty$, ($s \to -\infty$) and that vanish at the origin. Moreover it is strictly increasing and with asymptotic values at $\pm\infty$ equal to $\pm m_\beta$, to which it converges exponentially fast.

Finally, for any unit vector ν in $\mathbb{R}^d$, $m(r) := \bar{m}(r\nu)$ solves the equation

$$m(r) = \tanh\{\beta(J \star m(r)\} \qquad \textit{for all} \quad r \in \mathbb{R}^d\,. \tag{2.45}$$

In [12] it is also proven that in $d = 1$, the manifold of translations of the instanton is globally stable in the same class of functions where the uniqueness of the instanton is proven.

Theorem 2.5.3. *For any $\epsilon > 0$, $L > 0$ and $0 < \tau_0 < \tau_c$ there is γ_0 such that the following holds. For all $\gamma < \gamma_0$ there is a set $\mathcal{G}_\gamma^{(3)} \subset \{-1,1\}^{\mathbb{Z}^d}$ such that*

$$\mathbb{P}^\gamma_{\mu_0}\big(\sigma(\cdot,\lambda^{-2}\tau_0) \in \mathcal{G}_\gamma^{(3)}\big) > 1 - \epsilon \tag{2.46}$$

and the following holds.

There is a positive function R_γ, $\gamma \leqslant \gamma_0$ so that

$$\lim_{\gamma\to 0} R_\gamma = 0 \qquad \lim_{\gamma\to 0} \lambda^{-1} R_\gamma = \infty\,.$$

For any $\sigma \in \mathcal{G}_\gamma^{(3)}$,

$$u_\gamma(\xi) := m_\beta \;\mathrm{sign}\; \hat{\ell}(\xi|\sigma) \in \mathcal{U}\,.$$

Let then Σ be the interface of u_γ and $|d(\xi,\Sigma)|$ the distance of ξ from Σ. Then

$$|m_{\gamma,\lambda^{-2}\tau_0}(x,t^\star|\sigma) - u_\gamma(\gamma\lambda x)| \leqslant \epsilon \tag{2.47}$$

for all $\gamma \leqslant \gamma_0$ and all $|x| \leqslant L(\lambda\gamma)^{-1}$ and such that $|d(x\lambda\gamma,\Sigma)| \geqslant R_\gamma$.

Moreover, for any $\xi_0 \in \Sigma$, $|\xi_0| < L$, let ν be the unit vector normal to Σ at ξ_0 and pointing toward the region where $u(\xi) = m_\beta$. Then for all $\gamma < \gamma_0$,

$$\big|m_{\gamma,\lambda^{-2}\tau_0}\big([(\xi_0\lambda^{-1} + \nu s)\gamma^{-1}], t^\star|\sigma\big) - \bar{m}\big(s\big)\big| \leqslant \epsilon \qquad \textit{for all} \quad |s| \leqslant R_\gamma\lambda^{-1} \tag{2.48}$$

where $\bar{m}$ is the instanton solution, see definition 2.5.2.

3. The early stage of the decomposition

In this section we study the process until time $t_c = \lambda^{-2}\tau_c$ (see equation (2.11)). In particular, in section 3.1 we introduce the main definitions and notation. In section 3.2 we study the statistical solutions of (2.22). In section 3.3 we prove bounds on the v-functions and in section 3.4 that the process evolves deterministically according to (2.22) if it starts at some 'positive time' from a spin configuration in a 'nice set'. In section 3.5 we prove that this 'nice set' has a probability that goes to 1 as $\gamma \to 0$. At the end of the section we prove theorems 2.3.2 and 2.3.3, as a corollary of the previous analysis.

3.1. The time grid and the seminorms

In this subsection we state the main definitions and notation.

Definition 3.1.1. *The time grid.*
Given $a \in (0, \tau_c)$ we denote by N the smallest integer such that $(N+1)a \geqslant \tau_c$. We also set

$$t_{na} = na\lambda^{-2} \qquad \lambda^{-2} = \log \gamma^{-1}\,. \tag{3.1}$$

Definition 3.1.2. *The seminorms. We define $\psi : \mathbb{R}^d \to \mathbb{R}_+$ and, given $\gamma > 0$, $\phi_\gamma : \mathbb{R}^d \to \mathbb{R}_+$, as*

$$\psi(\xi) = (1 + |\xi|)^{-b} \qquad 0 < b < \tfrac{1}{160} \qquad \phi_\gamma(x) = \psi(\lambda\gamma x)\,. \tag{3.2}$$

We also set, for $h > 0$,

$$R_{h,\gamma} = \gamma^{-1}\big[\lambda^{-16} - h\lambda^{-8}\big] \tag{3.3}$$

$$\phi_{h,\gamma}(x) = \phi_\gamma(x)\mathbf{1}(|x| \leqslant R_{h,\gamma}) \tag{3.4}$$

and, for $a \in (0, \tau_c)$, with N as in definition 3.1.1, we then define for any $\zeta > 0$, $n \leqslant N+1$, $h \geqslant n$, and $f : \mathbb{Z}^d \to \mathbb{R}$

$$\|f\|_{h,n,\gamma,a,\zeta} = [\gamma^{-\zeta-\alpha a n + d/2}]^{-1} \sup_x \big\{\phi_{h,\gamma}(x)\big|(J_\gamma \circ f)(x)\big|\big\}\,. \tag{3.5}$$

We finally set

$$\pi^\gamma(x, y) = \mathrm{e}^{-c^\star t_a} \sum_{n \geqslant 1} \frac{(\beta t_a)^n}{n!} J_\gamma^n(x, y) \tag{3.6}$$

$$c^\star = \beta \hat{J}_{\gamma,0} = 1 + \alpha_\gamma \qquad J_\gamma^n = J_\gamma \circ \ldots \circ J_\gamma \qquad n\text{-times}\,. \tag{3.7}$$

See equation (2.27) for notation, and define, for n and h as above,

$$|||f|||_{h,n,\gamma,a} = [\lambda^{d/2}\gamma^{-\alpha a n + d/2}]^{-1} \sup_x \big\{\phi_{h,\gamma}(x)\big|(\pi^\gamma \circ f)(x)\big|\big\} \tag{3.8}$$

$$F_{\gamma,a,\zeta}(h, n, C) = \big\{f : \mathbb{Z}^d \to [-1, 1] \big| \|f\|_{h,n,\gamma,a,\zeta} \leqslant C\ ;\ |||f|||_{h,n,\gamma,a} \leqslant C\big\}\,. \tag{3.9}$$

For notational simplicity in the sequel σ also denotes a function on $\mathbb{Z}^d$ with values in $[-1, 1]$.

Recalling the definition of p_t^γ, (2.28) and (2.29), we have

$$p_{t_a}^\gamma(x, y) = \pi^\gamma(x, y) \qquad \text{if} \quad x \neq y\,.$$

In section 4 we will prove the following:

Lemma 3.1.3.
For all h, k, m positive there is c so that for all $\gamma > 0$

$$\sup_{t \leqslant 2t_c} \sum_{|x| \geqslant h\lambda^{-8}} p_t^\gamma(0, x) < c\gamma^k \tag{3.10}$$

$$\sup_{t \leqslant 2t_c} (1 + \gamma\lambda|x|)^{-m} \sum_y p_t^\gamma(x, y)(1 + \gamma\lambda|y|)^m \leqslant c\,. \tag{3.11}$$

We will next use lemma 3.1.3 to establish a relation between the $\|\cdot\|$ and $|||\cdot|||$ seminorms. We first introduce the parameter c_1, often used in the following:

$$c_1 = \sup_{\gamma, x, t \leqslant 2t_c} \sum_y p_t^\gamma(x, y) \left[1 + \frac{\phi_\gamma(x)}{\phi_\gamma(y)} + \sum_z J_\gamma(y, z) \frac{\phi_\gamma(x)^2}{\phi_\gamma(z)^2} \right] < \infty\,. \tag{3.12}$$

which is finite because of (3.11).

Lemma 3.1.4.
Let $a \in (0, \tau_c)$, N as in definition 3.1.1, $\zeta > 0$, $0 \leqslant n \leqslant N$, $h \in (n, n+1)$, $k > 0$. Then there is c so that for all $\gamma > 0$ and all $\sigma : \mathbb{Z}^d \to [-1, 1]$,

$$|||\sigma|||_{h,n,\gamma,a} \leqslant \left\{ \beta a c_1 \gamma^{-\zeta} \lambda^{-d/2-2} \|\sigma\|_{n,n,\gamma,a,\zeta} + c\gamma^k \right\} \tag{3.13}$$

and, for $t_{na} \leqslant t \leqslant t_{Na}$,

$$\phi_{h,\gamma}(x) \left| (p_{t-t_{na}}^\gamma \circ \sigma)(x) - \mathrm{e}^{-c^\star(t-t_{na})} \sigma(x) \right| \leqslant \mathrm{e}^{\alpha t_{na}} \gamma^{d/2-\zeta} \beta a c_1 \lambda^{-2} \|\sigma\|_{n,n,\gamma,a,\zeta} + c\gamma^k\,. \tag{3.14}$$

Finally, for $t_{na} \leqslant t \leqslant t_c$, calling q the integer such that $t_{qa} \leqslant t < t_{(q+1)a}$, for all γ small enough (how small depending on a),

$$\begin{aligned} \phi_{h,\gamma}(x) \left| \mathrm{e}^{\alpha_\gamma(t-t_{na})} (p_{t-t_{na}}^\gamma \circ \sigma)(x) - \mathrm{e}^{-(t-t_{na})} \sigma(x) \right| &\leqslant 2c_1 \mathrm{e}^{\alpha t} \gamma^{d/2} \lambda^{d/2} |||\sigma|||_{n,n,\gamma,a} \\ &+ \mathbf{1}_{\{t_{qa} < t\}} 2\, \mathrm{e}^{-c^\star(t_{qa}-t_{na})} \mathrm{e}^{\alpha t} \gamma^{d/2-\zeta} \beta a c_1 \lambda^{-2} \|\sigma\|_{n,n,\gamma,a,\zeta} + c\gamma^k\,. \end{aligned} \tag{3.15}$$

Proof. Let $|x| \leqslant R_{h,\gamma}$, then

$$(\pi^\gamma \circ \sigma)(x) = \mathrm{e}^{-c^\star t_a} \sum_{k \geqslant 0} \frac{(\beta t_a)^{k+1}}{(k+1)!} \sum_z J_\gamma^k(x, z)(J_\gamma \circ \sigma)(z)$$

$$\begin{aligned} [\lambda^{d/2} \gamma^{-\alpha a n + d/2}]^{-1} \phi_{h,\gamma}(x) \left| (\pi^\gamma \circ \sigma)(x) \right| &\leqslant [\lambda^{d/2} \gamma^{-\alpha a n + d/2}]^{-1} c^\star t_a \phi_{h,\gamma}(x) \sum_{|z| \geqslant R_{n,\gamma}} p_{t_a}^\gamma(x, z) \\ &+ (\beta t_a) \phi_{h,\gamma}(x) \sum_{|z| \leqslant R_{n,\gamma}} p_{t_a}^\gamma(x, z) \phi_{n,\gamma}(z)^{-1} \gamma^{-\zeta} \lambda^{-d/2} \|\sigma\|_{n,n,\gamma,a,\zeta}\,. \end{aligned}$$

The first term on the right-hand side is bounded using (3.10), then using (3.12) we derive (3.13). The proof of (3.14) is completely analogous and it is omitted.

We write

$$\begin{aligned} (p_{t-t_{na}}^\gamma \circ \sigma)(x) &= \mathrm{e}^{-c^\star(t-t_{na})} \sigma(x) + \sum_{i=n+1}^{q} \mathrm{e}^{-c^\star(t_{(i-1)a}-t_{na})} \sum_y p_{t-t_{ia}}^\gamma(x, y)(\pi^\gamma \circ \sigma)(y) \\ &+ \mathrm{e}^{-c^\star(t_{qa}-t_{na})} \left[\sum_y p_{t-t_{qa}}^\gamma(x, y)\sigma(y) - \mathrm{e}^{-c^\star(t-t_{qa})} \sigma(x) \right]. \end{aligned}$$

We use that $\exp\{\alpha_\gamma((t-t_{na}) - c^\star(t-t_{na})\} = \exp\{-(t-t_{na})\}$. Observing that $|\alpha_\gamma - \alpha| \leqslant c\gamma$ and using (3.14) we obtain (3.15), the proof of the lemma is then completed. □

3.2. Solutions of (2.22) for $t \leqslant t_c$

The main result in this subsection concerns the functions $m_{\gamma,t_0}(x,t|\sigma)$. Recall that they solve (2.22) for $t \geqslant t_0$ with the condition $m_{\gamma,t_0}(x,t_0|\sigma) = \sigma(x)$ for all $x \in \mathbb{Z}^d$.

Theorem 3.2.0. *For any $a \in (0,\tau_c)$, any ζ that satisfies (3.21) below and any C positive there is c so that the following holds. For all $n < N$, N as in definition 3.1.1, all $h > n$, all $\gamma > 0$, all $|x| \leqslant R_{h,\gamma}$ all $\sigma \in F_{\gamma,a,\zeta}(n,n,C)$ and all $t_{na} \leqslant t \leqslant t_{Na}$*

$$\left|m_{\gamma,t_{na}}(x,t|\sigma) - \mathrm{e}^{\alpha_\gamma(t-t_{na})}(p^\gamma_{t-t_{na}} \circ \sigma)(x)\right| \leqslant c\left(\mathrm{e}^{\alpha t}\gamma^{d/2-\zeta}\phi_\gamma(x)^{-1}\right)^2 \qquad (3.16)$$

while for $t_{(n+1)a} < t \leqslant t_c$:

$$\left|m_{\gamma,t_{na}}(x,t|\sigma) - \mathrm{e}^{\alpha_\gamma(t-t_{na})}(p^\gamma_{t-t_{na}} \circ \sigma)(x)\right| \leqslant c\,\mathrm{e}^{\alpha t}\gamma^{d/2}\left(\lambda^{3d/2}\phi_\gamma(x)^{-1}\right). \qquad (3.17)$$

Finally, as a consequence of (3.15), for $t_{na} \leqslant t \leqslant t_c$

$$\left|\mathrm{e}^{\alpha(t-t_{na})}(p^\gamma_{t-t_{na}} \circ \sigma)(x)\right| \leqslant c\Big[\mathrm{e}^{\alpha t}\gamma^{d/2}\lambda^{d/2}\phi_\gamma(x)^{-1} + \mathrm{e}^{-(t-t_{na})} + \mathrm{e}^{\alpha t}\gamma^{d/2-\zeta}\phi_\gamma(x)^{-1}\mathbf{1}_{t_{na}\leqslant t<t_{(n+1)a}}\Big] \qquad (3.18)$$

and for any $n+1 \leqslant q \leqslant N$, $h \geqslant q$:

$$|||m_{\gamma,t_{na}}(\cdot,t_{qa}|\sigma)|||_{h,q,\gamma,a} \leqslant c\,. \qquad (3.19)$$

As an application of theorem 3.2.0, let $\tau_0 \in [0,\tau_c)$ and set $t_0 = \lambda^{-2}\tau_0$. Then $m_{\gamma,t_0}(x,t|\sigma)$, $t_0 \leqslant t \leqslant t_c$ satisfies (2.37) if given a and n so that $na = \tau_0$, there is ζ so that (3.21) below holds and

$$\mathrm{e}^{\alpha t}\,\mathrm{e}^{\alpha t_0}\gamma^{d/2-\zeta} \leqslant \mathrm{e}^{-t} \qquad \textit{for all} \quad 0 \leqslant t \leqslant t_a \qquad (3.20)$$

is satisfied; moreover, σ should be in $F_{\gamma,a,\zeta}(n,n,C)$.

In such a case in fact the third term in (3.18) is bounded by the second one and (2.37) follows from (3.16) and (3.18).

To find a and ζ for which (3.20) holds, we observe that, by the definition of τ_c,

$$\gamma^{-\alpha\tau_0}\gamma^{d/2} = \gamma^{\alpha(\tau_c-\tau_0)}\,.$$

Then equation (3.20) is implied by

$$\gamma^{\alpha(\tau_c-\tau_0)} \leqslant \gamma^{(\alpha+1)a+\zeta}$$

which is satisfied if a and ζ are small enough.

When proving theorem 2.3.3 we will choose $\mathcal{G}^{(0)}_\gamma \subset F_{\gamma,a,\zeta}$ and show that the probability of the latter goes to 1 as $\gamma \to 0$.

Hereafter we require that, given $a \in (0,\tau_c)$ and N as in definition 3.1.1, ζ should be restricted to

$$0 < 2\zeta < \min\left\{a, \frac{d}{2} - \alpha a N\right\}. \qquad (3.21)$$

The definition is non-empty because $d/2 > \alpha a N > 0$ since

$$\alpha a N < \tau_c \text{ and } \gamma^{d/2-\alpha\tau_c} = 1\,.$$

Before beginning the proof of theorem 3.2.0 we state some basic properties of the evolution (2.22) that will be extensively used throughout the paper.

Lemma 3.2.1.

(i) *Barrier lemma. Given any* $\delta > 0$ *there are* c *and* c' *so that the following holds. Given any* $\gamma > 0$ *and* $T > 0$*, let* $u_\gamma(x,t)$ *and* $v_\gamma(x,t)$ *be two solutions of (2.22) for* $t \geqslant 0$ *such that*

$$|u_\gamma(x,0)| \leqslant 1 \qquad |v_\gamma(x,0)| \leqslant 1 \qquad \textit{for all} \quad x \tag{3.22}$$

$$u_\gamma(x,0) = v_\gamma(x,0) \qquad \textit{for all} \quad |x| \leqslant c\gamma^{-1}T\,. \tag{3.23}$$

Then

$$|u_\gamma(0,T) - v_\gamma(0,T)| \leqslant c'\,\mathrm{e}^{-\delta T}\,. \tag{3.24}$$

(ii) *Monotonicity. If* $u_\gamma(x,t)$ *and* $v_\gamma(x,t)$ *are solutions of (2.22) and* $u_\gamma(x,0) \geqslant v_\gamma(x,0)$ *for all* x*, then* $u_\gamma(x,t) \geqslant v_\gamma(x,t)$ *for all* x *and all* $t \geqslant 0$*.*

(iii) *If* $u_\gamma(x,t)$ *is a solution of (2.22) and* $u_\gamma(x,0) \leqslant 1$ *for all* x*, then* $u_\gamma(x,t) \leqslant 1$ *for all* x *and all* $t \geqslant 0$*.*

(iv) *Let* δ*,* c *and* c' *be as (1) and* u *and* v *be two solutions of (2.22) which verify (3.22). Suppose that*

$$u_\gamma(x,0) \geqslant v_\gamma(x,0) \qquad \textit{for all } x \textit{ such that } |x| \leqslant cT\gamma^{-1} \tag{3.25}$$

then

$$u_\gamma(0,T) \geqslant v_\gamma(0,T) - c'\,\mathrm{e}^{-\delta T} \tag{3.26}$$

(v) *The statements (i)–(iv) are valid also for the equation in the continuum (2.13).*

Proof. The proof of (v), that is the statements (i)–(iv) for the continuum equation (2.13) are given in [12]. The proofs of (i)–(iv) for (2.22) are completely analogous, in particular (iii) and (iv) follows from (i) and (ii). For completeness we report below the proof of (i).

Denote by

$$D_\gamma(x,t) = |u_\gamma(x,t) - v_\gamma(x,t)|$$

then, from (2.22) and the fact that $(\cosh \beta z)^{-2} \leqslant 1$ for all z it follows that

$$D_\gamma(x,t) \leqslant D_\gamma(x,0) + \int_0^t \mathrm{d}s \big\{ D_\gamma(x,s) + \beta (J_\gamma \circ D_\gamma)(x,s) \big\}\,.$$

Calling

$$d_\gamma(x,t) = \mathrm{e}^{-t} D_\gamma(x,t)$$

we then have

$$d_\gamma(x,t) \leqslant d_\gamma(x,0) + \int_0^t \mathrm{d}s\, \beta (J_\gamma \circ d_\gamma)(x,s) \leqslant \sum_n \frac{(\beta t)^n}{n!} (J_\gamma^n \circ d_\gamma)(x,0)$$

where for any function g,

$$(J_\gamma^n \circ g)(x) = \sum_{x_1,\dots,x_n} J_\gamma(x,x_1)\dots J_\gamma(x_{n-1},x_n) g(x_n)\,.$$

We now use two facts: the first one is that $D_\gamma(x,0) = 0$ for $|x| \leqslant c\gamma^{-1}T$, and $D_\gamma(x,0) \leqslant 2$ elsewhere, the second is that $J_\gamma(x,y) = 0$ if $|x-y| > \gamma^{-1}$. Then there are c and c' so that

$$D_\gamma(0,T) \leqslant 2 \sum_{n \geqslant cT} \frac{(\beta T \hat{J}_{\gamma,0})^n}{n!} \mathrm{e}^T \leqslant c'\,\mathrm{e}^{-\delta T}\,. \qquad \square$$

Given $t_0 \geqslant 0$, $\sigma : \mathbb{Z}^d \to [-1, 1]$, $\gamma > 0$, we define

$$\tilde{m}_{\gamma,t_0}(x, t|\sigma) = \sum_y J_\gamma(x, y) m_{\gamma,t_0}(y, t|\sigma) \tag{3.27}$$

and state the following lemma:

Lemma 3.2.2.
For any $a \in (0, \tau_c)$, ζ that satisfies (3.21), $0 \leqslant n < N$ and any $C > 0$, $k > 0$, $h > n$ there are $\gamma_0 > 0$ and c so that for all $\gamma \leqslant \gamma_0$, all $\sigma : \mathbb{Z}^d \to [-1, 1]$ such that $\|\sigma\|_{n,n,\gamma,a,\zeta} \leqslant C$ and all $t_{na} \leqslant t \leqslant t_{Na}$:

$$\gamma^{-d/2+\zeta} \phi_{h,\gamma}(x) |\tilde{m}_{\gamma,t_{na}}(x, t|\sigma)| \leqslant 3c_1 \, \mathrm{e}^{\alpha t} \|\sigma\|_{n,n,\gamma,a,\zeta} + c\gamma^k \tag{3.28}$$

where c_1 is defined in (3.12) and $\tilde{m}$ in (3.27) with $t_0 = t_{na}$. Moreover, if σ' satisfies the same bound as σ:

$$\gamma^{-d/2+\zeta} \phi_{h,\gamma}(x) \left|\tilde{m}_{\gamma,t_{na}}(x, t|\sigma) - \tilde{m}_{\gamma,t_{na}}(x, t|\sigma')\right| \leqslant 3c_1 \, \mathrm{e}^{\alpha t} \|\sigma - \sigma'\|_{n,n,\gamma,a,\zeta} + c\gamma^k . \tag{3.29}$$

Proof. As a shorthand we use $\tilde{m}(x, t)$ for $\tilde{m}_{\gamma,t_{na}}(x, t|\sigma)$, then

$$\frac{\mathrm{d}\tilde{m}}{\mathrm{d}t} = -\tilde{m} + J_\gamma \circ \tanh(\beta \tilde{m}) . \tag{3.30}$$

We use (1) of lemma 3.2.1 (valid also for (3.30)) to conclude that the solution $\hat{m}(x, t)$, $t \geqslant t_{na}$, of (3.30) with initial condition

$$\hat{m}(x, t_{na}) = \tilde{m}(x, t_{na}) \mathbf{1}(|x| \leqslant R_{n,\gamma}) \tag{3.31}$$

verifies the following. For any k there is c_k so that for all $t_{na} \leqslant t \leqslant t_{Na}$ and all $h > n$

$$\left|\tilde{m}(x, t) - \hat{m}(x, t)\right| \leqslant c_k \gamma^k \qquad \text{for all} \quad |x| \leqslant R_{h,\gamma} . \tag{3.32}$$

It will suffice to prove (3.28) for $\hat{m}$.

After some simple algebra,

$$\frac{\mathrm{d}\hat{m}(x, t)}{\mathrm{d}t} = \alpha_\gamma \hat{m}(x, t) + \beta[(J_\gamma \circ \hat{m})(x, t) - \hat{J}_{\gamma,0} \hat{m}(x, t)] + (J_\gamma \circ \Theta)(x, t) \tag{3.33}$$

with α_γ defined in (2.27) and

$$\Theta(x, t) = \tanh\{\beta \hat{m}(x, t)\} - \beta \hat{m}(x, t) . \tag{3.34}$$

We observe that

$$0 \leqslant x - \tanh x \leqslant x^2 \qquad \text{and also } x - \tanh x \leqslant x^3/3 \tag{3.35}$$

and we use the inequality

$$|\tanh x - x| \leqslant x^2 . \tag{3.36}$$

After writing (3.33) in integral form, we get

$$|\hat{m}(x, t)| \leqslant \mathrm{e}^{\alpha_\gamma (t - t_{na})} (p^\gamma_{t - t_{na}} \circ |\hat{m}|)(x, t_{na}) + \int_{t_{na}}^t \mathrm{d}s \, \mathrm{e}^{\alpha_\gamma (t-s)} (p^\gamma_{t-s} \circ J_\gamma \circ |\beta \hat{m}|^2)(x, s) . \tag{3.37}$$

Let

$$S(t) = [\gamma^{-\zeta - \alpha a n + d/2}]^{-1} \sup_x \phi^\star_{n,\gamma}(x) |\hat{m}(x, t)| \tag{3.38}$$

$$\phi^\star_{n,\gamma}(x) = \begin{cases} \phi_\gamma(x) & \text{if} \quad |x| \leqslant R_{n,\gamma} \\ \phi_\gamma(R_{n,\gamma}) & \text{otherwise} . \end{cases} \tag{3.39}$$

Then we have

$$S(t) \leqslant e^{\alpha_\gamma (t-t_{na})} A S(t_{na}) + \int_{t_{na}}^{t} ds\, e^{\alpha_\gamma (t-s)} \gamma^{-\zeta-\alpha a n + d/2} \beta^2 \phi_\gamma (R_{n,\gamma})^{-1} B S(s)^2 \tag{3.40}$$

where

$$A = \sup_{x,t\leqslant t_c} \sum_y p_t^\gamma (x, y) \frac{\phi^\star_{n,\gamma}(x)}{\phi^\star_{n,\gamma}(y)} \qquad B = \sup_{x,t\leqslant t_c} \sum_{y,z} p_t^\gamma (x, y) J_\gamma (y, z) \frac{\phi^\star_{n,\gamma}(x)^2}{\phi^\star_{n,\gamma}(z)^2} . \tag{3.41}$$

We next show that

$$A \leqslant c_1 \qquad B \leqslant c_1 . \tag{3.42}$$

First of all we observe that for $|x| \leqslant R_{n,\gamma}$

$$\sum_{|y|\leqslant R_{n,\gamma}} p_t^\gamma (x, y) \frac{\phi_\gamma(x)}{\phi_\gamma(y)} + \sum_{|y|> R_{n,\gamma}} p_t^\gamma (x, y) \frac{\phi_\gamma(x)}{\phi_\gamma(R_{n,\gamma})} \leqslant c_1$$

because we can bound the last factor by computing ϕ_γ at y, since $|y| > R_{n,\gamma}$ and ϕ_γ is a decreasing function.

For $|x| > R_{n,\gamma}$ we have

$$\sum_{|y|\leqslant R_{n,\gamma}} p_t^\gamma (x, y) \frac{\phi_\gamma(R_{n,\gamma})}{\phi_\gamma(y)} + \sum_{|y|> R_{n,\gamma}} p_t^\gamma (x, y) \leqslant 1$$

using again the monotonicity of ϕ_γ and the fact that $\sum_y p_t^\gamma(x, y) = 1$.

In an analogous way we prove the bounds on B. We then get from (3.40):

$$S(t) \leqslant e^{\alpha_\gamma (t-t_{na})} c_1 S(t_{na}) + \beta^2 c_1 \gamma^{-\zeta-\alpha a n + d/2} \lambda^{-15b} \int_{t_{na}}^{t} ds\, e^{\alpha_\gamma (t-s)} S(s)^2 . \tag{3.43}$$

By the definition of $\tilde{m}$ and (3.31), $S(t_{na}) = \|\sigma\|_{n,n,\gamma,a,\zeta}$. Now suppose that there is a first time T, $t_{na} \leqslant T \leqslant t_{Na}$, when

$$S(T) = 2\, e^{\alpha_\gamma (T-t_{na})} c_1 \|\sigma\|_{n,n,\gamma,a,\zeta} . \tag{3.44}$$

From equations (3.43) we then get

$$\begin{aligned} S(T) \leqslant\ & e^{\alpha_\gamma (T-t_{na})} c_1 \|\sigma\|_{n,n,\gamma,a,\zeta} \\ & + \beta^2 c_1 \gamma^{-\zeta-\alpha a n + d/2} \lambda^{-15b} \int_{t_{na}}^{T} ds\, e^{\alpha_\gamma (T-s)} [2\, e^{\alpha_\gamma (s-t_{na})} c_1 \|\sigma\|_{n,n,\gamma,a,\zeta}]^2 . \end{aligned}$$

The last term can be bounded by (recall that by hypothesis, $\|\sigma\|_{n,n,\gamma,a,\zeta} \leqslant C$),

$$\mathcal{I} \equiv \beta^2 c_1 \gamma^{-\zeta-\alpha a n + d/2} \lambda^{-15b}\, e^{\alpha_\gamma (T-t_{na})} (4 c_1 C) c_1 \|\sigma\|_{n,n,\gamma,a,\zeta} \int_{t_{na}}^{T} ds\, e^{\alpha_\gamma (s-t_{na})}$$

The factor

$$\gamma^{-\zeta-\alpha a n + d/2} \lambda^{-15b}\, e^{\alpha_\gamma (T-t_{na})} \leqslant \gamma^{-\zeta-\alpha a n + d/2} \lambda^{-15b}\, e^{\alpha_\gamma (t_{Na}-t_{na})}$$

vanishes when $\gamma \to 0$, because $|\alpha_\gamma - \alpha| \leqslant c\gamma$ for some constant c, and because ζ satisfies (3.21). Then there is $\gamma_0 > 0$, which depends on C, so that, for $\gamma \leqslant \gamma_0$,

$$\mathcal{I} < e^{\alpha_\gamma (T-t_{na})} c_1 \|\sigma\|_{n,n,\gamma,a,\zeta} . \tag{3.45}$$

We have thus reached a contradiction with the definition of T, which therefore implies that

$$S(t) \leqslant 2\, e^{\alpha_\gamma (t-t_{na})} c_1 \|\sigma\|_{n,n,\gamma,a,\zeta} \qquad \text{for all} \quad t_{na} \leqslant t \leqslant t_{Na} . \tag{3.46}$$

Recalling that $|\alpha_\gamma - \alpha| \leqslant c\gamma$ for a suitable constant c, we have, for all γ small enough,

$$2\,\mathrm{e}^{\alpha_\gamma(t-t_{na})} \leqslant 3\,\mathrm{e}^{\alpha(t-t_{na})} \qquad t_{na} \leqslant t \leqslant t_{Na}\,. \tag{3.47}$$

We have thus completed the proof of (3.28).

