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The optimal interface profile for a non-local model of phase separation*

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Abstract
We consider a non-local excess free energy functional, which arises in the description of the continuum limit of Ising spin system with Kac interaction and external random magnetic field. We study the functional for values of the parameters in the phase transition region and we characterize the optimal profile describing the interface between the two pure thermodynamic phases. We use a dynamic method to minimize the excess free energy in the class of profiles connecting the two stable phases. We namely characterize the optimal profiles as stationary solutions of a system of non-local equations proving global nonlinear stability results for the shape of the optimal profile and decay estimates uniform in the profile within a certain class of functions.

Mathematics Subject Classification: 60K35, 82B20, 82B43

1. Introduction

We consider the following functional

$$\begin{aligned}\mathcal{F}(m) &= \mathcal{F}(m_1, m_2)\\ &= \frac{1}{4}\iint J(x-y)[\tilde{m}(x)-\tilde{m}(y)]^2\,\mathrm{d}x\,\mathrm{d}y + \int [f_{\beta,\theta}(m_1(x), m_2(x))\\ &\quad - f_{\beta,\theta}(m_{\beta,1}, m_{\beta,2})]\,\mathrm{d}x \end{aligned} \tag{1.1}$$

where $m_i = m_i(x)$ for $i = 1, 2$ is a real-valued function on $\mathbb{R}$; $\tilde{m} = \frac{1}{2}(m_1 + m_2)$, β a positive number larger than 1; θ a positive number which we take suitably small, J a non-negative,

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even function, supported in the interval $[-1, 1]$, with an integral equal to 1 and $\int |J'(x)|\, \mathrm{d}x$ finite,

$$f_{\beta,\theta}(m_1, m_2) \equiv f_{\beta,\theta}(m) = \frac{-(m_1+m_2)^2}{8} - \frac{\theta}{2}(m_1 - m_2) - \frac{1}{2\beta}\left(I(m_1) + I(m_2)\right), \tag{1.2}$$

$$I(m) = -\frac{1+m}{2}\log\left(\frac{1+m}{2}\right) - \frac{1-m}{2}\log\left(\frac{1-m}{2}\right) \qquad \text{and} \qquad m_\beta = (m_{\beta,1}, m_{\beta,2})$$

is a minimum of $f_{\beta,\theta}(m_1, m_2)$. Thus, for each value of $\beta > 1$ and θ small enough, the precise condition will be given in the following, $f_{\beta,\theta}(m_1, m_2)$ has two absolute minima $m_\beta = (m_{\beta,1}, m_{\beta,2})$ and $Tm_\beta = (-m_{\beta,2}, -m_{\beta,1})$, see (1.5) and $f_{\beta,\theta}(m_\beta) = f_{\beta,\theta}(Tm_\beta)$. The functional (1.1) arises in the study of Ising spin systems with Kac type interactions and external magnetic field which randomly takes ± 1 values with equal probability, see [11] for a description of the model. The interaction J is the ferromagnetic Kac potential; the requirement to be positive is essential while the restriction to be of compact support is done for simplicity. All the results hold if J is taken to be exponentially fast decaying at infinity. We denote $\mathcal{F}(m)$ as the excess free energy functional. The m_1, (respectively m_2), is then interpreted as the magnetization density associated, through a block spin transformation, to the sites where the random magnetic field takes the value +1, (respectively -1). The parameter β^{-1} is the product of the absolute temperature and the Boltzmann constant. The parameter θ represents the strength (the variance) of the external random magnetic field. Notice that the functional (1.1) is well defined and positive, although it could take infinite values. In $\mathcal{F}(m)$ there are two mechanisms to penalize departures from equilibrium. The first one is ruled by the free energy density $f_{\beta,\theta}(m_1, m_2)$. Any value of $m(r)$ which is not a minimizer of $f_{\beta,\theta}(m_1, m_2)$, i.e. different from m_β and Tm_β, contributes to the total free energy proportionally to the space volume where it is attained. There is also some penalty for changing the minimizer in different regions of space, which is given by the non-local term. Thus the global minimizers of $\mathcal{F}$ are the functions that are constantly equal to the minimizer of $f_{\beta,\theta}(m_1, m_2)$, their free energy is 0 and they correspond to the two pure thermodynamic phases m_β and Tm_β. We are interested in determining the shape and properties of the optimal profile for the interfaces between a region m_β and Tm_β. In our case the optimal profile is a minimizer of $\mathcal{F}$ among all functions whose asymptotic values at $\pm\infty$ are m_β and Tm_β, respectively or the reverse. These two families of minimizers are well separated in all the metrics in which we work, and it suffices to consider only one of them. Actually, one is obtained by T-reflecting the other, where T is the map which associates at $m = (m_1, m_2)$, $Tm = (-m_2, -m_1)$. Knowledge of this particular type of stationary solutions is relevant for characterizing the interfaces appearing in the typical spin configurations of a RFKM [12].

The critical points of $f_{\beta,\theta}(m_1, m_2)$ are the two-dimensional vectors (m_1, m_2), which are solutions of the system of equations

$$\begin{aligned} m_1 &= \tanh\left(\beta\frac{(m_1+m_2)}{2} + \beta\theta\right) \\ m_2 &= \tanh\left(\beta\frac{(m_1+m_2)}{2} - \beta\theta\right). \end{aligned} \tag{1.3}$$

We assume that $\beta > 1$ and $\beta\theta$ satisfies $\tanh\beta\theta \leqslant \min(1/\sqrt{3}, (1-\beta^{-1})^{1/2})$. This implies that the system (1.3) has only three solutions, two of them being absolute minima and one the local maximum of $f_{\beta,\theta}(m)$. This can be easily proved by considering the equation obtained by summing the two equations of (1.3) obtaining

$$\tilde{m} = \tfrac{1}{2}\tanh\beta(\tilde{m}+\theta) + \tfrac{1}{2}\tanh\beta(\tilde{m}-\theta). \tag{1.4}$$

The previous condition implies that the derivative at the origin of the function on the right-hand side of (1.4) is bigger than one, and the function is concave on the positive real and convex on the negative real number. Moreover, if $\tilde{m}_\beta$ is the largest positive solution of (1.4), then the two absolute minima of $f_{\beta,\theta}(m)$ are of the form $m_\beta = (m_{\beta,1}, m_{\beta,2})$ and $Tm_\beta = (-m_{\beta,2}, -m_{\beta,1})$ where

$$\begin{aligned} m_{\beta,1} &= \tanh\beta(\tilde{m}_\beta + \theta) \\ m_{\beta,2} &= \tanh\beta(\tilde{m}_\beta - \theta). \end{aligned} \tag{1.5}$$

It is easy to see that the function $f_{\beta,\theta}(m)$ is quadratic around its minima. Moreover, there exists a constant $c(\beta,\theta)$ such that for all $m = (m_1, m_2)$,

$$f_{\beta,\theta}(m) - f_{\beta,\theta}(m_\beta) \geqslant c(\beta,\theta)\min(\|m - m_\beta\|_2^2, \|m - Tm_\beta\|_2^2). \tag{1.6}$$

Here $\|.\|_2$ is the Euclidean norm in $\mathbb{R}^2$. Consider the following sets of functions.

Definition 1.1. *We denote by $\mathcal{A}$ the set of functions $m = (m_1, m_2) \in L^\infty(\mathbb{R}) \times L^\infty(\mathbb{R})$, $\|m_i\|_\infty \leqslant 1$ for $i = 1, 2$ such that*

$$\liminf_{x\to+\infty} m_i(x) > 0 \qquad \text{and} \qquad \limsup_{x\to-\infty} m_i(x) < 0. \tag{1.7}$$

We have the following result.

Theorem 1.2. *Given $\beta > 1$, $\theta > 0$, $\beta\theta \leqslant \min, (1/\sqrt{3}, (1-\beta^{-1})^{\frac{1}{2}})$, there exists an unique (up translation) minimizer $\bar{m}$ of the functional $\mathcal{F}$, see (1.1), in the class $\mathcal{A}$. The minimizer $\bar{m}$ has the following properties: $\bar{m} = (\bar{m}_1, \bar{m}_2) \in C^\infty(\mathbb{R}) \times C^\infty(\mathbb{R})$, $\bar{m}_i$ for $i = 1, 2$ is strictly increasing, $\bar{m}_1(0) = \tanh\beta\theta$, $\bar{m}_2(0) = -\tanh\beta\theta$. Moreover, there exist two positive constants α and c depending only on β and θ such that for $i = 1, 2$*

$$|\bar{m}_i(x) - m_{\beta,i}| \leqslant c\mathrm{e}^{-\alpha x} \quad x \to \infty \qquad \text{and} \qquad |\bar{m}_i(x) + m_{\beta,3-i}| \leqslant c\mathrm{e}^{-\alpha|x|} \quad x \to -\infty.$$

Further $|\bar{m}_i^k(x)| \leqslant c_k \mathrm{e}^{-\alpha|x|}$, for $i = 1, 2$ where $\bar{m}^k(x)$ is the k derivative of $\bar{m}(x)$ and c_k are positive constants depending only on β and θ.

It is straightforward to see that theorem 1.2 holds in the class $\mathcal{A}$ without restriction on the L^∞ norm, namely the value of $\mathcal{F}$ over these functions is always bigger than the one on $\mathcal{A}$. We call the minimizer $\bar{m} = (\bar{m}_1, \bar{m}_2)$ an interface or an instanton. The constraint $\bar{m}_1(0) = -\bar{m}_2(0) = \tanh\beta\theta$ breaks the translational invariance symmetry of (1.1). In fact any shift of the instanton

$$(S_a\bar{m})(x) \equiv \bar{m}_a(x) = (\bar{m}_1(x-a), \bar{m}_2(x-a)) \qquad a \in \mathbb{R} \tag{1.8}$$

is still a minimizer and we call it the instanton with centre a. The centre of the instanton is characterized by the fact that $\bar{m}_a(a) = \bar{m}(0)$. The instanton $\bar{m}$ is then the instanton with the centre 0 and $\{\bar{m}_a, a \in \mathbb{R}\}$ is the manifold of the instantons.

The method we use to show existence, unicity up translations and properties of the optimal profile for the interface is a dynamic way. We consider the following system of integral equations

$$\begin{aligned} \frac{\partial m_1}{\partial t} &= -m_1 + \tanh\{\beta\,(J \star \tilde{m} + \theta)\} \\ \frac{\partial m_2}{\partial t} &= -m_2 + \tanh\{\beta\,(J \star \tilde{m} - \theta)\} \end{aligned} \tag{1.9}$$

where the $\star$ product denotes convolution, $(J \star \tilde{m})(x) = \int \mathrm{d}y J(x-y)\tilde{m}(y)$. A basic fact is that the functional (1.1) is decreasing along the evolution given by (1.9). Therefore proving that there is an unique, up to translations, stationary solution of (1.9), i.e. solution of the following system of equations

$$\begin{aligned} &m_1(x) = \tanh(\beta(J \star \tilde{m})(x) + \beta\theta) \\ &m_2(x) = \tanh(\beta(J \star \tilde{m})(x) - \beta\theta) \\ &\lim_{x\to\infty} m(x) = m_\beta, \qquad \lim_{x\to-\infty} m(x) = Tm_\beta \end{aligned} \tag{1.10}$$

and that any solution of (1.9) with the initial datum in the class $\mathcal{A}$ converges to the solution of (1.10), it is equivalent to prove the existence and unicity (up translations) of the minimum of $\mathcal{F}(m)$ over the set of functions $\mathcal{A}$. This is easy consequence of the monotonicity in time of $\mathcal{F}(m(t))$, see theorem 2.7, as well as of its lower semicontinuity with respect to convergence almost everywhere (Fatou's lemma).

The system (1.9) is closely related to the following integral equation

$$\begin{aligned} &\frac{\partial m}{\partial t} = -m + \tanh\{\beta J \star m\} \\ &\lim_{x\to\pm\infty} m(x) = \pm m_\beta. \end{aligned} \tag{1.11}$$

Here m_β plays the same role as the one defined earlier. Equation (1.11) models the continuum limit of Ising spin system with non-conservative dynamics, Kac potentials and without external magnetic field, see [8] for a description of the model. The existence, unicity (up reflections and translations), of the stationary solution and stability have been established in [9, 10]. When an external deterministic magnetic field is added to (1.11) travelling waves appear. The existence and stability of them have been studied in [7, 16]. Non-local equations like (1.11) have been studied by many authors, not only in physics, see [17], as they appear in several fields such as neural network, population dynamics, propagation of diseases, see for instance [1, 2, 14]. For a complete overview of results for equation (1.11) and more general problems of deriving the continuum equation from the statistical mechanics setting, see [18].

The model we are analysing shares several qualitative features with (1.11). In fact, one could try to analyse the equation obtained by summing the two equations in (1.9), this yields a closed equation for $\tilde{m}$. Proceeding in this way one cannot rely any more on the fact that the functional $\mathcal{F}(m)$, see (1.1), is a Lyapunov function, namely $f_{\beta,\theta}(m)$, and therefore $\mathcal{F}(m)$ cannot be written only in terms of $\tilde{m}$. Since the comparison principle still holds for the equation for $\tilde{m}$ the method presented in [6] can be applied to obtain existence, unicity up translations and L^∞ exponential convergence. The method is rather general, it holds for several types of one space dimensional nonlinear evolution equations and it does not use any of the variational structure of equation (1.9). We do not use the approach of [6] since our main interest is in determining the existence, uniqueness and properties of the minima of the functional (1.1) in the class of functions $\mathcal{A}$ and therefore the variational structure of (1.9) is essential to us. Moreover the method we are using could be applied to a more general system of the one in (1.9), obtained by multiplying by $\alpha > 0$ the right-hand side of the second equation of (1.9). The change will not allow the reduction of the system to a closed single equation, but since the comparison principle and the variational structure still holds it is possible to analyse it with the method we used. In the following we prove the existence of a solution of (1.10) with the properties stated in theorem 1.2.

Theorem 1.3 (existence). *There exists a solution $\bar{m} = (\bar{m}_1, \bar{m}_2)$ of (1.10) with the properties stated in theorem 1.2.*

Theorem 1.4 (unicity up translations). *Let $m = (m_1, m_2) \in \mathcal{A}$ solves (1.10), then there exists an a such that, for all x, $m(x) = \bar{m}(x - a)$.*

The proof of the unicity stated in theorem 1.4 is essentially based on proving that any stationary solution of (1.10) is trapped between two instantons and that the manifold of instanton is locally nonlinearly exponentially stable, see theorem 4.2. Moreover, we have an L^2 global exponential stability in the following class of functions

$$\mathcal{K}(M, N) = \left\{ m \;\middle|\; \sum_{i=1}^{2} \int_{\mathbb{R}} |m_i(x) - \chi_i(x)|^2 \, \mathrm{d}x \leqslant M, \, \|m\|_{\mathrm{H}^1} \leqslant N \right\} \tag{1.12}$$

where

$$\|m\|_{\mathrm{H}^1}^2 = \int_{\mathbb{R}} |m_1'(x)|^2 \, \mathrm{d}x + \int_{\mathbb{R}} |m_2'(x)|^2 \, \mathrm{d}x, \tag{1.13}$$

m_i' denotes the derivative of m_i,

$$\chi_1(x) = -m_{\beta,2}\mathbb{I}_{x\leqslant 0} + m_{\beta,1}\mathbb{I}_{x>0}, \qquad \chi_2(x) = -m_{\beta,1}\mathbb{I}_{x\leqslant 0} + m_{\beta,2}\mathbb{I}_{x>0}. \tag{1.14}$$

Theorem 1.5. *For all initial data $m(0) \in \mathcal{K}(M, N)$ there exists a positive constant $C(N, M)$ such that*

$$\|m(t) - \bar{m}_{a(t)}\|_{L^2} \leqslant \mathrm{e}^{-C(N,M)t} \|m(0) - \bar{m}_{a(0)}\|_{L^2}$$

where $\bar{m}_{a(t)}$ is any instanton minimizing the L^2 distance to $m(\cdot, t)$ among all the instantons $\bar{m}_b$, $b \in \mathbb{R}$ and $\|\cdot\|_{L^2}$ the norm in $L^2(\mathbb{R}, \mathrm{d}x) \times L^2(\mathbb{R}, \mathrm{d}x)$.

Notice that there is at least one instanton $\bar{m}_a$ minimizing $\|m(\cdot, t) - \bar{m}_b\|_{L^2}^2$, $b \in \mathbb{R}$, since this quantity is differentiable as a function of b, and tends to infinity as $b \to \pm\infty$. Hence, the distance is minimized at one value a at least and at this value of a, $\bar{m}_a'$ is orthogonal to $m - \bar{m}_a$ in L^2. Let $a(t)$ denote any such value.

The important point is that we obtain decay estimates which are *uniform in the profile* within the classes $\mathcal{K}(M, N)$, so that the relaxation is taking place at a uniform rate everywhere on the interface. These kinds of estimates were derived by Carlen *et al* [3] for a similar functional in the context of the non-conservative dynamics, see (1.11), and then applied to the more difficult problem of deriving local stability results in the context of conservative dynamics [4,5]. The method is very robust and relates the free energy functional $\mathcal{F}$ defined in (1.1) with the spectral analysis of the linear operator L, defined in (4.2), obtained by linearizing the system (1.9) around an instanton.

When the initial datum belongs to $\mathcal{A}$ we have the following.

Theorem 1.6. *Assume that $m \in \mathcal{A}$ and let $m(\cdot, t)$ solve (1.9) with $m(x, 0) = m(x)$, for all x. Then there is $a(\infty)$ so that*

$$\lim_{t\to\infty} \|m(\cdot, t) - \bar{m}(\cdot - a(\infty))\|_\infty = 0. \tag{1.15}$$

The convergence being exponentially fast.

Note that $a(\infty)$ is the same for both m_1 and m_2.

2. Basic properties of the evolution

In this section, we state and prove some basic properties of the evolution that will be constantly used in the sequel. For short notation we introduce the vector

$$\tanh_\theta(s) = \begin{pmatrix} \tanh \beta(s + \theta) \\ \tanh \beta(s - \theta) \end{pmatrix}. \tag{2.1}$$

We start from the integral representation of the solutions of (1.9). For all $x \in \mathbb{R}$ and all $t \geqslant 0$ we have

$$m(x,t) = \mathrm{e}^{-t} m(x,0) + \int_0^t \mathrm{d}s\, \mathrm{e}^{-(t-s)} \tanh_\theta\{\beta(J \star \tilde{m})(x,s)\}. \tag{2.2}$$

One basic tool which is very often used in the sequel is the fact that system (1.9) is order preserving.

Definition 2.1. *The function $v(x,t) = (v_1(x,t), v_2(x,t))$ is a subsolution of the Cauchy problem (1.9) with initial datum $m(\cdot,0) = (m_1(\cdot,0), m_2(\cdot,0))$, if $\|v_i(\cdot,t)\|_\infty \leqslant 1$ for all $t \geqslant 0$, $v_i(x,0) \leqslant m_i(x,0)$ for all x and $i = 1,2$; it is continuously differentiable with respect to t and satisfies, for all x and $t > 0$,*

$$\frac{\partial v(x,t)}{\partial t} \leqslant -v(x,t) + \tanh_\theta\{\beta(J \star \tilde{v})(x,t)\} \tag{2.3}$$

where the inequality is meant to hold componentwise. Analogously, the function $w(x,t) = (w_1(x,t), w_2(x,t))$ is a supersolution if it has the same regularity properties as above and it satisfies (2.3) with the reverse inequality and $w_i(x,0) \geqslant m_i(x,0)$ for $i = 1,2$.

From now on, inequalities among vectors are meant to hold always componentwise.

Theorem 2.2 (the Comparison theorem). *If $v(x,t) = (v_1(x,t), v_2(x,t))$, (respectively $w(x,t) = (w_1(x,t), w_2(x,t))$), is a subsolution, (respectively supersolution), of the Cauchy problem (1.9) with initial datum $m(\cdot,0)$ then for all x and all $t \geqslant 0$ and $i = 1,2$:*

$$v_i(x,t) \leqslant m_i(x,t) \leqslant w_i(x,t). \tag{2.4}$$

Proof. Given $T > 0$ let $\mathcal{N}$ be the space $L^\infty(\mathbb{R} \times [0,T]) \times L^\infty(\mathbb{R} \times [0,T])$, endowed with the sup norm. Let $G = (G_1, G_2)$ be the continuous map from $\mathcal{N}$ into itself defined, for $f \in \mathcal{N}$, by

$$(G(f))\,(x,t) = \mathrm{e}^{-t} f(x,0) + \int_0^t \mathrm{d}s\, \mathrm{e}^{-(t-s)} \tanh_\theta\{\beta(J \star \tilde{f})(x,s)\} \tag{2.5}$$

where $\tilde{f} = \frac{1}{2}(f_1 + f_2)$. It is easy to verify that G is monotone, i.e. if $f \in \mathcal{N}$ and $g \in \mathcal{N}$ with $f_i \geqslant g_i$ for $i = 1,2$ (pointwise in $\mathbb{R} \times [0,T]$) then $G_i(f) \geqslant G_i(g)$, $i = 1,2$. Moreover $(G_i(f))(x,0) = f_i(x,0)$. Furthermore, for $\beta(1 - \mathrm{e}^{-T}) < 1$, G is a contraction on any subset of functions of $\mathcal{N}$ with the same values at $t = 0$. Namely

$$\sum_{i=1}^2 |G_i(f)(x,t) - G_i(g)(x,t)| \leqslant \beta \int_0^t \mathrm{d}s\, \mathrm{e}^{-(t-s)} (J \star (\tilde{f} - \tilde{g}))(x,s)$$
$$\leqslant \beta(1 - \mathrm{e}^{-T}) \sup_{x,s} \sum_{i=1}^2 |f_i(x,s) - g_i(x,s)|.$$

Thus if $m(x,t)$ solves (1.9), we have

$$m_i = \lim_{n\to\infty} (G^n(m^0))_i, \quad m^0(x,t) = m(x,0) \qquad \text{in } \mathbb{R} \times [0,T].$$

Let $u = (u_1, u_2)$ be the solution of (1.9) and suppose $u_i^0 \leqslant m_i^0$ then $(G^n(u^0))_i \leqslant (G^n(m^0))_i$, hence $u_i \leqslant m_i$ in $\mathbb{R} \times [0,T]$.

Analogously, if v is a subsolution of (1.9), it is easy to see that $v_i \leqslant G_i(v_1, v_2)$, where G_i is defined in (2.5); hence $v_i \leqslant G_i^n(v)$ and therefore $v_i \leqslant z_i$, where $z_i = \lim_{n\to\infty} (G^n(v))_i$ and by the continuity of G, $z_i = G_i(z)$. Therefore, $z = (z_1, z_2)$ solves (1.9) in $\mathbb{R} \times [0,T]$ with an initial condition $z(\cdot,0) = v(\cdot,0)$. Then, for what was proven above, if $v_i(\cdot,0) \leqslant m_i(\cdot,0)$ we obtain $v_i \leqslant z_i \leqslant m_i$. The same argument applies to the supersolutions. We have thus proven (2.4) for $0 \leqslant t \leqslant T$. Then, we extend the result to $[T, 2T]$ by the same argument, since the estimate does not depend on the initial datum. Iterating we can complete the proof of the theorem. □

Proposition 2.3 (equicontinuity of the orbits). *Let* $\psi_i(x,t) := m_i(x,t) - \mathrm{e}^{-t} m_i(x,0)$ *and denote by* ψ_i' *its derivative with respect to* x*; then, for any* $t \geqslant 0$,

$$\|\psi_i'(\cdot,t)\|_\infty \leqslant \beta \|J'\|_1 := \beta \int \mathrm{d}x |J'(x)|. \tag{2.6}$$

Proof. From (2.2)

$$|\psi_1'(x,t)| \leqslant \int_0^t \mathrm{d}s\, \mathrm{e}^{-(t-s)} \left| \frac{\partial}{\partial x} \tanh\{\beta(J \star \tilde{m})(x,s) + \theta\} \right| \leqslant \int_0^t \mathrm{d}s\, \mathrm{e}^{-(t-s)} \beta(|J'| \star |\tilde{m}|)(x,s).$$

Similarly for ψ_2. This concludes the proof. □

Corollary 2.4 (limit points of the orbits). *Given any sequence* t_n *increasing to* ∞ *there is a function* $m^\star = (m_1^\star, m_2^\star) \in C_b(\mathbb{R}) \times C_b(\mathbb{R})$, $\|m_i^\star\|_\infty \leqslant 1$ *so that, uniformly on the compacts,*

$$\lim_{n\to\infty} m(x,t_n) = m^\star(x). \tag{2.7}$$

Proof. The family $m_i(x,t) - \mathrm{e}^{-t} m_i(x,0)$ is equicontinuous and equibounded in $\mathbb{R} \times \mathbb{R}_+$, so that, by the Ascoli Arzelà theorem, the statement is proven for x in a compact. Then, by a diagonalization procedure, (2.7) follows. □

To identify the limiting points of an orbit we use the (excess) free energy functional $\mathcal{F}(m)$ defined in (1.1). First, we prove that the following set

$$\mathcal{M} = \left\{(m_1, m_2) \in L^\infty(\mathbb{R}) \times L^\infty(\mathbb{R}), \|m_i\|_\infty \leqslant 1, m_i(x) - \chi_i(x) \in L^2(\mathbb{R})\right\} \tag{2.8}$$

is left invariant by the evolution for compact intervals of time; where, we recall, $\chi_1(x) = -m_{\beta,2}\mathbb{I}_{x\leqslant 0} + m_{\beta,1}\mathbb{I}_{x>0}$ and $\chi_2(x) = -m_{\beta,1}\mathbb{I}_{x\leqslant 0} + m_{\beta,2}\mathbb{I}_{x>0}$.

Lemma 2.5. *Assume that* $m \in \mathcal{M}$. *Then for* $i = 1, 2$ $\|m_i(\cdot,t) - \chi_i\|_{L^2}$ *is bounded for* t *in the compacts.*

Proof. We denote $v = m - \chi$ where $\chi = (\chi_1, \chi_2)$. Using (1.5) we have that $\chi(x) = \tanh_\theta\{\beta\tilde{\chi}(x)\}$ where $\tilde{\chi}(x) = -\tilde{m}_\beta \mathbf{1}_{x\leqslant 0} + \tilde{m}_\beta\, \mathbf{1}_{x>0}$. Therefore, see (2.2)

$$v(x,t) = \mathrm{e}^{-t} v(x,0) + \int_0^t \mathrm{d}s\, \mathrm{e}^{-(t-s)} \left(\tanh_\theta\{\beta(J \star \tilde{m})(x,s)\} - \tanh_\theta\{\beta\tilde{\chi}(x)\}\right).$$

Then we have

$$\|v(\cdot,t)\|_{L^2} \leqslant \mathrm{e}^{-t} \|v(\cdot,0)\|_{L^2} + \int_0^t \mathrm{d}s\, \mathrm{e}^{-(t-s)} \|\Lambda(\cdot,s)\|_{L^2}$$

where

$$\begin{aligned} \Lambda(x,s) = {} & |\tanh_\theta\{\beta(J \star \tilde{m})(x,s)\} - \tanh_\theta \beta\tilde{\chi}(x)| \\ & \leqslant \beta\, |(J \star \tilde{m})(x,s) - \tilde{\chi}(x)| \leqslant \beta |(J \star \tilde{m})(x,s) - (J \star \tilde{\chi})(x)| \\ & \quad + \beta |(J \star \tilde{\chi})(x) - \tilde{\chi}(x)|. \end{aligned}$$

Since the second term on the right-hand side is bounded by $\beta\tilde{m}_\beta$ when $|x| \leqslant 1$ and equal to 0 elsewhere, using Young inequality we obtain

$$\|\Lambda(\cdot,s)\|_{L^2} \leqslant \beta \|v(s)\|_{L^2} + \sqrt{2}\beta\tilde{m}_\beta$$

from which the lemma follows. □

We prove that $\mathcal{F}$ takes finite values on $\mathcal{M}$.

Proposition 2.6. *Assume that $m = (m_1, m_2) \in \mathcal{M}$. Then*

$$\mathcal{F}(m_1, m_2) \leqslant c(\beta, \theta)\left[\|m_1 - \chi_1\|_{L^2}^2 + \|m_2 - \chi_2\|_{L^2}^2\right] + \left[m_{\beta,1}^2 + m_{\beta,2}^2\right]$$

where $c(\beta, \theta)$ is a positive constant.

Proof. Set $v_i = m_i - \chi_i$, $i = 1, 2$ and $v = (v_1, v_2)$. For the energy term we have that

$$\left(\frac{m_1(r) + m_2(r)}{2} - \frac{m_1(r') + m_2(r')}{2}\right)^2$$
$$= \left(\frac{v_1(r) + \chi_1(r) + v_2(r) + \chi_2(r)}{2} - \frac{v_1(r') + \chi_1(r') + v_2(r') + \chi_2(r')}{2}\right)^2$$
$$\leqslant (v_1(r) - v_1(r'))^2 + (v_2(r) - v_2(r'))^2 + (\chi_1(r) - \chi_1(r'))^2 + (\chi_2(r) - \chi_2(r'))^2.$$

The last terms are equal to 0 when both r and r' are positive or both negative. Therefore,

$$\frac{1}{4}\iint J(|r - r'|)[m(r) - m(r')]^2\,\mathrm{d}r\,\mathrm{d}r' \leqslant \|m_1 - \chi_1\|_{L^2}^2 + \|m_2 - \chi_2\|_{L^2}^2 + [m_{\beta,1}^2 + m_{\beta,2}^2].$$

For the entropy term define

$$A = \left\{r \in \mathbb{R} : |m_1(r)| \geqslant \frac{1 + m_{\beta,1}}{2}\right\} \cup \left\{r \in \mathbb{R} : |m_2(r)| \geqslant \frac{1 + m_{\beta,1}}{2}\right\}.$$

Then on A one has $|m_1(r) - \chi_1(r)| \geqslant (1 - m_{\beta,1})/2$ or $|m_2(r) - \chi_2(r)| \geqslant (1 - m_{\beta,1})/2$, thus

$$|A| \leqslant \frac{4}{(1 - m_{\beta,1})^2}\left(\|m_1 - \chi_1\|_{L^2}^2 + \|m_2 - \chi_2\|_{L^2}^2\right).$$

Since for $|m_1| \leqslant 1$ and $|m_2| \leqslant 1$ there exists a finite constant $\mathrm{Osc}(f)$ such that $|f(m_1, m_2) - f(m_\beta)| \leqslant \mathrm{Osc}(f)$ we obtain the bound on A. On A^c, the complement of A we have

$$0 \leqslant f(m_1, m_2) - f(m_\beta) = f(m_1, m_2) - f(\chi_1, \chi_2) = \int_0^1 \mathrm{d}s \int_0^s \mathrm{d}\alpha \frac{\mathrm{d}^2}{\mathrm{d}\alpha^2} f(\chi + \alpha v)$$
$$\leqslant (|v_1|^2 + |v_2|^2)\int_0^1 \mathrm{d}s \int_0^s \mathrm{d}\alpha \left[1 + \frac{1}{4\beta}\frac{1}{1 - (\chi_1 + \alpha v_1)^2} + \frac{1}{4\beta}\frac{1}{1 - (\chi_2 + \alpha v_2)^2}\right]$$

where $0 < \alpha < 1$. Observe that on A^c, $|\chi_1(r) + \alpha v_1(r)| \leqslant \max\{m_{\beta,1}, (1 + m_{\beta,1})/2\}$ and $|\chi_2(r) + \alpha v_2(r)| \leqslant \max\{m_{\beta,1}, (1 + m_{\beta,1})/2\}$. □

Thus, we know that if $m \in \mathcal{M}$ the free energy functional $\mathcal{F}$ is well defined on the whole orbit $m(\cdot, t)$, $t \geqslant 0$. We will prove that $\mathcal{F}$ is a Lyapunov function for (1.9), namely that $\mathcal{F}(m(\cdot, t))$ decreases with t. We also give an explicit expression for its time derivative, which is well defined only when $|m_1(\cdot, t)| < 1$, $|m_2(\cdot, t)| < 1$. We shall prove that this condition could only fail at time 0. This proof uses the comparison theorem 2.2.