To prove (3.29) we denote by Θ' the expression in (3.34) with $\hat{m}$ replaced by $\hat{m}'$ (defined as $\hat{m}$, but starting from $m_{\gamma,t_0}(x,t|\sigma')$). We then have (using that $\tanh^2 x \leqslant x^2$),

$$\begin{aligned}|\Theta(x,t) - \Theta'(x,t)| &= \left| \int_{\beta\hat{m}(x,t)}^{\beta\hat{m}'(x,t)} \mathrm{d}z\ \tanh^2 z \right| \leqslant |\hat{m}'(x,t)^3 - \hat{m}(x,t)^3|\beta^3/3 \\ &\leqslant \beta^3 |\hat{m}'(x,t) - \hat{m}(x,t)| \big(\hat{m}'(x,t)^2 + \hat{m}(x,t)^2\big)\,.\end{aligned} \tag{3.48}$$

Analogously to (3.38), we define

$$D(t) = [\gamma^{-\zeta - \alpha a n + d/2}]^{-1} \sup_x \phi^\star_{n,\gamma}(x) |\hat{m}(x,t) - \hat{m}'(x,t)| \tag{3.49}$$

and we obtain for a suitable constant c

$$D(t) \leqslant \mathrm{e}^{\alpha_\gamma(t-t_{na})} c_1 D(t_{na}) + c[\gamma^{d/2-\zeta}\,\mathrm{e}^{\alpha t_{Na}}]^2 \lambda^{-15b} \int_{t_{na}}^t \mathrm{d}s\, \mathrm{e}^{\alpha_\gamma(t-s)} D(s)\,. \tag{3.50}$$

Since the square bracket is infinitesimal as $\gamma \to 0$, from (3.50) and (3.32) we get (3.29) and the lemma is proved. □

Corollary 3.2.3. *With the same notation as in lemma 3.2.2 and calling*

$$L_\gamma(x,t) = m_{\gamma,t_{na}}(x,t|\sigma) - m_{\gamma,t_{na}}(x,t|\sigma')$$

we have for all $t_{na} \leqslant t \leqslant t_{(n+1)a}$

$$\phi_{h,\gamma}(x) \big| L_\gamma(x,t) - \mathrm{e}^{-(t-t_{na})} L_\gamma(x,t_{na}) \big| \leqslant \beta\gamma^{d/2-\zeta} \big(3c_1\,\mathrm{e}^{\alpha t} \|\sigma - \sigma'\|_{n,n,\gamma,a,\zeta} + c\gamma^k\big)\,. \tag{3.51}$$

Moreover, for all $n < q \leqslant N$, $h \geqslant q$,

$$\|m_{\gamma,t_{na}}(\cdot, t_{qa}|\sigma)\|_{h,q,\gamma,a,\zeta} \leqslant 3c_1 \|\sigma\|_{n,n,\gamma,a,\zeta} + c\gamma^k\,. \tag{3.52}$$

Proof. We write

$$L_\gamma(x,t) = \mathrm{e}^{-(t-t_{na})} L_\gamma(x,t_{na}) + \int_{t_{na}}^t \mathrm{d}s\, \mathrm{e}^{-(t-s)} [\tanh\{\beta\tilde{m}(x,s)\} - \tanh\{\beta\tilde{m}'(x,s)\}]\,.$$

We then use lemma 3.2.2 and in this way we prove (3.51). Equation (3.52) follows directly from (3.28). □

Lemma 3.2.4. *Using the same notation and hypothesis as in lemma 3.2.2, there is a constant* c' *so that for all* $t_{na} \leqslant t \leqslant t_{Na}$

$$\phi_{h,\gamma}(x) \big| m_{\gamma,t_{na}}(x,t|\sigma) - \mathrm{e}^{\alpha_\gamma(t-t_{na})} (p^\gamma_{t-t_{na}} \circ \sigma)(x) \big| \leqslant c' \big(2c_1\,\mathrm{e}^{\alpha_\gamma t} \gamma^{d/2-\zeta} \|\sigma\|_{n,n,\gamma,a,\zeta} + c\gamma^k\big)^2\,. \tag{3.53}$$

Proof. Let Θ be as in (3.34) and let $m(x,t)$ stand for $m_{\gamma,t_{na}}(x,t|\sigma)$. Then

$$m(x,t) = \mathrm{e}^{\alpha_\gamma(t-t_{na})} (p^\gamma_{t-t_{na}} \circ \sigma)(x) + \int_{t_{na}}^t \mathrm{d}s\, \mathrm{e}^{\alpha_\gamma(t-s)} (p^\gamma_{t-s} \circ \Theta(\cdot, s))(x)\,.$$

Let $|x| \leqslant R_{h,\gamma}$ and let $n < h' < h$, then, by (3.36), with $\tilde{m}$ as in (3.27),

$$(p^\gamma_{t-s} \circ \Theta(\cdot,s))(x) \leqslant \sum_{|y| \leqslant R_{h',\gamma}} p^\gamma_{t-s}(x,y)[\beta\tilde{m}(y,s)]^2 + \sum_{|y| > R_{h',\gamma}} p^\gamma_{t-s}(x,y)\,.$$

The lemma then follows from lemma 3.2.2 and (3.10). □

Remarks. Using lemmas 3.2.4 and 3.1.3 we obtain the proof of the statements in theorem 3.2.0 relative to $t \leqslant t_{Na}$. In particular, (3.16) follows from (3.53) while the proof of (3.19) follows from (3.53) observing that, by lemma 3.1.3, since $q > n$,

$$\begin{aligned}
&[\lambda^{d/2}\gamma^{-\alpha a q+d/2}]^{-1}\phi_{h,\gamma}(x)\Big|\sum_y \pi^\gamma(x,y)\,\mathrm{e}^{\alpha_\gamma(t_{qa}-t_{na})}(p^\gamma_{t_{qa}-t_{na}}\circ\sigma)(y)\Big| \leqslant c\gamma^k\\
&\qquad +\phi_{h,\gamma}(x)\sum_{|y|\leqslant R_{n,\gamma}} p^\gamma_{t_{qa}-t_{na}}(x,y)\phi_\gamma(y)^{-1}\\
&\qquad \times\phi_{h,\gamma}(y)\left|\sum_z \pi^\gamma(y,z)[\lambda^{d/2}\gamma^{-\alpha a n+d/2}]^{-1}\sigma(z)\right|
\end{aligned}$$

and then using again lemma 3.1.3.

We conclude this subsection by studying the evolution in the time interval $[t_{Na}, t_c]$, thus completing the proof of theorem 3.2.0.

Using the notation of lemma 3.2.2, we define

$$M_\gamma(x,t) = \mathrm{e}^{\alpha_\gamma(t-t_{Na})}\sum_y p^\gamma_{t-t_{Na}}(x,y)m_{\gamma,t_{na}}(y,t_{Na}|\sigma) \qquad t_{Na}\leqslant t\leqslant t_c\,. \tag{3.54}$$

We then have:

Proposition 3.2.5. *Let* $\tau_0 \in (0,\tau_c)$ *and* a *be any number such that* $\tau_0 = na$, $n\in\mathbb{Z}_+$, *and* $(N-n)a > \alpha(\tau_c - Na)$, N *as in definition 3.1.1. Let* ζ *satisfy (3.21). Then for any* $C>0$ *and* $h>n$ *there is* c *so that for all* $\sigma\in F_{\gamma,a,\zeta}(n,n,C)$, *all* $t_{Na}\leqslant t\leqslant t_c$ *and all* $|x|\leqslant R_{h,x}$

$$|M_\gamma(x,t)| \leqslant c\lambda^{d/2}\,\mathrm{e}^{\alpha t}\gamma^{d/2}\phi_\gamma(x)^{-1} \tag{3.55}$$

$$\left|m_{\gamma,t_{na}}(x,t|\sigma) - M_\gamma(x,t)\right| \leqslant [c\lambda^{d/2}\,\mathrm{e}^{\alpha t}\gamma^{d/2}\phi_\gamma(x)^{-1}]^3 \tag{3.56}$$

$$\left|m_{\gamma,t_{na}}(x,t_c|\sigma) - M_\gamma(x,t_c)\right| \leqslant c[\lambda^{(d/2-16b)5} + \Omega_\gamma(x|\sigma)^3] \tag{3.57}$$

where

$$\Omega_\gamma(x|\sigma) = \sup_{\gamma|y-x|\leqslant(\ln\lambda^{-2})^2}\ \sup_{t_c^-\leqslant s\leqslant t_c}\{\mathrm{e}^{\alpha_\gamma(t_c-s)}|M_\gamma(y,s)|\} \qquad t_c^- \equiv t_c - (\ln\lambda^{-2})^2\,. \tag{3.58}$$

Remarks. Let $n < h' < h$, then by lemma 3.1.3 for any k there is c so that

$$\begin{aligned}
&\phi_{h,\gamma}(x)\left|M_\gamma(x,t) - \mathrm{e}^{\alpha_\gamma(t-t_{na})}(p^\gamma_{t-t_{na}}\circ\sigma)(x)\right| \leqslant c\gamma^k + \phi_{h,\gamma}(x)\,\mathrm{e}^{\alpha_\gamma(t-t_{Na})}\\
&\qquad\times\sum_{|y|\leqslant R_{h',\gamma}} p^\gamma_{t-t_{Na}}(x,y)\Big|m_{\gamma,t_{na}}(y,t_{Na}|\sigma)\\
&\qquad -\mathrm{e}^{\alpha_\gamma(t_{Na}-t_{na})}(p^\gamma_{t_{Na}-t_{na}}\circ\sigma)(x)\Big|\,.
\end{aligned}$$

By equation (3.56) and lemma 3.2.4 this proves (3.17). Since equation (3.16) has already been proved and (3.18) follows directly from (3.15), proposition 3.2.5 completes the proof of theorem 3.2.0.

The bounds (3.55) and (3.56) are optimal except close to the interface where the magnetization is atypically small. In lemma 4.2.5 and proposition 4.2.6, where we study the structure of the interfaces, we will in fact need the stronger estimate (3.57).

Proof. We write $m_\gamma(x,t)$ for $m_{\gamma,t_{na}}(x,t|\sigma)$ and set $n < h_1 < h_2 < h_3 < h$. Since $(N-n)a > \alpha(\tau_c - Na)$, there are $\delta > 0$ and c so that

$$\mathrm{e}^{-(t_{Na}-t_{na})} \leqslant c\,\mathrm{e}^{\alpha t_{Na}}\gamma^{d/2+\delta}\,.$$

Then by (3.53) and (3.15) there is c so that

$$\phi_{h_1,\gamma}(y)|m_\gamma(y, t_{Na})| \leqslant c\, \mathrm{e}^{\alpha t_{Na}}\gamma^{d/2}\lambda^{d/2} \tag{3.59}$$

equation (3.55) then easily follows using lemma 3.1.3 to bound the sum over $|y| > R_{h_1,\gamma}$.

We call $f_\gamma(x, t)$ the solution of (2.22) for $t \geqslant t_{Na}$ and such that $f_\gamma(x, t_{Na}) = m_\gamma(x, t_{Na}|\sigma)$ for all $|x| \leqslant R_{h_1,\gamma}$, while, elsewhere,

$$|f_\gamma(x, t_{Na})| \leqslant c\, \mathrm{e}^{\alpha t_{Na}}\gamma^{d/2}\lambda^{d/2}\lambda^{-16b}.$$

Then by (3.59) the above inequality holds everywhere. On the other hand, by the barrier lemma, see (i) in lemma 3.2.1, for any $k > 0$ there is c_k so that

$$\big|m_\gamma(x, t) - f_\gamma(x, t)\big| \leqslant c_k\gamma^k \tag{3.60}$$

for all $|x| \leqslant R_{h_2,\gamma}$ and all $t_{Na} \leqslant t \leqslant t_{(N+1)a}$.

We then define $f_\gamma^{\pm}(x, t)$ as the solutions of (2.22) for $t \geqslant t_{Na}$ and such that, for all x,

$$f_\gamma^{\pm}(x, t_{Na}) = \pm c\lambda^{d/2}\gamma^{-\alpha a N + d/2}\lambda^{-16b} = \pm c\lambda^{d/2-16b}\, \mathrm{e}^{\alpha(t_c - t_{Na})}. \tag{3.61}$$

Then $f_\gamma^{\pm}(x, t) \equiv f_\gamma^{\pm}(t)$ for all $t \geqslant t_{Na}$, they are thus independent of x. We can then easily integrate (2.22) obtaining

$$f_\gamma^{\pm}(x, t) = \mathrm{e}^{\alpha_\gamma(t - t_{Na})} f_\gamma^{\pm}(t_{Na}) + \int_{t_{Na}}^{t} \mathrm{d}s\, \mathrm{e}^{\alpha_\gamma(t-s)} \tanh \beta \hat{J}_{\gamma,0} f_\gamma^{\pm}(s)$$

and using (3.36) there is c so that

$$|f_\gamma^{\pm}(x, t)| \leqslant c\lambda^{d/2-16b}\, \mathrm{e}^{-\alpha(t_c - t)} \qquad t_{Na} \leqslant t \leqslant t_c. \tag{3.62}$$

Recall that by definition 3.1.2, $d/2 > 16b$. By the monotonicity properties of (2.22), see (ii) in lemma 3.2.1, $f_\gamma^{-}(t) \leqslant f_\gamma(x, t) \leqslant f_\gamma^{+}(t)$, hence

$$|f_\gamma(x, t)| \leqslant c\lambda^{d/2-16b}\, \mathrm{e}^{-\alpha(t_c - t)} \qquad |f_\gamma(x, t_c)| \leqslant c\lambda^{d/2-16b}. \tag{3.63}$$

From equation (3.60) and (3.63) there is a constant c' so that

$$\big|m_\gamma(x, t)\big| \leqslant c'\lambda^{d/2-16b}\, \mathrm{e}^{-\alpha(t_c - t)} \qquad \text{for all} \quad |x| \leqslant R_{h_2,\gamma} \quad \text{and} \quad t_{Na} \leqslant t \leqslant t_c. \tag{3.64}$$

We have

$$m_\gamma(x, t) - M_\gamma(x, t) = \int_{t_{Na}}^{t} \mathrm{d}s\, \mathrm{e}^{\alpha_\gamma(t-s)} (p_{t-s}^{\gamma} \circ \Theta)(x, s) \tag{3.65}$$

where

$$\Theta(x, s) = \tanh\left\{\beta \sum_y J_\gamma(x, y) m_\gamma(y, s)\right\} - \beta \sum_y J_\gamma(x, y) m_\gamma(y, s). \tag{3.66}$$

By equation (3.35)

$$\big|m_\gamma(z, s) - M_\gamma(z, s)\big| \leqslant \int_{t_{Na}}^{s} \mathrm{d}s'\, \mathrm{e}^{\alpha_\gamma(s-s')} \frac{\beta^3}{3} \sum_{z'} p_{s-s'}^{\gamma}(z, z') \prod_{i=1}^{3} \sum_{z_i'} J_\gamma(z', z_i') |m_\gamma(z_i', s')|. \tag{3.67}$$

Let $|z| \leqslant R_{h_3,\gamma}$, then by (3.64) for any k there is c so that

$$\big|m_\gamma(z, s) - M_\gamma(z, s)\big| \leqslant c\left\{\gamma^k + \frac{\hat{J}_{\gamma,0}^3}{2\alpha_\gamma}\, \mathrm{e}^{\alpha_\gamma t_a} [c'\, \mathrm{e}^{-\alpha(t_c - s)}\lambda^{d/2-16b}]^3\right\}. \tag{3.68}$$

Going back to (3.65), setting $|x| \leqslant R_{h,\gamma}$, we bound the last factor in (3.65) using (3.68) and we get

$$\left|(p^{\gamma}_{t-s} \circ \Theta)(x,s)\right| \leqslant \sum_y p^{\gamma}_{t-s}(x,y) \frac{\beta^3}{3} \prod_{i=1}^{3} \left\{ \sum_{z_i} J_\gamma(y,z_i)|M_\gamma(z_i,s)|\mathbf{1}(|z_i| \leqslant R_{h_3,\gamma}) \right\}$$
$$+c''[\mathrm{e}^{-\alpha(t_c-s)}\lambda^{d/2-16b}]^5 + c\gamma^k . \tag{3.69}$$

From equation (3.65) we then have for a suitable constant c

$$\left|m_\gamma(x,t_c) - M_\gamma(x,t_c)\right| \leqslant c\lambda^{(d/2-16b)5} + \mathcal{R} \tag{3.70}$$

where

$$\mathcal{R} = \frac{\beta^3}{3}\int_{t_{Na}}^{t_c} \mathrm{d}s\, \mathrm{e}^{\alpha_\gamma(t_c-s)} \sum_y p^{\gamma}_{t_c-s}(x,y) \prod_{i=1}^{3} \sum_{z_i} J_\gamma(y,z_i)|M_\gamma(z_i,s)|\mathbf{1}(|z_i| \leqslant R_{h_3,\gamma}) . \tag{3.71}$$

Let δ be as in lemma 4.3.5 supposing (without loss of generality) that $\delta \geqslant 1$. We then set

$$t_c^{(\delta)} = t_c - \frac{1}{2\delta}(\ln\lambda^{-2})^2 .$$

Then, using (3.55),

$$\mathcal{R} \leqslant \frac{\beta^3}{3}\int_{t_{Na}}^{t_c^{(\delta)}} \mathrm{d}s\, \mathrm{e}^{\alpha_\gamma(t_c-s)-3\alpha_\gamma(t_c-s)}\lambda^{3(d/2-16b)} + \frac{\beta^3}{3}\int_{t_c^{(\delta)}}^{t_c} \mathrm{e}^{-2\alpha_\gamma(t_c-s)}\Omega_\gamma(x|\sigma)^3$$
$$+\frac{\beta^3}{3}(\ln\lambda^{-2})^2\,\mathrm{e}^{\alpha_\gamma(t_c-t_{Na})}3 \sup_{t\leqslant t_c-t_c^{(\delta)}} \sum_z \sum_{\gamma|y|>(\ln\lambda^{-2})^2} p^{\gamma}_t(0,z)J_\gamma(z,y) . \tag{3.72}$$

By lemma 4.3.5

$$\sup_{t\leqslant t_c-t_c^{(\delta)}} \sum_{\gamma|y|>(\ln\lambda^{-2})^2-1} p^{\gamma}_t(0,y) \leqslant c\,\mathrm{e}^{-[(\ln\lambda^{-2})^2-1-\delta(t_c-t_c^{(\delta)})]}$$
$$\leqslant c\,\mathrm{e}^{-(\ln\lambda^{-2})^2/2+1} . \tag{3.73}$$

We have thus completed the bound on $\mathcal{R}$ which, inserted into (3.70), yields (3.57), therefore the proposition is proved. □

3.3. Bounds on the v-functions

In this subsection we prove that spin configurations and solutions of (2.22) are close in the sense of expectations. More precisely for any spin configuration σ, any $t^+ \geqslant 0$, any $t \geqslant t^+$, any integer $k \geqslant 1$, any subset $\underline{x}$ of k elements in $\mathbb{Z}^d$ (we call $\mathbb{Z}^{dk}_{\neq}$ the collection of such subsets), we define

$$v^\gamma(\underline{x},t|\sigma,t^+) = \mathbb{E}^{\gamma}_{\sigma,t^+}\left(\prod_{x\in\underline{x}}[\sigma(x,t) - m_{\gamma,t^+}(x,t|\sigma)]\right) \tag{3.74}$$

where $\mathbb{E}^{\gamma}_{\sigma,t^+}$ is the expectation of the process starting from σ at time t^+.

The main result in this subsection is:

Theorem 3.3.0. *There are* $\alpha^\star > \alpha$, $a^\star > 0$, $\delta > 0$ *and, for any integer* $k \geqslant 1$, c *so* that

$$\sup_\sigma \sup_{\underline{x}\in\mathbb{Z}^{dk}_{\neq}} \left|v^\gamma(\underline{x},t|\sigma,0)\right| \leqslant c[\gamma^{d/2}\,\mathrm{e}^{\alpha^\star t}]^k \qquad \textit{for all} \quad t \leqslant t_{a^\star} . \tag{3.75}$$

Moreover, for all $a > 0$ small enough, all $h > 0$, all ζ as in (3.21), all $C > 0$, all integers $k \geqslant 1$ and all integers $0 \leqslant n < N$, (N as in definition 3.1.1) there exists c so that

$$\left|v^\gamma(\underline{x}, t + t_{na}|\sigma, t_{na})\right| \leqslant c[(1 + C\gamma^\delta)\gamma^{d/2}\,\mathrm{e}^{\alpha t}]^k \qquad 0 \leqslant t \leqslant t_a \tag{3.76}$$

for all $\sigma \in F_{\gamma,a,\zeta}(n, n, C)$ and all subsets $\underline{x}$ of k elements in $\{x \in \mathbb{Z}^d : |x| \leqslant R_{n+h,\gamma}\}$.

Remarks. In the course of the proof of theorem 3.3.0 we will compute the leading term of the asymptotic expansion of v^γ as $\gamma \to 0$. This result is needed in section 4 to prove theorem 2.4.3.

Both bounds, (3.75) and (3.76), grow exponentially with time, the rate in (3.75) is $\alpha^\star$ that is improved to α in (3.76) after restricting the initial configuration to $F_{\gamma,a,\zeta}(n, n, C)$.

When $k = 1$ the v-function is just the difference between $m_{\gamma,t_{na}}(x, t|\sigma)$ and the expectation of $\sigma(x, t)$. The former at $t = t_{(n+1)a}$ has a factor growing as $\mathrm{e}^{\alpha t_{(n+1)a}}$, according to theorem 3.2.0, while the v-function, according to (3.76), only grows like $\mathrm{e}^{\alpha t_a}$. We thus see that spin configurations and statistical solutions are much closer to each other than their order of magnitude, provided that γ is small enough and $n > 0$. This latter reflects the statement in section 2 that after an initial time layer (here represented by t_a), the process is essentially deterministic.

We use the technique based on the analysis of the v-functions, developped in [9]. We recall in the definition below the main objects of our analysis and we refer to [13] for a more general discussion on the use of the v-functions in hydrodynamical and kinetic limits.

Definition 3.3.1. *The w functions.*
For any σ, $w^\gamma(x_1, x_2, t|\sigma)$, $x_1 \neq x_2$, $t \geqslant t_{na}$ is defined as the solution of

$$\frac{\mathrm{d}w^\gamma(x_1, x_2, t|\sigma)}{\mathrm{d}t} = \sum_{y\neq x_2} k_t(x_1, y)w^\gamma(y, x_2, t|\sigma) + \sum_{y\neq x_1} k_t(y, x_2)w^\gamma(x_1, y, t|\sigma) + \kappa_t(x_1, x_2) \tag{3.77}$$

$$w^\gamma(x_1, x_2, t_{na}|\sigma) = 0 \qquad \textit{for all} \quad x_1 \neq x_2 \tag{3.78}$$

where, writing $m_\gamma(x, t)$ for $m_{\gamma,t_{na}}(x, t|\sigma)$,

$$\kappa_t(x_1, x_2) = \beta J_\gamma(x_1, x_2)\big\{\cosh^{-2}\{\beta(J_\gamma \circ m_\gamma)(x_1, t)\} + \cosh^{-2}\{\beta(J_\gamma \circ m_\gamma)(x_2, t)\}\big\} \tag{3.79}$$

$$k_t(x, y) = -\mathbf{1}_{x=y} + \cosh^{-2}\{\beta(J_\gamma \circ m_\gamma)(x, t)\}\beta J_\gamma(x, y) \tag{3.80}$$

with $\mathbf{1}_{x=y}$ being the Kronecker symbol. Notice that, via m_γ, k_t depends on σ.

Analogously we define $w^\gamma(x, y, t)$ as the solution of (3.77), (3.78) with $k_t(x, y)$ and $\kappa_t(x, y)$ corresponding to $m_\gamma(x, t) \equiv 0$.

Finally, given any k and any $\underline{x} \in \mathbb{Z}^{d2k}$ we let

$$W^\gamma(\underline{x}, t|\sigma) = \sum_{\{(i_1,j_1),\ldots,(i_k,j_k)\}} \prod_{\ell=1}^{k} w^\gamma(x_{i_\ell}, x_{j_\ell}, t|\sigma) \qquad \underline{x} = (x_1, \ldots, x_{2k}) \tag{3.81}$$

where the sum is over all the partitions $\{(i_1, j_1), \ldots, (i_k, j_k)\}$ of $\{1, \ldots, 2k\}$ into k disjoint pairs. We complete the definition of W^γ by setting it equal to 0 when $|\underline{x}|$ is odd. Analogously we define $W^\gamma(\underline{x}, t)$ starting from $w^\gamma(\underline{x}, t)$.

In [9] it is proven that the main contribution to v^γ comes from w^γ, the result is recalled in proposition 3.3.3 below. We first study the behaviour of the w^γ functions.

Lemma 3.3.2. *Let $a \in (0, \tau_c)$ and ζ satisfy (3.21). Then the following holds.*

(i) *There are $\delta > 0$ and c so that for any $n < N$ and $h > n$*

$$|w^\gamma(x_1, x_2, t_{(n+1)a}|\sigma)| \leqslant c[1 + C^2\gamma^\delta]\gamma^{d-2\alpha a}\lambda^d \tag{3.82}$$

for all $\sigma \in F_{\gamma,a,\zeta}(n, n, C)$ and all

$$x_1 \neq x_2 : \qquad |x_i| \leqslant R_{h,\gamma} \quad i = 1, 2\,. \tag{3.83}$$

(ii) *Same bounds as in (1) hold for $w^\gamma(x_1, x_2, t)$. Moreover, for any positive $\zeta' < d$ there is $a > 0$ sufficiently small so that the following holds. There is a function $\Phi(|r|) \in L_1(\mathbb{R}^d, \mathbb{R}_+)$, such that for all $t \leqslant t_a$,*

$$\left| w^\gamma(x_1, x_2, t) - \int_0^t \mathrm{d}s\, \mathrm{e}^{2\alpha s}\big((p_s^\gamma \times p_s^\gamma) \circ 2\beta J_\gamma\big)(x_1, x_2)\right| \leqslant (\lambda\gamma)^{d-\zeta'}\Phi(\lambda\gamma|x_1 - x_2|) \tag{3.84}$$

where $p_s^\gamma \times p_s^\gamma$ denotes the product probability of p_s^γ with itself, thus being the transition probability of two independent random walks, each with transition probability p_s^γ.

Proof. We start by proving (i). We set

$$(J_\gamma \circ w^\gamma)_1(\underline{x}, t|\sigma) = \sum_{y_1} J_\gamma(x_1, y_1)w^\gamma(y_1, x_2, t|\sigma) \qquad \underline{x} = (x_1, x_2)\,. \tag{3.85}$$

Analogously we define $(J_\gamma \circ w^\gamma)_2(\underline{x}, t|\sigma)$.

Notice that if x_1 and x_2 are suitably close, then the sum on the right-hand side of (3.85) contains terms of the form $w^\gamma(x, x, t|\sigma)$ not yet defined. We do it now by setting

$$\frac{\mathrm{d}}{\mathrm{d}t}w^\gamma(x, x, t|\sigma) = 2\sum_{y\neq x} k_t(x, y)w^\gamma(x, y|\sigma) + 2\beta J_\gamma(x, x)\cosh^{-2}\beta\big\{(J_\gamma \circ m_\gamma)(x, t|\sigma)\big\}\,.$$

For $\underline{x} \in \mathbb{Z}^{d2}$ we have, in analogy with (3.33),

$$\begin{aligned}\frac{\mathrm{d}w^\gamma}{\mathrm{d}t} &= 2\alpha_\gamma w^\gamma + \beta J_\gamma(x_1, x_2)\sum_{i=1}^{2}[\cosh\{\beta(J_\gamma \circ m_\gamma)(x_i, t)\}]^{-2} \\ &\quad +\beta\sum_{i=1}^{2}[(J_\gamma \circ w^\gamma)_i - \hat{J}_{\gamma,0}w^\gamma] + \mathcal{R}_\gamma\end{aligned} \tag{3.86}$$

where

$$\begin{aligned}\mathcal{R}_\gamma(\underline{x}, t) = &-\beta J_\gamma(x_1, x_2)[w^\gamma(x_1, x_1, t|\sigma) + w^\gamma(x_2, x_2, t|\sigma)] \\ &+ \sum_{y\neq x_2}\big[\cosh^{-2}\{(J_\gamma \circ m_\gamma)(x_1, t)\} - 1\big]J_\gamma(x_1, y)w^\gamma(y, x_1|\sigma) \\ &+ \sum_{y\neq x_1}\big[\cosh^{-2}\{(J_\gamma \circ m_\gamma)(x_2, t)\} - 1\big]J_\gamma(x_2, y)w^\gamma(y, x_2|\sigma)\,.\end{aligned} \tag{3.87}$$

We then have

$$\begin{aligned}w^\gamma(\underline{x}, t|\sigma) &= \int_{t_{na}}^{t}\mathrm{d}s\, \mathrm{e}^{2\alpha_\gamma(t-s)}\big((p_{t-s}^\gamma \times p_{t-s}^\gamma) \circ [\phi_\gamma + \mathcal{R}_\gamma]\big)(x) \\ \phi_\gamma(y_1, y_2, s) &= \beta J_\gamma(y_1, y_2)\sum_{i=1}^{2}\cosh^{-2}\{(J_\gamma \circ m_\gamma)(y_i, s)\}\end{aligned} \tag{3.88}$$

where $(p_{t-s}^\gamma \times p_{t-s}^\gamma)$ denotes the product probability of p_{t-s}^γ with itself, thus being the transition probability of two independent random walks, each with transition probability p_{t-s}^γ.

In section 4.3 we shall prove that there is a constant $c^{(1)}$ so that for all $t \leqslant t_a$

$$\sup_{\underline{x}\in\mathbb{Z}^{d2}} \left|\left((p_t^\gamma \times p_t^\gamma) \circ \beta J_\gamma\right)(\underline{x})\right| \leqslant \frac{c^{(1)}}{(1+t)^{d/2}}\gamma^d . \tag{3.89}$$

On the other hand, if a is small enough then by (2.26) of [9], there are $c^{(2)}$ and $\zeta < d$ so that

$$|w^\gamma(\underline{x}, t|\sigma)| \leqslant c^{(2)}\gamma^{d-\zeta} \tag{3.90}$$

for all $t_{na} \leqslant t \leqslant t_{(n+1)a}$ and all $\underline{x} \in \mathbb{Z}^{d2}_{\neq}$, (but the result easily extends to all $\underline{x} = (x, x)$, if a is small enough). To bound the last two terms in (3.87) we use (3.46), (see equations (3.38), (3.31), (3.32) for notation), to conclude that if $|x| \leqslant R_{h',\gamma}$, $n < h' < h$, then

$$\left|\cosh^{-2}\{\beta(J_\gamma \circ m_\gamma)(x_i, t)\} - 1\right| \leqslant c^{(3)}\left[2\beta c_1 C\, \mathrm{e}^{\alpha_\gamma(t-t_{na})}\lambda^{-16}\gamma^{-\zeta-\alpha a n+d/2}\right]^2 \tag{3.91}$$

where $c^{(3)}$ is such that

$$|\cosh^{-2}\beta x - 1| \leqslant c^{(3)}x^2 .$$

From equation (3.88) we have that for $|x_i| \leqslant R_{h,\gamma}$, $i = 1, 2$,

$$|w^\gamma(\underline{x}, t|\sigma)| \leqslant I_1 + I_2 \tag{3.92}$$

where

$$I_1 := \int_{t_{na}}^{t} \mathrm{d}s\, \mathrm{e}^{2\alpha_\gamma(t-s)} \frac{2c^{(1)}}{(1+(t-s))^{d/2}}\gamma^d \tag{3.93}$$

$$I_2 := \int_{t_{na}}^{t} \mathrm{d}s\, \mathrm{e}^{2\alpha_\gamma(t-s)} 2\beta\gamma^{d-\zeta}\Big\{\sup_{x_1,x_2}|J_\gamma(x_1,x_2)|c^{(2)} + c^{(3)}\left[2\beta c_1 C\lambda^{-16}\gamma^{-\zeta-\alpha a(n+1)+d/2}\right]^2 c^{(2)}$$
$$+ \sup_{|x|\leqslant R''_{n,\gamma}}\ \sup_{|y|\geqslant R'_{n,\gamma}} p^\gamma_{t-s}(x,y)\hat{J}_{\gamma,0}c^{(2)}\Big\} . \tag{3.94}$$

We set $t = t_{(n+1)a}$ in (3.92) and start by the first integral: calling

$$t'_{na} = t_{na} + \tfrac{1}{2}a\lambda^{-2}$$

we have

$$\int_{t_{na}}^{t_{(n+1)a}} \mathrm{d}s\, \mathrm{e}^{2\alpha_\gamma(t_{(n+1)a}-s)} \frac{1}{(1+t_{(n+1)a}-s)^{d/2}} \leqslant \int_{t_{na}}^{t'_{na}} \mathrm{d}s\, \mathrm{e}^{2\alpha_\gamma(t_{(n+1)a}-s)} \frac{1}{(a/2\lambda^{-2})^{d/2}}$$
$$+ \int_{t'_{na}}^{t_{(n+1)a}} \mathrm{d}s\, \mathrm{e}^{2\alpha_\gamma(t_{(n+1)a}-s)} \leqslant \frac{1}{2\alpha_\gamma}\left[2^{d/2}\frac{\mathrm{e}^{2\alpha_\gamma t_a}}{(a/2\lambda^{-2})^{d/2}} + \mathrm{e}^{\alpha_\gamma t_a}\right]$$
$$\leqslant \bar{c}\left[\gamma^{-2\alpha a}\lambda^d + \gamma^{-\alpha a}\right].$$

The first term on the right-hand side of (3.92) is thus bounded by the right-hand side of (3.82), for a suitable value of the constant c.

The first term on the right-hand side of (3.94) is bounded by

$$\frac{1}{2\alpha_\gamma}\gamma^{-2\alpha_\gamma a}2\beta\gamma^d\|J\|_\infty c^{(2)}\gamma^{d-\zeta}$$

hence it is also bounded by the right-hand side of (3.82). The second term on the right-hand side of (3.94) is bounded by

$$2\beta\frac{1}{2\alpha_\gamma}\gamma^{-2\alpha_\gamma a}2c^{(3)}\left[2\beta c_1 C\lambda^{-16}\gamma^{-2\zeta-\alpha a N+d/2}\right]^2\gamma^{d-\zeta}$$

which is also bounded in agreement with (3.82).

Finally by lemma 3.1.3 also the last term in (3.94) is bounded by the right-hand side of (3.82), for a suitable c, hence the proof of (i) is completed.

Proof of (2). In this case we have (compare with (3.86) and (3.87))

$$\frac{\mathrm{d}w^\gamma}{\mathrm{d}t} = 2\alpha_\gamma w^\gamma + 2\beta J_\gamma(x_1, x_2) + \beta \sum_{i=1}^{2} [(J_\gamma \circ w^\gamma)_i - \hat{J}_{\gamma,0} w^\gamma]$$
$$-\beta J_\gamma(x_1, x_2)[w^\gamma(x_1, x_1, t) + w^\gamma(x_2, x_2, t)]\,. \tag{3.95}$$

Using arguments similar to those used previuosly, it is not difficult to prove (3.84). The prove of the lemma is thus completed. □

We define for any $t \geqslant 0$ and any $\underline{x} \in \mathbb{Z}^{dk}_{\neq}$

$$v^\gamma(\underline{x}, t) = \mathbb{E}^\gamma_{\mu_0}\left(\prod_{x \in \underline{x}} \sigma(x, t)\right) \tag{3.96}$$

where $\mathbb{E}^\gamma_{\mu_0}$ is the expectation of the process starting from μ_0 at time 0.

Proposition 3.3.3. *For any $\zeta' > 0$ there is $a > 0$ sufficiently small such that for all integer k there is c_k so that for all $t \in [0, t_a]$*

$$\sup_{\underline{x} \in \mathbb{Z}^{(2k+1)d}_{\neq}} (\gamma\lambda)^{-(k+\frac{1}{2})d} |v^\gamma(\underline{x}, t)| \leqslant c_k \gamma^{\zeta'} \qquad \sup_{\underline{x} \in \mathbb{Z}^{2kd}_{\neq}} (\gamma\lambda)^{-kd} |v^\gamma(\underline{x}, t) - W^\gamma(\underline{x}, t)| \leqslant c_k \gamma^{\zeta'} \tag{3.97}$$

where $v^\gamma(\underline{x}, t)$ is defined in (3.96) and $W^\gamma(\underline{x}, t)$ in definition 3.3.1. The same inequalities hold for $v^\gamma(\underline{x}, t|\sigma, t_{na})$ and $W^\gamma(\underline{x}, t|\sigma)$ with $t \in [t_{na}, t_{(n+1)a}]$ and $\sigma \in F_{\gamma,a,\zeta}(n, n, C)$, the parameters as in theorem 3.3.0.

Proof. The proof of (3.97) follows from theorem 2.6, lemmas 4.1 and 4.2 of [9]. We omit the details. □

Proof of theorem 3.3.0. The inequality (3.75) is proven in theorem 2.2 of [9]. From proposition 3.3.3 and lemma 3.3.2 it is not difficult to prove (3.76). Notice that from (3.75) it follows that, for any $\zeta' > 0$ there is a sufficiently small such that for all k there is c_k so that for all $\underline{x} \in \mathbb{Z}^{kd}_{\neq}$

$$\sup_{t \in [t_{na}, t_{(n+1)a}]} \sup_\sigma |v^\gamma(\underline{x}, t|\sigma)| \leqslant c_k \gamma^{(d-\zeta')\frac{k}{2}}\,. \tag{3.98}$$

□

3.4. The process until time t_c

In this subsection we improve the relation between spin trajectories and statistical solutions of (2.22) in two ways: by proving that the (empirical averages of the) two are close in probability and by extending the analysis to intervals longer than t_a.

More precisely, given $t \geqslant 0$, we set

$$K_{\gamma,t} = \max\{\mathrm{e}^{-\alpha t} \gamma^{-d/2}, 1\} \tag{3.99}$$

and, given $a \in (0, \tau_c)$, we call n_t the integer n such that $t_{(n-1)a} < t \leqslant t_{na}$. Then, given b_0 and h positive we set

$$R_{t,\gamma,h} = R_{n_t,\gamma} - h\gamma^{-1} \tag{3.100}$$

and define for $f : \mathbb{Z}^d \to \mathbb{R}$

$$A^\star_{\gamma,t,a,b_0,h}(f) = \sup_{|x|\leqslant R_{t,\gamma,h}} K_{\gamma,t}\frac{1}{|B_{\gamma,x}|}\Big|\sum_{y\in B_{\gamma,x}} f(y)\Big| \qquad B_{\gamma,x} = \{|y-x|\leqslant \gamma^{-b_0}\}\,. \tag{3.101}$$

The main result in this subsection is

Theorem 3.4.0. *For any $\tau_0 \in (0,\tau_c)$ there is $\delta > 0$ so that the following holds. For all $a > 0$ small enough and such that $\tau_0 = na$, $n \in \mathbb{Z}_+$, and, given a, for all ζ and $b'_0 = (1-b_0)$ both positive and small enough, for all h, C, k positive there is c so that for all $\gamma > 0$ and all $\sigma \in F_{\gamma,a,\zeta}(n,n,C)$*

$$\mathbb{P}^\gamma_{\sigma,t_{n_0a}}\bigg(\sup_{t_{na}\leqslant t\leqslant t_{(N+2)a}} A^\star_{\gamma,t,a,b_0,h}\big(\sigma(\cdot,t)-m_{\gamma,t_{n_0a}}(\cdot,t|\sigma)\big) > \gamma^\delta\bigg) \leqslant c\gamma^k \tag{3.102}$$

where N is as in definition 3.1.1.

Remarks. In equation (3.102), $\sigma(\cdot,t)$ and $m_{\gamma,t_{n_0a}}(\cdot,t|\sigma)$ are close to each other over times that exceed t_c. The proof exploits (3.75) that holds for arbitrary initial spin configurations that may not be small in the sense of the seminorms of definition 3.1.2.

In the next subsection we will show that the probability of $F_{\gamma,a,\zeta}(n,n,C)$ goes to 1 as $\gamma \to 0$: such a result, together with theorem 3.4.0 and theorem 3.2.0, will then complete the proof of theorem 2.3.3.

Lemma 3.4.1. *For any $a \in (0,\tau_c)$ sufficiently small the following holds. Let N be as in definition 3.1.1 and $0 \leqslant n < N$. Let δ and ζ be both positive and*

$$\delta < \zeta - (\alpha^\star - \alpha)a + \alpha a n\,. \tag{3.103}$$

Then for any $k > 0$ there is c so that for all spin configurations

$$\mathbb{P}^\gamma_{\sigma,t_{na}}\big(\|\sigma(\cdot,t_{(n+1)a}) - m_{\gamma,t_{na}}(\cdot,t_{(n+1)a}|\sigma)\|_{n+1,n+1,\gamma,a,\zeta} > \gamma^\delta\big) < c\gamma^k\,. \tag{3.104}$$

Moreover, for any C, k, any $\zeta > 0$ small enough and any $0 < \delta' < \zeta$, there is c so that for all $\|\sigma\|_{n,n,\gamma,a,\zeta} \leqslant C$,

$$\mathbb{P}^\gamma_{\sigma,t_{na}}\big(\|\sigma(\cdot,t_{(n+1)a}) - m_{\gamma,t_{na}}(\cdot,t_{(n+1)a}|\sigma)\|_{n+1,n+1,\gamma,a,\zeta} > \gamma^{\delta'}\big) < c\gamma^k\,. \tag{3.105}$$

Proof. Let

$$\tilde\sigma(x) = \sigma(x,t_{(n+1)a}) - m_\gamma(x,t_{(n+1)a}|\sigma)\,. \tag{3.106}$$

Setting

$$\epsilon = \delta - \zeta - \alpha a(n+1) + d/2 > 0 \tag{3.107}$$

and using the Chebishev inequality we have, for any positive integer ℓ.

$$\mathbb{P}^\gamma_{\sigma,t_{na}}\big(|(J_\gamma\circ\tilde\sigma)(x)| > \gamma^\epsilon\big) \leqslant \gamma^{-2\epsilon\ell}\sum_{\underline{y}\in\mathbb{Z}^{2d\ell}}\left[\prod_{i=1}^{2\ell} J_\gamma(x,y_i)\right]\mathbb{E}^\gamma_{\sigma,t_{na}}\big(\tilde\sigma(y_1)\cdots\tilde\sigma(y_{2\ell})\big)\,. \tag{3.108}$$

By distinguishing the sets $\underline{y} \in \mathbb{Z}^{2d\ell}_{\neq}$ from the others, we obtain, using (3.75),

$$\leqslant \max\big\{c(\gamma,\ell)\gamma^{(d-2\epsilon)\ell};\ [\mathrm{e}^{\alpha^\star a}\gamma^{d/2-\epsilon}]^{2\ell}\big\}$$

where $c(\gamma,\ell)$ is a bounded function of its arguments (that may be bounded in terms of the sup of $J(|r|)$ and of a combinatorial factor). Then there is $\delta' > 0$ so that, for a suitable constant c

$$\mathbb{P}^\gamma_{\sigma,t_{na}}\Big(|(J_\gamma\circ\tilde\sigma)(x)| > \gamma^\epsilon\Big) \leqslant c\gamma^{\delta'\ell}\,. \tag{3.109}$$

Then

$$\mathbb{P}^{\gamma}_{\sigma,t_{na}}\Big(\|\sigma(\cdot,t_{(n+1)a})-m_{\gamma}(\cdot,t_{(n+1)a}|\sigma)\|_{n+1,n+1,\gamma,a,\zeta}>\gamma^{\delta}\Big)$$
$$\leqslant[2R_{n+1,\gamma}+1]^d\sup_{|x|\leqslant R_{n+1,\gamma}}\mathbb{P}^{\gamma}_{\sigma,t_{na}}\big(|(J_{\gamma}\circ\tilde{\sigma})(x)|>\gamma^{\epsilon}\big)\,.$$

By equation (3.109) and the arbitrarity of ℓ we then obtain (3.104).

The same proof works as for (3.105) provided we use (3.76) instead of (3.75), the lemma is therefore proved. □

Corollary 3.4.2.
For any $\tau_0\in(0,\tau_c)$ and for any $a>0$ such that $\tau_0=n_0a$, $n_0\in\mathbb{Z}_+$, and so small that

$$0<\delta<\alpha\tau_0-(\alpha^\star-\alpha)a \tag{3.110}$$

the following holds. Let $\zeta<\delta$ be as in (3.21). Then for any C_{n_0} there are C_n, $n_0<n\leqslant N$, and for any k, c so that for all $\gamma>0$ and for all $\|\sigma\|_{n_0,n_0,\gamma,a,\zeta}\leqslant C_{n_0}$,

$$\mathbb{P}^{\gamma}_{\sigma,t_{n_0a}}\big(\|\sigma(\cdot,t_{na})\|_{n,n,\gamma,a,\zeta}>C_n\big)<c\gamma^k \tag{3.111}$$

$$\mathbb{P}^{\gamma}_{\sigma,t_{n_0a}}\big(\|\sigma(\cdot,t_{na})-m_{\gamma,t_{n_0a}}(\cdot,t_{na}|\sigma)\|_{n,n,\gamma,a,\zeta}>\gamma^{\delta}\big)<c\gamma^k\,. \tag{3.112}$$

If, moreover, $|||\sigma|||_{n_0,n_0,\gamma,a}\leqslant C_{n_0}$, for any $0\leqslant h<1$

$$\mathbb{P}^{\gamma}_{\sigma,t_{n_0a}}\big(\sigma(\cdot,t_{na})\in F_{\gamma,a,\zeta}(n+h,n,C_n)\big)\geqslant 1-c\gamma^k\,. \tag{3.113}$$

Proof. By lemma 3.4.1, letting δ as in (3.110), we have for $n>n_0$

$$\mathbb{P}^{\gamma}_{\sigma,t_{n_0a}}\big(\|\sigma(\cdot,t_{na})-m_{\gamma,t_{(n-1)a}}(\cdot,t_{na}|\sigma(\cdot,t_{(n-1)a})\|_{n,n,\gamma,a,\zeta}>\gamma^{\delta}\big)\ \leqslant\ c\gamma^k\,. \tag{3.114}$$

We call

$$H(n,\gamma)=\sum_{i=0}^{n-n_0-1}(3c_1)^i2\gamma^{\delta}+(3c_1)^{n-n_0}C_{n_0}\qquad H(n_0,\gamma)=C_{n_0}$$

and, using (3.114), we will show, by induction on n, that

$$\mathbb{P}^{\gamma}_{\sigma,t_{n_0a}}\big(\|\sigma(\cdot,t_{na})\|_{n,n,\gamma,a,\zeta}\leqslant H(n,\gamma)\big)\geqslant 1-(n-n_0)c\gamma^k \tag{3.115}$$

that implies (3.111). To prove (3.115) we first observe that by (3.52), for all γ small enough, we have that in the set $\{\|\sigma(\cdot,t_{(n-1)a})\|_{n-1,n-1,\gamma,a,\zeta}\leqslant H(n-1,\gamma)\}$

$$\|m_{\gamma,t_{(n-1)a}}\big(\cdot,t_{na}|\sigma(\cdot,t_{(n-1)a})\big)\|_{n,n,\gamma,a,\zeta}\leqslant 3c_1\|\sigma(\cdot,t_{(n-1)a})\|_{n-1,n-1,\gamma,a,\zeta}+\gamma^{\delta}$$

so that, by (3.114), the left-hand side of (3.115) is $\geqslant\mathbb{P}^{\gamma}_{\sigma,t_{n_0a}}(\mathcal{A})-c\gamma^k$ where

$$\mathcal{A}:=\big\{2\gamma^{\delta}+3c_1\|\sigma(\cdot,t_{(n-1)a})\|_{n-1,n-1,\gamma,a,\zeta}\leqslant H(n,\gamma)\big\}$$
$$\bigcap\big\{\|\sigma(\cdot,t_{(n-1)a})\|_{n-1,n-1,\gamma,a,\zeta}\leqslant H(n-1,\gamma)\big\}\,.$$

By the choice of $H(n,\gamma)$,

$$\mathcal{A}=\big\{\|\sigma(\cdot,t_{(n-1)a})\|_{n-1,n-1,\gamma,a,\zeta}\leqslant H(n-1,\gamma)\big\}\,.$$

By the induction hypothesis we then get (3.115) and (3.111) is therefore proven.

To prove (3.112) we use the relation

$$\|m_{\gamma,t_{(n-1)a}}(\cdot,t_{na}|\sigma(\cdot,t_{(n-1)a})-m_{\gamma,t_{n_0a}}(\cdot,t_{na}|\sigma)\|_{n,n,\gamma,a,\zeta}$$
$$\leqslant 3c_1\|\sigma(\cdot,t_{(n-1)a})-m_{\gamma,t_{n_0a}}(\cdot,t_{(n-1)a}|\sigma)\|_{n-1,n-1,\gamma,a,\zeta}+\gamma^{\delta} \tag{3.116}$$

which follows from (3.53) for γ small enough and when the right-hand side is bounded uniformly in γ. We then have, by an argument analogous to the previous one:

$$\mathbb{P}^{\gamma}_{\sigma,t_{n_0 a}}\left(\|\sigma(\cdot,t_{na}) - m_{\gamma,t_{n_0 a}}(\cdot,t_{na}|\sigma)\|_{n,n,\gamma,a,\zeta} > \sum_{i-0}^{n-n_0-1}(3c_1)^i 2\gamma^{\delta}\right) \leqslant (n-n_0)c\gamma^k \tag{3.117}$$

which proves (3.112).

By equation (3.112) and recalling (3.13), there is $\delta' > 0$ so that, for all γ small enough,

$$\begin{aligned}&\mathbb{P}^{\gamma}_{\sigma,t_{n_0 a}}\big(|||\sigma(\cdot,t_{na})|||_{n+h,n,\gamma,a} \leqslant C_n\big)\\ &\geqslant \mathbb{P}^{\gamma}_{\sigma,t_{n_0 a}}\big(|||m_{\gamma,t_{n_0 a}}(\cdot,t_{na}|\sigma)|||_{n+h,n,\gamma,a} \leqslant C_n - \gamma^{\delta'}\big) - c\gamma^k\,.\end{aligned}$$

By equation (3.29), if $|||m_{\gamma,t_{n_0 a}}(\cdot,t_{(n-1)a}|\sigma)|||_{n-1,n-1,\gamma,a} < C$, then

$$\begin{aligned}\phi_{n,\gamma}(x)|m_{\gamma,t_{n_0 a}}(x,t_{na}|\sigma)| &\leqslant \mathrm{e}^{\alpha_\gamma t_a}\big|\big(p^{\gamma}_{t_a}\circ m_{\gamma,t_{n_0 a}}(\cdot,t_{(n-1)a}|\sigma)\big)(x)\big|\phi_{n,\gamma}(x)\\ &\quad + c'\big(2c_1\,\mathrm{e}^{\alpha_\gamma t_{na}}\gamma^{d/2-\zeta}\|m_{\gamma,t_{n_0 a}}(\cdot,t_{(n-1)a}|\sigma)\|_{n,n-1,\gamma,a,\zeta} + c\gamma^k\big)^2\end{aligned} \tag{3.118}$$

which implies

$$|||m_{\gamma,t_{n_0 a}}(\cdot,t_{na}|\sigma)|||_{n,n,\gamma,a} \leqslant |||m_{\gamma,t_{n_0 a}}(\cdot,t_{(n-1)a}|\sigma)|||_{n-1,n-1,\gamma,a} + c''\,.$$

Thus

$$\begin{aligned}&\mathbb{P}^{\gamma}_{\sigma,t_{n_0 a}}\big(|||m_{\gamma,t_{n_0 a}}(\cdot,t_{na}|\sigma)|||_{n,n,\gamma,a} \leqslant C_n - \gamma^{\delta'}\big)\\ &\geqslant \mathbb{P}^{\gamma}_{\sigma,t_{n_0 a}}\big(|||m_{\gamma,t_{n_0 a}}(\cdot,t_{(n-1)a}|\sigma)|||_{n-1,n,\gamma,a} \leqslant C_n - \gamma^{\delta'} - c''\big)\,.\end{aligned}$$

Defining $C_n = C_{n-1} + c''$ and iterating the above procedure we then prove (3.113), thus completing the proof of the corollary. □

Proof of theorem 3.4.0. It is enough to prove (3.102) separately when $t_{na} \leqslant t \leqslant t_{(n+1)a}$, $n < N$, and when $t_{Na} \leqslant t \leqslant t_{(N+2)a}$. We begin with the former and, given $n < N$, we consider $t_{na} \leqslant t \leqslant t_{(n+1)a}$ and write

$$\sigma(x,t) - m_{\gamma,t_{n_0 a}}(x,t|\sigma) = [\sigma(x,t) - m_{\gamma,t_{na}}\big(x,t|\sigma(\cdot,t_{na})\big)] + L_\gamma(x,t) \tag{3.119}$$

$$L_\gamma(x,t) = m_{\gamma,t_{na}}\big(x,t|\sigma(\cdot,t_{na})\big) - m_{\gamma,t_{n_0 a}}(x,t|\sigma)\,. \tag{3.120}$$

By equation (3.51)

$$\begin{aligned}A^{\star}_{\gamma,t,a,b_0,h}\big(L_\gamma(\cdot,t)\big) &\leqslant A^{\star}_{\gamma,t,a,b_0,h}\big(\sigma(\cdot,t_{na}) - m_{\gamma,t_{n_0 a}}(\cdot,t_{na}|\sigma)\big)\\ &\quad + c\gamma^{-\zeta}\|\sigma(\cdot,t_{na}) - m_{\gamma,t_{n_0 a}}(\cdot,t_{na}|\sigma)\|_{n,n,\gamma,a,\zeta} + c\gamma^k\,.\end{aligned}$$

We choose $\zeta < \delta$, δ satisfying (3.110), so that, by (3.112),

$$\begin{aligned}&\mathbb{P}^{\gamma}_{\sigma,t_{n_0 a}}\bigg(\sup_{t_{na}\leqslant t\leqslant t_{(n+1)a}} A^{\star}_{\gamma,t,a,b_0,h}\big(L_\gamma(\cdot,t)\big) \leqslant \gamma^{\delta}\big)\\ &\geqslant \mathbb{P}^{\gamma}_{\sigma,t_{n_0 a}}\big(A^{\star}_{\gamma,t_{na},a,b_0,h}\big(\sigma(\cdot,t_{na}) - m_{\gamma,t_{n_0 a}}(\cdot,t_{na}|\sigma)\big) \leqslant 2\gamma^{\delta}\bigg) - c\gamma^k\,.\end{aligned}$$

Recalling (3.119) and using an inductive argument, we reduce the proof for showing that for any $n_0 \leqslant n < N$:

$$\mathbb{P}^{\gamma}_{\sigma,t_{n_0 a}}\left(\sup_{t_{na}\leqslant t\leqslant t_{(n+1)a}} A^{\star}_{\gamma,t,a,b_0,h}\big(\sigma(\cdot,t) - m_{\gamma,t_{na}}\big(\cdot,t|\sigma(\cdot,t_{na})\big)\big) > \gamma^{\delta}\right) \leqslant c\gamma^k\,. \tag{3.121}$$

To prove (3.121) we will use the following criterion. Given $\gamma > 0$ and $t \geqslant 0$, let $f_{\gamma,t} : \{-1,1\}^{\mathbb{Z}^d} \to \mathbb{R}_+$ depend only on less than γ^{-k_0}, $k_0 > 0$, spins of the configuration σ and suppose also that there is $\delta_0 > 0$ so that for all $\gamma > 0$ small enough

$$\sup_{\sigma \in \{-1,1\}^{\mathbb{Z}^d}} \sup_{x \in \mathbb{Z}^d} \left| f_{\gamma,t}(\sigma^x) - f_{\gamma,t}(\sigma) \right| \leqslant \gamma^{\delta_0} . \tag{3.122}$$

Lemma 3.4.3. *Let $f_{\gamma,t}$ be as above. Suppose that there are $\delta_1 \in (0, \delta_0)$, an interval T_γ in $[0, \gamma^{-k_1}]$, for some $k_1 > 0$, and, for any k, c so that, denoting by P^γ the law of the process in T_γ with some, unspecified, initial condition*

$$\sup_{t \in T_\gamma} P^\gamma\left(f_{\gamma,t}(\sigma(\cdot,t)) \geqslant \gamma^{\delta_1}\right) \leqslant c\gamma^k . \tag{3.123}$$

Then for any k there is c'_k so that

$$P^\gamma\left(\sup_{t \in T_\gamma} f_{\gamma,t}(\sigma(\cdot,t)) \geqslant \gamma^{2\delta_1}\right) \leqslant c'_k \gamma^k . \tag{3.124}$$

We omit the proof of this elementary lemma and refer to the proof of proposition 4.6 in [8] for a similar statement. For the reader convenience we give however a short outline of the proof.

Outline of the proof. We divide the time interval $[0, T_\gamma]$ into equal subintervals that are sufficiently short. They are such that the probability of the following set $\mathcal{B}$ is smaller than $c''_k \gamma^k$, for any given k. $\mathcal{B}$ is the set where in some of the subintervals at least two spins flip, among those on which f_γ depends. By our assumptions we can make the number of such subintervals grow only like some power of γ^{-1}. Here we have also used that the spin flip intensity is bounded.

Then with large probability we may reduce to trajectories where there is at most one 'relevant' spin flip in each subinterval. Recalling (3.122) and since $\delta_1 < \delta_0$, the proof of (3.124) follows from the assumption (3.123). □

We take

$$f_{\gamma,t} = \left| A^\star_{\gamma,t,a,b_0,h}\left(\sigma(\cdot,t) - m_{\gamma,t_{na}}(\cdot,t|\sigma)\right)\right|$$

and $T_\gamma = [t_{na}, t_{(n+1)a}]$. The variation of $f_{\gamma,t}$ due to a spin flip is bounded by

$$2\gamma^{-d/2}\gamma^{\alpha a}[c(d)\gamma^{-b_0 d}]^{-1}$$

where the square bracket is the volume of $B_{\gamma,x}$. Choosing b_0 close enough to 1, there is $\delta_0 > 0$ so that

$$2\gamma^{-d/2}\gamma^{\alpha a}[c(d)\gamma^{-b_0 d}]^{-1} \leqslant \gamma^{\delta_0} .$$

The spins on which $f_{\gamma,t}$ depends are contained in $R_{M,\gamma}$, with M fixed and large enough. Thus we can take $k_0 = 2d$, (for γ small enough). To prove (3.121) we thus need to check (3.123) with $P^\gamma = P^\gamma_{\sigma,t_{na}}$. Given $t \in [t_{na}, t_{(n+1)a}]$, we have

$$\begin{aligned} &\mathbb{P}^\gamma_{\sigma,t_{na}}\left(\left|A^\star_{\gamma,t,a,b_0}\left(\sigma(\cdot,t) - m_{\gamma,t_{na}}(\cdot,t|\sigma)\right)\right| > \gamma^\delta\right) \\ &\quad \leqslant \sup_{x \in R_{M,\gamma}} c(R^\star_{n+1,\gamma})^d \mathbb{P}^\gamma_{\sigma,t_{na}}\left(\left|A_{\gamma,x}\left(\sigma(\cdot,t) - m_{\gamma,t_{na}}(\cdot,t|\sigma)\right)\right| > K^{-1}_{\gamma,t}\gamma^\delta\right) . \end{aligned} \tag{3.125}$$

We bound the probability using the Chebishev inequality with power $2k$ and obtain the bound:

$$c[K_{\gamma,t}\gamma^{-\delta}]^{2k} \max\left\{[\gamma^{d/2}\, \mathrm{e}^{\alpha(t-t_{na})}]^{2k}; |B_{\gamma,x}|^{-k}\right\} \tag{3.126}$$

where c is a suitable constant. This term is bounded by

$$c' \max\left\{ e^{-\alpha t_{na}}\gamma^{-\delta}]^{2k}; [\gamma^{d/2-\alpha na-\delta}\gamma^{(db_0)/2}]^{2k}\right\} \tag{3.127}$$

(the second term is obtained by computing $K_{\gamma,t}$ at $t = t_{na}$ that gives an upper bound. We thus see that given any $a > 0$, by choosing $\delta > 0$ small enough and b_0 close enough to 1 we may bound (3.127) as $c'' e^{-Bk}$ with $B > 0$. This concludes the proof of (3.121).