Theorem 2.7. *Suppose $m(\cdot, 0) \in \mathcal{M}$, then $\mathcal{F}(m_1(\cdot, t), m_2(\cdot, t))$ is well defined for all $t \geqslant 0$, it is differentiable with respect to t if $t > 0$ and*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(m_1(\cdot, t), m_2(\cdot, t)) = -I(m_1(\cdot, t), m_2(\cdot, t)) \leqslant 0 \tag{2.9}$$

where, for any $(h_1, h_2) \in L^\infty(\mathbb{R}) \times L^\infty(\mathbb{R})$, $\|h_i\|_\infty < 1$

$$I(h_1(\cdot, t), h_2(\cdot, t)) = \int_{\mathbb{R}} \mathrm{d}x[(J \star \tilde{h})(x) + \theta - \beta^{-1}\tanh^{-1} h_1(x)]$$
$$\times[\tanh\beta((J \star \tilde{h})(x) + \theta) - h_1(x)]$$
$$+ \int_{\mathbb{R}} \mathrm{d}x[(J \star \tilde{h})(x) - \theta - \beta^{-1}\tanh^{-1} h_2(x)]$$
$$\times[\tanh\beta((J \star \tilde{h})(x) - \theta) - h_2(x)]. \tag{2.10}$$

The integrand in $I(h)$ is a non-negative function which is in $L^1(\mathbb{R})$ when $h_i = m_i(\cdot, t)$. Finally, for all $t_0 \geqslant 0$ and all $t \geqslant t_0$

$$\mathcal{F}(m_1(\cdot, t), m_2(\cdot, t)) - \mathcal{F}(m_1(\cdot, t_0), m_2(\cdot, t_0)) = -\int_{t_0}^{t} \mathrm{d}s\, I(m_1(\cdot, s), m_2(\cdot, s)) \leqslant 0. \quad (2.11)$$

Proof. Assume first that, given $t > 0$, there is $\epsilon > 0$ such that $\|m_1(\cdot, s)\|_\infty \leqslant 1 - \epsilon$ and $\|m_2(\cdot, s)\|_\infty \leqslant 1 - \epsilon$ when s varies in a small finite interval Δ containing t. For $s \in \Delta$ we write

$$\mathcal{F}(m_1(\cdot, s), m_2(\cdot, s)) := \int \mathrm{d}x\, \phi(x, s), \qquad I(m_1(\cdot, s), m_2(\cdot, s)) := \int \mathrm{d}x\, \iota(x, s).$$

By lemma 2.5 for any $s \in \Delta$, $\iota(\cdot, s) \in L^1(\mathbb{R})$ and $\sup_{s\in\Delta} \|\iota(\cdot, s)\|_1 < \infty$. Moreover for each x, $\phi(x, s)$ is differentiable in s with $\iota(x, s)$ as partial derivative hence $\sup_{s\in\Delta} \|(\partial/\partial s)\phi(\cdot, s)\|_1 < \infty$. It then follows that the time derivative of $\mathcal{F}(m_1(\cdot, t), m_2(\cdot, t))$ is $I(m_1(\cdot, t), m_2(\cdot, t))$, hence (2.9) is proven for any $t > 0$, provided $\|m_i(\cdot, s)\| < 1$ uniformly when s is in some finite interval containing t. We next prove that this assumption holds for any $t > 0$. In fact if $m_i(x, 0) \leqslant 1$ for all x and if we call $\lambda_i(x, t)$ the solution of (1.9) such that $\lambda_i(x, 0) \equiv 1$, then $\lambda_i(x, t) \equiv \lambda_i(t)$ where

$$\frac{\mathrm{d}\lambda(t)}{\mathrm{d}t} = -\lambda(t) + \tanh_\theta\{\beta\tilde{\lambda}(t)\}. \quad (2.12)$$

Thus $\lambda_i(t)$, for $i = 1, 2$ is strictly less than 1 for $t > 0$. Therefore, by theorem 2.2, we have that $m_i(x, t) \leqslant \lambda_i(t) < 1$ for all x. Repeating the same argument starting from the inequality $m_i(x, 0) \geqslant -1$, we then prove that $|m_i(x, t)| \leqslant \lambda_i(t)$ for all x and all t, hence (2.9) and (2.10). Equation (2.11) then holds for $t_0 > 0$ and by the continuity of $\mathcal{F}(m_1(\cdot, t), m_2(\cdot, t))$ for $t \geqslant 0$ it also holds for $t_0 = 0$. □

3. Existence of the instanton

In this section we prove theorem 1.3. We denote by $l(x) = (l_1(x), l_2(x))$ the following function:

$$l_1(x) = \begin{cases} -m_{\beta,2} & \text{for } x \leqslant -1 \\ m_{\beta,1} & \text{for } x \geqslant 1 \\ \dfrac{m_{\beta,1} + m_{\beta,2}}{2}(x - 1) + m_{\beta,1} & \text{for } |x| < 1 \end{cases}$$

$$l_2(x) = \begin{cases} -m_{\beta,1} & \text{for } x \leqslant -1 \\ m_{\beta,2} & \text{for } x \geqslant 1 \\ \dfrac{m_{\beta,1} + m_{\beta,2}}{2}(x - 1) + m_{\beta,2} & \text{for } |x| < 1. \end{cases}$$

Let $l_1(x, t)$, $l_2(x, t)$ be the solution of (1.9) such that $l_i(x, 0) = l_i(x)$. Then $l_i(x, t)$ is non-decreasing as a function of x for any $t \geqslant 0$. In fact let $b \in \mathbb{R}$, $b > 0$, then $l_i(x+b) \geqslant l_i(x)$, since the $l_i(x)$ are increasing. Denote by $u_i(x, t)$ the solution of (1.9) with initial datum $l_i(x + b)$. Then from theorem 2.2, $u_i(x, t) = l_i(x + b, t) \geqslant l_i(x, t)$. Since b is chosen arbitrarily we prove that $l_i(x, t)$ is non-decreasing as a function of x for any $t \geqslant 0$.

By lemma 2.5 for any compact interval of time $l(\cdot, t) \in \mathcal{M}$ and by proposition 2.6 $\mathcal{F}(l_1(t), l_2(t)) < \infty$. Moreover, by the comparison theorem, $-m_{\beta,2} \leqslant l_1(x, t) \leqslant m_{\beta,1}$ and $-m_{\beta,1} \leqslant l_2(x, t) \leqslant m_{\beta,2}$ because that happens at $t = 0$ and the functions constantly equal either to m_β or to Tm_β are solutions of (1.9). Therefore, by theorem 2.7, $I(l_1(t), l_2(t))$ is well defined and from (2.11) it follows that

$$\liminf_{t\to\infty} I\,(l_1(\cdot,t), l_2(\cdot,t)) = 0 \tag{3.1}$$

since otherwise $\mathcal{F}\,(l_1(\cdot,t), l_2(\cdot,t)) < 0$ for some t which, by (1.1), is impossible. Therefore, there is a sequence t_n increasing to infinity, such that

$$\lim_{n\to\infty} I\,(l_1(\cdot,t_n), l_2(\cdot,t_n)) = 0. \tag{3.2}$$

By corollary 2.4 there is a continuous function $(\bar m_1(\cdot), \bar m_2(\cdot))$, with the sup norm bounded by 1 and a subsequence of t_n: $s_n \to \infty$ so that for any $\ell > 0$

$$\lim_{n\to\infty} \sup_{|x|\leqslant\ell} \sum_{i=1,2} |l_i(x,s_n) - \bar m_i(x)| = 0. \tag{3.3}$$

By Fatou's lemma $I\,(\bar m_1(\cdot), \bar m_2(\cdot)) = 0$, hence, by the continuity of $\bar m_i$, $\bar m_i$ for $i = 1, 2$ solves (1.10) everywhere and is non-decreasing. Moreover, again by Fatou's lemma, if $\mathcal{F}$ is lower semicontinuous with respect to the convergence almost everywhere, then

$$\lim_{n\to\infty} \inf \mathcal{F}(l_1(s_n), l_2(s_n)) \geqslant \mathcal{F}(\bar m_1, \bar m_2). \tag{3.4}$$

Therefore, $\mathcal{F}(\bar m)$ takes a finite positive value and we will prove that this implies

$$\begin{aligned} &\lim_{x\to+\infty} \bar m_1(x) = m_{\beta,1} \qquad &&\lim_{x\to-\infty} \bar m_1(x) = -m_{\beta,2} \\ &\lim_{x\to+\infty} \bar m_2(x) = m_{\beta,2} \qquad &&\lim_{x\to-\infty} \bar m_2(x) = -m_{\beta,1}. \end{aligned} \tag{3.5}$$

We show (3.5) by contradictions. Suppose (3.5) is false. Then, since $\bar m_1$ and $\bar m_2$ are increasing functions we would have

$$\begin{aligned} &\lim_{x\to\infty} \bar m_1(x) = \alpha_1, \qquad &&\lim_{x\to-\infty} \bar m_1(x) = -\alpha_2 \\ &\lim_{x\to\infty} \bar m_2(x) = \alpha_3, \qquad &&\lim_{x\to-\infty} \bar m_2(x) = -\alpha_4 \end{aligned} \tag{3.6}$$

where at least one $\alpha_i > 0$ is different from the limits value in (3.5). Since $f_{\beta,\theta}(\bar m(\cdot)) - f_{\beta,\theta}(m_\beta)$ is a continuous function which is strictly positive when $\bar m(x) \neq m_\beta$ or $\bar m(x) \neq Tm_\beta$, see (1.6), we obtain

$$\int [f_{\beta,\theta}(\bar m(x)) - f_{\beta,\theta}(m_\beta)]\,\mathrm{d}x = \infty.$$

This implies that $\mathcal{F}(\bar m) = \infty$ and this is impossible.

To prove that the $\bar m \in C^\infty$, we differentiate the right-hand side of (1.10) with respect to x, obtaining, since $J' \in L^1(\mathbb{R})$

$$\begin{aligned} &\frac{\partial}{\partial x}[\tanh\{\beta(J \star \bar{\tilde{m}} + \theta)] = \beta(1 - \bar m_1^2) J' \star \bar{\tilde{m}} \\ &\frac{\partial}{\partial x}[\tanh\{\beta(J \star \bar{\tilde{m}} - \theta)] = \beta(1 - \bar m_2^2) J' \star \bar{\tilde{m}}. \end{aligned} \tag{3.7}$$

This implies that $\bar m_i \in C^1(\mathbb{R})$ for $i = 1, 2$. Iterating we get that $\bar m_i \in C^\infty(\mathbb{R})$.

To show that the $\bar m_i$ is strictly increasing we assume that for some x, $\bar m_i'(x) = 0$. Then by (3.7), integrating by parts, we get

$$\int \mathrm{d}y\, J(y-x) \bar{\tilde{m}}'(y) = 0$$

where recall $\bar{\tilde{m}}(x) = \frac{1}{2}(\bar m_1(x) + \bar m_2(x))$. Since $J \geqslant 0$ it then follows that $\bar{\tilde{m}}'(y) = 0$. By iteration, $\bar{\tilde{m}}'(y)$ must vanish on the set

$$\left\{ y \in R : \sum_{n\geqslant 1} J^{\star n}(y-x) > 0 \right\}$$

which is readily seen to coincide with the whole line, because J is even. Therefore, $\tilde{\bar{m}}(y)$ is constant and by the structure of (1.10), this implies that $\bar{m}_1$ and $\bar{m}_2$ are constant as well. This contradicts (3.6).

We consider now, the function $\tilde{l}(x,t) = \frac{1}{2}(l_1(x,t) + l_2(x,t))$ obtained by summing up the components of $l(x,t)$ defined above. The function $\tilde{l}(x,t)$ is then the solution of the equation obtained by summing up and dividing by two the equations in (1.9)

$$\frac{\partial \tilde{l}}{\partial t} = -\tilde{l} + \frac{1}{2}\tanh\{\beta(J \star \tilde{l} + \theta)\} + \frac{1}{2}\tanh\{\beta(J \star \tilde{l} - \theta)\} \tag{3.8}$$

with initial datum $\tilde{l}(x,0) = \frac{1}{2}(l_1(x) + l_2(x))$. It is easy to see that (3.8) leaves the class of antisymmetric function unchanged. Then since $\tilde{l}(x)$ is antisymmetric, $\tilde{l}(x,t)$ is antisymmetric as well for $t \geqslant 0$. Therefore, $\tilde{\bar{m}} = \frac{1}{2}(\bar{m}_1 + \bar{m}_2)$ is antisymmetric and from (1.10) $\bar{m}_1(0) = -\bar{m}_2(0) = \tanh\beta\theta$.

To prove the exponential convergence to the asymptotes of $\bar{m}$ it is enough to prove that there exists $c > 0$ and $\alpha > 0$ such that

$$|\tilde{\bar{m}}(x) \mp \tilde{m}_\beta| \leqslant c\,\mathrm{e}^{-\alpha|x|}. \tag{3.9}$$

Namely if (3.9) is satisfied then, for $x > 0$

$$|\bar{m}_1(x) - m_{\beta,1}| = |\tanh\beta(J \star \tilde{\bar{m}}(x) + \theta) - \tanh\beta(\tilde{m}_\beta + \theta)| \leqslant \beta c\,\mathrm{e}^{-\alpha x} \tag{3.10}$$

and when $x < 0$

$$|\bar{m}_1(x) + m_{\beta,2}| = |\tanh\beta(J \star \tilde{\bar{m}}(x) + \theta) - \tanh\beta(-\tilde{m}_\beta + \theta)| \leqslant \beta c\,\mathrm{e}^{-\alpha|x|}. \tag{3.11}$$

Similarly for $\bar{m}_2$. The bounds in (3.9) can be obtained similarly as done in [9]. We recall it and apply to our case. Using only that $\bar{m}_i'(x) > 0$ and J has support on the unit interval, we have that

$$\tilde{\bar{m}}(x-1) \leqslant (J \star \tilde{\bar{m}})(x). \tag{3.12}$$

Let $\Phi(s) = \frac{1}{2}\tanh\beta(s+\theta) + \frac{1}{2}\tanh\beta(s-\theta)$, then $\Phi(s)$ is increasing

$$\Phi(s) \leqslant \Phi(s') \qquad s \leqslant s' \tag{3.13}$$

and

$$s \leqslant \Phi(s) \qquad \text{when } s \in [0, \tilde{m}_\beta]. \tag{3.14}$$

Then from (3.13) and (3.12) we have that

$$\Phi(\tilde{\bar{m}}(x-1)) \leqslant \Phi((J \star \tilde{\bar{m}})(x)) = \tilde{\bar{m}}(x). \tag{3.15}$$

Repeating the argument k times we have

$$\Phi^k(\tilde{\bar{m}}(x)) \leqslant \tilde{\bar{m}}(x+k) \leqslant \tilde{m}_\beta.$$

Let us prove that the orbit $\Phi^k(s)$ converges exponentially fast to $\tilde{m}_\beta$. Indeed $\tilde{m}_\beta$ is the only fixed point of the map $\Phi(s)$ which is exponentially stable since $(\mathrm{d}/\mathrm{d}s)\Phi(s)$ is computed in $\tilde{m}_\beta$, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}s}\Phi(\tilde{m}_\beta) = \frac{1}{2}\beta(1 - m_{\beta,1}^2) + \frac{1}{2}\beta(1 - m_{\beta,2}^2) = \beta\left(1 - \frac{1}{2}(m_{\beta,1}^2 + m_{\beta,2}^2)\right) < 1. \tag{3.16}$$

Namely $\Phi(s) > s$ for $s \in (0, \tilde{m}_\beta)$, $\Phi(0) = 0$, $\Phi(\tilde{m}_\beta) = \tilde{m}_\beta$.

Since (3.16), let $x_0 > 0$ such that $(\mathrm{d}/\mathrm{d}s)\Phi(\tilde{\bar{m}}(x_0)) < 1$. Denote $\tilde{\bar{m}}(x_0) = s_0$. Then there exists some positive α and c such that

$$0 < \tilde{m}_\beta - \tilde{\bar{m}}(x_0 + k) \leqslant \tilde{m}_\beta - \Phi^k(s_0) \leqslant c\,\mathrm{e}^{-\alpha k}.$$

Then for any $x > x_0$, such that $x - x_0$ is integer we obtain the bound (3.9) with $c = e^{\alpha(x_0+1)}$. Since $\tilde{\bar{m}}' > 0$ we can interpolate obtaining the bounds (3.9).

We start proving that the first derivative $\bar{m}'$ of $\bar{m}$ decays exponentially fast. From (3.7) we obtain

$$\begin{aligned} \bar{m}_1'(x) &= \beta(1 - \bar{m}_1(x)^2)(J' \star \tilde{\bar{m}})(x) \\ &= \beta(1 - \bar{m}_1(x)^2)[(J' \star \tilde{\bar{m}})(x) - (J' \star \tilde{\chi})(x)] + \beta(1 - \bar{m}_1(x)^2)(J' \star \tilde{\chi})(x) \end{aligned} \tag{3.17}$$

where $\tilde{\chi}(x) = -\tilde{m}_\beta \mathbb{I}_{x\leqslant 0} + \tilde{m}_\beta \mathbb{I}_{x\geqslant 0}$. For $|x| > 1$, since $\beta(1 - \bar{m}_1(x)^2)(J' \star \tilde{\chi})(x) = 0$ we have that

$$\bar{m}_1'(x) = \beta(1 - \bar{m}_1(x)^2)(J' \star \tilde{\bar{m}})(x) \tag{3.18}$$

$$= \beta(1 - \bar{m}_1(x)^2) \int_{x-1}^{x+1} J'(x - y)[\tilde{\bar{m}}(y) - \tilde{\chi}(y)]\,\mathrm{d}y$$

$$\leqslant \beta \|J'\|_1 c\, e^{-\alpha|x|} = c_1\, e^{-\alpha|x|}. \tag{3.19}$$

By increasing c_1 if necessarily the result holds for $x \in \mathbb{R}$. Similarly for $\bar{m}_2'$. Iterating the argument, i.e. deriving with respect to x

$$\bar{m}_1'(x) = \beta(1 - \bar{m}_1(x)^2)(J \star \tilde{\bar{m}}')(x)$$

and using the exponentially decay of $\bar{m}'$ the exponentially decay of $\bar{m}''$ follows. This argument can be iterated k times obtaining the exponentially decay of the $\bar{m}^k$. So the proof of theorem 1.3 is concluded.