To complete the proof of theorem 3.4.0 we need to consider the time interval $t_{Na} \leqslant t \leqslant t_{(N+2)a}$. We use again (3.119) and 3.120 with $n = N$. We have

$$L_\gamma(x, t_{Na}) = \sigma(\cdot, t_{Na}) - m_{\gamma,t_{n_0a}}(x, t_{Na}|\sigma) \qquad |x| \leqslant R_{N+h,\gamma}\,. \tag{3.128}$$

We have already proven that this term is bounded by γ^δ with probability larger than $1-c\gamma^k$. In section 4 of [8] it is proven that if (3.128) holds then there are a', h' and δ' positive so that for $|x| \leqslant R_{N+h',\gamma}$

$$\sup_{t_{Na}\leqslant t\leqslant t_{(N+2)a}} A^\star_{\gamma,t,a,b_0,h}\big(L_\gamma(\cdot,t)\big) \leqslant e^{-\alpha t_{Na}}\gamma^{-d/2+\delta'}$$
$$= e^{\alpha(t_c - t_{Na})}\gamma^{\delta'} \leqslant e^{\alpha t_a}\gamma^{\delta'} \leqslant \gamma^{\delta''}$$

with $\delta'' > 0$ for a small enough.

It thus remain to consider the first term on the right-hand side of (3.119), with $n + N$. This is dealt with exactly like when $n < N$, provided $2a < a^\star$, $a^\star$ as in theorem 3.3.0. We have thus completed the proof of theorem 3.4.0. □

3.5. Probability estimates on the seminorms

In this subsection we will prove the following:

Theorem 3.5.0. *For any $a \in (0, \tau_c)$, any $0 \leqslant n \leqslant N$, N as in definition 3.1.1, any $C > 1$ and any $k > 0$ there is c so that*

$$P^\gamma_{\mu_0}\big(\{\sigma(\cdot, t_{na}) \in F_{\gamma,a,\zeta}(n, n, C)\}\big) \geqslant 1 - cC^{-k}\,. \tag{3.129}$$

By theorems 3.2.0 and 3.4.0, if $\sigma \in F_{\gamma,a,\zeta}(1, 1, C)$, then it is also in $F_{\gamma,a,\zeta}(n, n, C')$, $1 \leqslant n \leqslant N$, with probability larger than $1-c\gamma^k$, if C' is suitably large. The whole problem is therefore the proof of (3.129) with $n = 1$. The special role played by the value $n = 1$ has already been remarked after theorem 3.4.0.

We will easily see that (3.129) holds when $n = 0$ and indeed we will show that it is possible to replace the measure μ_0 by one supported by the single configuration σ, provided $\sigma \in F_{\gamma,a,\zeta}(0, 0, C)$, for some C.

Let

$$X(x) = (\pi^\gamma \circ \tilde\sigma)(x, t_a) \qquad \tilde\sigma(x, t_a) = \sigma(x, t_a) - m_\gamma(x, t_a|\sigma) \tag{3.130}$$

and consider the process starting from σ at time 0, (call $\mathbb{P}^\gamma_\sigma$ its law). We suppose that

$$\sigma \in F_{\gamma,a,\zeta}(0, 0, C) \tag{3.131}$$

with C independent of γ. Calling

$$\Gamma = \lambda^{d/2}\gamma^{-\alpha a+d/2} \tag{3.132}$$

we have

Lemma 3.5.2. *For any k there is c so that for any $C > 0$, $u \geqslant 1$ and all $\sigma \in F_{\gamma,a,\zeta}(0, C)$*

$$\mathbb{P}^\gamma_\sigma\big(|X(x)| > u\Gamma\big) \leqslant c[1 + C^{2k}\gamma^\delta]u^{-k} \qquad \textit{for all} \quad x \in \mathbb{Z}^d\,. \tag{3.133}$$

Proof. As in the proof of lemma 3.4.1 we get

$$\mathbb{P}^{\gamma}_{\sigma}\big(|X(x)| > u\Gamma\big) \leqslant (u\Gamma)^{-2\ell} \sum_{\underline{y}\in\mathbb{Z}^{d2\ell}} \left[\prod_{i=1}^{2\ell} \pi^{\gamma}(x, y_i)\right] \mathbb{E}_{\sigma}\left(\prod_{i=1}^{2\ell} \tilde{\sigma}(x, y_i)\right)$$
$$\leqslant (u\Gamma)^{-2\ell} \max\left\{c(\gamma, \ell)\gamma^{d\ell}; [c\{1 + C\gamma^{\delta}\}\Gamma]^{2\ell}\right\} \tag{3.134}$$

which proves the lemma. □

The bound (3.133) is inadequate to bound $|||\tilde{\sigma}|||_{1,1,\gamma}$, because of the sup involved in the definition of the seminorm. When the process starts at time t_{na}, $n \geqslant 1$, then we gain a factor $\gamma^{\delta'\ell}$, $\delta' > 0$, see equation (3.109), by which we control the sup over regions with γ^{-k} sites, for arbitrary k, after choosing ℓ large enough. Here we do not have any extra power of γ from the estimate (3.133), and we cannot repeat the previous proof. We then use an argument from [14, 20].

Let

$$L_{\gamma} = (\lambda\gamma)^{-1} \qquad \mathbb{Z}^d_{\gamma} = \{L_{\gamma}x, \ \ x \in \mathbb{Z}^d\} \tag{3.135}$$

then, by (3.133), for all $\sigma \in F_{\gamma,a,\zeta}(0, 0, C)$

$$\mathbb{P}^{\gamma}_{\sigma}\left(\sup_{x\in\mathbb{Z}^d_{\gamma}} \psi_{1,\gamma}(x)\big|X(x)\big| > u\Gamma\right) \leqslant \sum_{x\in\mathbb{Z}^d} \mathbf{1}(L_{\gamma}|x| \leqslant R_{1,\gamma})\mathbb{P}^{\gamma}_{\sigma}\big(\big|X(L_{\gamma}x)\big| > \Gamma u(1 + |x|)^b\big)$$
$$\leqslant \sum_{x\in\mathbb{Z}^d} c[1 + C^{2k}\gamma^{\delta}]u^{-k}(1 + |x|)^{-bk} \tag{3.136}$$

(recall that $\psi_{1,\gamma}(x) = 0$ if $|x| > R_{1,\gamma}$). For k large enough the series converges, hence

$$\mathbb{P}^{\gamma}_{\sigma}\left(\sup_{x\in\mathbb{Z}^d_{\gamma}} \psi_{1,\gamma}(|x|)\,\big|X(x)\big| > \Gamma u\right) \leqslant c[1 + C^{2k}\gamma^{\delta}]u^{-k} \tag{3.137}$$

with c a suitable constant.

To have the sup also over $x \in \mathbb{Z}^d \setminus \mathbb{Z}^d_{\gamma}$, we use (3.137) and an estimate on the variation of $\tilde{\sigma}$ in cubes of side L_{γ}. For any $x = (x_1, \ldots, x_d) \in \mathbb{Z}^d_{\gamma}$ let

$$\Lambda_x = \{y \in \mathbb{Z}^d : \ x_i \leqslant y_i \leqslant x_i + L_{\gamma}, \ \ i = 1, \ldots, d\}\,. \tag{3.138}$$

We will control the variations of $\tilde{\sigma}$ in Λ_x by a Sobolev-like inequality, with an integral norm involving $\tilde{\sigma}$ and its 'derivatives'.

We explain the idea in the continuum with true derivatives. Let $\Pi_{0,R}$, $R \in \mathbb{R}^d$, be the parallelepiped in $\mathbb{R}^d$ with extreme points 0 and R; i.e. the set of points r such that $0 \leqslant r_i \leqslant R_i$, we are supposing that all $R_i > 0$. Let $f \in C^d(\mathbb{R}^d)$ and denote by

$$f^{(J)} = \frac{\partial^{|J|} f}{\partial x_{j_1} \ldots \partial x_{j_\ell}} \qquad J = (j_1, \ldots, j_\ell) \subseteqq I = \{1, \ldots, d\}\,. \tag{3.139}$$

We define $(r_J, 0)$ as the element $r' \in \mathbb{R}^d$ such that $r'_i = r_i$ for all $i \in J$ and 0 otherwise. We then set

$$\Delta_J = \{(r_J, 0) : (r_J, 0) \in \Pi_{0,R}\} \tag{3.140}$$

and we have, $R = (R_1, \ldots, R_d)$ below,

$$f(R) - f(0) = \sum_{\emptyset\neq J\subseteqq I} \int \mathrm{d}r_J\, \mathbf{1}((r_J, 0) \in \Delta_J) f^{(J)}((r_J, 0)) \tag{3.141}$$

equation (3.141) can be proven.

Proof of (3.141). The proof is by induction on d, being obviously true for $d = 1$. We then suppose it true for $d - 1$ and prove it for d. We write

$$f(R) - f(R_1, \dots, R_{d-1}, 0) = \int_0^{R_d} \mathrm{d}r\, f^{(\{d\})}(R_1, \dots, R_{d-1}, r)\,.$$

We then use the induction assumption on the functions:

$g(R_1, \dots, R_{d-1}) := f(R_1, \dots, R_{d-1}, 0)$ and, given
$r, h(R_1, \dots, R_{d-1}) := f^{(\{d\})}(R_1, \dots, R_{d-1}, r)$

and obtain (3.141). □

The identity (3.141) holds in the discrete case a well. Let $f(x)$, $x \in \mathbb{Z}^d$, and first define

$$f^{(i)}(x) = (\lambda\gamma)^{-1}[f(x + e_i) - f(x)] \qquad e_i = (\delta_{j,i},\ \ j = 1, \dots, d) \tag{3.142}$$

and then $f^{(J)}$, by an iterative procedure. $\Pi_{0,y}$, $(x_J, 0)$ and Δ_J are defined as in the continuous case. We set

$$\int \mathrm{d}x_J\, f((x_j, 0)) = L_\gamma^{-|J|} \sum_{x_J} f((x_j, 0))\,. \tag{3.143}$$

We then have, with $y_+ = (y_i + 1,\ i = 1, \dots, d)$,

$$f(y_+) - f(0) = \sum_{\emptyset \neq J \subseteq I} \int \mathrm{d}x_J\, \mathbf{1}((x_j, 0) \in \Delta_J) f^{(J)}((x_j, 0))\,. \tag{3.144}$$

We will bound the integrals on the right-hand side by using the Cauchy Schwartz inequality. We thus set for $x \in \mathbb{Z}_\gamma^d$:

$$N_{\gamma,x}(f)^2 = \sum_{\emptyset \neq J \subseteq I} \int \mathrm{d}y_J\, \mathbf{1}\big((y_j, 0) \in \Delta_J(x)\big)[f^{(J)}((y_J, 0))]^2 \tag{3.145}$$

where the set $\Delta_J(x)$ is the set Δ_J corresponding to the cube Λ_x. From equation (3.144) we then have

$$\sup_{y \in \Lambda_x} |f(y) - f(x)| \leqslant c N_{\gamma,x}(f) \tag{3.146}$$

where c^2 bounds the 'integrals' over $\Delta_J(x)$ of $\mathrm{d}x_J$, for all J. We next prove the following:

Lemma 3.5.3. *With the above notation, given any $C > 0$ and any $k > 0$ there is c so that for any σ satisfying (3.131)*

$$\mathbb{P}_\sigma^\gamma\left(\sup_{x \in \mathbb{Z}_\gamma^d} \phi_{1,\gamma}(|x|) N_{\gamma,x}(X(\cdot)) > u\Gamma\right) \leqslant c[1 + C^{2k}\gamma^\delta]u^{-k} \tag{3.147}$$

for all $u \geqslant 1$, ($X(x)$ being defined in (3.130)).

Proof. The basic ingredient of the proof is the following estimate on π^γ which is proven in lemma 4.3.7: let $\pi^\gamma(x) \equiv \pi^\gamma(0, x)$, then

$$(\pi^\gamma)^{(J)}(x) \leqslant (\lambda\gamma)^d P(\lambda\gamma|x|) \tag{3.148}$$

where $P(|x|)$ satisfies

$$\int \mathrm{d}r\, P(|r|) < \infty \qquad \sup_r P(|r|) < \infty\,.$$

As in lemma 3.5.2, we have

$$\mathbb{P}^{\gamma}_{\sigma}\big(N_{\gamma,x}(X(\cdot)) > u\Gamma\big) \leqslant (u\Gamma)^{-2\ell} \sum_{J_1\dots J_\ell} \prod_{i=1}^{\ell} \int \mathrm{d}z^i_{J_i}\, \mathbf{1}((z^i_{J_i},0)\in \Delta_{J_i}(x))$$
$$\times \sum_{\underline{y}} \Bigg[\prod_{i=1}^{\ell} (\pi^{\gamma})^{(J_i)}(y_{2i}-x_i)(\pi^{\gamma})^{(J_i)}(y_{2i+1}-x_i)\Bigg] \mathbb{E}_{\sigma}\big(\tilde\sigma(y_1)\dots\tilde\sigma(y_{2\ell})\big) \tag{3.149}$$

where $\underline{y}=(y_1,\dots,y_{2\ell})$ and $x_i=(z^i_{J_i},0)$. As before the leading term is when $\underline{y}\in\mathbb{Z}^{d2\ell}_{\neq}$ and since the integral is finite, recall the definition (3.143), we obtain an estimate as in (3.133). We then conclude the proof of the lemma, proceeding like in the proof of (3.137), we omit the details. □

In the next lemma we complete the bounds on the seminorms.

Lemma 3.5.4. *For any $k>0$ there is c so that for all $u\geqslant 1$:*

$$\mu_0\big(F_{\gamma,a,\zeta}(0,0,u)\big) \;\geqslant\; 1-cu^{-k} \tag{3.150}$$

and for all $\sigma\in F_{\gamma,a,\zeta}(0,C)$ and γ small enough

$$\mathbb{P}^{\gamma}_{\sigma}\big(|||\tilde\sigma|||_{1,1,\gamma,a} > u\big) \;\leqslant\; c[1+C^{2k}\gamma^{\delta}]u^{-k}\,. \tag{3.151}$$

More precisely,

$$P^{\gamma}_{\sigma}\big(|||\sigma|||_{1,1,\gamma,a} > 3c_1u\big)\leqslant c[1+C^{2k}\gamma^{\delta}]u^{-k} \qquad P^{\gamma}_{\sigma}\big(\|\sigma\|_{1,1,\gamma,a,\zeta} > 4c_1C\big)\leqslant c\gamma^{k} \tag{3.152}$$

Proof. Equation (3.151) is an obvious corollary of (3.133) and (3.147). The first inequality in (3.152) follows from (3.151) and (3.19). The second inequality in (3.152) follows from (3.105) and (3.52).

In the proof of (3.150) the bound on $\|\sigma\|_{0,0,\gamma}$ is easy and it is omitted. To bound the probability that $|||\sigma|||_{0,0,\gamma} > u$ we use the same procedure as in lemmas 3.5.2 and 3.5.3. The bounds on the w and v functions are not necessary here because we know the measure explicitly (μ_0), which is a product measure. Except for such a simplifying feature the proof is unchanged and it is omitted. □

We conclude this subsection by proving a bound needed in section 4, namely (4.26).

Lemma 3.5.5. *For any $a\in(0,\tau_c)$ small enough there is $\delta^\star>0$ and for all $n<q\leqslant N$, all $\zeta>0$ small enough, all C and all k there is c so that for all $\sigma\in F_{\gamma,a,\zeta}(n,n,C)$*

$$\mathbb{P}^{\gamma}_{\sigma,t_{na}}\big(|||\sigma(\cdot,t_{qa})-m_\gamma(\cdot,t_{qa}|\sigma)|||_{q,q,\gamma,a} > \gamma^{\delta^\star}\big) \;\leqslant\; c\gamma^{k}\,. \tag{3.153}$$

Proof. Equation (3.153) follows from (3.112) and (3.13). □

3.6. Proof of theorems 2.3.2 and 2.3.3

Proof of theorem 2.3.2. To prove (2.34) we take $a>0$ small enough and distinguish $\tau\leqslant a$ from $\tau>a$. In the former case we use the Chebishev inequality as in the proof of lemma 3.4.1. and then proposition 3.3.3. The proof when $\tau=a$ is a consequence of theorem 2.3.3 that we will prove at the end of the section.

We fix $a\in(0,\tau_c)$ small enough and let N be as in definition 3.1.1. We want to prove that for any $k\geqslant 1$,

$$\lim_{\gamma\to 0}\sup_{\underline{x}\in\mathbb{Z}^{dk}_{\neq}} \mathbf{1}_{\{|x_i|\leqslant R_{N+1,\gamma},\ i=1,\dots,k\}} \sup_{t\leqslant t_c}\left|\mathbb{E}^{\gamma}_{\mu_0}\left(\prod_{i=1}^{k}\sigma(x_i,t)\right)\right| = 0 \tag{3.154}$$

which proves (2.35).

Recalling proposition 3.3.3 and lemma 3.3.2 part (2), there is $\delta > 0$ and for any k c so that

$$\sup_{\underline{x}\in\mathbb{Z}^{dk}_{\neq}} \sup_{t\leqslant t_{2a}} \left|\mathbb{E}^{\gamma}_{\mu_0}\left(\prod_{x\in\underline{x}}\sigma(x,t)\right)\right| = \sup_{\underline{x}\in\mathbb{Z}^{dk}_{\neq}} \sup_{t\leqslant t_{2a}} \left|v^{\gamma}(\underline{x},t)\right| \leqslant c\gamma^{\delta k} \tag{3.155}$$

which proves (3.154) with the sup limited to $t \leqslant t_{2a}$.

For $t \geqslant t_{2a}$, let $n \geqslant 1$ be such that $t_{(n+1)a} \leqslant t \leqslant t_{(n+2)a}$, then

$$\begin{aligned}\left|\mathbb{E}^{\gamma}_{\mu_0}\left(\prod_{x\in\underline{x}}\sigma(x,t)\right)\right| &= \left|\mathbb{E}^{\gamma}_{\mu_0}\left(\mathbb{E}^{\gamma}_{\sigma(\cdot,t_{na}),t_{na}}\left(\prod_{x\in\underline{x}}\sigma(x,t)\right)\right)\right| \\ &\leqslant 1 - \mathbb{P}^{\gamma}_{\mu_0}\big(\sigma(\cdot,t_{na}) \in F_{\gamma,a,\zeta}(n,n,C)\big) + \sup_{\underline{x}\in\mathbb{Z}^{dk}_{\neq}} \sup_{\sigma} \sup_{t_a\leqslant t\leqslant t_{2a}} |v^{\gamma}(\underline{x},t|\sigma,t_a)| \\ &\quad + k2^{k-1} \sup_{\sigma(\cdot,t_{na})\in F_{\gamma,a,\zeta}(n,n,C)} \sup_{t_{(n+1)a}\leqslant t\leqslant t_{(n+2)a}} \sup_{|x|\leqslant R_{n+1,\gamma}} |m_{\gamma,t_{na}}(x,t|\sigma(\cdot,t_{na}))| .\end{aligned} \tag{3.156}$$

To derive (3.156) we have written $\sigma(x,t) = \tilde{\sigma}(x) + m_{\gamma}(x)$ where $\tilde{\sigma}(x)$ is a shorthand form of $\sigma(x,t) - m_{\gamma}(x)$ and $m_{\gamma}(x)$ stands for $m_{\gamma,t_{na}}\big(x,t|\sigma(\cdot,t_{na})\big)$. Then

$$\prod_{i=1}^{k}\left[\tilde{\sigma}(x_i) + m_{\gamma}(x_i)\right] = \prod_{i=1}^{k}\tilde{\sigma}(x_i) + \sum_{i=1}^{k-1}\left[\prod_{\ell=1}^{i}\tilde{\sigma}(x_{\ell})\right] m_{\gamma}(x_{i+1}) \left[\prod_{j=i+2}^{k}\sigma(x_j)\right].$$

The first term gives v^{γ}. The other k terms are bounded as on the right-hand side of (3.156), recalling that both σ and m_{γ} are bounded by 1.

In equation (3.156) we take the lim sup as $\gamma \to 0$, then, by (3.97), the contribution of the term with v^{γ} vanishes. Also the term with m_{γ} in (3.156) vanishes as $\gamma \to 0$, by (3.16) and (3.18). We thus have

$$\limsup_{\gamma\to 0} \sup_{t_{2a}\leqslant t\leqslant t_{Na}} \left|\mathbb{E}^{\gamma}_{\mu_0}\left(\prod_{x\in\underline{x}}\sigma(x,t)\right)\right| \leqslant \limsup_{\gamma\to 0} |\big[1 - \mathbb{P}^{\gamma}_{\mu_0}\big(\sigma(\cdot,t_{na}) \in F_{\gamma,a,\zeta}(n,n,C)\big)\big] .$$

By equations (3.113), (3.150) and (3.151) the right-hand side vanishes if we let $C \to \infty$. For $t_{Na} < t \leqslant t_c$ we proceed as before, with $n = N - 1$ and use proposition 3.2.5 to show that $m_{\gamma}\big(x,t|\sigma(\cdot,t_{na})\big)$ vanishes as $\gamma \to 0$, we omit the details. □

Proof of theorem 2.3.3. We take a small enough and such that $\tau_0 = na$, $n \in \mathbb{Z}_+$. Given C large enough we choose

$$\mathcal{G}^{(0)}_{\gamma} = F_{\gamma,a,\zeta}(n,n,C)$$

then (2.36) follows from theorem 3.5.0.

Equation (2.37) is proved in the remarks following theorem 3.2.0. Finally equation (2.38) is proved by theorem 3.4.0. □

4. The geometry of the interfaces

This is an intermediate section between the previous one where we studied the process until time t_c and the next one where we will extend the analysis through time $t^{\star}$. We have seen in section 3 that if a configuration σ at time $t_{na} < t_c$ is in $F_{\gamma,a,\zeta}(n,n,C)$, then the magnetization in the time interval $[t_{na}, t_c]$ is infinitesimal as $\gamma \to 0$, see theorem 2.3.2. To prove that at time $t^{\star}$ the magnetization is instead finite and the clusters mentioned in section 2 have developed, we need to show that the magnetization at time t_{na} is not too small. Since the set $F_{\gamma,a,\zeta}(n,n,C)$ is defined in terms of upper bounds, we need new conditions to control the magnetization from below. This lower bound cannot be uniform in space away from 0, because in such a case the magnetization, being continuous (as it is defined in terms of averages) would have a definite sign in the whole space, while regions that are far apart are certainly uncorrelated. Thus the magnetization will be 0 somewhere and the main goal of this section is to characterize the region where this happens. Before outlining the specific contents of this section we recall some definitions and notation.

Given $\tau_0 \in (0,\tau_c)$, we set a (the time-grid parameter of definition 3.1.1) so that

$$na = \tau_0 \qquad n \text{ a positive integer}. \tag{4.1}$$

Let N be as in definition 3.1.1, and recall that

$$Na < \tau_c = \frac{d}{2\alpha} \qquad t_{Na} = Na\lambda^{-2} < t_c = \tau_c\lambda^{-2} = \tau_c \log\gamma^{-1}. \tag{4.2}$$

Hereafter

$$\sigma \text{ is a configuration at time } t_{na} \equiv \tau_0\lambda^{-2}. \tag{4.3}$$

We will use the bound (3.18) for $t = t_{Na}$. We then require a so small that

$$\mathrm{e}^{-(t_{Na}-t_{na})} \leqslant \mathrm{e}^{\alpha t_{Na}}\gamma^{d/2}\lambda^{d/2}. \tag{4.4}$$

Recalling that $\gamma^{d/2}\mathrm{e}^{\alpha t_c} = 1$, equation (4.4) is valid if

$$\delta' := (N-n)a - \alpha(\tau_c - Na) > 0. \tag{4.5}$$

We also recall that $m_{\gamma,t_{na}}(x,t|\sigma)$ is the solution of (2.22) for $t \geqslant t_{na}$ with $m_{\gamma,t_{na}}(x,t_{na}|\sigma) = \sigma(x)$ and

$$\ell_\gamma(r|\sigma) = \lambda^{-d/2}\,\mathrm{e}^{\alpha(t_c-t_{Na})}\int \mathrm{d}r'\, q_{t_c-t_{Na}}(r-r')m_{\gamma,\tau_0}([\gamma^{-1}r'],t_{Na}|\sigma) \qquad r\in\mathbb{R}^d \tag{4.6}$$

where $q_t(r)$ is defined in (2.31). We finally recall

$$\hat{\ell}_\gamma(\xi|\sigma) = \ell_\gamma(\lambda^{-1}\xi|\sigma). \tag{4.7}$$

Given any $L > 0$, we denote by

$$S(0,\lambda^{-1}L) \subset \mathbb{R}^d \text{ the sphere of centre 0 and radius } \lambda^{-1}L. \tag{4.8}$$

We divide this section in three subsections. In section 4.1 we prove that $\hat{\ell}_\gamma$ converges to a Gaussian process, theorem 2.4.3. In section 4.2 we prove that the zeros of $\hat{\ell}_\gamma$ are close to the zeros of $m_{\gamma,\tau_0}(\cdot,t_c|\sigma)$. This result will be used in section 5.1 to characterize the development of the interfaces associated to $m_{\gamma,\tau_0}(\cdot,t^\star|\sigma)$. In section 4.3 we prove a central limit estimate for p_t^γ (see equation (2.29)) and its convergence to q_t.

4.1. The central limit theorem

We will prove the central limit theorem in the Sobolev spaces defined below.

Definition 4.1.1. *The spaces* $(H^m_{\mathrm{loc}},\mathcal{P}_{\gamma,\tau_0})$.
The space H^m_{loc} is the Sobolev space of functions on $\mathbb{R}^d$, with m generalized derivatives all in $L_2(\mathbb{R}^d,d\xi)$-local. For each $\gamma > 0$ and τ_0 as in (4.1), we denote by $\mathcal{P}_{\gamma,\tau_0}$ the probability on H^m_{loc} induced by $\mu^\gamma_{\tau_0\lambda^{-2}}$ via the map which associates to a configuration σ the function $\hat{\ell}_\gamma(\xi|\sigma)$ defined in (4.7).

By a Sobolev inequality, convergence in H^m implies convergence in a bounded region of the sup norms of the first $(m-d)$ derivatives, see [24], so that the above result yields a control of all derivatives of ℓ_γ in the limit $\gamma \to 0$. This will be used in section 4.2.

In the next proposition we prove the convergence of $\hat{\ell}_\gamma$ to the Gaussian process of theorem 2.4.3.

4.1.2 Proposition. *Let* $0 < \tau_0 < \tau_c$ *and* $\hat{\ell}_\gamma$ *be as in (4.7). Denote by* $\tilde{\mathcal{P}}$ *the Gaussian process in* H^m_{loc} *with zero average and covariance kernel*

$$C(\xi, \xi') = \left(1 + \frac{1}{\alpha}\right)\left(\frac{\alpha}{\pi\beta D d}\right)^{d/2} \mathrm{e}^{-\alpha(\xi-\xi')^2/(d\beta D)} \tag{4.9}$$

Then,

$$(\mathcal{P}_{\gamma,\tau_0}, H^m_{loc}) \to (\tilde{\mathcal{P}}, H^m_{loc}) \qquad as \quad \gamma \to 0\,. \tag{4.10}$$

Proof. We first prove tightness and then that any limit process is equal to $(\tilde{\mathcal{P}}, H^m_{loc})$. We thus start by proving tightness.

In section 4.3, see (4.151), we prove that for any $M > 0$ and any multindex I,

$$I = (i_1, \cdots, i_d) \qquad i_j \geqslant 0 \quad \text{for all} \quad j = 1, \ldots, d \ |I| = \sum_{j=1}^{d} i_j \tag{4.11}$$

there is a constant c so that

$$\left|\lambda^{-|I|} \frac{\partial^I q_{\tau_\lambda}(r)}{\partial r^I}\right| \leqslant c\, \frac{\lambda^d}{1 + (\lambda |r|^M)} \qquad \tau_\lambda = \lambda^{-2}(\tau_c - Na)\,. \tag{4.12}$$

We next take the I-derivative of both sides of (4.6) and use (4.12) on the right-hand side. If, for some $C > 0$, $\sigma \in F_{\gamma,a,\zeta}(n, n, C)$ (see equation (3.9) for notation) then from (3.17) it follows that there is c' so that for any $r \in \mathbb{R}^d$,

$$\begin{aligned} &\left| m_{\gamma,t_{na}}([\gamma^{-1} r], t_{Na}|\sigma) - \mathrm{e}^{\alpha_\gamma(t_{Na}-t_{na})}\big(p^\gamma_{t_{Na}-t_{na}} \circ \sigma\big)([\gamma^{-1} r])\right| \\ &\qquad \leqslant c'\, \mathrm{e}^{\alpha t_{Na}} \gamma^{d/2} (1 + \lambda |r'|)^b \gamma^{-\alpha N a + d/2 - \zeta} \end{aligned} \tag{4.13}$$

and from (3.18) and (4.5)

$$\mathrm{e}^{\alpha_\gamma(t_{Na}-t_{na})}\big(p^\gamma_{t_{Na}-t_{na}}([\gamma^{-1} r]) \leqslant \lambda^{d/2} \gamma^{-\alpha a N + d/2} (1 + \lambda |r|)^b \{1 + \lambda^{-d/2}\}\,. \tag{4.14}$$

We then have

$$\begin{aligned} \left|\lambda^{-|I|} \frac{\partial^I \ell_\gamma(r|\sigma)}{\partial r^I}\right| &\leqslant \lambda^{-d/2}\, \mathrm{e}^{\alpha(t_c - t_{Na})} \int \mathrm{d}r'\, \frac{c\lambda^d}{1 + (\lambda |r - r'|)^M} \big[\mathbf{1}(|r'| > R_{N,\gamma}) \\ &\quad + \lambda^{d/2} \gamma^{-\alpha a N + d/2} (1 + \lambda |r'|)^b \{1 + \lambda^{-d/2} + c' \gamma^\delta\}\big]\,. \end{aligned}$$

By letting M be large enough and recalling that $|r| \leqslant L\lambda^{-1}$, we see that the above derivative is bounded uniformly in γ. Since this is the derivative of $\hat{\ell}_\gamma$ (because of the prefactor $\lambda^{-|I|}$) we have concluded the proof that the probability $\mathcal{P}_{\gamma,\tau_0}$ in H^m_{loc} is tight, for any positive m. We shall consider hereafter m so large that the consideration below apply.

Identification of the limit laws. Let $(\mathcal{P}_{\tau_0}, H^m_{loc})$ be any limit law. This is identified by the marginals of $\mathcal{P}_{\tau_0}$ on the variables $\{\hat{\ell}(\xi_1|\sigma), \ldots \hat{\ell}(\xi_k|\sigma)\}$, $k \geqslant 1$, thought of as functions in H^m_{loc}.

We denote by $\sigma' = \sigma(\cdot, t_a)$, the configuration at time t_a and by $\sigma = \sigma(\cdot, t_{na})$ the configuration at time t_{na}. If, for some $C > 0$, $\sigma \in F_{\gamma,a,\zeta}(n, n, C)$, then, by (4.13), there are $\delta > 0$ and $\hat{c}$ so that

$$\sup_{|x| \leqslant R_{N,\gamma}} |m_\gamma(x, t_{Na}|\sigma) - \mathrm{e}^{\alpha_\gamma(t_{Na}-t_{na})}(p^\gamma_{t_{Na}-t_{na}} \circ \sigma)(x)| \leqslant \hat{c}\, \mathrm{e}^{-\alpha(t_c - t_{Na})} \gamma^\delta\,. \tag{4.15}$$

We have

$$e^{\alpha_\gamma(t_{Na}-t_{na})}(p^\gamma_{t_{Na}-t_{na}}\circ\sigma)(x) - e^{\alpha_\gamma(t_{Na}-t_a)}(p^\gamma_{t_{Na}-t_a}\circ\sigma')(x) = e^{\alpha_\gamma(t_{Na}-t_{na})}(p^\gamma_{t_{Na}-t_{na}}\circ g_\gamma)(x) \tag{4.16}$$

where

$$g_\gamma(x) = \sigma(x) - e^{\alpha_\gamma(t_{na}-t_a)}(p^\gamma_{t_{na}-t_a}\circ\sigma')(x)\,. \tag{4.17}$$

Similarly to (4.15), if $\sigma' \in F_{\gamma,a,\zeta}(1,1,C)$,

$$\sup_{|x|\leqslant R_{N,\gamma}} |g_\gamma(x) - \tilde\sigma(x)| \leqslant \hat c\, e^{-\alpha(t_c-t_{na})}\gamma^\delta \qquad \tilde\sigma(x) = \sigma(x) - m_{\gamma,t_a}(x, t_{na}|\sigma')\,. \tag{4.18}$$

We will then use the following fact.

Recalling the definition of $p^\gamma_t(x,y)$ given in (2.28) and defining $\pi^\gamma_t(x,y)$ analogously to (3.6), that is

$$\pi^\gamma_t(x,y) = e^{-c^\star t}\sum_{n=1}^\infty \frac{(\beta t)^n}{n!} J^n_\gamma(x,y) \tag{4.19}$$

we have

$$\begin{aligned} e^{\alpha_\gamma(t_{Na}-t_{na})}&\left|\sum_y \left(p^\gamma_{t_{Na}-t_{na}}(x,y) - \pi^\gamma_{t_{Na}-t_{na}}(x,y)\right)\tilde\sigma(y)\right| \\ &\leqslant |\tilde\sigma(x)|\, e^{\alpha_\gamma(t_{Na}-t_{na})}\, e^{-c^\star(t_{Na}-t_{na})} \leqslant 2\, e^{-(t_{Na}-t_{na})} \\ &\leqslant 2\, e^{-\alpha(t_c-t_{Na})}\gamma^{\delta'}\,. \end{aligned} \tag{4.20}$$

In the second inequality on the right-hand side of (4.20) we have used that $c^\star - 1 = \alpha_\gamma$, (see equation (2.27)), $|\alpha - \alpha_\gamma| \leqslant c\gamma$, and in the third one (4.5).

From equations (4.15), (4.16), (4.18) and (4.20) it then follows that if $\sigma' \in F_{\gamma,a,\zeta}(1,1,C)$ and $\sigma \in F_{\gamma,a,\zeta}(n,n,C)$, then there is a constant $c_1^\star$ so that

$$\begin{aligned} \sup_{|x|\leqslant R_{N,\gamma}} |m_\gamma(x,t_{Na}|\sigma) - e^{\alpha_\gamma(t_{Na}-t_a)}(p^\gamma_{t_{Na}-t_a}\circ\sigma')(x)| &\leqslant c_1^\star\, e^{-\alpha(t_c-t_{Na})}[\gamma^{\delta'} + 2\gamma^\delta] \\ + e^{\alpha(t_{Na}-t_{na})} \sup_{|x|\leqslant R_{N,\gamma}} |(\pi^\gamma_{t_{Na}-t_{na}}\circ\tilde\sigma)(x)|\,. \end{aligned} \tag{4.21}$$

Defining $\tilde\sigma$ as in (4.18), we let $\mathcal{G}$ be the following set (in the space of trajectories of the spin configurations),

$$\mathcal{G} = \left\{ \sup_{|x|\leqslant R_{N,\gamma}} |(\pi^\gamma_{t_{Na}-t_{na}}\circ\tilde\sigma)(x)| \leqslant e^{-\alpha(t_c-t_{na})}\gamma^{\delta''}\} \cap \{\sigma' \in F_{\gamma,a,\zeta}(1,1,C) \right\} \cap \{\sigma \in F_{\gamma,a,\zeta}(n,n,C)\}\,. \tag{4.22}$$

We next prove that for any $\delta'' > 0$ small enough,

$$\liminf_{C\to\infty}\liminf_{\gamma\to 0} \mathbb{P}^\gamma_{\mu_0}(\mathcal{G}) = 1\,. \tag{4.23}$$

We are going to prove that each of the three sets whose intersection defines $\mathcal{G}$ have full probability in the limit as $C\to\infty$ and $\gamma\to 0$.

For the first one we use the following inequality:

$$|(\pi^\gamma_{t_{(N-n)a}}\circ\tilde\sigma)(x)| \leqslant c[\lambda^{d/2}\gamma^{-\alpha a n + d/2}]\phi_\gamma(|x|)^{-1}|||\tilde\sigma|||_{n,n,\gamma} + c\gamma^k\,. \tag{4.24}$$

We then get for γ small enough and taking $\delta''' > \delta''$,

$$\mathbb{P}^{\gamma}_{\mu_0}\left(\sup_{|x|\leqslant R_{n,\gamma}} \left|(\pi^{\gamma}_{t_{(N-n)a}} \circ \tilde{\sigma})(x)\right| \leqslant \mathrm{e}^{-\alpha(t_c - t_{na})}\gamma^{\delta''}\right) \geqslant \mathbb{P}^{\gamma}_{\mu_0}\big(|||\tilde{\sigma}|||_{n,n,\gamma} \leqslant \gamma^{\delta'''}\big)$$
$$= \mathbb{E}^{\gamma}_{\mu_0}\big(\mathbb{P}^{\gamma}_{\sigma',t_a}\big(|||\sigma(\cdot, t_{na}) - m_{\gamma,t_a}(\cdot, t_{na}|\sigma')|||_{n,n,\gamma,a} \leqslant \gamma^{\delta'''}\big)\big)\,. \tag{4.25}$$

We then choose ζ and $\delta^\star$ as in lemma 3.5.5 to conclude that for $\delta''' < \delta^\star$,

$$\mathbb{E}^{\gamma}_{\mu_0}\big(\mathbf{1}_{F_{\gamma,a,\zeta}(1,1,C)}\mathbb{P}^{\gamma}_{\sigma',t_a}\big(|||\sigma(\cdot, t_{na}) - m_{\gamma,t_a}(\cdot, t_{na}|\sigma')|||_{n,n,\gamma,a} \leqslant \gamma^{\delta'''}\big)\big) \geqslant 1 - c\gamma^k\,. \tag{4.26}$$

From theorem 3.5.0 and (4.26) the probability on the right-hand side of (4.25) goes to 1 as $\gamma \to 0$.

By theorem 3.5.0 the probability of the second and third set in $\mathcal{G}$ goes to 1 as $C \to \infty$ uniformly in γ. We have thus completed the proof of (4.23).

From equation (4.21) it follows that if a spin trajectory is in $\mathcal{G}$, then there are $c_2^\star$ and $\hat{\delta}$ so that

$$\sup_{|x|\leqslant R_{N,\gamma}} |m_\gamma(x, t_{Na}|\sigma) - e^{\alpha_\gamma(t_{Na}-t_a)}(p^{\gamma}_{t_{Na}-t_a} \circ \sigma')(x)| \leqslant c_2^\star\, \mathrm{e}^{-\alpha(t_c - t_{Na})}\gamma^{\hat{\delta}}\,. \tag{4.27}$$

This (recalling that $|\alpha_\gamma - \alpha| \leqslant c\gamma$) implies that if the trajectory $\{\sigma(\cdot, t),\ t \geqslant 0\}$ belongs to $\mathcal{G}$, then there is $\tilde{\delta}$ so that for $\xi = \gamma\lambda x$ and $|x| \leqslant R_{N+1,\gamma}$,

$$|\hat{\ell}_\gamma(\xi, |\sigma) - \mathrm{e}^{\alpha(t_c - t_a)}\lambda^{-d/2}(Q_\gamma \circ \sigma')(\xi)| \leqslant c_2^\star\gamma^{\tilde{\delta}} \tag{4.28}$$

where

$$Q_\gamma(\xi, y) = \int \mathrm{d}r'\, q_{t_c - t_{Na}}(\lambda^{-1}\xi - r')\, p^{\gamma}_{t_{(N-1)a}}([\gamma^{-1}r'], y)\,. \tag{4.29}$$

We denote by

$$G_\gamma(x, \tau) = (\lambda\gamma)^d(2\pi\beta D\tau)^{-d/2}\exp\left\{-\frac{x^2}{2\tau\beta D}\right\} \tag{4.30}$$

where D is defined in (2.42). We have

$$\mathbb{P}^{\gamma}_{\mu_0}\left(\left|\hat{\ell}_\gamma(\xi|\sigma) - \mathrm{e}^{\alpha(t_c - t_a)}\lambda^{-d/2}\sum_y G_\gamma(\xi - \lambda\gamma y, \tau_c - a)\sigma'(y)\right| > \epsilon + c_2^\star\gamma^{\tilde{\delta}}\right)$$
$$\leqslant \mathbb{P}^{\gamma}_{\mu_0}\left(\mathrm{e}^{\alpha(t_c - t_a)}\lambda^{-d/2}\left|\sum_y [G_\gamma(\xi - \lambda\gamma y, \tau_c - a) - Q_\gamma(\xi, y)]\sigma'(y)\right| > \epsilon\right)$$
$$+[1 - \mathbb{P}^{\gamma}_{\mu_0}(\mathcal{G})]\,. \tag{4.31}$$

By equation (4.23) the square brackets goes to 0, so that it only remains to bound the first term on the right-hand side of (4.31). By the Chebichev inequality we have

$$\mathbb{P}^{\gamma}_{\mu_0}\big(\mathrm{e}^{\alpha(t_c - t_a)}\lambda^{-d/2}\big|\sum_y [G_\gamma(\xi - \lambda\gamma y, \tau_c - a) - Q_\gamma(\xi, y)]\sigma'(y)\big| > \epsilon\big) \leqslant A + B \tag{4.32}$$

where

$$A = \epsilon^{-2}\big(\mathrm{e}^{\alpha(t_c - t_a)}\lambda^{-d/2}\big)^2 \sum_y |G_\gamma(\xi - \lambda\gamma y, \tau_c - a) - Q_\gamma(\xi, y)|^2 \tag{4.33}$$

and

$$B = \epsilon^{-2}(\mathrm{e}^{\alpha(t_c - t_a)}\lambda^{-d/2})^2 \sum_{y\neq z} |G_\gamma(\xi - \lambda\gamma y, \tau_c - a) - Q_\gamma(\xi, y)|$$
$$\times|G_\gamma(\xi - \lambda\gamma z, \tau_c - a) - Q_\gamma(\xi, z)||\mathbb{E}^{\gamma}_{\mu_0}(\sigma(y, t_a)\sigma(z, t_a))|\,. \tag{4.34}$$

In equation (4.130) below we prove that there are c and ζ, so that for all $x, y \in \mathbb{Z}^d$,

$$|p^{\gamma}_{t_{(N-1)a}}(x, y) - \delta_{x,y}\, e^{-c^\star t_{(N-1)a}} - \gamma^d q_{t_{(N-1)a}}(\gamma x, \gamma y)| \leqslant \gamma^{d+\zeta} \lambda^d P(\lambda\gamma|x-y|) \tag{4.35}$$

where the function P is uniformly bounded and, for all γ,

$$\sum_y (\lambda\gamma)^d P(\lambda\gamma|y|) \leqslant c\,. \tag{4.36}$$

From this we have for $r = \gamma x$,

$$\left| Q_\gamma(r, y) - \gamma^d q_{t_c - t_a}(r, \gamma y) - e^{-c^\star t_{(N-1)a}} \int dr'\, q_{t_c - t_{Na}}(r, r') \mathbf{1}_{\{[\gamma^{-1}r']=y\}} \right| \\ \leqslant \gamma^{d+\zeta}\lambda^d P\big(\lambda|r-\gamma y|\big)\,. \tag{4.37}$$

At the end of section 4.3, we will prove that for all $\tau > 0$ there is P (that, without loss of generality, we can suppose equal to the previous one) so that

$$\left| q_{\lambda^{-2}\tau}(0, r) - \frac{\lambda^d}{(2\pi\beta D\tau)^{d/2}} \exp\left\{-\frac{(\lambda r)^2}{2\tau\beta D}\right\}\right| \leqslant \lambda^{d+1} P(\lambda|r|)\,. \tag{4.38}$$

Calling

$$\tilde{P}(r, y) = \int dr'\, \mathbf{1}_{\{[\gamma^{-1}r']=y\}} \Bigg(\lambda^{d+1} P(\lambda|r-r'|) \\ + e^{-c^\star t_{(N-1)a}} \frac{\lambda^d}{(2\pi\beta D\tau)^{d/2}} \exp\left\{-\frac{(\lambda|r-r'|)^2}{2\tau\beta D}\right\}\Bigg)$$

we have, from (4.37), for $r = \gamma x$

$$\big|G_\gamma(r - \lambda\gamma y, \tau_c - a) - Q^\gamma(r, y)\big| \leqslant (\gamma^d\lambda^{d+1} + \gamma^{d+\zeta}\lambda^d) P(\lambda|r-\gamma y|) + \tilde{P}(r, y)\,. \tag{4.39}$$

For γ small we have $\gamma^{d+\zeta}\lambda^d < \gamma^d \lambda^{d+1}$ and, for a suitable constant c':

$$2 \sup P(|r|) \leqslant c' \tag{4.40}$$

$$\sup_{r,y} \tilde{P}(r, y) \leqslant c'(\lambda\gamma)^d[\lambda + e^{-c^\star t_{(N-1)a}}] \tag{4.41}$$

$$\sum_y \tilde{P}(r, y) \leqslant c'(e^{-c^\star t_{(N-1)a}} + \lambda)\,. \tag{4.42}$$

From equations (4.36) and (4.40)–(4.42) we get (taking the square of the right-hand side of (4.39) and using the inequality $(a+b)^2 \leqslant 2(a^2+b^2)$)

$$A \leqslant \epsilon^{-2}\big[e^{\alpha(t_c - t_a)}\lambda^{-d/2}\big]^2 2\big(c'c\gamma^d\lambda^{d+1} + (\lambda\gamma)^d[\gamma^{c^\star(N-1)a} + \lambda]^2 (c')^2\big) \tag{4.43}$$

which vanishes when $\gamma \to 0$ because $e^{2\alpha t_c}\gamma^d = 1$.

In order to estimate B, we use proposition 3.3.3 and lemma 3.3.2. Then there is c'' so that for all $y \neq z$

$$(e^{\alpha(t_c - t_a)}\lambda^{-d/2})^2 |\mathbb{E}^\gamma_{\mu_0}(\sigma(y, t_a)\sigma(z, t_a))| \leqslant c''\,. \tag{4.44}$$

Therefore, using (4.40)–(4.42) we have

$$B \leqslant c''\epsilon^{-2}\big[2\lambda c + c'(\gamma^{c^\star(N-1)a} + \lambda)\big]^2 \leqslant c'''\epsilon^{-2}\lambda^2\,. \tag{4.45}$$

From equations (4.32), (4.43) and (4.45), we then have that the probability on the left-hand side of (4.31) vanishes in the limit $\gamma \to 0$. Hence any limit point is the same as that of the variables $\{\tilde{\ell}_\gamma(\xi_1|\sigma) \ldots \tilde{\ell}_\gamma(\xi_k|\sigma)\}$, where

$$\tilde{\ell}_\gamma(\xi|\sigma) = e^{\alpha(t_c - t_a)}\lambda^{-d/2} \frac{(\lambda\gamma)^d}{(2\pi\beta D(\tau_c - a))^{d/2}} \sum_y \exp\left\{-\frac{(\xi - \lambda\gamma y)^2}{2(\tau_c - a)\beta D}\right\}\sigma'(y)\,. \tag{4.46}$$

To complete the proof of the proposition we thus are left with the proof that

$$\lim_{\gamma\to 0} \mathbb{E}^{\gamma}_{\mu_0}\left(\prod_{i=1}^{k} \tilde{\ell}_{\gamma}(\xi_i|\sigma)\right) = \tilde{\mathbb{E}}\left(\prod_{i=1}^{k} \tilde{\ell}(\xi_i)\right) \tag{4.47}$$

where $\tilde{\mathbb{E}}$ is the expectation with respect to the law $\tilde{P}$ of the mean zero Gaussian process with covariance defined in (4.9) and $\tilde{\ell}(\xi_i)$ are the canonical variables for this process.

To prove (4.47) we use (2) of lemma 3.3.2 and (3.97) of proposition 3.3.3. We first observe that, using (4.30), (4.46) can be written as

$$\tilde{\ell}_{\gamma}(\xi|\sigma) = \mathrm{e}^{-\alpha t_a} \sum_{y} (\lambda\gamma)^{-d/2} G_{\gamma}(\xi - \lambda\gamma y, \tau_c - a)\sigma(y, t_a)\,. \tag{4.48}$$

We first compute (4.47) for $k = 2$. This gives

$$\mathbb{E}^{\gamma}_{\mu_0}\left(\prod_{i=1}^{2} \tilde{\ell}_{\gamma}(\xi_i|\sigma)\right) = I_1(\xi_1, \xi_2) + I_2(\xi_1, \xi_2) \tag{4.49}$$

where (see the definition (3.96)),

$$I_1(\xi_1, \xi_2) = \mathrm{e}^{-2\alpha t_a} \sum_{\underline{y}\in\mathbb{Z}^{2d}_{\neq}} (\lambda\gamma)^{-d} G_{\gamma}(\xi_1 - \lambda\gamma y_1, \tau_c - a) G_{\gamma}(\xi_2 - \lambda\gamma y_2, \tau_c - a) v^{\gamma}(\underline{y}, t_a) \tag{4.50}$$

$$I_2(\xi_1, \xi_2) = \mathrm{e}^{-2\alpha t_a} \sum_{y} (\lambda\gamma)^{-d} G_{\gamma}(\xi_1 - \lambda\gamma y, \tau_c - a) G_{\gamma}(\xi_2 - \lambda\gamma y, \tau_c - a)\,. \tag{4.51}$$

Notice that there is a constant c so that

$$\sup_{\xi_1,\xi_2} |I_2(\xi_1, \xi_2)| \leqslant c\gamma^{2\alpha a}\,. \tag{4.52}$$

In order to estimate I_1 we write $v^{\gamma}(\underline{y}, t_a) = w^{\gamma}(\underline{y}, t_a) + [v^{\gamma}(\underline{y}, t_a) - w^{\gamma}(\underline{y}, t_a)]$ and we use (3.97) with $k = 1$. We are then left with the estimate of the right-hand side of (4.50) with $w^{\gamma}(\underline{y}, t_a)$ in place of $v^{\gamma}(\underline{y}, t_a)$. For this last one we use (3.84) with $\zeta < 2\alpha a$ and therefore we get that there are $c > 0$ and $\delta > 0$ so that for all ξ_1 and ξ_2,

$$|I_1(\xi_1, \xi_2) - \tilde{I}_1(\xi_1, \xi_2)| \leqslant c\gamma^{\delta} \tag{4.53}$$

where, see (3.84),

$$\tilde{I}_1(\xi_1, \xi_2) = \sum_{\underline{y}\in\mathbb{Z}^{2d}_{\neq}} G_{\gamma}(\xi_1 - \lambda\gamma y_1, \tau_c - a) G_{\gamma}(\xi_2 - \lambda\gamma y_2, \tau_c - a) \times (\lambda\gamma)^{-d} \int_0^{t_a} \mathrm{d}s\, \mathrm{e}^{-2\alpha(t_a - s)} \big(p^{\gamma}_s \times p^{\gamma}_s) \circ 2\beta J_{\gamma}\big)(y_1, y_2)\,. \tag{4.54}$$

From equation (4.128) below and (4.38) there is a constant $\bar{c}$ so that for all $x, y \in \mathbb{Z}^d$ and all $s \in [0.t_a]$

$$\big|p^{\gamma}_s(x, y) - G_{\gamma}(\lambda\gamma(x - y), \lambda^2 s)\big| \leqslant \big|p^{\gamma}_s(x, y) - \gamma^d q_s(\gamma(x - y))\big| + \big|\gamma^d q_s(\gamma(x - y)) - G_{\gamma}(\lambda\gamma(x - y), \lambda^2 s)\big| \leqslant \bar{c}(\gamma\lambda)^d[\gamma^{\zeta} + \lambda]\,. \tag{4.55}$$

Using equation (4.55) there is $\bar{c}'$ so that

$$\left|(\lambda\gamma)^{-d} \int_0^{t_a} \mathrm{d}s\, \mathrm{e}^{-2\alpha(t_a - s)} \big(p^{\gamma}_s \times p^{\gamma}_s) \circ 2\beta J_{\gamma}\big)(y_1, y_2) - \int_0^{t_a} \mathrm{d}s\, \mathrm{e}^{-2\alpha(t_a - s)} K_{\lambda^2 s}(y_1, y_2)\right| \leqslant \bar{c}'\lambda \tag{4.56}$$

where

$$K_{\lambda^2 s}(y_1, y_2) = (\lambda\gamma)^{-d} \sum_{z_1, z_2} G_\gamma(\lambda\gamma(y_1 - z_1, \lambda^2 s) G_\gamma(\lambda\gamma(y_2 - z_2, \lambda^2 s) 2\beta J_\gamma(z_1, z_2)\,. \quad (4.57)$$

There is a constant $\bar{c}''$ so that

$$\sup_{y_1, y_2} \left| K_{\lambda^2 s}(y_1, y_2) - K_a(y_1, y_2) \right| \leqslant \bar{c}''\lambda \qquad \text{for all} \quad s \in [0, t_a]\,. \quad (4.58)$$

Using the fact that (see equation (3.7))

$$2\beta \sum_{z_2} J_\gamma(z_1, z_2) = 2\beta\gamma^d \sum_z J_\gamma(\gamma|z|) = 2\beta \hat{J}_\gamma(0) = 2(\alpha_\gamma + 1)$$

from (4.57), (4.58) and the fact that $|\alpha_\gamma - \alpha| \leqslant c\gamma$, we get

$$\left| K_{\lambda^2 s}(y_1, y_2) - 2(\alpha + 1)(\lambda\gamma)^{-d} \sum_z G_\gamma(\lambda\gamma(y_1 - z), a) G_\gamma(\lambda\gamma(y_2 - z), a) \right| \leqslant \bar{c}'''\lambda\,. \quad (4.59)$$

Since

$$\sum_y G_\gamma(\xi - \lambda\gamma y, \tau_c - a) G_\gamma(\lambda\gamma(y - z), a) = G_\gamma(\xi - \lambda\gamma z, \tau_c)$$

from (4.54), (4.56), (4.59) and the fact that $\tau_c = d/2\alpha$, we then get

$$\begin{aligned} \lim_{\gamma\to 0} I_1(\xi_1, \xi_2) &= \lim_{\gamma\to 0} \frac{\alpha + 1}{\alpha} (\lambda\gamma)^{-d} \sum_z G_\gamma(\xi_1 - \lambda\gamma z, \tau_c) G_\gamma(\xi_2 - \lambda\gamma z, \tau_c) \\ &= \left(1 + \frac{1}{\alpha}\right) \frac{\alpha^{d/2}}{(\pi\beta D d)^{d/2}} \, \mathrm{e}^{-\alpha(\xi_1 - \xi_2)^2/d\beta D}\,. \end{aligned} \quad (4.60)$$

The proof of (4.47) for any $k \geqslant 2$ uses similar arguments together with both inequalities in (3.97). We omit the details. □

4.2. The interfaces

The motivation for definitions 4.2.1 and 4.2.3 below is technical, it will become clear in section 5.

Definition 4.2.1. *We denote by C_1 and c_1' the parameters which, in (3.23) and (3.24), respectively, correspond to $\delta = 1$. Given any $k > 0$ we define*

$$\mathcal{A}_k' = \mathcal{A}_+'(1) \cup \mathcal{A}_-'(1) \cup \mathcal{A}_0'(k) \quad (4.61)$$

where

$$\mathcal{A}_+'(1) = \left\{ r : \ell_\gamma(r'|\sigma) \geqslant \tfrac{3}{2}\lambda \log \lambda^{-2} \quad \text{for all} \quad r' : |r - r'| \leqslant C_1(\log \lambda^{-2})^2 \right\}. \quad (4.62)$$

$\mathcal{A}_-'(1)$ is defined with the reversed inequality. Then, given any k, we set

$$\begin{aligned} \mathcal{A}_0'(k) = \{ r :\ & \text{there are } r_0, \theta \in \left(\frac{1}{k}, k\right) \text{ and a unit vector } \nu \\ & \text{such that (4.64), (4.65) below hold} \} \end{aligned} \quad (4.63)$$

$$|r - r_0| \leqslant 2C_1(\log \lambda^{-2})^2 \quad (4.64)$$

$$|\ell_\gamma(r'|\sigma) - \theta\lambda(r' - r_0)\nu| \leqslant \lambda\epsilon_\gamma \qquad \textit{for all} \quad r' : |r' - r_0| \leqslant 5C_1(\log \lambda^{-2})^2 \quad (4.65)$$

where ϵ_γ is defined as follows. In $d = 1$

$$\epsilon_\gamma = k\lambda^{1-80b} \quad (4.66)$$

$b > 0$ is the parameter entering in the definition of the seminorms, see definition 3.1.2. In $d \geqslant 2$

$$\epsilon_\gamma = k\lambda(\log\lambda^{-2})^5\,. \tag{4.67}$$

Proposition 4.2.2. *For any $L > 0$*

$$\liminf_{k\to\infty}\ \liminf_{\gamma\to 0}\ \mathbb{P}^{\gamma}_{\mu_0}\big(\{S(0,\lambda^{-1}L)\subset \mathcal{A}'_k\}\big) = 1 \tag{4.68}$$

with the same notation as in definition 4.2.1.

Proof. Let $\mathcal{G}(L,k)$ be the following set in H^m_{loc}, $m > d+3$, (we want the elements of $\mathcal{G}(L,k)$ to be in C^3)

$$\mathcal{G}(L,k) = \left\{ g\in H^m_{loc} : \inf_{r:|r|\leqslant L}\big[|g(r)| + |\nabla g(r)|\big] > \frac{1}{k} \ \text{ and } \sup_{r:|r|\leqslant L}|\nabla g(r)| \leqslant k \right\}. \tag{4.69}$$

Then in [21] it has been proven that for any $L > 0$

$$\lim_{k\to\infty}\tilde{\mathcal{P}}\big(\mathcal{G}(L,k)\big) = 1\,. \tag{4.70}$$

From proposition 4.1.2 it then follows that there is a sequence ζ_k decreasing to 0 as $k\to\infty$, and, for any k, there is γ_0 so that for all $\gamma \leqslant \gamma_0$

$$\mathcal{P}_{\gamma,\tau_0}\big(\mathcal{G}(L,k)\big) \geqslant 1-\zeta_k\,. \tag{4.71}$$

Assuming that $\hat{\ell}_\gamma(\cdot|\sigma)\in\mathcal{G}(L,k)$ (where $\hat{\ell}_\gamma$ is defined in (4.7)) we now prove that for γ sufficiently small $S(0,\lambda^{-1}L)\subset\mathcal{A}'_k$ and from this and (4.71) the proposition will follow.

In what follows we omit the dependence on σ of the functions ℓ_γ and $\hat{\ell}_\gamma$.

Given any $r\in S(0,\lambda^{-1}L)$, either $r\in\mathcal{A}'_+(1)\cup\mathcal{A}'_-(1)$ or the following holds. There is a $\bar r$ such that

$$|\bar r - r| < C_1(\log\lambda^{-2})^2 \qquad \text{and} \qquad |\ell_\gamma(\bar r)| < \tfrac{3}{2}\lambda\log\lambda^{-2}\,. \tag{4.72}$$

If $\ell_\gamma(\bar r) = 0$, then, since $\hat{\ell}_\gamma(\lambda\bar r) = 0$ and $\hat{\ell}_\gamma\in\mathcal{G}(L,k)$, we have from (4.69) that

$$\theta \equiv |\nabla\hat{\ell}_\gamma(\lambda\bar r)| > \frac{1}{k} \tag{4.73}$$

and so for any r' such that $|r'-\bar r| \leqslant 5C_1(\log\lambda^{-2})^2$

$$|\ell_\gamma(r') - \theta\lambda(r'-\bar r)\nu| \leqslant \lambda\big[M\lambda(5C_1(\log\lambda^{-2})^2)^2/2\big] \leqslant \lambda\epsilon_\gamma \tag{4.74}$$

where

$$M = \sup_{|r-\lambda\bar r|\leqslant 5C_1\lambda(\log\lambda^{-2})^2}\ \sum_{i,j}\left|\frac{\partial^2\hat{\ell}_\gamma(r)}{\partial r_i\partial r_j}\right|. \tag{4.75}$$

Therefore (4.64) and (4.65) hold and that implies $r\in\mathcal{A}'_0(k)$.