4. Local nonlinear stability

The linearization of the evolution equation (1.9) around $\bar{m} = (\bar{m}_1, \bar{m}_2)$ is, for i=1,2 the following

$$\frac{\partial v_i}{\partial t} = -v_i + \beta(1 - \bar{m}_i^2)\frac{1}{2} J \star (v_1 + v_2). \tag{4.1}$$

We denote by L the linear operator equal to the right hand side of (4.1); namely

$$Lv \equiv ((Lv)_1, (Lv)_2) \tag{4.2}$$

where

$$(Lv)_i = -v_i + \beta(1 - \bar{m}_i^2)\tfrac{1}{2} J \star (v_1 + v_2). \tag{4.3}$$

Define for $i = 1, 2$ the measures

$$\mathrm{d}\nu_i(x) = \frac{1}{(1 - \bar{m}_i^2(x))}\,\mathrm{d}x \tag{4.4}$$

that are equivalent to the Lebesgue measure and call $\mathcal{H}$ the Hilbert space $\mathcal{H} = L^2(\mathbb{R}, \mathrm{d}\nu_1) \times L^2(\mathbb{R}, \mathrm{d}\nu_2)$. On $\mathcal{H}$ the operator L is self-adjoint and this simplifies the analysis. We denote by $\|\cdot\|_{\mathcal{H}}$ and $\langle \cdot, \cdot \rangle_{\mathcal{H}}$, respectively, the norm and the scalar product in $\mathcal{H}$; by $\|\cdot\|_{L^2}$ and $(\cdot;\cdot)_{L^2}$ the norm and the scalar product in $L^2(\mathbb{R}, \mathrm{d}x) \times L^2(\mathbb{R}, \mathrm{d}x)$. Whenever there is no ambiguity, we will short-hand notation denoting $\|\cdot\|_{\mathcal{H}} \equiv \|\cdot\|$ and $\langle \cdot, \cdot \rangle_{\mathcal{H}} = \langle \cdot, \cdot \rangle$. Moreover, the norm and the scalar product in each single component $L^2(\mathbb{R}, \mathrm{d}\nu_i)$ for $i = 1, 2$ and in $L^2(\mathbb{R}, \mathrm{d}x)$ will be denoted in the same way as in the full vectorial space. Therefore, if $v = (v_1, v_2) \in \mathcal{H}$ and $u = (u_1, u_2) \in \mathcal{H}$ we have

$$\langle v, u \rangle_{\mathcal{H}} \equiv \langle v, u \rangle = \langle v_1, u_1 \rangle_{\mathcal{H}} + \langle v_2, u_2 \rangle_{\mathcal{H}} \tag{4.5}$$

and

$$\|v\|_{\mathcal{H}}^2 \equiv \|v\| = \|v_1\|_{\mathcal{H}}^2 + \|v_2\|_{\mathcal{H}}^2. \tag{4.6}$$

The quadratic form associated to L has the simple expression

$$\langle g, Lv\rangle_{\mathcal{H}} = -\langle g, v\rangle_{\mathcal{H}} + \tfrac{1}{2}\beta\,(g_1 + g_2;\, J\star(v_1+v_2))_{L^2} \tag{4.7}$$

which proves the symmetry of L in $\mathcal{H}$. In spite of the different signs on the right-hand side of (4.7) it is possible to prove that the spectrum of L lies on the negative axis. We in fact obtain the following.

Theorem 4.1. *L is a bounded, self-adjoint operator in $\mathcal{H}$. It is negative semidefinite,* 0 *is a simple eigenvalue with $\bar{m}' = (\bar{m}_1', \bar{m}_2')$ its eigenfunction. It has a spectral gap α, $\alpha > 0$, namely*

$$\langle v, Lv\rangle_{\mathcal{H}} \leqslant -\alpha\|v\|_{\mathcal{H}}^2 \qquad \textit{for all } v \in (\mathrm{Null}\, L)^{\perp}. \tag{4.8}$$

Proof. L is clearly a bounded self-adjoint operator having 0 as eigenvalue and eigenfunction $\bar{m}' = (\bar{m}_1', \bar{m}_2')$. This follows easily differentiating, with respect to x, (1.10). To prove that the spectrum lies on R^-, we represent the quadratic form in another way. We set $\tilde{\bar{m}} = \frac{1}{2}(\bar{m}_1 + \bar{m}_2)$ and use that for $i = 1, 2$

$$\bar{m}_i' = \beta(1 - \bar{m}_i^2)(J \star \tilde{\bar{m}}'). \tag{4.9}$$

Substitute in (4.7), we obtain

$$\langle v, Lv\rangle = -\beta \int \mathrm{d}x\,\mathrm{d}y\, J(x-y)\tilde{\bar{m}}'(y)\tilde{\bar{m}}'(x)\left[\frac{v_1(x)^2}{\bar{m}_1'(x)\tilde{\bar{m}}'(x)} + \frac{v_2(x)^2}{\bar{m}_2'(x)\tilde{\bar{m}}'(x)}\right.$$
$$\left. -\frac{1}{2}\frac{(v_1(x)+v_2(x))\,(v_1(y)+v_2(y))}{\tilde{\bar{m}}'(y)\tilde{\bar{m}}'(x)}\right]. \tag{4.10}$$

Since J is invariant in the exchange of x and y, we can write (4.10) as follows

$$\langle v, Lv\rangle = -\beta \int \mathrm{d}x\,\mathrm{d}y\, J(x-y)\tilde{\bar{m}}'(y)\tilde{\bar{m}}'(x)$$
$$\times\frac{1}{2}\left[\frac{v_1(x)^2}{\bar{m}_1'(x)\tilde{\bar{m}}'(x)} + \frac{v_2(x)^2}{\bar{m}_2'(x)\tilde{\bar{m}}'(x)} + \frac{v_1(y)^2}{\bar{m}_1'(y)\tilde{\bar{m}}'(y)} + \frac{v_2(y)^2}{\bar{m}_2'(y)\tilde{\bar{m}}'(y)}\right.$$
$$\left. -\frac{(v_1(x)+v_2(x))\,(v_1(y)+v_2(y))}{\tilde{\bar{m}}'(y)\tilde{\bar{m}}'(x)}\right]. \tag{4.11}$$

By some straightforward algebra, we obtain

$$\langle v, Lv\rangle = -\beta \int \mathrm{d}x\,\mathrm{d}y\,J(x-y)\left\{\frac{1}{4}\bar{m}_1'(y)\bar{m}_1'(x)\left[\frac{v_1(x)}{\bar{m}_1'(x)} - \frac{v_1(y)}{\bar{m}_1'(y)}\right]^2\right.$$
$$+\frac{1}{4}\bar{m}_2'(y)\bar{m}_2'(x)\left[\frac{v_2(x)}{\bar{m}_2'(x)} - \frac{v_2(y)}{\bar{m}_2'(y)}\right]^2$$
$$+\frac{1}{4}\bar{m}_1'(x)\bar{m}_2'(y)\left[\frac{v_1(x)}{\bar{m}_1'(x)} - \frac{v_2(y)}{\bar{m}_2'(y)}\right]^2$$
$$\left.+\frac{1}{4}\bar{m}_2'(x)\bar{m}_1'(y)\left[\frac{v_2(x)}{\bar{m}_2'(x)} - \frac{v_1(y)}{\bar{m}_1'(y)}\right]^2\right\} \leqslant 0. \tag{4.12}$$

This proves that 0 is a simple eigenvalue with eigenfunction $\bar{m}' = (\bar{m}_1', \bar{m}_2')$; $\bar{m}' \in \mathcal{H}$ since theorem 1.3. To prove the gap property we consider the transformation from $\mathcal{H}$ to $\mathcal{H}$ and its inverse defined as follows

$$U = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad U^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}. \tag{4.13}$$

We denote $\tilde{L} = U^{-1}LU$ the operator obtained under the transformation U. With simple algebra it is easy to show that

$$\tilde{L} = \begin{pmatrix} \mathcal{L}_1 & 0 \\ -\mathcal{L}_2 & -\mathbb{I} \end{pmatrix} \tag{4.14}$$

where $\mathbb{I}$ is the identity operator on $\mathcal{H}_2$, $\mathcal{L}_1$ and $\mathcal{L}_2$ are the operators on $\mathcal{H}_1$ such that

$$\mathcal{L}_1 u_1 = -u_1 + \beta\left[1 - \frac{\bar{m}_1^2 + \bar{m}_2^2}{2}\right] J \star u_1 \tag{4.15}$$

$$\mathcal{L}_2 u_1 = \beta\left[\frac{\bar{m}_1^2 - \bar{m}_2^2}{2}\right] J \star u_1. \tag{4.16}$$

Denote by $\mathcal{R}(\tilde{L})$ the resolvent set of $\tilde{L}$ and by $\mathcal{R}_\lambda(\tilde{L}) = (\tilde{L} - \lambda\mathbb{I})^{-1}$ with $\lambda \in \mathcal{R}(\tilde{L})$. It is easy to see that the spectrum of L coincides with the spectrum of $\tilde{L}$ since

$$\mathcal{R}_\lambda(\tilde{L}) = (\tilde{L} - \lambda\mathbb{I})^{-1} = (U^{-1}LU - \lambda U^{-1}U)^{-1} = \left(U^{-1}(L - \lambda\mathbb{I})\,U\right)^{-1} = U^{-1}\mathcal{R}_\lambda(L)U.$$

Moreover, if we show that $\mathcal{R}(\tilde{L}) = \mathcal{R}(\mathcal{L}_1)$ then the operator $\tilde{L}$ has a gap in the spectrum if $\mathcal{L}_1$ has a gap and therefore the operator L acting on $\mathcal{H}$ has a gap as well. We first prove that $\mathcal{R}(\tilde{L}) = \mathcal{R}(\mathcal{L}_1)$. Supposing $\lambda \in \mathcal{R}(\mathcal{L}_1)$, we want to prove $\lambda \in \mathcal{R}(\tilde{L})$. We can formally compute $\mathcal{R}_\lambda(\tilde{L})$ obtaining the following operator

$$\mathcal{R}_\lambda(\tilde{L}) = \begin{pmatrix} \mathcal{R}_\lambda(\mathcal{L}_1) & 0 \\ \frac{1}{(1+\lambda)}\mathcal{L}_2\left(\mathcal{R}_\lambda(\mathcal{L}_1)\right) & -\frac{1}{(1+\lambda)}\mathbb{I} \end{pmatrix}. \tag{4.17}$$

The previous representation of $\mathcal{R}_\lambda(\tilde{L})$ makes sense if $\lambda \neq -1$ and $\lambda \in \mathcal{R}(\mathcal{L}_1)$. We will see in the following that $-1 \neq \mathcal{R}(\mathcal{L}_1)$, therefore for (4.17) to hold it is enough that $\lambda \in \mathcal{R}(\mathcal{L}_1)$ and therefore $\lambda \in \mathcal{R}(\tilde{L})$.

On the other hand, if $\lambda \in \mathcal{R}(\tilde{L})$ then there will exist well-defined operator γ_i, $i = 1, \ldots, 4$ such that

$$\mathcal{R}_\lambda(\tilde{L}) = \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_3 & \gamma_4 \end{pmatrix}. \tag{4.18}$$

Then we have that

$$\mathbb{I} = (\tilde{L} - \lambda\mathbb{I})\mathcal{R}_\lambda(\tilde{L}) = \begin{pmatrix} \mathcal{L}_1 - \lambda\mathbb{I} & 0 \\ \mathcal{L}_2 & -(1+\lambda)\mathbb{I} \end{pmatrix} \times \begin{pmatrix} \gamma_1 & \gamma_2 \\ \gamma_3 & \gamma_4 \end{pmatrix} = \begin{pmatrix} (\mathcal{L}_1 - \lambda\mathbb{I})\gamma_1 & \eta_2 \\ \eta_3 & \eta_4 \end{pmatrix} \quad (4.19)$$

where we denoted by η_i, $i = 2, \ldots, 4$ the operators obtained by multiplying the two matrix operators. Since $(\mathcal{L}_1 - \lambda\mathbb{I})\gamma_1 = \mathbb{I}$ then $\gamma_1 = \mathcal{R}_\lambda(\mathcal{L}_1)$ and $\lambda \in \mathcal{R}(\mathcal{L}_1)$. The proof that $\mathcal{L}_1$ has a gap is quite standard, see [9]. We set $p = \beta[1 - (\bar{m}_{\beta,1}^2 + \bar{m}_{\beta,2}^2)/2] < 1$, $\omega(x) = \beta[1 - (\bar{m}_1^2(x) + \bar{m}_2^2(x))/2] - p$ and write $\mathcal{L}_1 = \mathcal{L}_0 + K$, where

$$\mathcal{L}_0 u = -u + pJ \star u \quad (4.20)$$

$$Ku = \omega(x)J \star u. \quad (4.21)$$

Since $p < 1$, see (3.16), by Fourier analysis, the spectrum of $\mathcal{L}_0$ lies in the interval $[-1-p, -1+p]$ which is strictly contained in $(-\infty, 0)$.

Moreover, the bounded operator K is compact since it maps the bounded sets of $L^2(\mathbb{R}, \mathrm{d}\nu_1)$ into relatively compact sets in the same space. These properties are easily proved using the regularity of the convolution term and the fact that $\omega(x)$, vanishing exponentially fast at infinity, has a bounded derivative. Since the essential spectrum is conserved under compact perturbations, see [15], we conclude the proof of theorem 4.1. □

Given $\bar{m}_a(x) = (\bar{m}_1(x-a), \bar{m}_2(x-a))$ we denote by $\mathcal{H}_a$ the Hilbert space

$$\mathcal{H}_a = L_2\left(\mathbb{R}, \frac{\mathrm{d}x}{1 - \bar{m}_1^2(x-a)}\right) \times L_2\left(\mathbb{R}, \frac{\mathrm{d}x}{1 - \bar{m}_2^2(x-a)}\right) \quad (4.22)$$

$\langle \cdot, \cdot \rangle_a$ and $\| \cdot \|_a$ the scalar product and the norm defined in $\mathcal{H}_a$. We have the following result.

Theorem 4.2 (local nonlinear stability). *There exist positive constants c_0, δ, α such that if $\|m_0 - \bar{m}_{a(0)}\|_{a(0)} \leqslant \delta$, then there exists $a(t)$ such that*

$$\|m(t) - \bar{m}_{a(t)}\|_{a(t)} \leqslant c_0 \,\mathrm{e}^{-\alpha t} \quad (4.23)$$

where $m(t)$ is the solution of (1.9) with initial datum m_0. Moreover, there exists $a(\infty)$ such that

$$\lim_{t \to \infty} a(t) = a(\infty) \quad (4.24)$$

the convergence being exponentially fast and $|a(\infty) - a(0)| \leqslant c_1 \|m_0 - \bar{m}_{a_0}\|_{a_0}^2$ where c_1 is a suitable constant.

Proof. We represent the evolving profile $m(\cdot, t)$ as a solution of (1.9) in terms of a moving instanton and the corresponding variation, by writing

$$m(\cdot, t) = \bar{m}_{a(t)}(\cdot) + v(\cdot, t) \quad (4.25)$$

where $\bar{m}_{a(t)}(x) = (\bar{m}_1(x - a(t)), \bar{m}_2(x - a(t)))$ and the variation part $v = (v_1, v_2)$ is orthogonal to $\bar{m}'_{a(t)}$ in $\mathcal{H}_{a(t)}$, i.e.

$$\langle v, \bar{m}'_{a(t)} \rangle_{a(t)} = 0. \quad (4.26)$$

It can be proved that any profile m in a suitable small neighbourhood of the manifold of the instanton can be uniquely represented as in (4.25). The proof is similar to the one already given in the appendix of [9] and we therefore omit it. In the following, for notational simplicity, we do not write the explicit dependence on time of $a(t)$ when no confusion arises. Whenever the representation (4.25) holds we have

$$\frac{\partial(m(t) - \bar{m}_{a(t)})}{\partial t} = -(\bar{m}_{a(t)} + v(t)) + \tanh_\theta \beta J \star \left(\tilde{v} + \tilde{\bar{m}}_a\right) + \dot{a}(t)\bar{m}'_a. \quad (4.27)$$

Recalling (4.2), we write

$$\frac{\partial v}{\partial t} = L_a v + \dot{a}\bar{m}'_a + R[v] \tag{4.28}$$

where L_a is the operator obtained linearizing (1.9) around $\bar{m}_a$

$$R[v] = \tanh_\theta(J \star \tilde{v}) - \bar{m}(\cdot - a(t)) - (1 - \bar{m}(\cdot - a(t)))\beta J \star \tilde{v}. \tag{4.29}$$

Applying the Taylor formula easily we obtain that

$$\|R[v]\|_{L^2} \leqslant c(\beta, \theta, J)\|\tilde{v}\|^2_{L^2} \tag{4.30}$$

where $c(\beta, \theta, J)$ is a positive constant depending only on β , θ and J. Differentiating with respect to time (4.26) we obtain

$$\langle \partial_t v, \bar{m}'_a\rangle_a + \dot{a}(v; \Phi_a)_{L^2} = 0 \tag{4.31}$$

where for $i = 1, 2$,

$$\Phi_{a,i}(x) = \partial_a\left(\frac{\bar{m}'_i(x - a(t))}{1 - \bar{m}^2_i(x - a(t))}\right). \tag{4.32}$$

By taking the scalar product in $\mathcal{H}_a$ on both sides of (4.28) with $\bar{m}'_a$, taking care that $\bar{m}'_a \in$ Null L and (4.31) we obtain

$$\dot{a}[\|\bar{m}'_a\|^2 + (v; \Phi_a)_{L^2}] = -\langle R[v], \bar{m}'_a\rangle_a. \tag{4.33}$$

By taking the scalar product in $\mathcal{H}$ on both sides of (4.28) with $2v$ we obtain, since $\langle v, \bar{m}'_{a(t)}\rangle_a = 0$ the following

$$2\langle \partial_t v, v\rangle_a = 2\langle v, L_a v\rangle_a + 2\langle v, R[v]\rangle_a. \tag{4.34}$$

Note that

$$2\langle \partial_t v, v\rangle_a = \frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2_a - 2\dot{a}\left[(v_1^2; \Psi_1)_{L^2} + (v_2^2; \Psi_2)_{L^2}\right] \tag{4.35}$$

where for $i = 1, 2$, $\Psi_i(x) = (\bar{m}_i(x - a)\bar{m}'_i(x - a))/(1 - \bar{m}^2_i(x - a))^2$ is a bounded continuous function. We therefore obtain a system of ordinary differential inequalities

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2_a \leqslant -2\alpha\|v\|^2_a + k_1\|v\|^3_a + |\dot{a}|k_1\|v\|^2_a \\ &|\dot{a}|\left|\|\bar{m}'_a\|^2_a - |(v; \Phi)_{L^2}|\right| \leqslant k_3\|v\|^2_a. \end{aligned} \tag{4.36}$$

We take the initial datum m_0 so that $v_0 = m_0 - \bar{m}_{a(0)}$ satisfies the bound

$$\|v_0\|_{a(0)} \leqslant \tfrac{1}{4}\|\Phi_{a(0)}\|^{-1}_{L^2}\|\bar{m}'_{a(0)}\|^2_{a(0)}. \tag{4.37}$$

Note that $\|\bar{m}'_{a(t)}\|^2_{a(t)} = \|\bar{m}'\|^2_{\mathcal{H}}$ and $\|\Phi_{a(t)}\|_{L^2} = \|\Phi_{a(0)}\|_{L^2}$. We then denote by

$$t^* = \sup\{t : \|v(\cdot, t)\|_{a(t)} \leqslant \tfrac{1}{2}\|\Phi_{a(0)}\|^{-1}_{L^2}\|\bar{m}'\|^2_{\mathcal{H}}\}. \tag{4.38}$$

Then by (4.36) we obtain that there is a suitable constant c_0 such that for all $t \leqslant t^*$

$$\|v(\cdot, t)\|^2_{a(t)} \leqslant c_0\,\mathrm{e}^{-2\alpha t}\|v_0\|^2_{a(0)}. \tag{4.39}$$

This implies that $t^* = \infty$ and that (4.39) holds for all $t \in \mathbb{R}$ provided $\|v_0\|_{a(0)}$ is taken sufficiently small, as in (4.37). Moreover, from the estimate on $|\dot{a}(t)|$ we have that $a(t)$ is a Cauchy sequence and therefore $\lim_{t\to\infty} a(t) = a(\infty)$, the convergence being exponentially fast. Namely, since (4.36)

$$|a(t) - a(0)| \leqslant \int_0^t |\dot{a}(s)|\,\mathrm{d}s \leqslant c_0\int_0^t \|v(\cdot, s)\|^2_{a(s)}\,\mathrm{d}s.$$

Then from (4.39) we obtain the exponentially convergence. Moreover, $|a(\infty) - a(0)| \leqslant c_0\|m_0 - \bar{m}_{a(0)}\|^2_{a(0)}$. □

5. *A priori* estimates

In this section, we show that any solution of (1.9) $m(\cdot, t)$ with an initial datum in $\mathcal{A}$ gets eventually trapped between two instantons. This is an essential ingredient in the proof of theorems 1.4 and 1.6. These estimates, already established in [10, 13], can be extended in our case. For completeness we give the main details.

Throughout this section, $m(x,t) = (m_1(x,t), m_2(x,t))$ and $u(x,t) = (u_1(x,t), u_2(x,t))$ denote two solutions of (1.9) with initial data $m(\cdot,0) = (m_1(\cdot,0), m_2(\cdot,0))$ and $u(\cdot,0) = (u_1(\cdot,0), u_2(\cdot,0))$ $m_i(\cdot,0) \in L^\infty(\mathbb{R})$, $u_i(\cdot,0) \in L^\infty(\mathbb{R})$; $\|m_i(\cdot,0)\|_\infty \leqslant 1$, $\|u_i(\cdot,0)\|_\infty \leqslant 1$, for $i = 1, 2$. Then, as outlined at the beginning of the introduction, the $m_i(\cdot,t)$ and $u_i(\cdot,t)$ are continuous with sup norm bounded by 1, for all $t > 0$.

Lemma 5.1 (Barrier lemma). *There is a constant $C > 0$ such that for any two solutions $m(t)$ and $u(t)$ of (1.9), for all $t > 0$ and all $V > e^2\beta$*

$$|m_i(0,t) - u_i(0,t)| \leqslant \mathrm{e}^{(\beta-1)t} \sup_{|x|\leqslant Vt} \sum_{i=1}^{2} |m_i(x,0) - u_i(x,0)| + C\,\mathrm{e}^{-t\log(V/\mathrm{e}^2\beta)} \tag{5.1}$$

$$|m_i(x,t) - u_i(x,t)| \leqslant C\,\mathrm{e}^{-t\log(V/\mathrm{e}^2\beta)} \qquad |x| \geqslant Vt. \tag{5.2}$$

Proof. Denote by $d_i(x,t) = |m_i(x,t) - u_i(x,t)|$ for $i = 1, 2$, $\tilde{d}(x,t) = \frac{1}{2}(d_1(x,t) + d_2(x,t))$ and $J^{\star n}$ the n-fold convolution of J with itself. Then from (2.1) we obtain

$$d_i(x,t) \leqslant \mathrm{e}^{-t} d_i(x,0) + \int_0^t \mathrm{d}s\, \mathrm{e}^{-(t-s)} \beta (J \star \tilde{d})(x,s)$$

hence by iterating we get

$$d_i(x,t) \leqslant \mathrm{e}^{-t} \sum_{n\geqslant 0} \frac{(\beta t)^n}{n!} (J^{\star n} \star \tilde{d})(x,0). \tag{5.3}$$

We write $\tilde{d} = \tilde{d}_+ + \tilde{d}_-$, where $\tilde{d}_- = \tilde{d}\mathbf{1}_{|x|\leqslant Vt}$ and $\tilde{d}_+ = \tilde{d}\mathbf{1}_{|x|>Vt}$, with $\mathbf{1}_A$ the indicator function of the set A. We set $x = 0$ in (5.3) and notice that

$$(J^{\star n} \star \tilde{d}_+)(0,0) = 0 \qquad \text{if } n < Vt. \tag{5.4}$$

In fact the support of $J(x - x')$ is $|x - x'| \leqslant 1$, therefore the support of $J^{\star n}$, when $x = 0$, is $[-n, n]$ from which we get (5.4). Then from (5.3)

$$d_i(x,t) \leqslant \mathrm{e}^{-t} \sum_{n\geqslant 0}^{Vt} \frac{(\beta t)^n}{n!} (J^{\star n} \star \tilde{d}_-)(x,0) + \mathrm{e}^{-t} \sum_{n\geqslant Vt} \frac{(\beta t)^n}{n!} 2. \tag{5.5}$$

For the second-term we have used that $|\tilde{m}(x,t)| \leqslant 1$ and $|\tilde{u}(x,t)| \leqslant 1$. Setting $L = Vt$ it is easy to check that

$$\mathrm{e}^{-t} \sum_{n\geqslant L} \frac{(\beta t)^n}{n!} 2 \leqslant 2 \frac{(\beta t)^L}{L!} \mathrm{e}^{(\beta-1)t} \leqslant C\,\mathrm{e}^{-L\log(L/e^2\beta t)}.$$

The proof of the proposition is immediate. □

Lemma 5.2. *Let $m \in \mathcal{A}$ and $m(t,\cdot)$ be the solution of (1.9) with initial datum m. Then for any $\epsilon > 0$, there are t_0 and $\ell > 0$ so that*

$$m(x,t_0) \leqslant m_\beta + \epsilon \quad \textit{for all } x \qquad \textit{and} \qquad m(x,t_0) \geqslant m_\beta - \epsilon \quad \textit{for all } x \geqslant \ell. \tag{5.6}$$

$$m(x,t_0) \geqslant Tm_\beta - \epsilon \quad \textit{for all } x \qquad \textit{and} \qquad m(x,t_0) \leqslant Tm_\beta + \epsilon \quad \textit{for all } x \leqslant -\ell. \tag{5.7}$$

Proof. Denote by $T_t(u)$ the solution at time t of the Cauchy problem (1.9) with initial datum u. Since $-1 \leqslant m_i(x) \leqslant 1$ for $i = 1, 2$ then by the theorem 2.2 $T_t(-1) \leqslant T_t(m) \leqslant T_t(1)$. We prove first (5.6), the proof of (5.7) is similar. Note that when the initial datum is constant equation (1.9) becomes a system of ordinary differential equations and it is easy to see that

$$\|T_t(1) - m_\beta\|_\infty \leqslant c\,\mathrm{e}^{-\omega t} \tag{5.8}$$

where c and ω are positive constants. We take t_0 such that $c\,\mathrm{e}^{-\omega t_0} = \epsilon$, so that $T_{t_0}(m) \leqslant T_{t_0}(1) \leqslant m_\beta + \epsilon$. To prove the second statement in (5.6) we note that, since $m \in \mathcal{A}$ then there exists $\zeta = (\zeta_1, \zeta_2)$, $\zeta_i > 0$, $i = 1, 2$ and x_0 such that $m(x) \geqslant \zeta$ for $x \geqslant x_0$. Let t_0 be so large that $T_{t_0}(\zeta) \geqslant m_\beta - \epsilon/2$. Such t_0 always exists since $\zeta > 0$ and $\|T_t(\zeta) - m_\beta\|_\infty \leqslant c\,\mathrm{e}^{-\omega t}$. Define now $u(x) = m(x)$ for $x \geqslant x_0$ and $u(x) = m(x_0)$ for $x \leqslant x_0$, so that $u(x) \geqslant \zeta$ everywhere. Then $u(x, t_0) \geqslant m_\beta - \epsilon/2$ everywhere. Applying the (5.2) we have that

$$\sum_{i=1}^{2} |m_i(x, t_0) - u_i(x, t_0)| \leqslant 2C\,\mathrm{e}^{-t_0 \log(V/\beta e^2)} \qquad \text{for all } x \geqslant x_0 + Vt_0. \tag{5.9}$$

We take V in (5.9) such that $2C\,\mathrm{e}^{-t_0 \log(V/\beta e^2)} \leqslant \frac{1}{2}\epsilon$. Hence

$$m(x, t_0) \geqslant u(x, t_0) - \frac{\epsilon}{2} \geqslant m_\beta - \epsilon \qquad \text{for all } x \geqslant \ell = x_0 + Vt_0$$

thus proving (5.6). □

Lemma 5.3. *Let $m \in \mathcal{A}$ and $m(t, \cdot)$ be the solution of (1.9) with initial datum m. For any $\delta \in \mathbb{R}$ such that $m_{\beta,2} > \delta > 0$ there exist t_0, $a_1 > 0$, $a_2 > 0$ and $0 \leqslant q_0 \leqslant \delta$ so that*

$$\bar{m}_{a_2}(x) - q_0 \leqslant m(x, t_0) \leqslant \bar{m}_{a_1}(x) + q_0 \qquad \textit{for all } x \in \mathbb{R}. \tag{5.10}$$

Proof. We will prove the lower bound in (5.10), the proof of the upper bound is similar. From theorem 1.1 there exists $a > 0$ be such that $\bar{m}(x) \leqslant Tm_\beta + \epsilon$ for all $x \leqslant -a$. Then

$$\bar{m}_{\ell+a}(x) = \bar{m}(x - [\ell + a]) \leqslant Tm_\beta + \epsilon, \qquad \text{for } x \leqslant \ell. \tag{5.11}$$

Then from the first inequality in (5.7) $m(x, t_0) \geqslant \bar{m}_{\ell+a}(x) - 2\epsilon$ for $x \leqslant \ell$. By the second inequality in (5.6) and theorem 1.1 we have that $m(x, t_0) \geqslant \bar{m}_{\ell+a}(x) - 2\epsilon$ for $x \geqslant \ell$. Hence

$$m(x, t_0) \geqslant \bar{m}_{\ell+a}(x) - 2\epsilon \qquad \text{for all } x. \tag{5.12}$$

Denote $2\epsilon = q_0$ and $a_2 = \ell + a$ we obtain the lower bound. □

Let $\delta \in \mathbb{R}$ such that $m_{\beta,2} > \delta > 0$, denote by

$$\begin{aligned}\mathcal{B}_\delta = \{&m = (m_1, m_2) \in C_b(\mathbb{R}) \times C_b(\mathbb{R});\ \|m_i\|_\infty \leqslant 1; \exists a_2 = a_2(m),\ a_1 = a_1(m),\\ &q_0 = q_0(m)\ \bar{m}_{a_2}(x) - q_0 \leqslant m(x) \leqslant \bar{m}_{a_1}(x) + q_0,\ 0 \leqslant q_0 \leqslant \delta,\ \text{for all } x \in \mathbb{R}\big\}.\end{aligned} \tag{5.13}$$

From the previous lemma we obtain that for any $m \in \mathcal{A}$ there exists a suitable time t_0 such that $m(\cdot, t_0) \in \mathcal{B}_\delta$ where $m(\cdot, t_0)$ is the solution of (1.9) with initial datum $m \in \mathcal{A}$.