If instead $\ell_\gamma(\bar r) > 0$, or $\ell_\gamma(\bar r) < 0$, then we want to prove that there is a r_0, suitably close to $\bar r$ so that (4.64) and (4.65) hold. To be definite we assume $\ell_\gamma(\bar r) > 0$. From equation (4.72) and the fact that $\hat{\ell}_\gamma(\cdot|\sigma)\in\mathcal{G}(L,k)$ it follows that

$$|\nabla\hat{\ell}_\gamma(\lambda\bar r)| > \frac{1}{k} - \frac{3}{2}\lambda\log\lambda^{-2}\,. \tag{4.76}$$

Let $r(t)$ be the solution of the following equation:

$$\dot r = -\nabla\hat{\ell}_\gamma(r(t)) \qquad r(0) = \bar r \tag{4.77}$$

then

$$\frac{d\hat{\ell}_\gamma(r(t))}{dt} = -[\nabla\hat{\ell}_\gamma(r(t))]^2 \tag{4.78}$$

so that $\hat{\ell}_\gamma$ decreases along the curve (4.77). Observe that this, together with the fact that $\hat{\ell}_\gamma(\cdot|\sigma) \in \mathcal{G}(L,k)$, implies that

$$|\nabla\hat{\ell}_\gamma(r(t))| \geqslant \frac{1}{k} - \frac{3}{2}\lambda \log \lambda^{-2} \tag{4.79}$$

up to the first t when $\hat{\ell}_\gamma$ becomes equal to 0. At this time, call it t_0 and let $r(t_0) = \lambda r_0$, we have $\hat{\ell}_\gamma(\lambda r_0) = 0$, and $\nabla\hat{\ell}_\gamma(\lambda r_0) > 1/k$. To evaluate t_0 and $|\bar{r} - r_0|$, we observe that from (4.78), (4.79) and (4.76) it follows that

$$\frac{3}{2}\lambda \log \lambda^{-2} \geqslant \hat{\ell}_\gamma(\lambda\bar{r}) - \hat{\ell}_\gamma(\lambda r_0) = \int_0^{t_0} dt\, [\nabla\hat{\ell}_\gamma(r(t)]^2 > t_0\left(\frac{1}{k} - \frac{3}{2}\lambda \log \lambda^{-2}\right)^2 . \tag{4.80}$$

From equation (4.77), recalling that, by (4.69), the gradient is bounded by k, we have, for any λ smaller than some value λ_0, which depends on k,

$$|\lambda r_0 - \lambda\bar{r}| \leqslant t_0 k \leqslant k\frac{3}{2}\lambda \log \lambda^{-2}\left(\frac{1}{k} - \frac{3}{2}\lambda \log \lambda^{-2}\right)^{-2} \leqslant C_1\lambda(\log \lambda^{-2})^2 . \tag{4.81}$$

By equation (4.81), $|r - r_0| \leqslant |r - \bar{r}| + |\bar{r} - r_0| \leqslant 2C_1(\log \lambda^{-2})^2$ so that (4.64) holds. We define θ and ν so that

$$\nabla\hat{\ell}_\gamma(\lambda r_0) = \theta\nu . \tag{4.82}$$

An argument analogous to the one given for proving (4.74) shows that (4.65) holds. This, in turns, implies that $r \in \mathcal{A}'_0(k)$, thus concluding the proof of the proposition. □

We next prove the analogue of proposition 4.2.2 for the function $m_{\gamma,t_{na}}$. To this purpose we need to define the sets $\mathcal{A}$ analogously to definition 4.2.1.

Definition 4.2.3. *Let C_1 and c'_1 be as in definition 4.2.1. Furthermore if f is a function on $\mathbb{Z}^d$, (or on $\mathbb{R}^d$), we define f_γ as the function on $\gamma\mathbb{Z}^d$ such that $f_\gamma(\gamma x) = f(x)$. Then for any function f as above and any number $\omega > 0$, we introduce*

$$\begin{aligned}\mathcal{A}_+(\omega, f, \gamma) = \big\{ & r \in \gamma\mathbb{Z}^d : f_\gamma(r') \geqslant \lambda^{1+d/2}\log \lambda^{-2},\ \text{for all } r' \in \gamma\mathbb{Z}^d \text{ such that} \\ & |r - r'| \leqslant C_1\omega(\log \lambda^{-2})^2 \big\}\end{aligned} \tag{4.83}$$

and set

$$\mathcal{A}_-(\omega, f, \gamma) = \mathcal{A}_+(\omega, -f, \gamma) .$$

Given na as in (4.1) we set

$$\mathcal{A}_\pm(\omega) = \mathcal{A}_\pm(\omega, m_{\gamma,t_{na}}(\cdot, t_c|\sigma), \gamma) . \tag{4.84}$$

Given f, f_γ and C_1 as above, for any $k > 1$ we define

$$\begin{aligned}\mathcal{A}_0(k, f, \gamma) = \{ & r \in \gamma\mathbb{Z}^d : \text{there are } r_0, \theta \in \left(\frac{1}{k}, k\right) \text{ and a unit vector } \nu \\ & \text{such that (4.85)–(4.86) below hold}\}\end{aligned}$$

$$|r_0 - r| \leqslant 2C_1(\log \lambda^{-2})^2 \tag{4.85}$$

$$|f_\gamma(r') - \theta\lambda^{1+d/2}(r' - r_0)\nu| \leqslant \lambda^{1+d/2}\epsilon_\gamma \qquad \textit{for all} \quad r' : |r' - r_0| \leqslant 5C_1(\log \lambda^{-2})^2 \tag{4.86}$$

where ϵ_γ is defined in (4.66) and (4.67).

We call

$$\mathcal{A}_0(k) = \mathcal{A}_0(k, m_{\gamma,t_{na}}(\cdot, t_c|\sigma), \gamma) \,.$$

Lemma 4.2.4. *Let $L > 0$ and $C > 0$, assume that $\sigma \in F_{\gamma,a,\zeta}(n, n, C)$ and that $|x| \leqslant L(\gamma\lambda)^{-1}$, $\gamma x \in \mathcal{A}'_+(1)$, then $\gamma x \in \mathcal{A}_+(1)$ for all γ small enough.*

Proof. We will show that if $\ell_\gamma(\gamma x|\sigma) \geqslant (3/2)\lambda \log \lambda^{-2}$, then, for λ small enough

$$m_{\gamma,t_{na}}(x, t_c|\sigma) \geqslant \lambda^{1+d/2} \log \lambda^{-2}$$

and this will prove the lemma.

We recall the definition (3.54) of M_γ:

$$M_\gamma(x, t_c) = \mathrm{e}^{\alpha_\gamma(t_c - t_{Na})} \sum_y p^\gamma_{t_c - t_{Na}}(x, y) m_{\gamma,t_{na}}(y, t_{Na}|\sigma) \,. \tag{4.87}$$

We then use (3.56) which gives, recalling that $\lambda\gamma|x| \leqslant L$,

$$|m_\gamma(x, t_c|\sigma) - M_\gamma(x, t_c|\sigma)| \leqslant c'\lambda^{\frac{3d}{2}}[1 + L]^b \leqslant c\lambda^{3d/2} \,. \tag{4.88}$$

We next estimate the difference between M_γ and ℓ_γ. We will prove below that there is $\tilde{c}$ so that

$$\begin{aligned} \left|\lambda^{-d/2} M_\gamma(x, t_c) - \ell_\gamma(\gamma x, t_c|\sigma)\right| &\leqslant \lambda^{-d/2}\, \mathrm{e}^{\alpha(t_c - t_{Na})}\, \mathrm{e}^{(\alpha_\gamma - \alpha)(t_c - t_{Na})} \\ &\quad \times \left| \sum_y p^\gamma_{t_c - t_{Na}}(x, y) m_{\gamma,t_{na}}(y, t_{Na}|\sigma) \right| + A_\gamma \\ &\leqslant \tilde{c}\lambda^{-16b}\gamma^{\delta_1} + A_\gamma \qquad \delta_1 := \tau_c - Na \end{aligned} \tag{4.89}$$

where

$$A_\gamma = \lambda^{-d/2}\, \mathrm{e}^{\alpha(t_c - t_{Na})} \sum_y |m_{\gamma,t_{na}}(y, t_{Na}|\sigma)| D_\gamma(x, y) \tag{4.90}$$

with

$$D_\gamma(x, y) = \left| p^\gamma_{t_c - t_{Na}}(x, y) - \int \mathrm{d}r'\, \mathbf{1}_{\{[\gamma^{-1} r'] = y\}} q_{t_c - t_{Na}}(\gamma x, r') \right| . \tag{4.91}$$

The proof of (4.89) easily follows from estimates below, used to bound A_γ, and it is omitted. By equations (4.13) and (4.14) (recall equation (4.5)) we derive the following bound on $m_{\gamma,t_{na}}(y, t_{Na}|\sigma)$:

$$\begin{aligned} \sup_{|y| \leqslant R_{N,\gamma}} \lambda^{-d/2}\, \mathrm{e}^{\alpha(t_c - t_{Na})} |m_{\gamma,t_{na}}(y, t_{Na}|\sigma)| &= \sup_{|y| \leqslant R_{N,\gamma}} [\lambda^{d/2}\gamma^{-\alpha Na + d/2}]^{-1} |m_{\gamma,t_{na}}(y, t_{Na}|\sigma)| \\ &\leqslant c\gamma^\delta \lambda^{-16b} + \lambda^{-16b} \leqslant c'\lambda^{-16b} \,. \end{aligned} \tag{4.92}$$

We then have

$$A_\gamma \leqslant c'\lambda^{-16b} \sum_{|y| \leqslant R_{N,\gamma}} D_\gamma(x, y) + \lambda^{-d/2}\, \mathrm{e}^{\alpha(t_c - t_{Na})} \sum_{|y| > R_{N,\gamma}} D_\gamma(x, y) \,. \tag{4.93}$$

We use (4.128) to bound the first sum on the right-hand side of (4.93):

$$c'\lambda^{-16b} \sum_{|y| \leqslant R_{N,\gamma}} D_\gamma(x, y) \leqslant \hat{c}_1\big(\gamma^\zeta \lambda^{-16b} + \mathrm{e}^{-c^\star(t_c - t_{Na})}\big) \leqslant \hat{c}_1 \gamma^{\zeta_1}$$

for suitable $\hat{c}_1$ and $\zeta_1 > 0$. For the second sum on the right-hand side of (4.93) we use (4.129):

$$\lambda^{-d/2}\, \mathrm{e}^{\alpha(t_c - t_{Na})} \sum_{|y| > R_{N,\gamma}} D_\gamma(x, y) \leqslant \lambda^{-d/2}\, \mathrm{e}^{\alpha(t_c - t_{Na})} 2c\, \mathrm{e}^{-|\gamma R_{N,\gamma} - L|}\, \mathrm{e}^{\delta(t_c - t_{Na})} \leqslant c_2 \gamma^{\zeta_2}$$

where c_2 and $\zeta_2 > 0$ are suitably chosen.

Collecting the above estimates, we conclude that there are $\hat{c}$ and $\hat{\zeta} > 0$ so that

$$\left|\lambda^{-d/2}M_\gamma(x,t_c) - \ell_\gamma(\gamma x, t_c|\sigma)\right| \leqslant \left[\tilde{c}\lambda^{-16b}\gamma^{\delta_1} + \hat{c}_1\gamma^{\zeta_1} + c_2\gamma^{\zeta_2}\right] \leqslant \hat{c}\gamma^{\hat{\zeta}}\,. \tag{4.94}$$

From equation (4.88) and (4.94), recalling that by hypothesis $\ell_\gamma(\gamma x|\sigma) \geqslant 3/2\lambda\log\lambda^{-2}$, we have that for all γ small enough

$$m_\gamma(x,t_c|\sigma) \geqslant \lambda^{d/2}\ell_\gamma(\gamma x,\sigma) - 2c\lambda^{3d/2} - \hat{c}\lambda^{d/2}\gamma^{\hat{\zeta}} \geqslant \tfrac{3}{2}\lambda^{1+d/2}\log\lambda^{-2} - \tfrac{1}{2}\lambda^{1+d/2}\log\lambda^{-2} \tag{4.95}$$

From this the lemma follows. □

The analogous result holds for $\gamma x \in \mathcal{A}'_-(1)$, it only remains to consider the case where $\gamma x \in \mathcal{A}'_0(1)$. The proof is similar to that of lemma 4.2.4, but we have to modify the bound (4.30) since the *a priori* estimate given in (3.55) is too rough. We do that in the following lemma.

Lemma 4.2.5. *Let $C > 0$ and assume that $\sigma \in F_{\gamma,a,\zeta}(n,n,C)$, then , for any $L > 0$, there is c so that*

$$\sup_{\lambda\gamma|x|\leqslant L} |M_\gamma(x,t_c) - \Omega_\gamma(x|\sigma)| \leqslant c\lambda^{1+d/2}\left(\log\lambda^{-2}\right)^3 \tag{4.96}$$

where M_γ and Ω_γ are defined in (3.54) and (3.58), respectively.

Proof. The proof is based on the fact that Ω_γ is a sup of M_γ over a 'small' spacetime interval. Since p^γ is smooth, more precisely it is close to a differentiable function (as proven in the next section), then the difference on the left-hand side of (4.96) is bounded by a factor proportional to the derivative of M_γ, i.e. to $\lambda^{d/2+1}$, times the size of the space interval, actually slightly larger than that to take into account errors.

In fact, from lemma 4.3.4

$$\sup_{\lambda\gamma|x|\leqslant L} |M_\gamma(x,t_c|\sigma) - \Omega_\gamma(x|\sigma)| \leqslant \sup_{x,y\in S^\star}\ \sup_{t_c^-\leqslant s\leqslant t_c} \mathrm{e}^{\alpha_\gamma(t_c-s)}|M_\gamma(x,s|\sigma) - M_\gamma(y,s|\sigma)| \tag{4.97}$$

where

$$S^\star = \{x : |\gamma x| \leqslant \lambda^{-1}L + \left(\log\lambda^{-2}\right)^3\}\,. \tag{4.98}$$

We then use (3.54), lemma 4.3.6, (4.13) and (4.14) to bound $m_{\gamma,t_{na}}(\cdot,t_{Na}|\sigma)$. From this we obtain (4.96), we omit the details. □

Proposition 4.2.6. *Let $L > 0$, $C > 0$, $k > 1$ and assume that $\sigma \in F_{\gamma,a,\zeta}(n,n,C)$. Let $|x| \leqslant (\lambda\gamma)^{-1}L$ and $\gamma x \in \mathcal{A}'_0(k)$, then there is k' (independent of γ) so that $\gamma x \in \mathcal{A}_0(k')$ with the same parameters θ, ν and r_0.*

Proof. By the definition of $\mathcal{A}'_0(k)$, there are r_0, $|\gamma x - r_0| \leqslant 2c_1(\log\lambda^{-2})^2$, $\theta \in (k^{-1},k)$ and ν so that for all $|r - r_0| \leqslant 5c_1(\log\lambda^{-2})^2$,

$$|\ell_\gamma(r|\sigma) - \theta\lambda(r-r_0)\nu| \leqslant \lambda\epsilon_\gamma \tag{4.99}$$

where ϵ_γ is defined in (4.66) for $d = 1$ and in (4.67) for $d \geqslant 2$. As in the proof of lemma 4.2.4 we write for all y so that $|\gamma y - r_0| \leqslant 5c_1(\log\lambda^{-2})^2$,

$$\begin{aligned}\left|m_{\gamma,t_{na}}(y,t_c|\sigma) - \theta\lambda^{1+d/2}(\gamma y - r_0)\nu\right| &\leqslant \left|M_\gamma(y,t_c) - m_{\gamma,t_{na}}(y,t_c|\sigma)\right| \\ &+\lambda^{d/2}\left|\lambda^{-d/2}M_\gamma(y,t_c) - \ell_\gamma(\gamma y|\sigma)\right| + \lambda^{d/2}\left|\ell_\gamma(\gamma y|\sigma) - \theta\lambda(\gamma y - r_0)\nu\right|\,.\end{aligned} \tag{4.100}$$

For the second term on the right-hand side of (4.100), we use (4.94), while for the third term we use (4.99). We then have that

$$\left|m_{\gamma,t_{na}}(y,t_c|\sigma) - -\theta\lambda^{1+d/2}(\gamma y - r_0)\nu\right| \leqslant \lambda^{d/2}\hat{c}\gamma^{\hat{\zeta}} + \lambda^{1+d/2}\epsilon_\gamma \\ +\left|M_\gamma(y,t_c) - m_{\gamma,t_{na}}(y,t_c|\sigma)\right|. \tag{4.101}$$

Using equations (3.57) and (4.96) we have, for a suitable constant $\hat{c}_1$,

$$\left|M_\gamma(y,t_c) - m_{\gamma,t_{na}}(y,t_c|\sigma)\right| \leqslant \hat{c}_1\left\{\lambda^{\frac{5}{2}d-80b} + \left[|M_\gamma(y,t_c)| + \lambda^{1+d/2}(\log\lambda^{-2})^3\right]^3\right\}. \tag{4.102}$$

We call

$$\psi_\gamma(y) = \left|M_\gamma(y,t_c) - m_{\gamma,t_{na}}(y,t_c|\sigma)\right|. \tag{4.103}$$

In equation (4.88) we have proven that

$$\psi_\gamma(y) \leqslant 2c\lambda^{\frac{3}{2}d}. \tag{4.104}$$

From equation (4.102), using (4.101), we then have

$$\psi_\gamma(y) \leqslant \hat{c}_1\Big\{\lambda^{\frac{5}{2}d-80b} + \Big[\psi_\gamma(y) + |\theta\lambda^{1+d/2}(\gamma y - r_0)\nu| + \psi_\gamma(y) + \hat{c}\lambda^{d/2}\gamma^{\hat{\zeta}} \\ +\lambda^{1+d/2}\epsilon_\gamma + \lambda^{1+d/2}(\log\lambda^{-2})^3\Big]^3\Big\}. \tag{4.105}$$

We next bound the terms in the square bracket on the right-hand side of (4.105). By equation (4.104) and the fact that $|\gamma y - r_0| \leqslant 5c_1(\log\lambda^{-2})^2$, there is a constant c_2 so that

$$2\psi_\gamma(y) + |\theta\lambda^{1+d/2}(\gamma y - r_0)\nu| \leqslant 4c\lambda^{3d/2} + k\lambda^{1+d/2}5c_1(\log\lambda^{-2})^2 \leqslant c_2 k\lambda^{1+d/2}(\log\lambda^{-2})^2. \tag{4.106}$$

We have used that $\theta < k$ and that $3d/2 \geqslant 1 + d/2$.

Then, using the definition (4.66) for ϵ_γ, we bound the other terms in the square bracket on the right-hand side of (4.105) as

$$\hat{c}\lambda^{d/2}\gamma^{\hat{\zeta}} + \lambda^{1+d/2}\epsilon_\gamma + \lambda^{1+d/2}(\log\lambda^{-2})^3 \leqslant kc_3\lambda^{1+d/2}(\log\lambda^{-2})^3 \tag{4.107}$$

where c_3 is a suitable constant. From equations (4.105), (4.106) and (4.107) it follows that there is c_4 so that

$$\psi_\gamma(y) \leqslant c_4\left(\lambda^{5d/2-80b} + k^3\lambda^{3+3d/2}(\log\lambda^{-2})^9\right). \tag{4.108}$$

Using equations (4.101), (4.103) and (4.108) we then get

$$\left|m_{\gamma,t_{na}}(y,t_c|\sigma) - \theta\lambda^{1+d/2}(\gamma y - r_0)\nu\right| \leqslant c_4\left[\lambda^{5d/2-80b} + k^3\lambda^{3+3d/2}(\log\lambda^{-2})^9\right] \\ +\hat{c}\lambda^{d/2}\gamma^{\hat{\zeta}} + \lambda^{1+d/2}\epsilon_\gamma \\ \leqslant \lambda^{1+d/2}\left[c_4\lambda^{2d-1-80b} + c_4k^3\lambda^{d+2}(\log\lambda^{-2})^9 + \hat{c}\lambda^{-1}\gamma^{\hat{\zeta}} + \epsilon_\gamma\right]. \tag{4.109}$$

Then, from (4.109), for a suitable constant k',

$$\left|m_{\gamma,t_{na}}(y,t_c|\sigma) - \theta\lambda^{1+d/2}(\gamma y - r_0)\nu\right| \leqslant k'/k\ \lambda^{1+d/2}\epsilon_\gamma \tag{4.110}$$

which proves the proposition. □

4.3. The random walk with jump intensity $J_\gamma(x, y)$

In this subsection we study a single random walk in $\mathbb{Z}^d$ which jumps with intensity $J_\gamma(x, y)$, proving that it behaves essentially as a jump process on $\mathbb{R}^d$ with jump intensity $J(|r-r'|)\,\mathrm{d}r'$, provided we make the correspondence $x \to r = \gamma x$. From that the properties stated in the previous sections easily follow. Our analysis is based on classical arguments in central limit theorems, but due to the specificity of our problem with mixed limits $\gamma \to 0$ and $t \to \infty$, we have not been able to refer to the literature.

We use the shorthand

$$p_t^\gamma(x) = p_t^\gamma(0, x) \tag{4.111}$$

$\gamma > 0$, $t \leqslant t^\star$. We then set

$$\hat{p}_t^\gamma(k) = \sum_x \mathrm{e}^{\mathrm{i}kx} p_t^\gamma(x) \qquad -\pi \leqslant k_i \leqslant \pi \qquad i = 1, \dots d\,. \tag{4.112}$$

Thus

$$p_t^\gamma(x) = \frac{1}{(2\pi)^d} \int \mathrm{d}k\, \mathrm{e}^{-\mathrm{i}kx}\, \mathrm{e}^{-c^\star t} \sum_{n=0}^{\infty} \frac{(\beta t \hat{J}_{\gamma,k})^n}{n!} \tag{4.113}$$

$$\hat{J}_{\gamma,k} = \sum_y \mathrm{e}^{\mathrm{i}ky} J_\gamma(0, y), \qquad c^\star = \beta \hat{J}_{\gamma,0}\,. \tag{4.114}$$

As usual in central limit theorems, we distinguish different regions of values of k. We start from 'large k's', i.e. $|k_i| > \gamma^{1-b}$, $b > 0$, we will be interested in b small enough. As $J_\gamma(0, y) = J(|\gamma y|)$, with J a smooth function, the values $|k| \gg \gamma$ give small contribution, as we are going to prove. Let $I(x)$ be a function on $\mathbb{Z}$ with compact support and call

$$I^{(1)}(y) = \gamma^{-1}[I(y) - I(y+1)] \qquad I^{(\ell)}(y) = \gamma^{-1}[I^{(\ell-1)}(y) - I^{(\ell-1)}(y+1)]\,. \tag{4.115}$$

Integration by parts on the lattice yields:

Lemma 4.3.1. *Let $k \neq 0$, then, for any $n \geqslant 1$,*

$$\sum_y \mathrm{e}^{\mathrm{i}ky} I(y) = \left(\frac{\gamma}{\mathrm{i}k}\right)^n \left(\frac{\mathrm{i}k}{1-\,\mathrm{e}^{-\mathrm{i}k}}\right)^n \sum_y \mathrm{e}^{\mathrm{i}ky} I^{(n)}(y)\,. \tag{4.116}$$

Proof. Since (4.116) is obviously true for $n = 0$ the proof follows by induction on n. □

Going back to (4.113) we define for $b \in (0, 1)$

$$p_t^{\gamma,>}(x) = \frac{1}{(2\pi)^d} \int_{|k_i| > \gamma^{1-b}} \mathrm{d}k\, \mathrm{e}^{-\mathrm{i}kx} \sum_{n \geqslant 1} \mathrm{e}^{-c^\star t} \frac{(\beta \hat{J}_{\gamma,k})^n}{n!} \tag{4.117}$$

where the integral is extended to all k's for which there exists $i \in \{1, \dots, d\}$ such that $|k_i| > \gamma^{1-b}$. We shall hereafter use such a shorthand notation without further mention. We first estimate $p_t^{\gamma,>}$, the contribution of the other values of k and n will be analysed later.

Lemma 4.3.2. *Given any $b > 0$, (see (4.117)) for any integer m there is c so that for all x and $\gamma > 0$*

$$|p_t^{\gamma,>}(x)| \leqslant c\gamma^m t\,. \tag{4.118}$$

Proof. Since $J_\gamma(0, y) \geqslant 0$, $\beta|\hat{J}_{\gamma,k}| \leqslant \beta \hat{J}_{\gamma,0} = c^\star$, recalling that $|k_i| \leqslant \pi$, $i = 1, \dots, d$, we get

$$|p_t^{\gamma,>}(x)| \leqslant \frac{d}{(2\pi)} \int_{|k_1| > \gamma^{1-b}} \mathrm{d}k_1\, \beta t \sum_{y_2 \dots y_d} \Big| \sum_{y_1} \mathrm{e}^{\mathrm{i}k_1 y_1} J_\gamma(0, y)\Big|\,.$$

Keeping $y_2, \ldots, y_d$ fixed, we set $I(y_1) = J_\gamma(0, y)$, with y_1 the first coordinate of y. For any ℓ, $|I^{(\ell)}(y)|$ is bounded uniformly in γ, hence, using (4.116) we derive (4.118). □

We write

$$p_t^\gamma(x) = p_t^{\gamma,0}(x) + p_t^{\gamma,<}(x) + p_t^{\gamma,>}(x) \tag{4.119}$$

where

$$p_t^{\gamma,0}(x) = \delta_{x,0}\, \mathrm{e}^{-c^\star t} \tag{4.120}$$

$$p_t^{\gamma,<}(x) = (2\pi)^{-d} \int_{|k_i| \leqslant \gamma^{1-b}} \mathrm{d}k\, \mathrm{e}^{-\mathrm{i}kx}\, \mathrm{e}^{-c^\star t} \sum_{n \geqslant 1} \frac{(\beta t \hat{J}_{\gamma,k})^n}{n!} \tag{4.121}$$

where, in agreement with the previous notation, the above integral is extended to the set $\{|k_i| \leqslant \gamma^{1-b}\}$, for all $i = 1, \ldots, d$. In the next lemma we bound $p_t^{\gamma,<}(x)$: our bound is 'good' only when $\gamma^{1-2b} t \to 0$, as in our case where $t \leqslant c \log \gamma^{-1}$, for some c. For larger values of t we need a more accurate analysis.

Lemma 4.3.3. *For any $b \in (0, 1)$ there is c so that*

$$\left| p_t^{\gamma,<}(x) - \frac{\gamma^d}{(2\pi)^d} \int_{|k_i| \leqslant \gamma^{-b}} \mathrm{d}k\, \mathrm{e}^{-\mathrm{i}k\gamma x}\, \mathrm{e}^{-c^\star t} \sum_{n \geqslant 1} \frac{(\beta t \hat{J}(k))^n}{n!} \right| \leqslant \gamma^{d-bd} \beta c t \gamma^{1-b}\, \mathrm{e}^{\beta c t \gamma^{1-b}} \tag{4.122}$$

where

$$\hat{J}(k) = \int \mathrm{d}r\, \mathrm{e}^{\mathrm{i}kr} J(|r|)\,. \tag{4.123}$$

Proof. After the change of variables $k \to \gamma k$, (4.121) becomes

$$p_t^{\gamma,<}(x) = \frac{\gamma^d}{(2\pi)^d} \int_{|k_i| \leqslant \gamma^{-b}} \mathrm{d}k\, \mathrm{e}^{-\mathrm{i}kx}\, \mathrm{e}^{-c^\star t} \sum_{n \geqslant 1} \frac{(\beta t \hat{J}_{\gamma,\gamma k})^n}{n!}\,. \tag{4.124}$$

There is c so that for all $|k| \leqslant \gamma^{-b}$

$$\left| \hat{J}(k) - \hat{J}_{\gamma,\gamma k} \right| \leqslant c\gamma^{1-b} \tag{4.125}$$

and, for any $n \geqslant 1$,

$$\left| \hat{J}(k)^n - \hat{J}^n_{\gamma,\gamma k} \right| \leqslant n \big(|\hat{J}_{\gamma,0}| + c\gamma^{1-b} \big)^{n-1} c\gamma^{1-b}\,. \tag{4.126}$$

The left-hand side of (4.122) is then bounded by

$$\frac{\gamma^d}{(2\pi)^d} \int_{|k_i| \leqslant \gamma^{-b}} \mathrm{d}k\, \mathrm{e}^{-c^\star t} (\beta t c \gamma^{1-b})\, \mathrm{e}^{c^\star t + \beta t c \gamma^{1-b}}\,.$$

The lemma is therefore proven. □

The following theorem just summarizes the results in (4.118) and (4.122):

Theorem 4.3.4. *For any $b \in (0, 1)$ there are $\zeta > 0$ and c so that for all x and all $t \leqslant t^\star$*

$$\left| p_t^\gamma(x) - \mathrm{e}^{-c^\star t} \delta_{x,0} - \frac{\gamma^d}{(2\pi)^d} \int_{|k_i| \leqslant \gamma^{-b}} \mathrm{d}k\, \mathrm{e}^{-\mathrm{i}k\gamma x}\, \mathrm{e}^{-c^\star t} \sum_{n \geqslant 1} \frac{(\beta t \hat{J}(k))^n}{n!} \right| \leqslant c\gamma^{d+\zeta} \tag{4.127}$$

$$\left| p_t^\gamma(x) - \mathrm{e}^{-c^\star t} \delta_{x,0} - \gamma^d q_t(\gamma x) \right| \leqslant c\gamma^{d+\zeta} \tag{4.128}$$

where $q_t(\cdot)$ is defined in definition 2.3.1.

Remarks. As already mentioned, (4.127) is proven by (4.118) and (4.122); to prove (4.128) we have used that (4.118) holds also when $\hat{J}(k)$ replaces $\hat{J}_{\gamma,\gamma k}$.

We next prove bounds on $p_t^\gamma(x)$, the first one is trivial, it does not require what done so far:

Lemma 4.3.5. *There are $\delta > 0$ and c so that for all ℓ and t*

$$\sum_{|y|>\gamma^{-1}\ell} p_t^\gamma(y) \leqslant c\, \mathrm{e}^{-(\ell-\delta t)} \qquad \int_{|r|\geqslant \ell} \mathrm{d}r\, q_t(r) \leqslant c\, \mathrm{e}^{-(\ell-\delta t)} . \tag{4.129}$$

Proof. Since the random walk jumps at most by γ^{-1}, the left-hand side in the first inequality in (4.129) is bounded by

$$\mathrm{e}^{-c^\star t} \sum_{n\geqslant \ell t} \frac{(c^\star t)^n}{n!}$$

from which the first inequality follows; the second one is proven similarly. □

Remarks. Using lemma 4.3.5 we can improve (4.128), proving the following estimate used previously. There is $\zeta > 0$ and for any m, c', we have that for all x and all $t \leqslant t^\star$,

$$\left| p_t^\gamma(x) - \mathrm{e}^{-c^\star t}\delta_{x,0} - \gamma^d q_t(\gamma x) \right| \leqslant c'\gamma^\zeta \frac{(\lambda\gamma)^d}{1+(\lambda\gamma|x|)^m} . \tag{4.130}$$

In fact for $|x| \leqslant Ct^\star$, equation (4.130) is implied by (4.128) with ζ in (4.130) smaller than the parameter ζ in (4.128). For $|x| > Ct^\star$, we use (4.129). We then get the following condition on c':

$$c'(\lambda\gamma)^d\gamma^\zeta \leqslant \sup_{|x|\geqslant Ct^\star} c\, \mathrm{e}^{-(|x|-\delta t^\star)}(\lambda\gamma|x|)^m .$$

Recalling that $t^\star > \tau_c \log\gamma^{-1}$, by choosing C large enough, we see that the above condition can be satisfied with a finite c' uniformly in γ, hence concluding the proof of (4.130).

Many of the bounds of section 3 are a straight consequence of lemma 4.3.5. The bounds in (3.12), however, require some more care.

Lemma 4.3.6. *For any $m > 0$ there is c so that for all x and all $t \leqslant t^\star$*

$$(1+\gamma\lambda|x|)^{-m} \sum_y p_t^\gamma(y-x)(1+\gamma\lambda|y|)^m \leqslant c . \tag{4.131}$$

Proof. Equation (4.131) is easily proven for $t \leqslant 1$, so that we suppose, hereafter in this proof, that $t \geqslant 1$. If a, b and m are all positive

$$(a+b)^m \leqslant 2^m(a^m+b^m) \tag{4.132}$$

so that, calling A the left-hand side of (4.131),

$$A \leqslant 2^m + 2^{2m}(1+\gamma\lambda|x|)^{-m} \sum_y p_t^\gamma(y-x)(\gamma\lambda)^m[|x|^m + |y-x|^m] . \tag{4.133}$$

We thus reduce the proof of (4.131) to proving that A_1 is bounded, where

$$A_1 := (\gamma\lambda)^m \sum_y p_t^\gamma(y)|y|^m . \tag{4.134}$$

Since $t \leqslant c\lambda^{-2}$, for some c, using (4.129), for L large enough and γ small,

$$A_1 \leqslant 2 + (\gamma\lambda)^m \sum_{1\leqslant\lambda\gamma|y|\leqslant\lambda^{-1}L} p_t^\gamma(y)|y|^m =: 2 + A_2 \tag{4.135}$$

with A_2 being defined by the last equality. We have excluded the values $\lambda\gamma|y| < 1$ to avoid the divergence in (4.138) below for $\gamma\lambda|y| \to 0$. By equation (4.127)

$$A_2 \leqslant A_3 + c\gamma^{d+\zeta}(\gamma\lambda)^m \sum_{\lambda\gamma|y|\leqslant\lambda^{-1}L} |y|^m \tag{4.136}$$

$$A_3 := \left| \frac{\gamma^d}{(2\pi)^d} \int_{|k_i|\leqslant\gamma^{-b}} \mathrm{d}k\,(\gamma\lambda)^m \sum_{1\leqslant\gamma\lambda|y|\leqslant\lambda^{-1}L} \mathrm{e}^{-\mathrm{i}k\gamma y}|y|^m\,\mathrm{e}^{-c^\star t} \sum_{n\geqslant 1} \frac{(\beta t\hat{J}(k))^n}{n!} \right|. \tag{4.137}$$

The second term on the right-hand side of (4.136) is bounded by

$$c\gamma^{d+\zeta}(\gamma\lambda)^m(L\gamma^{-1}\lambda^{-2})^{m+d} = c\gamma^\zeta\lambda^{-m-2d}L^{m+d}$$

which vanishes as $\gamma \to 0$ for any fixed L, because $\lambda^{-2} = \log\gamma^{-1}$. It thus remains to show that A_3 is bounded.

Let m' be an even integer larger than $m + d + 2$, then, integrating by parts with respect to $\mathrm{e}^{-\mathrm{i}k\gamma x}$, we get

$$A_3 \leqslant \frac{\gamma^d}{(2\pi)^d} \int_{|k_i|\leqslant\gamma^{-b}} \mathrm{d}k\,\lambda^{m'} \sum_{1\leqslant\gamma\lambda|y|\leqslant\lambda^{-1}L} (\lambda\gamma|y|)^{m-m'} \sum_{I\in\mathcal{I}_{m'}} c_I \left| \frac{\partial^I}{\partial k^I}\,\mathrm{e}^{-c^\star t} \sum_{n\geqslant 1} \frac{(\beta t\hat{J}(k))^n}{n!} \right| \tag{4.138}$$

where

$$\mathcal{I}_{m'} = \left\{ I = (i_1, \ldots, i_d) \in \mathbb{Z}^d : i_j \geqslant 0,\ \ j = 1, \ldots, d; \sum_{j=1}^d i_j = m' \right\} \tag{4.139}$$

and the coefficients c_I are such that

$$|y|^{m'} = \sum_{I\in\mathcal{I}_{m'}} c_I y^I \qquad y^I = \prod_{j=1}^d y_j^{i_j} \qquad y = (y_1, \ldots, y_d) \tag{4.140}$$

(recall that m' is an even integer);

$$\frac{\partial^I}{\partial k^I} = \frac{\partial^{i_1}}{\partial k_1^{i_1}} \cdots \frac{\partial^{i_d}}{\partial k_d^{i_d}}\,. \tag{4.141}$$

Observe that

$$\sum_{I\in\mathcal{I}_{m'}} c_I \frac{\partial^I}{\partial k^I} = \nabla^{m'}\,.$$

Recalling that $m' > d + 2$, the sum over y is finite. This is like the Riemann sum of the corresponding integral, if we had the volume element $(\lambda\gamma)^d$, but we only have γ^d in (4.138). We are then reduced to prove that there is a constant c so that

$$\int_{|k_i|\leqslant\gamma^{-b}} \mathrm{d}k\,\lambda^{m'-d} \left| \frac{\partial^I}{\partial k^I} \sum_{n\geqslant 1} \mathrm{e}^{-c^\star t} \frac{(\beta t\hat{J}(k))^n}{n!} \right| \leqslant c \tag{4.142}$$

for all $I \in \mathcal{I}_{m'}$.

Since $\nabla\hat{J}(0) = 0$ (J depends on $|r|$) and $|\hat{J}(k)| < 1$ for $k \neq 0$, (by the positivity of J), for any $\delta > 0$ there is $c_0 > 0$ so that

$$|\hat{J}(k)| \leqslant \hat{J}(0) - c_0k^2 \qquad |k| \leqslant \delta\,. \tag{4.143}$$

Furthermore, by the smoothness of $J(\cdot)$, for any $n > 1$ there is c so that $|\hat{J}(k)| \leqslant c|k|^{-n}$, hence, given any $\delta > 0$ there is $\epsilon = \epsilon(\delta) > 0$ so that

$$|\hat{J}(k)| \leqslant \hat{J}(0) - \epsilon \qquad |k| > \delta\,. \tag{4.144}$$

(Later in this section, we will also need the following: there is $\epsilon_0 > 0$ so that for all δ small enough, $\epsilon(\delta) = \epsilon_0\delta^2$.) Finally, there are functions $\phi_p^I(k)$ so that, for all n,

$$\frac{\partial^I}{\partial k^I}\hat{J}(k)^n = \sum_{p=1}^{m'} \hat{J}(k)^{n-p}\phi_p^I(k)n(n-1)\dots(n-p+1) \tag{4.145}$$

(the terms with $p > n$ are therefore 0). The functions $\phi_p^I(k)$ are proportional to products of derivatives of $\hat{J}(k)$, their only property we are going to use is that there are coefficients $c_{p,m'}$ such that

$$|\phi_p^I(k)| \leqslant c_{p,m'}\frac{|k|^{(2p-m')_+}}{1+|k|^{m'+d+2}} \qquad n_+ = \max\{n, 0\}\,. \tag{4.146}$$

To derive (4.146) we use the decay properties of $\hat{J}(k)$ and its derivatives, hence the denominator in (4.146). The exponent $(2p-m)_+$ bounds from below the number of factors $\partial\hat{J}(k)/\partial k_i$, $i \in \{1,\dots,d\}$, present in $\phi_p^I(k)$. Since for k small, $\partial\hat{J}(k)/\partial k_i$ goes like k_i, we obtain the numerator in (4.146), we omit the details and give (4.146).

Using equation (4.145) we have

$$\frac{\partial^I}{\partial k^I}\sum_{n\geqslant 1}\frac{(\beta t\hat{J}(k))^n}{n!} = \sum_{p=1}^{m'}\phi_p^I(k)(\beta t)^p\,\mathrm{e}^{\beta t\hat{J}(k)}\,. \tag{4.147}$$

We fix $\delta > 0$ and then split the integral in (4.142) over $|k| \leqslant \delta$ and $|k| > \delta$. The latter is bounded, using (4.144) and (4.146), by

$$\lambda^{m'-d}\int \mathrm{d}k\,\mathbf{1}\big(|k| > \delta, |k_i| \leqslant \gamma^{-b}\big)\sum_{p=1}^{m'}|\phi_p^I(k)|(\beta t)^p\,\mathrm{e}^{-\epsilon\beta t - c^\star t + \hat{J}(0)\beta t} \leqslant c\lambda^{m'-d} \tag{4.148}$$

which vanishes because $m' > d$. In deriving (4.148) we have used that

$$\beta\hat{J}(0)t = c^\star t + \beta t(\hat{J}(0) - \hat{J}^0_{\gamma,0}) \qquad \big|\beta t(\hat{J}(0) - \hat{J}_{\gamma,0})\big| \leqslant c\gamma\lambda^{-2}$$

for some $c > 0$.

The integral in (4.142) extended to $|k| \leqslant \delta$ is bounded, using (4.147) and (4.146), by

$$\begin{aligned}
\lambda^{m'-d}&\int_{|k|\leqslant\delta}\mathrm{d}k\sum_{p=1}^{m'}(\beta t)^p c_{p,m'}|k|^{(2p-m')_+}\,\mathrm{e}^{-c_0k^2\beta t - c^\star t + \beta\hat{J}(0)t} \\
&\leqslant c'\sum_{p=1}^{m'/2}\lambda^{m'-d}t^{p-d/2} + c''\sum_{p=m'/2+1}^{m'}\lambda^{m'-d}t^{p-(p-m'/2)-d/2} \\
&\leqslant c'''(\lambda t^{1/2})^{m'-d}
\end{aligned}$$

(recall that since the beginning of the proof we have restricted ourselves to $t \geqslant 1$). Since $\lambda t^{1/2}$ is bounded, the proof of the lemma is completed. □

Remark. Observe that when proving lemma 4.3.6, we have actually shown that for any $a > 0$ and any m there is c so that

$$p^\gamma_{t_a}(x) \leqslant \frac{c(\gamma\lambda)^d}{(1+\gamma\lambda|x|)^m} \qquad \text{for all} \quad x \neq 0\,. \tag{4.149}$$

We next turn to the proof of (3.148). The proof is essentially similar to that of lemma 4.3.6, but we give a few details, for the reader's convenience.

Lemma 4.3.7. *Let $\pi^\gamma(x) \equiv \pi^\gamma(0,x)$ be as in (3.6) (with $a>0$); let m' be an even integer larger than $d+2$. Then, for any multindex I, (see (4.140) for notation) there is c so that for all x*

$$\left|\frac{\partial^I}{\partial x^I}\,\pi^\gamma(x)\right| \leqslant c\,\frac{(\lambda\gamma)^d}{1+(\lambda\gamma|x|)^{m'}}\,. \tag{4.150}$$

The discrete derivative on the left-hand side of (4.150) is defined in (3.139). Analogously if $\tau>0$, there is c so that

$$\left|\frac{\partial^I}{\partial x^I}\,q_{\lambda^{-2}}(r)\right| \leqslant c\,\frac{\lambda^{d+|I|}}{1+(\lambda r)^{m'}}\,. \tag{4.151}$$

Proof. The proof of (4.151) is completely analogous to that of (4.150) and we omit it. We shorthand $t\equiv t_a = a\lambda^{-2}$. By (4.129), recalling $p^\gamma_{t_a}(x)=\pi^\gamma(x)$, $x\neq 0$,

$$\pi^\gamma(x) \leqslant c\,\mathrm{e}^{-\gamma|x|+\delta t}$$

hence (4.150) for $|x|\geqslant \gamma^{-1}L\lambda^{-2}$, with L large enough. We shall then restrict, hereafter in this proof, to $|x|<\gamma^{-1}\lambda^{-2}L$.

We recall that

$$\pi^\gamma(x) = \frac{1}{(2\pi)^d}\int \mathrm{d}k\,\mathrm{e}^{-\mathrm{i}kx}\sum_{n\geqslant 1}\frac{(\beta t\hat{J}_{\gamma,k})^n}{n!}\,.$$

If we estimated $\pi^\gamma(x)$ using (4.127), the error on the right-hand side would give a divergent bound for the left-hand side of (4.150), we thus need a more accurate analysis. We go back to the decomposition (4.119). Using lemma 4.3.2 we have that for any $b>0$ and any k there is c so that for all x

$$\left|\frac{\delta^I}{\delta x^I}[\pi^\gamma(x)-p^{\gamma,<}_t(x)]\right| \leqslant c\gamma^k\,. \tag{4.152}$$

By equation (4.152) it is therefore sufficient to bound the derivative of $p^{\gamma,<}_t$ and for that we use the representation (4.124). Using the bound

$$\left|\frac{\delta^I}{\delta x^I}\,\mathrm{e}^{-\mathrm{i}k\gamma x}\right| \leqslant c'\lambda^{-|I|}\gamma^{-|I|b} \qquad |k|\leqslant\gamma^{-b}$$

we get, proceeding as in the proof of lemma 4.3.3,

$$\left|\frac{\delta^I}{\delta x^I}\,p^{\gamma,<}_t(x) - \frac{\gamma^d}{(2\pi)^d}\int_{|k_i|\leqslant\gamma^{-b}}\mathrm{d}k\left[\frac{\delta^I}{\delta x^I}\,\mathrm{e}^{-\mathrm{i}k\gamma x}\right]\mathrm{e}^{-c^\star t}\sum_{n\geqslant 1}\frac{(\beta t\hat{J}(k))^n}{n!}\right|$$
$$\leqslant c'\lambda^{-|I|}\gamma^{-|I|b}\gamma^{d-bd}\beta ct\gamma^{1-b}\,\mathrm{e}^{\beta ct\gamma^{1-b}}\,. \tag{4.153}$$

The right-hand side is bounded by a constant times

$$\lambda^{-|I|}\gamma^{-b|I|}\gamma^{d-bd-b+1}\lambda^{-2} \leqslant \gamma^{d+\zeta} \qquad \zeta>0$$

if b is chosen small enough, i.e. $b(|I|+d+1)<1$.

Extending the second term on the left-hand side of (4.153) to a function of $x\in\mathbb{R}^d$, we can replace the discrete derivative by integrals of 'continuous derivatives'. We have an extra factor $(\lambda\gamma)^{-|I|}$, so that we are left with the estimate of

$$B := \frac{\gamma^d}{(2\pi)^d}\int_{|k_i|\leqslant\gamma^{-b}}\mathrm{d}k\,(\gamma\lambda)^{-|I|}\frac{\partial^I}{\partial x^I}\,\mathrm{e}^{\mathrm{i}k\gamma x}\,\mathrm{e}^{-c^\star t}\sum_{n\geqslant 1}\frac{(\hat{J}(k)\beta t)^n}{n!}\,. \tag{4.154}$$

We distinguish whether $|\gamma\lambda x| \leqslant 1$ or $\geqslant 1$ and consider explicitly the latter case, where the argument has extra difficulties. Let $m' > d+2$ be an even integer, then

$$B = (-\mathrm{i}\gamma\lambda|x|)^{-m'}\lambda^{m'}\frac{\gamma^d}{(2\pi)^d}\int_{|k_i|\leqslant\gamma^{-b}} \mathrm{d}k\,\lambda^{-|I|}k^I\left[\sum_{I'\in\mathcal{I}_{m'}} c_{I'}\frac{\partial^{I'}}{\partial k^{I'}}\,\mathrm{e}^{-\mathrm{i}k\gamma x}\right] \times \mathrm{e}^{-c^\star t}\sum_{n\geqslant 1}\frac{(\hat{J}(k)\beta t)^n}{n!} \tag{4.155}$$

see (4.138) and (4.140) for notation. Recalling that $\gamma\lambda|x| \geqslant 1$, the first factor in (4.155) plays the role of P in (3.148).

It remains to prove that there is c so that for any $I' \in \mathcal{I}_{m'}$

$$B_1 := \lambda^{-d+m'}\int_{|k_i|\leqslant\gamma^{-b}} \mathrm{d}k\,\lambda^{-|I|}\left|\frac{\partial^{I'}}{\partial k^{I'}}\left\{k^I\,\mathrm{e}^{-c^\star t}\sum_{n\geqslant 1}\frac{(\hat{J}(k)\beta t)^n}{n!}\right\}\right| \leqslant c \tag{4.156}$$

(the factor λ^{-d} comes from having reconstructed the volume element $(\lambda\gamma)^d$ present in (3.148), we only had γ^d in (4.155)). The proof of (4.156) is now very similar to that of lemma 4.3.5, see (4.145)–(4.149). Let $I'' \leqslant I$ and $I'' \leqslant I'$, then we need to show that there is c (independent of I, I', I'') so that

$$B_2 := \lambda^{-d+m'}\int_{|k_i|\leqslant\gamma^{-b}} \mathrm{d}k\,\lambda^{-|I|}|k^{I-I''}|\left|\frac{\partial^{I'-I''}}{\partial k^{I'-I''}}\,\mathrm{e}^{-c^\star t}\sum_{n\geqslant 1}\frac{(\hat{J}(k)\beta t)^n}{n!}\right| \leqslant c\,. \tag{4.157}$$

By equation (4.147) we then have, calling $\ell = |I' - I''|$,

$$B_2 \leqslant \lambda^{-d+m'}\int_{|k_i|\leqslant\gamma^{-b}} \mathrm{d}k\,\lambda^{-|I|}|k^{I-I''}|\sum_{p=1}^{\ell}\left|\phi_p^\ell(k)(\beta t)^p\,\mathrm{e}^{\beta t\hat{J}(k)-c^\star t}\right|. \tag{4.158}$$

Using equation (4.144) and recalling that $t = a\lambda^{-2}$, the integral over $|k| \geqslant \delta$ is bounded and vanishingly small as $k \to 0$. We use (4.146) and (4.143) to bound the integral over $|k| \leqslant \delta$, which is then bounded by

$$\begin{aligned}\lambda^{-d+m'-|I|}&\int_{|k_i|\leqslant\delta} \mathrm{d}k\,\left|k^{I-I''}|\sum_{p=1}^{|I'-I''|}|k|^{2p-|I'-I''|}(\beta t)^p\,\mathrm{e}^{c_0k^2\beta t-c^\star t+\beta\hat{J}(0)t}\right| \\ &\leqslant c'\sum_{p\leqslant|I'-I''|/2}\lambda^{-d+m'-|I|}t^{-|I-I''|/2-d/2+p} \\ &\quad+c''\sum_{p>|I'-I''|/2}\lambda^{-d+m'-|I|}t^{-|I-I''|/2-d/2}t^{-p+|I'-I''|/2}t^p\,.\end{aligned}$$

Recalling that $t^{-1/2} = a^{-1/2}\lambda$, we have that the last expression is bounded by a constant and the lemma is proven. □

Proof of (4.38). The proof is essentially that of the local central limit theorem, (after having explicited the dependence on γ by using theorem 4.3.4). For the reader's convenience, we give some details.

We call

$$T = t_c - t_{Na} = \lambda^{-2}(\tau_c - Na)\,. \tag{4.159}$$

Recall that

$$q_T(r) = \frac{1}{(2\pi)^d}\int \mathrm{d}k\,\mathrm{e}^{-\mathrm{i}kr}\,\mathrm{e}^{-\beta\hat{J}(0)T}\sum_{n\geqslant 1}\frac{(\beta T\hat{J}(k))^n}{n!}\,.$$

We call $q_T(r| \leqslant \delta)$ the integral extended to $|k| \leqslant \delta$ and we choose $\delta = \delta_T = T^{-(1/2-\zeta)}$, with $0 < \zeta < \frac{1}{6}$. We then have

$$\left| q_T\big(r| \leqslant \delta\big) - \frac{1}{(2\pi)^d} \int_{|k|\leqslant\delta_T} \mathrm{d}k\, \mathrm{e}^{-ikr}\, \mathrm{e}^{\beta T(\hat{J}(k)-\hat{J}(0))} \right| \leqslant \delta_T^d\, \mathrm{e}^{-\beta\hat{J}(0)T} . \tag{4.160}$$

By a Taylor expansion we have for a suitable constant c

$$\big|\hat{J}(k) - \hat{J}(0) + \tfrac{1}{2}Dk^2\big| \leqslant ck^3 \qquad \text{for all} \quad |k| \leqslant \delta_T . \tag{4.161}$$

We also have

$$\begin{aligned}
\left| \int_{|k|\leqslant\delta_T} \mathrm{d}k\, \mathrm{e}^{-\mathrm{i}kr}[\mathrm{e}^{-\beta T(\hat{J}(0)-\hat{J}(k))} - \mathrm{e}^{-\beta TDk^2/2}] \right| &\leqslant \int_{|k|\leqslant\delta_T} \mathrm{d}k\, \mathrm{e}^{-\beta TDk^2/2}\big|1 - \mathrm{e}^{c\beta Tk^3}\big| \\
&\leqslant \int_{|k|\leqslant\delta_T} \mathrm{d}k\, \mathrm{e}^{-\beta TDk^2/2} c'|Tk^3| \leqslant \int_{|k|\leqslant T^{\zeta}} \mathrm{d}k\, T^{-d/2}\, \mathrm{e}^{-\beta Dk^2/2} c' T^{-1/2}|k|^3 \\
&\leqslant \left(\int \mathrm{d}k\, \mathrm{e}^{-\beta Dk^2/2}|k^3| \right) c' T^{-d/2-1/2} \leqslant c'' T^{-d/2-1/2} .
\end{aligned} \tag{4.162}$$

From equations (4.160), (4.161) and (4.162)

$$\begin{aligned}
&\left| q_T\big(r| \leqslant \delta_T\big) - \frac{1}{(2\pi)^d} \int \mathrm{d}k\, \mathrm{e}^{-\mathrm{i}kr-\beta TDk^2/2} \right| \\
&\qquad \leqslant \delta_T^d\, \mathrm{e}^{-\beta\hat{J}(0)T} + c'' T^{-d/2-1/2} + \int_{|k|\geqslant\delta_T} \mathrm{d}k\, \mathrm{e}^{-\beta TDk^2/2} \\
&\qquad \leqslant c''' T^{-d/2-1/2} .
\end{aligned} \tag{4.163}$$

Calling $q_T(r| > \delta_T)$ the contribution to q_T coming from the integral extended to $|k| > \delta_T$, we have

$$\begin{aligned}
\big|q_T\big(r| > \delta_T\big)\big| &\leqslant \int_{|k|>\delta_T} \beta T|\hat{J}(k)|\, \mathrm{e}^{\beta T(\hat{J}(k)-\hat{J}(0))} \\
&\leqslant \int_{|k|>\delta_T} \beta T|\hat{J}(k)|\, \mathrm{e}^{-\beta T\epsilon_0\delta_T^2/2} \leqslant \int \beta T|\hat{J}(k)|\, \mathrm{e}^{-\beta T\epsilon_0 T^{2\zeta}/2}
\end{aligned} \tag{4.164}$$

where ϵ_0 is defined below (4.144).

We have thus proven (4.38) for $|r|T^{-1/2} \leqslant 1$. For $|r| > cT$, c sufficiently large, we can use (4.129), we are thus left with $T^{1/2} \leqslant |r| \leqslant cT$. We can repeat the previous analysis for all terms except for (4.162) where we need a few extra considerations. We rewrite (4.161) as

$$\hat{J}(k) - [\hat{J}(0) - \tfrac{1}{2}Dk^2] = k^3\tilde{J}(k) \tag{4.165}$$

with $\tilde{J}(k)$ a C^∞ function with fast decay. The left hand side of (4.162) is equal to

$$A := \left| \int_{|k|\leqslant\delta_T} \mathrm{d}k\, \mathrm{e}^{-\mathrm{i}kr}\, \mathrm{e}^{-\beta TDk^2/2}[\, \mathrm{e}^{\beta Tk^3\tilde{J}(k)} - 1] \right| . \tag{4.166}$$

Let m' be an even integer, then

$$A \leqslant |r|^{-m'} \left| \int_{|k|\leqslant\delta_T} [\nabla^{m'}\, \mathrm{e}^{-\mathrm{i}kr}]\, \mathrm{e}^{-\beta TDk^2/2}[\, \mathrm{e}^{\beta Tk^3\tilde{J}(k)} - 1] \right| \tag{4.167}$$

where the gradients is with respect to k. Integration by parts yields

$$A \leqslant |r|^{-m'} \int_{|k|\leqslant\delta_T} \big|\nabla^{m'}\{\, \mathrm{e}^{-\beta TDk^2/2}[\, \mathrm{e}^{\beta Tk^3\tilde{J}(k)} - 1]\}\big| + R \tag{4.168}$$

where the remainder R is a sum of integrals extended to $|k| = \delta_T$: since $\delta_T^3 T \to 0$ because $\zeta < \frac{1}{6}$, the remainder is bounded by

$$R \leqslant c\, \mathrm{e}^{-\beta D T^{2\zeta}/2} \tag{4.169}$$

for a suitable constant c (we omit the details). By the change of variables $k \to T^{1/2}k$, we obtain from (4.168)

$$A \leqslant |rT^{-1/2}|^{-m'} T^{-d/2} \int_{|k| \leqslant T^{\zeta}} \mathrm{d}k \left|\nabla^{m'}\{\mathrm{e}^{-\beta D k^2/2}[\mathrm{e}^{\beta T^{-1/2} k^3 \tilde{J}(kT^{-1/2})} - 1]\}\right| .$$

Recalling the definition of T, see (4.159), $T^{-1/2}$ is proportional to λ, it is then easy to see that

$$A \leqslant c(\lambda|r|)^{-m'} \lambda^{d+1}$$

for a suitable constant c, we omit the details.

As has already been said, the estimates of the other terms which contribute to q_T can be treated as when $|r| \leqslant \lambda^{-1}$, the proof of (4.38) is thus completed. □

5. The development of the interfaces

In this section we study the process in the time interval $[t_c, t^\star]$. We will see that the statistical solutions $m_{\gamma,t_0}(\cdot, t^\star|\sigma)$, $t_0 = \lambda^{-2}\tau_0$, $\tau_0 \in (0, \tau_c)$, describe, with large probability, clusters of fully developed phases, separated from each other by interfaces. This result is proven when σ is in a set of $\mathcal{G}_\gamma^{(3)} \subset \{-1, 1\}^{\mathbb{Z}^d}$, see theorem 2.5.3. Since in the limit $\gamma \to 0$, the empirical spin averages (see (2.43)) are with large probability close to the same averages of $m_{\gamma,t_0}(\cdot, \cdot|\sigma)$, see theorem 3.4.0, this will complete the proof of theorem 2.5.1.

The problem of the development of the interfaces is a well known problem in the PDE literature, de Mottoni and Schatzman [7], and Chen [3] have solved it for the Allen–Cahn equation (2.18). We extend the results obtained by de Mottoni–Schatzman and Chen to an evolution defined by (2.13), which is a result interesting in its own right. We actually prove it for the evolution (2.22), but the extension to (2.13) is then straightforward.

Analysis of the statistical solutions in the time interval $[t_c, t^\star]$

Let $\tau_0 \in (0, \tau_c)$ and $a > 0$ be such that $na = \tau_0$, n a positive integer. Let $C > 0$, and $\sigma \in F_{\gamma,a,\zeta}(n, n, C)$. We then consider the function $m_{\gamma,t_{na}}(x, t|\sigma)$ defined in definition 2.2.1. From proposition 4.2.2, lemma 4.2.4 and proposition 4.2.6, we know that for any $L > 0$,

$$\liminf_{k\to\infty} \liminf_{\gamma\to 0} \mathbb{P}^{\gamma}_{\mu_0}\big(S(0, \lambda^{-1}L) \subset \mathcal{A}_k\big) = 1$$

where the set $\mathcal{A}_k \subset \mathbb{R}^d$ is determined by the behaviour of $m_{\gamma,t_{na}}(\cdot, t|\sigma)$:

$$\mathcal{A} = \mathcal{A}_+(1) \cup \mathcal{A}_-(1) \cup \mathcal{A}_0(k) \tag{5.1}$$

with $\mathcal{A}_\pm(1)$ and $\mathcal{A}_0(k)$ defined in definition 4.2.3. The points in $\mathcal{A}_\pm$ are called the *easy ones*, in fact, in a suitably large neighbourhood of each of them, the function $m_{\gamma,t_{na}}(\cdot, t_c|\sigma)$ is bounded away from 0. Using of the barrier lemma (see lemma 3.2.1) we will see that $m_{\gamma,t_{na}}(\cdot, t^\star|\sigma)$ goes to $\pm m_\beta$, as $\gamma \to 0$. In a neighbourhood of the points in $\mathcal{A}_0(k)$ $m_{\gamma,t_{na}}(\cdot, t_c|\sigma)$ is not bounded away from 0, and we will prove that in these points $m_{\gamma,t_{na}}(\cdot, t^\star|\sigma)$ approaches the instanton solution (see definition 2.5.2).

To understand the definitions in the sequel let us imagine for simplicity that $m_{\gamma,t_{na}}(x, t_c|\sigma)$ is replaced by a function $v(r)$, with $r = \gamma x$. Assume, moreover, that

$v(r) = \lambda^{d/2}u(\lambda r)$, with u a smooth function and then ignore that r is a discrete variable, letting $r \in \mathbb{R}^d$. We then denote by $v(r,t)$ the solution to (2.13) with inital datum $v(r)$.

Given $k > 0$, r is then called an *easy point* if

$$\text{for all } r' \text{ such that } |r - r'| \leqslant C_1(\log\lambda^{-2})^2 \qquad \text{either } v(r') \geqslant \lambda^k \text{ or } v(r') \leqslant -\lambda^k \tag{5.2}$$

where C_1 is defined in definition 4.2.2.

Using the barrier lemma we will prove convergence of $v(r,t^\star)$ to $\pm m_\beta$ whenever r is an *easy point*, for any choice of k. There are, however, in general, also points which are not easy, as for instance in a small neighbourhood of a point r_0 such that $v(r_0) = 0$. Assume for notational simplicity that $d = 1$ and suppose that $u'(\lambda r_0) \neq 0$. Then

$$r = r_0 + 2C_1(\log\lambda^{-2})^2$$

is already an *easy point*, at least for λ small enough and k sufficiently large.
In fact if $|\lambda r - \lambda r_0| \leqslant \delta$ for some $\delta > 0$, then

$$v(r) \approx \lambda^{1+d/2}u'(\lambda r_0)(r - r_0)\,.$$

Therefore for all r' such that $|r - r'| \leqslant C_1(\log\lambda^{-2})^2$, we have that $|\lambda r' - \lambda r_0| \leqslant \delta$, if λ is small enough. So if we take $k \geqslant d/2 + 1$ (due to the presence of the term $(\log\lambda^{-2})^2$ the equality is also allowed), (5.2) holds hence r is an *easy point*.