Proposition 5.4. *Let $m(x, t)$ the solution of (1.9) with initial datum $m \in \mathcal{B}_\delta$, where δ is taken small enough. There exist positive constants $b = b(\delta)$, $d = d(\delta)$ and $\lambda = \lambda(\delta)$ such that denoting $q_1(t) = q_0[\mathrm{e}^{-t} + d(\mathrm{e}^{-\lambda t} - \mathrm{e}^{-t})]$, $q_2(t) = q_0[\mathrm{e}^{-t} + (2 - d)(\mathrm{e}^{-\lambda t} - \mathrm{e}^{-t})]$, $q(t) = (q_1(t), q_2(t))$, $a_2(t) = a_2 + bq_0(1 - \mathrm{e}^{-\lambda t})$, $a_1(t) = a_1 + bq_0(1 - \mathrm{e}^{-\lambda t})$ we have that for all $x \in \mathbb{R}$, $t \in \mathbb{R}_+$*

$$\bar{m}(x - a_1(t)) - q(t) \leqslant m(x, t) \leqslant \bar{m}(x - a_2(t)) + q(t). \tag{5.14}$$

Proof. We start showing the lower bound. The upper bound is done similarly. We set $a_1(t) = a(t)$. It will be sufficient to prove that, $v(x,t) = \bar{m}(x - a(t)) - q(t)$ is a subsolution of (1.9) provided the parameters b, d and λ are suitably chosen. Observe that $v(x, 0) \leqslant m(x, 0) = m(x)$. Then (5.14) will follow from theorem 2.2 once we verify that for all $(x,t) \in \mathbb{R} \times \mathbb{R}^+$, $v(x,t)$ satisfies

$$\frac{\partial v}{\partial t} \leqslant -v + \tanh_\theta\{\beta\,(J \star \tilde{v})\}. \tag{5.15}$$

We differentiate $v(\cdot, t)$ with respect to t getting

$$\frac{\partial v(x,t)}{\partial t} = -\bar{m}'(x - a(t))\dot{a}(t) - \dot{q}(t). \tag{5.16}$$

We then need to show that

$$-\bar{m}'(x - a(t))\dot{a}(t) - \dot{q}(t) \leqslant -[\bar{m}(x - a(t)) - q(t)] + \tanh_\theta\{\beta(J \star \tilde{\bar{m}}(x - a(t)) - \tilde{q}(t))\} \tag{5.17}$$

where $\tilde{q}(t) = \frac{1}{2}[q_1(t) + q_2(t)]$. To prove (5.17) note that $a(t)$ is increasing, so that the contribution of the first term of (5.17) will be always negative, as $\bar{m}'$ is always strictly positive. We shall take advantage of that, but this will be not sufficient since $\lim_{|x|\to\infty} \bar{m}'(x) = 0$, i.e. when $\lim_{x\to\infty} \bar{m}(x) = m_\beta$ and $\lim_{x\to-\infty} \bar{m}(x) = Tm_\beta$. We therefore use different arguments depending on the value of $\bar{m}(x)$. More precisely, given $t > 0$ and $I = \{[m_{\beta,1} - \epsilon, m_{\beta,1}] \times [m_{\beta,2} - \epsilon, m_{\beta,2}]\} \cup \{[-m_{\beta,2}, -m_{\beta,2} + \epsilon] \times [-m_{\beta,1}, -m_{\beta,1} + \epsilon]\}$ where $\epsilon > 0$ will be fixed later, we consider all the values of x such that $(J \star \bar{m}_1(x - a(t)), J \star \bar{m}_2(x - a(t))) \in I$. We use short-hand notation

$$u = J \star \tilde{\bar{m}}(x - a(t)) \tag{5.18}$$

and therefore $u \in [\tilde{m}_\beta - \epsilon, \tilde{m}_\beta] \cup [-\tilde{m}_\beta, -\tilde{m}_\beta + \epsilon]$. We then need to show that, for $i = 1, 2$

$$-\dot{q}_i(t) \leqslant F_i(u, q_1, q_2) \tag{5.19}$$

where

$$\begin{aligned} F_1(u, q_1, q_2) &= -[\tanh\beta(u + \theta) - q_1] + \tanh\beta(u - \tilde{q} + \theta) \\ F_2(u, q_1, q_2) &= -[\tanh\beta(u - \theta) - q_2] + \tanh\underline{(}u - \tilde{q} - \theta) \end{aligned} \tag{5.20}$$

having used (1.10) to write the first term in the right-hand side of (5.17).

Since $\dot{q}_1(t) = -q_1(t) + d(1-\lambda)\tilde{q}(t)$ and $\dot{q}_2(t) = -q_2(t) + (2-d)(1-\lambda)\tilde{q}(t)$ proving (5.19), is equivalent to show that

$$\begin{aligned} q_1 - d(1-\lambda)\tilde{q} &\leqslant F_1(u, q_1, q_2) \\ q_2 - (2-d)(1-\lambda)\tilde{q} &\leqslant F_2(u, q_1, q_2) \end{aligned} \tag{5.21}$$

for all $u \in [\tilde{m_\beta} - \epsilon, \tilde{m_\beta}] \cup [-\tilde{m_\beta}, -\tilde{m_\beta} + \epsilon]$, $0 \leqslant q_1 \leqslant \delta$ and $0 \leqslant q_2 \leqslant \delta$. We have that, since $F_1(u, 0, 0) = F_2(u, 0, 0) = 0$

$$F_1(u, q_1, q_2) \geqslant q_1 - \frac{\beta}{\cosh^2\beta(\tilde{m}_\beta - \epsilon + \theta - \delta)}\tilde{q}$$

and

$$F_2(u, q_1, q_2) \geqslant q_2 - \frac{\beta}{\cosh^2\beta(\tilde{m}_\beta - \epsilon - \theta - \delta)}\tilde{q}.$$

Therefore to satisfy (5.21) it is enough to show that there is a suitable choice of d and λ such that

$$\begin{aligned} q_1 - d(1-\lambda)\tilde{q} &\leqslant q_1 - \frac{\beta}{\cosh^2\beta(\tilde{m}_\beta - \epsilon + \theta - \delta)}\tilde{q} \\ q_2 - (2-d)(1-\lambda)\tilde{q} &\leqslant q_2 - \frac{\beta}{\cosh^2\beta(\tilde{m}_\beta - \epsilon - \theta - \delta)}\tilde{q} \end{aligned}. \tag{5.22}$$

System (5.22) is satisfied provided we take

$$d = \frac{\beta}{\cosh^2 \beta(\tilde{m}_\beta - \epsilon + \theta - \delta)} \frac{1}{(1-\lambda)} \qquad \text{and} \qquad \lambda > 0$$

such that

$$0 < \lambda \leqslant 1 - \frac{1}{2}\frac{\beta}{\cosh^2 \beta(\tilde{m}_\beta - \epsilon - \theta - \delta)} - \frac{1}{2}\frac{\beta}{\cosh^2 \beta(\tilde{m}_\beta - \epsilon + \theta - \delta)}. \tag{5.23}$$

Since

$$1 - \frac{1}{2}\frac{\beta}{\cosh^2 \beta(\tilde{m}_\beta - \theta)} - \frac{1}{2}\frac{\beta}{\cosh^2 \beta(\tilde{m}_\beta + \theta)} > 0,$$

provided we take ϵ and δ small enough (5.23) is satisfied.

For the other values of (x, t) there exists $c_1 > 0$ such that $\bar{m}_i'(x) \geqslant c_1$ for $i = 1, 2$. In fact from theorem 1.3 $\bar{m}'$ is strictly positive when x varies in a compact set, and the sets

$$\{x : -m_{\beta,2} + \epsilon \leqslant J \star \bar{m}_1(x - a(t)) \leqslant m_{\beta,1} - \epsilon\}$$

and

$$\{x : -m_{\beta,1} + \epsilon \leqslant J \star \bar{m}_2(x - a(t)) \leqslant m_{\beta,2} - \epsilon\}$$

are compact sets.

Moreover, there is $\gamma > 0$ so that $F_1(u, q_1, q_2) \geqslant -\gamma\tilde{q}$ and $F_2(u, q_1, q_2) \geqslant -\gamma\tilde{q}$, since $F_1(u, 0, 0) = F_2(u, 0, 0) = 0$ and the derivatives with respect to q_1 and q_2 are bounded. Hence we need to verify that

$$-c_1\dot{a}(t) - \dot{q}_i(t) + \gamma\tilde{q}(t) \leqslant 0 \tag{5.24}$$

for $i = 1, 2$. Equation (5.24) is satisfied provided b in the definition of $a(t)$ is chosen sufficiently large. □

6. Unicity in $\mathcal{A}$

In this section, we prove theorem 1.4; i.e. the unicity, up translations, of the instanton in the class $\mathcal{A}$. The proof is based essentially on the nonlinear local stability results proven in section 4 and on the estimates proven in section 5. In fact, using estimates proven in section 5 it is possible to show that starting with an initial datum in $\mathcal{A}$ there will be a time, not necessarily finite, such that the difference between the solution and the manifold of the instantons is in L^2. We then apply the nonlinear local stability results. To pursue in this way we need to show that there exists only one instanton which minimizes the L^2 distance between the solution and the manifold of the instantons whenever we are close enough to it. This is the content of the next lemma.

Lemma 6.1. *Let $m - \bar{m} \in L^2$. Then there exists a such that*

$$\|m - \bar{m}_a\|_{L^2} = \inf_{b\in\mathbb{R}} \{\|m - \bar{m}_b\|_{L^2}\}. \tag{6.1}$$

Further, there exists δ_0 such that if $\|m - \bar{m}_b\|_{L^2} \leqslant \delta_0$ then there exists only one a which satisfies (6.1).

Proof. There is at least one instanton $\bar{m}_a$ minimizing $\|m - \bar{m}_b\|_{L^2}^2$ since this quantity is differentiable as a function of b and tends to infinity as $b \to \pm\infty$. Hence the distance is minimized at one value at least, which we denote as a. At this value $\bar{m}'_a$ is orthogonal to $m - \bar{m}_a$ in L^2. To show the local unicity call

$$d(b) = \|m - \bar{m}_b\|^2_{L^2}. \tag{6.2}$$

Differentiating with respect to b one obtains

$$d'(b) = 2(m; \bar{m}'_b)_{L^2}$$

and

$$d''(b) = -2(m; \bar{m}''_b)_{L^2} = -2(m - \bar{m}_a; \bar{m}''_b)_{L^2} + 2(\bar{m}_a; \bar{m}'_b)_{L^2}. \tag{6.3}$$

Since

$$(m - \bar{m}_a; \bar{m}''_b)_{L^2} \leqslant \|m - \bar{m}_a\|_{L^2}\|\bar{m}''_b\|_{L^2} \leqslant \delta_0\|\bar{m}''_b\|_{L^2}$$

where δ_0 will be suitably chosen in the following. By continuity, there exists a positive constant $c = c(\beta, \theta, J)$ such that if $b \in (a - c, a + c)$

$$(\bar{m}'_a; \bar{m}'_b)_{L^2} \geqslant \tfrac{1}{2}\|\bar{m}'_a\|^2_{L^2}.$$

Then

$$d''(b) \geqslant \tfrac{1}{2}\|\bar{m}'_a\|^2_{L^2} - 2\delta_0\|\bar{m}''_b\|_{L^2}.$$

Denote by $A(\beta, \theta, J) = \inf_{b\in(a-c,a+c)} \|\bar{m}''_b\|_{L^2}$. Then if

$$\delta_0 \leqslant \frac{1}{4}\frac{\|\bar{m}'_a\|^2_{L^2}}{A(\beta, \theta, J)} \tag{6.4}$$

we have that $d''(b) > 0$ and therefore the minimum is unique. □

We want to stress that, if $m - \bar{m} \in L^2$ and $\bar{m}_a$ minimizes the L^2 distance then

$$(m - \bar{m}_a; \bar{m}'_a)_{L^2} = 0. \tag{6.5}$$

In the following whenever we have a function such that $m - \bar{m} \in L^2$ we simply says that m has centre a if (6.5) holds. Note that (6.5) can be extended to any function $m \in \mathcal{A}$ since the exponential decay to zero of $\bar{m}'$.

Lemma 6.2. *Let $\epsilon_0 > 0$ small enough and v a continuous function such that $\bar{m}(x) \geqslant v \geqslant \bar{m}(x - \epsilon_0)$. Then if $v \neq \bar{m}$, there exists $a(\infty) > 0$ such that $\|T_t(v) - \bar{m}_{a(\infty)}\|_{L^2} \leqslant C\,\mathrm{e}^{-\alpha t}$.*

Proof. Let $\lambda \in [0, 1]$ and denote

$$v_\lambda = \bar{m} + \lambda[v - \bar{m}]. \tag{6.6}$$

Then if $\epsilon_0 > 0$ is small enough, $T_t(v_\lambda)$ the solution of (1.9) with initial datum v_λ converges to an instanton denoted $\bar{m}_{a_\lambda(\infty)}$. Since v_λ is a decreasing function of λ, by the comparison theorem $a_\lambda(\infty)$ is a non-decreasing function of λ, then $a_\lambda(\infty) \leqslant a_1(\infty)$. Since $v_1 = v$ lemma 6.2 will follow once we show that if $v \neq \bar{m}$ then there exists λ such that $a_\lambda(\infty) > 0$. By theorem 4.2

$$|a_\lambda(\infty) - a_\lambda(0)| \leqslant C\|v_\lambda - \bar{m}_{a_\lambda(0)}\|^2_{L^2}. \tag{6.7}$$

Then from (6.6) we have that

$$\|v_\lambda - \bar{m}_{a_\lambda(0)}\|^2_{L^2} \leqslant 2\|\bar{m} - \bar{m}_{a_\lambda(0)}\|^2_{L^2} + 2\lambda^2\|v - \bar{m}\|^2_{L^2}. \tag{6.8}$$

Similarly as in (6.22)

$$\|\bar{m} - \bar{m}_{a_\lambda(0)}\|^2_{L^2} \leqslant Ca^2_\lambda(0) \leqslant C\lambda^2. \tag{6.9}$$

The last inequality in (6.9) follows the next lemma 6.3, see (6.12). Therefore we obtain that, see (6.7)

$$|a_\lambda(\infty) - a_\lambda(0)| \leqslant C\lambda^2. \tag{6.10}$$

Again from the next lemma 6.3, see (6.13)

$$-\lambda \frac{([v - \bar{m}]; \bar{m}')_{L^2}}{(\bar{m}'; \bar{m}')_{L^2}} - C\lambda^2 \leqslant a_\lambda(0).$$

Since $\bar{m}(x) \geqslant v(x)$ and $\bar{m} - v \neq 0$ we obtain that

$$-\lambda \frac{([v - \bar{m}]; \bar{m}')_{L^2}}{(\bar{m}'; \bar{m}')_{L^2}} > 0. \tag{6.11}$$

Then $a_\lambda(0) \geqslant \lambda C$ and the lemma is proved. □

Lemma 6.3. *Let* $v_\lambda = \bar{m} + \lambda[v - \bar{m}]$. *Denote by* a_λ *the centre of* v_λ. *Then there exists* λ_0 *such that for* $\lambda \leqslant \lambda_0$

$$|a_\lambda| \leqslant C\lambda \tag{6.12}$$

and

$$\left| a_\lambda + \lambda \frac{([v - \bar{m}]; \bar{m}')_{L^2}}{(\bar{m}'; \bar{m}')_{L^2}} \right| \leqslant C\lambda^2 \tag{6.13}$$

where C *is a positive constant.*

Proof. Denote by $f(\xi) = (v_\lambda - \bar{m}_\xi; \bar{m}'_\xi)_{L^2}$. We differentiate with respect to ξ the function $f(\xi)$ obtaining

$$f'(\xi) = -(\bar{m}; \bar{m}''_\xi)_{L^2} - \lambda([v - \bar{m}]; \bar{m}''_\xi)_{L^2}.$$

Note that

$$-(\bar{m}; \bar{m}''_\xi)_{L^2} = (\bar{m}'; \bar{m}'_\xi)_{L^2} > 0$$

which is strictly positive for all values of $\xi \in \mathbb{R}$. Denote by $\gamma_0 = \inf_\xi (\bar{m}'; \bar{m}'_\xi)_{L^2}$. Since $\bar{m}''_\xi$ is exponentially decaying to zero, see theorem 1.3

$$|([v - \bar{m}]; \bar{m}''_\xi)_{L^2}| \leqslant K$$

where K is a positive constant. We can therefore take λ_0 such that $\lambda_0 K \leqslant \gamma_0/2$ obtaining that

$$f'(\xi) \geqslant \frac{\gamma_0}{2} \qquad \xi \in \mathbb{R}. \tag{6.14}$$

Since $|f(0)| \leqslant 2\lambda m_\beta$ and (6.14) holds, by geometrical argument we obtain that

$$|a_\lambda| \leqslant \frac{2}{\gamma_0} \lambda m_\beta.$$

Denoting by $C = (2/\gamma_0) m_\beta$ (6.12) follows. Next, since by definition a_λ is a centre of v_λ when $f(a_\lambda) = 0$ we have that

$$f(a_\lambda) = (\bar{m} - \bar{m}_{a(\lambda)}; \bar{m}'_{a(\lambda)})_{L^2} + \lambda(v - \bar{m}; \bar{m}'_{a(\lambda)})_{L^2} = 0. \tag{6.15}$$