We are then left with the points $|r - r_0| < 2C_1(\log\lambda^{-2})^2$. We have a separate argument which allows to control the solution at $|r - r_0| \leqslant \epsilon(\log\lambda^{-2})^2$ with $\epsilon > 0$ small enough. We can then use the barrier lemma for the points at distance $|r - r_0| \geqslant \epsilon(\log\lambda^{-2})^2$ up to time $\epsilon C_1(\log\lambda^{-2})^2$, this time interval is long enough to reach equilibrium, thus completing the analysis of all points such that $|r - r_0| \leqslant 2C_1(\log\lambda^{-2})^2$.

Lemma 5.1.2. *There are δ' and c so that for all ω and all $\gamma x \in \mathcal{A}_\pm(\omega)$,*

$$\left|m_{\gamma,t_{na}}\left(x, t_c + \omega(\log\lambda^{-2})^2|\sigma\right) \mp m_\beta\right| \leqslant c\,\mathrm{e}^{-\omega\delta'(\log\lambda^{-2})^2} \tag{5.3}$$

Proof. Given any ω and any $\gamma x \in \mathcal{A}_+(\omega)$ we define

$$g_\gamma(y) = \begin{cases} m_\gamma(y, t_c|\sigma) & \text{if} \quad |y - x| \leqslant \gamma^{-1}C_1\omega(\log\lambda^{-2})^2 \\ \lambda^{1+d/2}\log\lambda^{-2} & \text{elsewhere}\,. \end{cases}$$

By the hypothesis on x, we then have

$$g_\gamma(y) \geqslant \lambda^{1+d/2}\log\lambda^{-2} \qquad \text{for all } y\,.$$

We let $g_\gamma(\cdot,t)$ the solution to (2.22) with initial datum g_γ.

Denote by $z_\gamma(t)$, $t \geqslant 0$ the solution of

$$\frac{\mathrm{d}z_\gamma}{\mathrm{d}t} = -z_\gamma + \tanh\beta\hat{J}_{\gamma,0}z_\gamma \qquad z_\gamma(0) = \lambda^{1+d/2}\log\lambda^{-2}\,.$$

Then, by the monotonicity property of (2.22) (see (2) of lemma 3.2.1), for all y

$$g_\gamma(y, \omega(\log\lambda^{-2})^2) \geqslant z_\gamma(\omega(\log\lambda^{-2})^2)\,.$$

Moreover, z_γ converges exponentially to $m_{\beta,\gamma}$, the positive solution of

$$m_{\beta,\gamma} = \tanh\{\beta\hat{J}_{\gamma,0}m_{\beta,\gamma}\}\,.$$

Since $|\hat{J}_{\gamma,0} - 1| \leqslant c'\gamma$ (see (2.6)), there is c so that $|m_\beta - m_{\beta,\gamma}| \leqslant c\gamma$. Therefore there are $\zeta > 0$ and $c_2 > 0$ such that for all y

$$g_\gamma(y, \omega(\log\lambda^{-2})^2) \geqslant m_\beta - c_2\,\mathrm{e}^{-\zeta(\log\lambda^{-2})^2\omega}\,.$$

We have used that $\gamma = \exp(-\lambda^{-2})$ so that given $\zeta\omega$ there is some c' so that

$$|m_\beta - m_{\beta,\gamma}| \leqslant c' \, \mathrm{e}^{-\zeta(\log\lambda^{-2})^2\omega} .$$

On the other hand $g_\gamma(y) \leqslant 1$, for all y and again $z_\gamma(t)$ starting from $z_\gamma(0) = 1$, converges exponentially fast to $m_{\beta,\gamma}$, so that the bound (5.3) is proven for g_γ in the place of m_γ. By the barrier lemma we have

$$|m_{\gamma,t_{na}}(x, t_c + \omega(\log\lambda^{-2})^2|\sigma) - g_\gamma(x, \omega(\log\lambda^{-2})^2)| \leqslant c_1' \, \mathrm{e}^{-(\log\lambda^{-2})^2\omega}$$

equation (5.3) is therefore proved. The proof when $\gamma x \in \mathcal{A}_-(\omega)$ is completely analogous, hence the lemma is proven. □

We next consider the third set $\mathcal{A}_0$. Notice that if $r \in \mathcal{A}_0$ then (4.86) hold. This inequality can be rewritten as follows:

$$\theta\lambda^{1+d/2}[\gamma x - (r_0 + \epsilon_\gamma \nu)]\nu \leqslant m_{\gamma,t_{na}}(x, t_c|\sigma)) \leqslant \theta\lambda^{1+d/2}[\gamma x - (r_0 - \epsilon_\gamma \nu)]\nu \tag{5.4}$$

for all x such that $|\gamma x - r_0| \leqslant 5C_1(\log\lambda^{-2})^2$.

ϵ_γ therefore has the meaning of the displacement along ν necessary for bounding (locally) $m_{\gamma,t_{na}}$ in terms of a linear function.

Proposition 5.1.3. *There is c so that for any $k \geqslant 1$, if $\gamma y \in \mathcal{A}_0(k)$, and r_0 and ν are the corresponding parameters as in (4.85), then*

$$|m_{\gamma,t_{na}}(x, t^\star|\sigma) - \bar{m}((\gamma x - r_0)\nu)| \leqslant c\epsilon_\gamma \qquad \textit{for all} \quad f x : |\gamma x - r_0| \leqslant 2C_1(\log\lambda^{-2})^2 \tag{5.5}$$

where $\bar{m}$ is the instanton, defined in definition 2.5.2, and ϵ_γ is given by (4.66) and (4.67), respectively, when $d = 1$ and $d > 1$.

Proof. We first prove a lower bound, then an upper bound which, together, yield (5.5). We start from the lower bound and divide the proof into several parts.

Step 1 (Reduction to $d = 1$). Let $\hat{u}(z, t)$, $z \in \mathbb{R}$, $t \geqslant 0$, solve (2.13) in $d = 1$ with J replaced by $\hat{J}$, i.e.

$$\frac{\partial \hat{u}(z, t)}{\partial t} = -\hat{u}(z, t) + \tanh\beta(\bar{J} \star \hat{u})(z, t) \tag{5.6}$$

where

$$\bar{J}(|z|) = \int_{\mathbb{R}^{d-1}} \mathrm{d}r \, J(|z^2 + r^2|^{1/2}) \tag{5.7}$$

and with initial condition

$$\hat{u}_0(z) = \begin{cases} \theta\lambda^{1+d/2} z & \text{if} \quad |z| \leqslant 4C_1(\log\lambda^{-2})^2 \\ \theta\lambda^{1+d/2} 4C_1(\log\lambda^{-2})^2 & \text{if} \quad z > 4C_1(\log\lambda^{-2})^2 \\ -\theta\lambda^{1+d/2} 4C_1(\log\lambda^{-2})^2 & \text{if} \quad z < -4C_1(\log\lambda^{-2})^2 . \end{cases} \tag{5.8}$$

Let x, r_0, ν and C_1 be as in (5.5), then

$$m_{\gamma,t_{na}}(x, t^\star|\sigma) \geqslant \hat{u}(z, t^\star - t_c) - \hat{c}\gamma^\zeta - c_1' \, \mathrm{e}^{-(\log\lambda^{-2})^2} \qquad z = (\gamma x - r_0^\star) \cdot \nu \tag{5.9}$$

where $r_0^\star = r_0 + (\epsilon_\gamma + \gamma)\nu$ and $\hat{c}$, c_1' and $\zeta > 0$ are coefficients which will be specified in the course of the proof.

Proof of step 1. We refer to theorem 2.1.8 of [8] for the proof of the following statement:

Statement 1.
There are $\zeta > 0$, $\hat{c}$ *and* $a > 0$ *so small that the following holds. If* $m(r,t)$, $r \in \mathbb{R}^d$ *and* $m_\gamma(x,t)$, $x \in \mathbb{Z}^d$ *solve respectively (2.13) and (2.22) for* $t \geqslant 0$ *and*

$$m(r,0) = m_\gamma([\gamma^{-1}r], 0) \qquad \text{for all } r$$

then

$$|m(r,t) - m_\gamma([\gamma^{-1}r], t)| \leqslant \hat{c}\gamma^\zeta \qquad \text{for all } r \text{ and for all } t \leqslant a\log\lambda^{-1}\,. \tag{5.10}$$

We define

$$u_0(r) = m_\gamma([\gamma^{-1}r], t_c|\sigma)\,. \tag{5.11}$$

We then denote by $u(r,t)$, $t \geqslant t_c$, the solution to (2.13) with

$$u(r, t_c) = u_0(r)\,. \tag{5.12}$$

From equation (5.10) it then follows that

$$m_{\gamma,t_{na}}(x, t^\star|\sigma) \geqslant u(\gamma x, t^\star) - \hat{c}\gamma^\zeta\,. \tag{5.13}$$

We therefore need a lower bound on $u(\cdot, t^\star)$. Let r_0, θ and ν be the parameters corresponding to $\gamma y \in \mathcal{A}_0(k)$. We then have, by the definition of $\mathcal{A}_0(k)$, see equation (5.4),

$$u_0(r) \geqslant \theta\lambda^{1+d/2}[r - (r_0 + \epsilon'_\gamma\nu)]\nu \qquad \text{for all} \quad r : |r - r_0| \leqslant 5C_1(\log\lambda^{-2})^2 \tag{5.14}$$

where

$$\epsilon'_\gamma = \epsilon_\gamma + \gamma\,. \tag{5.15}$$

We define

$$\tilde{u}_0(r) = \begin{cases} \theta\lambda^{1+d/2} r\cdot\nu & \text{if} \quad |r\nu| \leqslant 4C_1(\log\lambda^{-2})^2 \\ \theta\lambda^{1+d/2}4C_1(\log\lambda^{-2})^2 & \text{if} \quad r\nu > 4C_1(\log\lambda^{-2})^2 \\ -\theta\lambda^{1+d/2}4C_1(\log\lambda^{-2})^2 & \text{if} \quad r\nu < -4C_1(\log\lambda^{-2})^2 \end{cases} \tag{5.16}$$

and we denote by $\tilde{u}(r,t)$ the solution to (2.13) with $\tilde{u}(r,0) = \tilde{u}_0(r)$. We let

$$r_0^\star = r_0 + \epsilon'_\gamma\nu \tag{5.17}$$

and we observe that

$$|r_0 - r_0^\star| \leqslant \epsilon'_\gamma < C_1(\log\lambda^{-2})^2\,. \tag{5.18}$$

Then from (5.14), (5.16) and (5.18) it follows that

$$u_0(r) \geqslant \tilde{u}_0(r - r_0^\star) \qquad \text{for all} \quad r : |r - r_0^\star| \leqslant 4C_1(\log\lambda^{-2})^2\,. \tag{5.19}$$

We next apply (4) of lemma 3.2.1, with $\delta = 1$ and $T = (\log\lambda^{-2})^2 = t^\star - t_c$ as we are going to explain.

Let r be such that

$$|r - r_0| \leqslant 2C_1(\log\lambda^{-2})^2$$

and consider $r' \in S(r, C_1(\log\lambda^{-2})^2)$, i.e. the sphere of center r and radius $C_1(\log\lambda^{-2})^2$, then from (5.18) we have

$$|r' - r_0^\star| \leqslant |r' - r| + |r - r_0| + |r_0 - r_0^\star| \leqslant 4C_1(\log\lambda^{-2})^2$$

so that for all $r' \in S(r, C_1(\log\lambda^{-2})^2)$, $u_0(r') \geqslant \tilde{u}_0(r' - r_0^\star)$, then from (3.26), we have that

$$u(r', t^\star) \geqslant \tilde{u}(r' - r_0^\star, t^\star - t_c) - c'_1\,\mathrm{e}^{-(\log\lambda^{-2})^2}\,. \tag{5.20}$$

We are therefore reduced to the analysis of $\tilde{u}(r,t)$. We let

$$z = r\nu$$

and we observe that $\tilde{u}_0$ is a function of z alone. Therefore, given $\hat{u}_0(z)$ as in (5.8), we denote by $\hat{u}(z,t)$ the solution to (5.6). It is not difficult to see that $\hat{u}(r\cdot\nu,t)$ considered as a function of $r \in \mathbb{R}^d$ and $t \geqslant 0$, solves (2.13) (in its original version, with d and J), hence, by uniqueness,

$$\tilde{u}(r,t) = \hat{u}(r\cdot\nu,t)\,. \tag{5.21}$$

This concludes the proof of step 1.

We are now reduced to a one-dimensional problem with the antisymmetric monotonically non-decreasing initial datum $\hat{u}_0$.

We are going to use the following properties proven in [12]

Statement 2.
Let $f(z,0)$, $z \in \mathbb{R}$, be any antisymmetric function non-negative for $z \geqslant 0$ and not identically 0. Let $f(z,t)$, $t \geqslant 0$, solve (5.6) with initial datum $f(z,0)$. Then the following holds.
(i) $f(z,t)$ is antisymmetric for all $t \geqslant 0$ and $f(z,t) \geqslant 0$ for all $z \geqslant 0$ and $t \geqslant 0$.
(ii) There are c and c', which depend on $f(\cdot,0)$, such that for all $t \geqslant 0$

$$\sup_z |f(z,t) - \bar{m}(z)| \leqslant c'\,\mathrm{e}^{-ct}\,. \tag{5.22}$$

Because of (ii) of statement 2, we know that $\hat{u}(z,t)$ converges exponentially fast to $\bar{m}$. This result, however, does not help us directly, because the rate of convergence (i.e. the constant c in (5.22)) does depend on the initial datum, which, in our case, depends on λ. Therefore the convergence to $\bar{m}$ may, in principle, occur much later than in (5.5). To solve this problem we use the barrier lemma and lemma 5.1.2 to prove that $|\hat{u}(z,t)|$ grows to finite values (bounded away from 0 independently of λ) except for a 'short space interval' of length $(\epsilon\log\lambda^{-2})^2$, $\epsilon > 0$.

Step 2. Let $\hat{u}(z,t)$, $t \geqslant 0$, be the solution of (5.6) with $\hat{u}(z,0) = \hat{u}_0(z)$ given in (5.8). Then $\hat{u}(z,t)$ is antisymmetric and monotonically non-decreasing. Furthermore, given $\epsilon > 0$, let

$$z' = 2\epsilon C_1(\log\lambda^{-2})^2 \qquad t' = \epsilon(t^\star - t_c) \tag{5.23}$$

then

$$\hat{u}(z,t') \geqslant \begin{cases} 0 & \text{for} \quad z \in (0,z') \\ m_\beta/2 & \text{for} \quad z \geqslant z'\,. \end{cases} \tag{5.24}$$

To prove (5.24) we go back to $\mathbb{R}^d$ and, recalling (5.21), we easily check that

$$z'\nu \in \mathcal{A}_+\big(\epsilon, \tilde{u}(\cdot,t_c), \gamma\big)$$

for γ small enough, cf definition 4.2.3. The inequality (5.24) then follows from lemma 5.1.2, hence step 2 is completed.

We then define $v(z,t)$, $t \geqslant t'$, as the solution of (5.6) with $v(z,t')$ an antisymmetric function of z equal to the right-hand side of (5.24) for $z \geqslant 0$. Notice that $\hat{u}(z,t') \geqslant v(z,t')$ for all $z \geqslant 0$ and that the reverse inequality holds for $z \leqslant 0$, therefore we cannot use, at least directly, the monotonicity properties of (5.6) to conclude that $\hat{u}(z,t) \geqslant v(z,t)$ for $t > t'$, not even when $z \geqslant 0$. Nonetheless this happens to be true if $\bar{J}$ is a monotonic non-increasing function.

Step 3. $\hat{u}(z,t) \geqslant v(z,t)$ for all $z \geqslant 0$ and all $t \geqslant t'$.

Step 3 is a corollary of the following:

Statement 3.
Let $f(z,t)$ and $g(z,t)$ be two antisymmetric functions which solve (5.6). Then

$$f(z,0) \geqslant g(z,0), \text{ for all } z \geqslant 0, \text{ implies that } f(z,t) \geqslant g(z,t), \text{ for all } z \geqslant 0 \text{ and } t \geqslant 0\,.$$

Proof of statement 3. Since $f(z,t)$ is antisymmetric, the function $f(z,t)$ restricted to $\xi \geqslant 0$ still obeys a closed equation. To make it explicit, we rewrite the non-local term in (5.6) for $z \geqslant 0$ as

$$\int_{\mathbb{R}} \mathrm{d}z'\,\hat{J}(|z'-z|)f(z') = \int_0^\infty \mathrm{d}z'\,\hat{J}(|z'-z|)f(z') - \int_0^\infty \mathrm{d}z'\,\hat{J}(|z'+z|)f(z')\,.$$

Since $z'+z > |z'-z|$, if $z > 0$ and $z' \geqslant 0$,

$$K(z,z') = \hat{J}(|z'-z|) - \hat{J}(|z'+z|) \geqslant 0$$

because $\hat{J}$ is monotonic non-increasing.

We then have that for $z \geqslant 0$

$$\frac{\partial f}{\partial t} = -f + \tanh\{\beta K \star f\}\,. \tag{5.25}$$

Since K is non-negative, equation (5.25) has the same monotonicity property as (5.6), statement 3 is thus proven.

We are now in better shape than after step 1, since we 'only' need a lower bound on $v(z,t)$ for $z \geqslant 0$. Recall that v solves the $d = 1$ problem for $t \geqslant t'$ and that it is antisymmetric and no longer infinitesimal with λ, as $v(z,t') \geqslant m_\beta/2$ for all $z \geqslant z'$. Unfortunately, we are still far from the end, since $v(z,t') = 0$ in the 'long space interval' $0 \leqslant z < z'$ with $z' = \epsilon(\log\lambda^{-2})^2$. We cannot use lemma 5.1.2 in $[0,z']$, because in that interval $v(\cdot,t') = 0$. We will exploit at this point the other mechanism of growth: the 'infection'. We shall see that the positive values of v at $z \geqslant z'$ spread with finite velocity, and they invade the positive real axis in a time proportional to z'.

We are going to use the following:

Statement 4. Let $f(z,t)$ and $g(z,t)$, $t \geqslant 0$, be two solutions of (5.6). Assume that
(i) $f(z,0)$ is antisymmetric and non-negative for $z \geqslant 0$
(ii) there is $M \geqslant 1$ such that $g(z+M,0)$ is an antisymmetric function of z, non-negative for $z+M \geqslant 0$.
(iii) $f(z,0) \geqslant g(z,0)$ for $z \geqslant 0$
Then $f(z,t) \geqslant g(z,t)$ for all $z \geqslant 0$ and all $t \geqslant 0$.

Proof. First of all observe that from (i) of statement 2 and the translation invariance of (5.6), it follows that $g(z+M,t)$ is antisymmetric for all $t \geqslant 0$.

Let $\psi(z,t)$ solve the equation:

$$\frac{\mathrm{d}\psi(z,t)}{\mathrm{d}t} = -\psi(z,t) + \tanh\{\beta(\hat{J}\star\psi)(z,t)\} \qquad z \geqslant M \qquad t \geqslant 0\,. \tag{5.26}$$

To specify ψ we need to impose both the initial value and the boundary conditions, namely $\psi(z,0)$ for all z and $\psi(z,t)$ for all $z < M$ and all $t \geqslant 0$. Observe that if we are only interested in $\psi(z,t)$ with $z \geqslant M$, then it is enough to specify the boundary condition in $M-1 \leqslant z < M$ (and the initial datum in $z \geqslant M$). We set

$$\begin{aligned} &\psi(z,t) = 0 \qquad \text{for all} \quad t \geqslant 0 \quad \text{and all} \quad z < M \\ &\psi(z,0) = f(z,0) \qquad \text{for all} \quad z \geqslant M\,. \end{aligned} \tag{5.27}$$

Since $f(z,t) \geqslant 0$ for all $z \geqslant 0$ and $t \geqslant 0$, $f(z,t)$ solves (5.26) with boundary condition $f(z,t) \geqslant \psi(z,t)$ for $M-1 \leqslant z < M$. Since (5.26) has the same monotonicity properties as (5.6), we conclude that

$$f(z,t) \geqslant \psi(z,t) \qquad \text{for all } z \geqslant 0 \text{ and for all } t \geqslant 0\,.$$

On the other hand

$$g(z,t) \leqslant 0 \qquad \text{for all } z \in (M-1, M) \text{ and for all } t \geqslant 0$$

because it is antisymmetric around M. Hence by the same monotonicity argument

$$\psi(z,t) \geqslant g(z,t) \qquad \text{for all } z \geqslant 0 \text{ and for all } t \geqslant 0\,.$$

The statement 4 is therefore proven.

Step 4. There are $T > 0$ *and* $L' > 0$ *so that for all* $z \geqslant 0$

$$v(z, t'+T) \geqslant v_1(z, t'+T)$$

where $v_1(z, t'+nT)$ *is antisymmetric and such that*

$$v_1(z, t'+T) = \begin{cases} 0 & \textit{for} \quad 0 \leqslant z < z' - L' \\ m_\beta/2 & \textit{for} \quad z \geqslant z' - L'\,. \end{cases} \tag{5.28}$$

To prove step 4 we use (ii) of statement 2. Let L' and L be such that

$$\bar{m}(L') = \tfrac{2}{3} m_\beta \qquad L = 2L'\,. \tag{5.29}$$

We then define $w_1(z)$ as

$$w_1(z + z' - L) = \begin{cases} 0 & \text{if} \quad -L \leqslant z \leqslant L \\ m_\beta/2 & \text{if} \quad z \geqslant L \\ -m_\beta/2 & \text{if} \quad z \leqslant -L\,. \end{cases} \tag{5.30}$$

Observe that w_1 is antisymmetric around $z' - L$. We then denote by $w_1(z,t)$, $t \geqslant 0$, the solution to (5.6) with initial datum given by (5.30).

By statement 2, for all $t \geqslant 0$,

$$w_1(z + z' - L, t) \geqslant \bar{m}(z) - c' \,\mathrm{e}^{-ct}\,. \tag{5.31}$$

Observe that, due to the translational invariance of (5.6), $w_1(z,t)$ can be obtained by solving (5.6) with initial datum as on the right-hand side of (5.30) and then translating it by $z' - L$. As the right-hand side of (5.30) does not depend on λ, the constants c and c' in (5.31) are also independent of λ. Choosing T so that

$$c' \,\mathrm{e}^{-cT} = \tfrac{1}{6} m_\beta \tag{5.32}$$

we have from (5.29), (5.31) with $z = L'$ and (5.32)

$$w_1(z' - L', T) \geqslant \tfrac{1}{2} m_\beta\,. \tag{5.33}$$

Choosing $M = z' - L'$, $f = v$ and $g = w_1$ in statement 4, we have

$$v(z, t'+t) \geqslant w_1(z,t) \qquad \text{for all} \quad z \geqslant z' - L' \text{ and for all } t\,.$$

The proof of step 4 is thus concluded.

By iterating the previous proof we easily get

$$v(z, t'+nT) \geqslant v_n(z, t'+nT) \qquad z \geqslant 0 \tag{5.34}$$

where the functions $v_n(z, t' + nT)$ are antisymmetric and such that

$$v_n(z, t' + nT) = \begin{cases} 0 & \text{for} \quad 0 \leqslant z < z' - nL' \\ m_\beta/2 & \text{for} \quad z \geqslant z' - nL' \end{cases} \tag{5.35}$$

for all $n \leqslant N$ where

$$z' - NL' = 2L' + 1 \qquad N \leqslant \epsilon C_1 \frac{(\log \lambda^{-2})^2}{L'} . \tag{5.36}$$

We have thus proven that

$$\hat{u}(z, t' + NT) \geqslant \psi(z) \qquad \text{for all} \quad z \geqslant 0 \tag{5.37}$$

where $\psi(z)$ is antisymmetric and

$$\psi(z) = \begin{cases} 0 & \text{if} \quad z \in [0, 2L' + 1) \\ \frac{1}{2} m_\beta & \text{if} \quad z \geqslant 2L' + 1 . \end{cases} \tag{5.38}$$

Since for $z \geqslant 0$,

$$\psi(z) \leqslant v_N(z, t' + NT) . \tag{5.39}$$

From the monotonicity property it follows that

$$\hat{u}(z, t) \geqslant \psi(z, t) \qquad \text{for all} \quad z \geqslant 0, \ \text{for all } t \geqslant t' + NT \tag{5.40}$$

where $\psi(z, t)$ solves (5.6) with $\psi(z, 0) = \psi(z)$. Then, since t' is given by (5.23),

$$(t^\star - t_c) - (t' + NT) \geqslant (\log \lambda^{-2})^2 \left[1 - \epsilon - \epsilon C_1 \frac{T}{L'} \right] \geqslant \frac{1}{2} (\log \lambda^{-2})^2$$

for a suitable choice of ϵ. We then have

$$\hat{u}(z, t^\star - t_c) \geqslant \psi\big(z, (t^\star - t_c) - (t' + NT)\big) \qquad \text{for all} \quad z \geqslant 0 . \tag{5.41}$$

By statement 2 with c and c' corresponding to ψ, we then conclude that

$$\hat{u}(\xi, t^\star - t_c) \geqslant m^{dd}(\xi) - c' \, \mathrm{e}^{-c(\log \lambda^{-2})^2/2} \qquad \text{for all} \quad \xi \geqslant 0 \tag{5.42}$$

and from step 1

$$m_\gamma(x, t^\star, |\sigma) \geqslant m^{dd}(\xi) - \hat{c}\gamma^\zeta - c_1' \, \mathrm{e}^{-(\log \lambda^{-2})^2} - c' \, \mathrm{e}^{-c(\log \lambda^{-2})^2/2} \qquad \xi = (\gamma x - r_0^\star)\nu . \tag{5.43}$$

We have thus completed the proof of the lower bound for $z \geqslant 0$.

An upper bound for $m_{\gamma, t_{na}}(x, t^\star, |\sigma)$ is easier to prove. Defining $u_0(r)$ as in (5.11), and using (5.4), we have that

$$u_0(r) \leqslant \theta \lambda^{1+d/2} [r - (r_0 - \epsilon_\gamma)] \nu \qquad \text{for all} \quad r : |r - r_0| \leqslant 5c_1 (\log \lambda^{-2})^2 \tag{5.44}$$

therefore letting $\tilde{u}_0(r)$ be as in (5.16) we have

$$u_0(r) \leqslant \tilde{u}_0(r - \hat{r}_0) \qquad \text{for all} \quad r : |r - r_0| \leqslant 4c_1 (\log \lambda^{-2})^2 \tag{5.45}$$

where

$$\hat{r}_0 = r_0 - \epsilon_\gamma \nu . \tag{5.46}$$

The analogue of (5.13) and (5.20) is

$$m_\gamma(x, t^\star, |\sigma) \leqslant \hat{c}\gamma^\zeta + \tilde{u}(\gamma x - \hat{r}_0, t^\star - t_c) + c_1' \, \mathrm{e}^{-(\log \lambda^{-2})^2} . \tag{5.47}$$

Like before $\tilde{u}(r, t) = \hat{u}(z, t)$ where $r \cdot \nu = z$. By statement 3

$$\hat{u}(z, t) \leqslant \phi(z, t) \qquad \text{for all} \quad z \geqslant 0 \tag{5.48}$$

where $\phi(z,t)$ is the antisymmetric function equal to m_β for all $z > 0$. By statement 2, there are c'' and c''' so that for all z

$$|\phi(z, t^\star - t_c) - \bar{m}(z)| \leqslant c''' e^{-c''(\log \lambda^{-2})^2} . \tag{5.49}$$

From equations (5.47), (5.48) and (5.49)

$$m_{\gamma,t_{na}}(x, t^\star, |\sigma) \leqslant \bar{m}(z) + \hat{c}\gamma^\zeta + c_1' \,\mathrm{e}^{-(\log \lambda^{-2})^2} + c''' \,\mathrm{e}^{-c''(\log \lambda^{-2})^2} \tag{5.50}$$

equation (5.5) follows from (5.43) and (5.50), thus the proof of proposition 5.1.3 is concluded. □

5.1. Proof of theorems 2.5.1 and 2.5.3

Proof of theorem 2.5.3. We take a so that $\tau_0 = na$ and so small that $(N-n)a > \alpha(\tau_c - Na)$. We specify the set $\mathcal{G}_\gamma^{(3)}$ in theorem 2.5.3 as the intersection of $F_{\gamma,a,\zeta}(n, C)$, with the set $\{\ell_\gamma(\cdot|\sigma) \in \mathcal{G}(L, k)\}$, see (4.69). By (4.71), 2.46 is true with this definition of $\mathcal{G}_\gamma^{(3)}$.

We specify the parameter R_γ in theorem 2.5.3 as $R_\gamma = 2\lambda C_1(\log \lambda^{-2})^2$, C_1 as in definition 4.2.1. We also set: $u_\gamma(\xi) = m_\beta$ sign $\hat{\ell}_\gamma(\xi|\sigma)$, hence $\Sigma = \{\xi : \hat{\ell}_\gamma(\xi|\sigma) = 0\}$.

In the proof of proposition 4.2.2, it is shown that for all γ small enough, if $\ell_\gamma(\cdot|\sigma) \in \mathcal{G}(L, k)$, then $S(0, \lambda^{-1}L) \subset \mathcal{A}'_k$ (see definition 4.2.1). Let us assume γ small enough, then, if $\gamma x \in \mathcal{A}'_\pm(1)$, by lemma 4.2.4, $\gamma x \in \mathcal{A}_\pm(1)$ and, by (5.3) (with $\omega = 1$), we get (2.47). Therefore all the points x such that $\gamma x \in \mathcal{A}'_\pm(1)$ and $d(\lambda\gamma x, \Sigma) \geqslant R_\gamma$, verify (2.47).

If $\gamma x \notin \mathcal{A}'_\pm(1)$, then, by (4.72) and (4.81), there is r_0 so that $\lambda r_0 \in \Sigma$ and $|\lambda\gamma x - \lambda r_0| \leqslant R_\gamma$. By proposition 4.2.6, $\gamma x \in \mathcal{A}_0(k)$, with same parameters θ, ν and r_0. Then (5.5) gives (2.48), for γ small enough). Notice that if $\gamma x \notin \mathcal{A}'_\pm(1)$, and $d(\lambda\gamma x, \Sigma) > R_\gamma$, then (5.5) gives (2.47), because $\bar{m}(s)$ goes exponentially fast to m_β as $s \to \infty$, as proven in [12]. We have thus completed the proof of theorem 2.5.3 □

Proof of theorem 2.5.1. We choose a so that $\tau_0 = na$ and we set $\mathcal{G}_\gamma^{(2)} = F_{\gamma,a,\zeta}(n, C)$ then from theorem 3.5.0 we have

$$\mathbb{P}_{\mu_0}^\gamma\big(\sigma(\cdot, \lambda^{-2}\tau_0) \in \mathcal{G}_\gamma^{(2)}\big) > 1 - \epsilon .$$

Equation (2.43) follows from theorem 3.4.0. □

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