By the Taylor formula, taking in account that $(\bar{m}_d; \bar{m}'_d)_{L^2} = 0$ for any $d \in \mathbb{R}$, as it follows by integration, we have

$$\begin{aligned}
(\bar{m} - \bar{m}_{a(\lambda)}; \bar{m}'_{a(\lambda)})_{L^2} &= (\bar{m}; \bar{m}'_{a(\lambda)})_{L^2} \\
&= (\bar{m}; \bar{m}')_{L^2} - a(\lambda)(\bar{m}; \bar{m}'')_{L^2} + \frac{1}{2} a(\lambda)^2 \sum_{i=1}^{2} \int_{\mathbb{R}} \bar{m}'_i(x) \int_{x-\xi}^{\xi} \bar{m}'''_i(s)\, ds \\
&= a(\lambda)(\bar{m}', \bar{m}') + \frac{1}{2} a(\lambda)^2 \sum_{i=1}^{2} \int_{\mathbb{R}} \bar{m}'_i(x) \int_{x-\xi}^{\xi} \bar{m}'''_i(s)\, ds.
\end{aligned} \tag{6.16}$$

Again, by the Taylor formula,

$$(v - \bar{m}; \bar{m}'_{a(\lambda)})_{L^2} = (v - \bar{m}; \bar{m}')_{L^2} - a(\lambda) \sum_{i=1}^{2} \int \mathrm{d}x [v_i(x) - \bar{m}_i(x)] \int_{x-\xi}^{\xi} \bar{m}''_i(s)\, \mathrm{d}s. \tag{6.17}$$

Inserting (6.16) and (6.17) in (6.15) we get that

$$\left| a(\lambda) + \lambda \frac{(v - \bar{m}; \bar{m}')_{L^2}}{(\bar{m}'; \bar{m}')_{L^2}} \right| \leqslant C a(\lambda)[a(\lambda) + \lambda] \leqslant C\lambda^2 \tag{6.18}$$

where we have used (6.12) at the last step. □

Proof of theorem 1.4. Let $m^* \in \mathcal{A}$ be a solution of (1.10). Then m^* is a stationary solution of (1.9), $m^*(x,t) = m^*(x,0) = m^*(x)$. By taking the limit as $t \to \infty$ from proposition 5.4 we deduce the existence of a_1 and a_2 such that

$$\bar{m}_{a_1} \leqslant m^* \leqslant \bar{m}_{a_2}. \tag{6.19}$$

Let a_1^* and a_2^* be such that

$$\bar{m}_{a_1^*} \leqslant m^* \leqslant \bar{m}_{a_2^*} \tag{6.20}$$

and

$$\text{if for all } x,\ m^*(x) \geqslant \bar{m}(x - c) \qquad \text{then } c \geqslant a_1^* \tag{6.21}$$

with analogous property holding for a_2^*. Note that

$$\|\bar{m}_{a_1^*} - \bar{m}_{a_2^*}\|_{L^2} \leqslant C|a_1^* - a_2^*|. \tag{6.22}$$

Since

$$\|\bar{m}_{a_1^*} - \bar{m}_{a_2^*}\|_{L^2}^2 = \sum_{i=1}^{2} \int \mathrm{d}x \left[\int_{x-a_2^*}^{x-a_1^*} \bar{m}'_i(s)\, \mathrm{d}s \right]^2 \leqslant c_1 \sum_{i=1}^{2} \int \mathrm{d}x \left[\int_{x-a_2^*}^{x-a_1^*} \mathrm{e}^{-\alpha|s|}\, \mathrm{d}s \right]^2$$
$$\leqslant 2c_1 |a_1^* - a_2^*|^2 \int_{\mathbb{R}} \mathrm{d}x\, \mathrm{e}^{-\alpha|x|} \leqslant C|a_1^* - a_2^*|^2$$

where we have used $|\bar{m}'_i(s)| \leqslant c_1\, \mathrm{e}^{-\alpha|s|}$, for $i = 1, 2$ that follows from theorem 1.3.

If $|a_1^* - a_2^*|$ is sufficiently small such that $\|\bar{m}_{a_1^*} - \bar{m}_{a_2^*}\|_2 \leqslant \delta$, then theorem 1.4 follows from the local nonlinear stability, see theorem 4.2. Otherwise denote by $a = a_1^* - \epsilon_0$. Note that $\bar{m}(x - a) = \bar{m}(x - a_1^* + \epsilon_0) > \bar{m}(x - a_1^*)$. Define for all x,

$$v(x) = \min\{m^*(x), \bar{m}(x - a)\} \tag{6.23}$$

and let $v(x,t)$ be the solution of (1.9) starting from $v(x)$. Since from (6.22) we have that

$$\|v - \bar{m}_{a_1^*}\|_{L^2} \leqslant \|\bar{m}_{a_1^*} - \bar{m}_a\|_{L^2} \leqslant C\epsilon_0$$

theorem 4.2 applies: there exists $a(\infty)$ such that $\|v(t) - \bar{m}_{a(\infty)}\|_{L^2} \leqslant c\, \mathrm{e}^{-\alpha t}$.

Since $m^*(x) \geqslant v(x)$, by the comparison theorem 2.2, $m^*(x) \geqslant \bar{m}_{a(\infty)}(x)$, therefore by the definition of $\bar{m}_{a_1^*}$ we have that $a(\infty) \geqslant a_1^*$. On the other hand, $v(x) \geqslant \bar{m}(x - a_1^*)$ and therefore by the comparison theorem $\bar{m}(x - a(\infty)) \geqslant \bar{m}(x - a_1^*)$ and hence $a(\infty) \leqslant a_1^*$. One then has that $a(\infty) = a_1^*$. In lemma 6.2 we proved that this implies that $m^*(x) = \bar{m}(x - a_1^*)$. Then theorem 1.4 is proved. □

7. Global Stability in $\mathcal{K}(M, N)$

In this section we prove that the manifold of the instanton is asymptotic stable in the class of functions $\mathcal{K}(M, N)$, see (1.12). The method to prove L^2-stability relates the free energy functional $\mathcal{F}$ defined in (1.1) with the spectral analysis of the linear operator L defined in (4.2) in the following way. Let us denote by $\mathcal{F}'\big(m\big)v$ and $\mathcal{F}''\big(m\big)(v, w)$ respectively the first and second Frechet derivatives of $\mathcal{F}$ computed at $m \in \mathcal{H}$ in the directions $v, w \in \mathcal{H}$. It is easy to see that for $\|m(\cdot)\|_\infty \leqslant c < 1$ the Frechet derivatives exist with

$$\mathcal{F}'(m)v = \frac{1}{2}\int_{\mathbb{R}}\left[\frac{1}{\beta}\tanh^{-1} m_1(x) - \theta - J * \tilde{m}(x)\right]v_1(x)\,\mathrm{d}x \\ +\frac{1}{2}\int_{\mathbb{R}}\left[\frac{1}{\beta}\tanh^{-1} m_2(x) + \theta - J * \tilde{m}(x)\right]v_2(x)\,\mathrm{d}x \tag{7.1}$$

and

$$\mathcal{F}''\big(m\big)(v, w) = \frac{1}{2}\int_{\mathbb{R}}\left[\frac{1}{\beta}\frac{w_1(x)}{(1 - m_1^2(x))} - J * \tilde{w}(x)\right]v_1(x)\,\mathrm{d}x \\ = \frac{1}{2}\int_{\mathbb{R}}\left[\frac{1}{\beta}\frac{w_2(x)}{(1 - m_2^2(x))} - J * \tilde{w}(x)\right]v_2(x)\,\mathrm{d}x. \tag{7.2}$$

Note that $\mathcal{F}'\big(\bar{m}\big) = 0$ and

$$\mathcal{F}''\big(\bar{m}\big)(v, w) = -\frac{1}{2\beta}\langle Lw, v\rangle_{\mathcal{H}}. \tag{7.3}$$

Hence the spectral properties of L govern the local convexity of the free energy. The analysis carried out in [3] can be adapted to our context obtaining the following results.

Theorem 7.1. *For any function $m \in \mathcal{K}(M, N)$ with $\|m_i(\cdot)\|_\infty \leqslant c < 1$, there are constants C_1 and C_2 depending only on β, θ, c, N and M so that*

$$C_1\|m - \bar{m}_a\|_{L^2}^2 \geqslant \mathcal{F}\big(m\big) - \mathcal{F}\big(\bar{m}_a\big) \geqslant C_2\|m - \bar{m}_a\|_{L^2}^2 \tag{7.4}$$

where $\bar{m}_a$ is any instanton which minimizes the L^2 distance to m among all instantons.

The bounds we obtain depend on certain regularity properties of the magnetization. This is necessarily the case since the free energy functional is not convex and has two distinct minima. Indeed, one can 'patch together' two instantons to obtain a 'plateau' of arbitrarily large width, but whose free energy is only a bit more than twice the free energy of the instanton. Hence, without some condition to control the width of such 'plateau', the inequality of theorem 7.1 could not hold.

Theorem 7.2. *For all initial data $m_0 \in \mathcal{K}(M, N)$ for some finite constants N and M, with $\|m_0(\cdot)\|_\infty \leqslant c < 1$, there is a constant C_3 depending only on β, θ, c, N and M so that the corresponding solution of (1.9) satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{F}\big(m(\cdot, t)) - \mathcal{F}(\bar{m}_{a(t)}(\cdot))) \leqslant -C_3\|m(t) - \bar{m}_{a(t)}\|_{L^2}^2 \tag{7.5}$$

for all $t > 0$ where $\bar{m}_{a(t)}$is any instanton minimizing the L^2 distance to $m(\cdot, t)$ among all instantons.

To apply these results we need the result stated in theorem 7.3 which says that for initial data in one of these sets $\mathcal{K}(M, N)$, there are other M' and N' such that the solution of (1.9) stays in $\mathcal{K}(M', N')$ for all time. This is what allows us to apply theorems 7.1 and 7.2 to the long-time behaviour of solutions of (1.9).

Theorem 7.3. *For all initial data $m \in \mathcal{K}(M, N)$, and for any finite M and N, there are finite M' and N' such that if $m(\cdot, t)$ is the solution of (1.9) with m as initial data, then for all $t \geqslant 0$*

$$m(\cdot, t) \in \mathcal{K}(M', N') \tag{7.6}$$

Thus, combining the theorems, one sees that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{F}(m(\cdot, t)) - \mathcal{F}(\bar{m}_{a(t)}(\cdot))) \leqslant -\frac{C_3}{C_1}(\mathcal{F}(m(\cdot, t)) - \mathcal{F}(\bar{m}_{a(t)}(\cdot)))$$

so that the free energy decays exponentially fast. The inequality (7.4) then implies that $\|m(t) - \bar{m}_{a(t)}\|^2_{L^2}$ decays to zero exponentially fast and therefore theorem 1.5 is proven.

Proof of theorem 7.1. We denote in the following the instanton $\bar{m}_a$ minimizing the L^2 distance simply by $\bar{m}$. We can represent

$$\begin{aligned}\mathcal{F}(m) - \mathcal{F}(\bar{m}) &= \int_0^1 \mathcal{F}'(\bar{m} + \tau(m - \bar{m}))(m - \bar{m})\,\mathrm{d}\tau \\ &= \int_0^1 \int_0^\tau \mathcal{F}''(\bar{m} + s(m - \bar{m}))(m - \bar{m}, m - \bar{m})\,\mathrm{d}s\,\mathrm{d}\tau.\end{aligned}$$

In order to get a lower bound for the last term above we expand $\mathcal{F}''(\bar{m} + s(m - \bar{m}))(m - \bar{m}, m - \bar{m})$ around $s = 0$ obtaining

$$\begin{aligned}&\mathcal{F}''(\bar{m} + s(m - \bar{m}))(m - \bar{m}, m - \bar{m}) \\ &\qquad = \mathcal{F}''(\bar{m})(m - \bar{m}, m - \bar{m}) + \mathcal{F}'''(v)(m - \bar{m}, m - \bar{m}, m - \bar{m})\end{aligned}$$

where $v = \bar{m} + s_0(m - \bar{m})$ for some s_0 between 0 and 1 by the mean value theorem. From (7.3) and the estimate (4.8) we obtain that

$$\mathcal{F}''\big(\bar{m})(m - \bar{m}, m - \bar{m}) = -\frac{1}{2\beta}\langle L(m - \bar{m}), m - \bar{m}\rangle_{\mathcal{H}} \geqslant \frac{1}{2\beta}\alpha\|P(m - \bar{m})\|^2_{\mathcal{H}} \tag{7.7}$$

where P is the orthogonal projection in $\mathcal{H}$ onto the orthogonal complement, in $\mathcal{H}$, of Null L. Thus we have that $(m - \bar{m}; \bar{m}')_{L^2} = 0$ since $\bar{m}$ is the profile which minimizes the L^2 distance from m but this does not imply $\langle m - \bar{m}, \bar{m}'\rangle_{\mathcal{H}} = 0$. To estimate from below $\|P(m - \bar{m})\|_{\mathcal{H}}$ we note that

$$\begin{aligned}\langle m - \bar{m}, \bar{m}'\rangle_{\mathcal{H}} &= \int_{\mathbb{R}} (m_1(x) - \bar{m}_1(x))\bar{m}_1'(x)\frac{\mathrm{d}x}{1 - \bar{m}_1^2(x)} \\ &\quad + \int_{\mathbb{R}} (m_2(x) - \bar{m}_2(x))\bar{m}_2'(x)\frac{\mathrm{d}x}{1 - \bar{m}_2^2(x)} \\ &= \int_{\mathbb{R}} (m_1(x) - \bar{m}_1(x))\bar{m}_1'(x)\frac{1 - \bar{m}_1^2(x) + \bar{m}_1^2(x)}{1 - \bar{m}_1^2(x)}\,\mathrm{d}x \\ &\quad + \int_{\mathbb{R}} (m_2(x) - \bar{m}_2(x))\bar{m}_2'(x)\frac{1 - \bar{m}_2^2(x) + \bar{m}_2^2(x)}{1 - \bar{m}_2^2(x)}\,\mathrm{d}x \\ &= \int_{\mathbb{R}} (m_1(x) - \bar{m}_1(x))\bar{m}_1'(x)\frac{\bar{m}_1^2(x)}{1 - \bar{m}_1^2(x)}\,\mathrm{d}x \\ &\quad + \int_{\mathbb{R}} (m_2(x) - \bar{m}_2(x))\bar{m}_2'(x)\frac{\bar{m}_2^2(x)}{1 - \bar{m}_2^2(x)}\,\mathrm{d}x \\ &\leqslant m^2_{\beta,1}\|m - \bar{m}\|_{\mathcal{H}}\|\bar{m}'\|_{\mathcal{H}}.\end{aligned} \tag{7.8}$$

Therefore, if $P^\perp$ is the orthogonal projection in $\mathcal{H}$ onto the span of $\bar{m}'$,

$$\|P^\perp(m - \bar{m})\|_{\mathcal{H}} \leqslant m^2_{\beta,1}\|m - \bar{m}\|_{\mathcal{H}}$$

and then

$$\|P(m-\bar{m})\|_{\mathcal{H}} \geqslant \sqrt{1-m_{\beta,1}^4}\,\|m-\bar{m}\|_{\mathcal{H}}. \tag{7.9}$$

Going back to (7.7) we have then

$$\mathcal{F}''(\bar{m})(m-\bar{m},m-\bar{m}) \geqslant \frac{1}{2\beta}\alpha\|P(m-\bar{m})\|_{\mathcal{H}}^2 \geqslant \frac{1}{2\beta}\alpha(1-m_{\beta,1}^4)\|m-\bar{m}\|_{\mathcal{H}}^2. \tag{7.10}$$

Now we need a lower bound on the term involving the third derivative of the free energy. But by direct computation,

$$\begin{aligned}|\mathcal{F}'''(v)(m-\bar{m},m-\bar{m},m-\bar{m})| &= \frac{1}{\beta}\left|\int_{\mathbb{R}}\frac{v_1}{(1-v_1^2)^2}(m_1(x)-\bar{m}_1(x))^3\,\mathrm{d}x\right.\\ &\quad\left.+\frac{1}{\beta}\int_{\mathbb{R}}\frac{v_2}{(1-v_2^2)^2}(m_2(x)-\bar{m}_2(x))^3\,\mathrm{d}x\right|\\ &\leqslant K\left[\int_{\mathbb{R}}|m_1(x)-\bar{m}_1(x)|^3\,\mathrm{d}x+\int_{\mathbb{R}}|m_2(x)-\bar{m}_2(x)|^3\,\mathrm{d}x\right]\\ &\leqslant \tilde{K}(\|m-\bar{m}\|_{\mathcal{H}})^{5/2}\end{aligned} \tag{7.11}$$

for some constant $K=K_{\beta,\theta,c}$ and $\tilde{K}=\tilde{K}_{\beta,\theta,c,N}$ where c is the bound on $\|m(\cdot)\|_\infty$. The last inequality in (7.11) is obtained since $\|f\|_\infty^2 \leqslant 2\|f\|_{L^2}\|f\|_{H^1}$ and elements in $\mathcal{K}(M,N)$ satisfy the bound $\|m-\bar{m}\|_{\mathrm{H}^1}\leqslant N$. If $\|m-\bar{m}\|_{\mathcal{H}}$ is sufficiently small, since $\frac{5}{2}>2$, we obtain that

$$\mathcal{F}''(\bar{m}+s(m-\bar{m}))(m-\bar{m},m-\bar{m}) \geqslant \alpha D\|m-\bar{m}\|_{\mathcal{H}}^2$$

where $D=D_{\beta,\theta,c}$ is a constant depending only on β, θ and c. Suppose $\epsilon>0$ is a number so that this is true whenever $\|m-\bar{m}\|_{\mathcal{H}}\leqslant\epsilon$. Then we have the desired lower bound for such m.

Next consider the functional Φ

$$\Phi(m)=\frac{\mathcal{F}(m)-\mathcal{F}(\bar{m})}{\|m-\bar{m}\|_{\mathcal{H}}^2} \tag{7.12}$$

on the set $\mathcal{K}(M,N)\cap\{m\,:\,\|m-\bar{m}\|_{\mathcal{H}}^2\geqslant\epsilon\}$. Since we are excluding in $\mathcal{K}(M,N)$ a neighbourhood of $\bar{m}$ we have

$$\inf\left\{\frac{\mathcal{F}(m)-\mathcal{F}(\bar{m})}{\|m-\bar{m}\|_{\mathcal{H}}^2} : m\in\mathcal{K}(M,N)\cap\{m\mid\|m-\bar{m}\|_{\mathcal{H}}^2\geqslant\epsilon\}\right\}=B>0.$$

From this we conclude that, for all $m\in\mathcal{K}(M,N)$,

$$\mathcal{F}(m)-\mathcal{F}(\bar{m})\geqslant\min(B,\alpha D)\|m-\bar{m}\|_{\mathcal{H}}^2. \tag{7.13}$$

The upper bound in theorem 7.1 follows by the same integration of second derivative simply using the boundedness of L. □

Proof of theorem 7.2. Recall from theorem 2.7 that $\mathcal{F}(m(\cdot,t))$ is well defined for all $t\geqslant 0$, it is differentiable with respect to t if $t>0$ and $(\mathrm{d}/\mathrm{d}t)\mathcal{F}(m(\cdot,t))=-I(m(\cdot,t))\leqslant 0$ where $I(h(t))$ is given in (2.10). Let $I'(m)v$ and $I''(m)(v,w)$ denote respectively the first and the second Frechet derivatives of I. With simple, however lengthy calculation we find that $I(\bar{m})=I'(\bar{m})=0$ and

$$I''(\bar{m})(v,v)=\frac{1}{\beta}\langle Lv,Lv\rangle_{\mathcal{H}}\geqslant\frac{1}{\beta}\alpha^2\|v\|_{\mathcal{H}}^2 \tag{7.14}$$

for all $v\in\mathrm{Null}(L)^\perp$. As in our previous analysis of $\mathcal{F}$, we have

$$\begin{aligned}I(m)&=\int_0^1 I'(\bar{m}+\tau(m-\bar{m}))(m-\bar{m})\,\mathrm{d}\tau\\ &=\int_0^1\int_0^\tau I''(\bar{m}+s(m-\bar{m}))(m-\bar{m},m-\bar{m})\,\mathrm{d}s\,\mathrm{d}\tau.\end{aligned}$$

One can directly compute the third derivative I''', and setting $v = \bar{m} + s(m - \bar{m})$, one finds that

$$|\mathcal{I}'''(v)(m - \bar{m}, m - \bar{m}, m - \bar{m})| \leqslant \bar{K}\,(\|m - \bar{m}\|_{\mathcal{H}})^{5/2} \tag{7.15}$$

for some constant $\bar{K} = \bar{K}_{\beta,\theta,c,N}$. As before, if $\|m - \bar{m}\|_{\mathcal{H}}$ is sufficiently small, from (7.14) we obtain that

$$I''(\bar{m} + s(m - \bar{m}))(m - \bar{m}, m - \bar{m}) \geqslant (d\alpha^2)\|m - \bar{m}\|^2_{\mathcal{H}}$$

where $d = d_{\beta,c,\theta}$ is a positive constant. Fix $\epsilon > 0$ such that this is true whenever $\|m - \bar{m}\|_{\mathcal{H}} \leqslant \epsilon$. Then we have the result for such m. Next consider the functional

$$m \to \frac{I(m)}{\|m - \bar{m}\|^2_{\mathcal{H}}} \tag{7.16}$$

on the set $\mathcal{K}(M, N) \cap \{m \mid \|m - \bar{m}\|^2_{\mathcal{H}} \geqslant \epsilon\}$. Since we are excluding in $\mathcal{K}(M, N)$ a neighbourhood of $\bar{m}$ we have that

$$\inf\left\{\frac{I(m)}{\|m - \bar{m}\|^2_{\mathcal{H}}} \mid m \in \mathcal{K}(M, N) \cap \{m \mid \|m - \bar{m}\|^2_{\mathcal{H}} \geqslant \epsilon\}\right\} = B_1 > 0.$$

We conclude from this that

$$I(m) \geqslant \min(B_1, (d\alpha^2))\|m - \bar{m}\|^2_{\mathcal{H}}$$

for all $m \in \mathcal{K}(M, N)$. This completes the proof. □

Proof of theorem 7.3. We begin with the bound on the the L^2 norm. In lemma 2.5 such a bound is proved for *compact time intervals* but this will not be enough for our purposes.

Observe that for any a and b

$$\bar{m}_b - (\bar{m}_b - m)_+ \leqslant m \leqslant \bar{m}_a + (m - \bar{m}_a)_+$$

where $(\cdot)_+$ denotes the positive part function.

Suppose first that there exists $\epsilon_0 > 0$, which will be chosen later, such that $Tm_\beta - \epsilon_0 \leqslant m(x) \leqslant m_\beta + \epsilon_0$. Since $m \in \mathcal{K}(M, N)$ there exist $\delta = \delta(\epsilon_0) > 0$ and b such that

$$\|(\bar{m}_b - m)_+\|_{L^2} \leqslant \delta. \tag{7.17}$$

In the same way, we can choose a so that

$$\|(m - \bar{m}_a)_+\|_{L^2} \leqslant \delta. \tag{7.18}$$

Next we use the fact that the time evolution specified by (1.9) is order preserving. Hence if $m(x, t)$ is the solution of (1.9) with initial data $m(x)$, and if $m_1(x, t)$ and $m_2(x, t)$ are the solutions of (1.9) with initial data $\bar{m}_b(x) - (\bar{m}_b(x) - m(x))_+$ and $\bar{m}_a(x) + (m(x) - \bar{m}_a(x))_+$ respectively, then

$$m_1(x, t) \leqslant m(x, t) \leqslant m_2(x, t)$$

almost everywhere in x for all $t \geqslant 0$, and hence, setting $\chi = (\chi_1, \chi_2)$ where we recall $\chi_1(x) = -m_{\beta,2}\mathbb{I}_{x\leqslant 0} + m_{\beta,1}\mathbb{I}_{x>0}$ and $\chi_2(x) = -m_{\beta,1}\mathbb{I}_{x\leqslant 0} + m_{\beta,2}\mathbb{I}_{x>0}$ we have

$$m_1(x, t) - \chi(x) \leqslant m(x, t) - \chi(x) \leqslant m_2(x, t) - \chi(x).$$

Then

$$\|m(\cdot, t) - \chi(\cdot)\|_{L^2} \leqslant \|m_1(\cdot, t) - \chi(\cdot)\|_{L^2} + \|m_2(\cdot, t) - \chi(\cdot)\|_{L^2}. \tag{7.19}$$

By theorem 4.2 there is a value of δ and therefore of ϵ_0 so that the conditions (7.17) and (7.18) imply that

$$\sup_{t\geqslant 0} \|m_1(\cdot, t) - \bar{m}_b\|_{L^2} \leqslant 1 \qquad \text{and} \qquad \sup_{t\geqslant 0} \|m_2(\cdot, t) - \bar{m}_a\|_{L^2} \leqslant 1. \tag{7.20}$$

This together with (7.19) yields for all $t \geqslant 0$

$$\|m(\cdot, t) - \chi(\cdot)\|_{L^2} \leqslant \|\bar{m}_a - \chi(\cdot)\|_{L^2} + \|\bar{m}_b - \chi(\cdot)\|_{L^2} + 2.$$

For any other $m \in \mathcal{K}(M, N)$, since lemma 5.2, we have that for any given $\epsilon_0 > 0$, there exists a finite time t_0, depending only on ϵ_0, β and θ such that $m(x, t_0) \leqslant m_\beta + \epsilon_0$ and $m(x, t_0) \geqslant Tm_\beta - \epsilon_0$. Since lemma 2.5, $\|m(\cdot, t_0) - \chi(\cdot)\|_{L^2} \leqslant \bar{M}$, where $\bar{M} > 0$ depends on ϵ_0 and M. We then choose the ϵ_0 such that (7.20) is satisfied. Applying the previous argument to $m(\cdot, t_0)$ we found the bound we seek.

We now prove an uniform bound on $\|m'(\cdot, t)\|_{L^2}$. Recall that in proposition 2.3 we proved that the Lipschitz norm of $m(\cdot, t)$, actually $\|m'(\cdot, t)\|_{L^\infty}$, is bounded uniformly in t. This result will be used here. Let $\bar{m}$ denote any fixed instanton, and define

$$\phi(x, t) = m(x, t) - \bar{m}(x).$$

Then rewriting (1.9) as an integral equation one obtains

$$\phi(x, t) = \mathrm{e}^{-t}\phi(x, 0) + \int_0^t \mathrm{e}^{s-t}[\tanh_\theta(\beta[(J \star \tilde{m})(x, s)] - \bar{m}(x)]\, \mathrm{d}s. \qquad (7.21)$$

By (1.10) the integral on the right of (7.21) is equal to

$$\phi(x, t) = \mathrm{e}^{-t}\phi(x, 0) + \int_0^t \mathrm{e}^{s-t}[\tanh_\theta \beta[(J \star \tilde{m})(x, s)] - \tanh_\theta \beta[(J \star \tilde{\bar{m}})(x)]]\, \mathrm{d}s. \qquad (7.22)$$

Then applying the identity $\tanh(A) - \tanh(B) = (1 - \tanh(A)\tanh(B))\tanh(A - B)$ to the integrand in (7.22) and differentiating, we obtain

$$[\tanh(\beta[(J \star \tilde{m})(x, s) + \theta] - \tanh(\beta[(J \star \tilde{\bar{m}})(x) + \theta]]$$
$$\leqslant (|\beta J \star \tilde{m}'| + |\beta J \star \tilde{\bar{m}}'|)|\beta J \star \phi_1| + 2|\beta J' \star \phi_1|$$

and similarly for the other component. By the uniform bound on $\|m'(\cdot, t)\|_{L^\infty}$ proved in proposition 2.3,

$$\sup_{t>0}\{|\beta J \star \tilde{m}'(\cdot, t)| + |\beta J \star \tilde{\bar{m}}'(\cdot)|\} \leqslant K$$

for some finite constant K. From the Young inequality $\|\beta J' \star \phi\|_{L^2} \leqslant \beta\|J'\|_{L^1}\|\phi\|_{L^2}$ we obtain

$$\|\phi'(\cdot, t)\|_{L^2} \leqslant \mathrm{e}^{-t}\|\phi'(\cdot, 0)\|_{L^2} + (1 - \mathrm{e}^{-t})\beta(K + 2\|J'\|_{L^1})\|\phi(\cdot, t)\|_{L^2}.$$

By the first part of the proof, $\|\phi(\cdot, t)\|_{L^2}$ is bounded uniformly in t, and hence we have the needed bound. □

8. Global Stability in $\mathcal{A}$

In this last section we prove theorem 1.6. Let $m(\cdot, t)$ be the solution of (1.9) with initial datum $m \in \mathcal{A}$. By corollary 2.4 there exists $m^\star \in C_b(\mathbb{R}) \times C_b(\mathbb{R})$ and a sequence $t_n \to \infty$ such that, uniformly on the compacts,

$$\lim_{n\to\infty} m(x, t_n) = m^\star(x). \qquad (8.1)$$

By lemma 5.3 and proposition 5.4 we have, see (5.14), that

$$\bar{m}(x - a_1(t_n)) - q(t_n) \leqslant m(x, t_n) \leqslant \bar{m}(x - a_2(t_n)) + q(t_n) \qquad (8.2)$$

then letting $t_n \to \infty$ we obtain that there are $a_1(\infty)$ and $a_2(\infty)$ such that

$$\bar{m}(x - a_1(\infty)) \leqslant m^\star(x) \leqslant \bar{m}(x - a_2(\infty)). \qquad (8.3)$$

Then since proposition 2.6 $\mathcal{F}(m^\star) < \infty$. Denote by $m^*(t, \cdot)$ the solution of (1.9) with initial datum m^*, then by theorem 2.7

$$\liminf_{t\to\infty} I(m^\star(\cdot, t)) = 0$$

otherwise $\mathcal{F}(m^\star(\cdot, t)) < 0$ for some t which by (1.1) is impossible. There are therefore a function $v \in C_b(\mathbb{R}) \times C_b(\mathbb{R})$ and a sequence $s_n \to \infty$ such that, uniformly on the compacts,

$$\lim_{s_n\to\infty} m^\star(x, s_n) = v(x) \tag{8.4}$$

and $I(v) = 0$. Then v satisfies (1.10) and by theorem 1.4 $v = \bar{m}_a$, for some a. Moreover, $m^\star(x, s_n)$ is trapped between two instantons since (8.3) and therefore, from (8.4), for any $\delta > 0$ we can find $s_n(\delta)$, such that for $s_n \geqslant s_n(\delta)$

$$\|m^\star(s_n) - \bar{m}_a\|_{L^2} \leqslant \delta. \tag{8.5}$$

From the local L^2 stability result, see theorem 4.2 we obtain that there exists $a(\infty)$ such that

$$\|m^\star(s_n + s) - \bar{m}_{a(\infty)}\|_{L^2} \leqslant c\, \mathrm{e}^{-\alpha s}. \tag{8.6}$$

Since

$$\|f\|_\infty^3 \leqslant 3\|f'\|_\infty \|f\|_{L^2}^2 \tag{8.7}$$

and proposition 2.3 we obtain

$$\|m^\star(s_n + s) - \bar{m}_{a(\infty)}\|_\infty \leqslant c\, \mathrm{e}^{-(2/3)\alpha s}. \tag{8.8}$$

Since (8.1) there exists t_n so that $\|m(t_n + s_n) - m^\star(s_n)\|_\infty \leqslant \epsilon$ therefore,

$$\|m(t_n + s_n + s) - \bar{m}_{a(\infty)}\|_\infty \leqslant \epsilon + c\mathrm{e}^{-(2/3)\alpha s}.$$

Provided we take s large enough

$$\|m(t_n + s_n + s) - \bar{m}_{a(\infty)}\|_\infty \leqslant 2\epsilon. \tag{8.9}$$

To obtain the exponential rate of convergence we perform a suitable surgery as in [13]. We briefly recall it. We will show that the solution starting from an initial datum in $\mathcal{A}$ becomes exponentially close to a function which is flat ouside a finite (t dependent) interval and equal to the asymptotic values m_β , Tm_β . This is close in L^2 to some instanton hence we have exponentially convergence in L^2 and in L^∞. From proposition 5.4 and the exponential convergence to m_β , Tm_β of $\bar{m}$ we obtain that there exists positive constants α and c such that

$$\sum_{i=1}^{2} |m_i(x, t) - \chi_i(x)| \leqslant c[\mathrm{e}^{-\alpha|x|} + \mathrm{e}^{-\alpha t}], \qquad (x, t) \in \mathbb{R} \times \mathbb{R}^+ \tag{8.10}$$

with χ_i defined in (1.14). To modify the tails of the solutions starting in $\mathcal{A}$ we define $\pi_+(x)$ a non-increasing $C^\infty(\mathbb{R})$ function equal to 1 when $x \leqslant 0$, to 0 when $x \geqslant 1$ and such that for $x \in [-\frac{1}{2}, \frac{1}{2}]$, $\pi_+(x + \frac{1}{2}) - \frac{1}{2}$ is antisymmetric. We also define, for all x, $\pi_-(x) := \pi_+(-x)$, so that $\pi_-(x - 1) + \pi_+(x) = 1$. Then, given $t \geqslant 0$, the (π, t) regularization of a function $f(\cdot) \in C_b(R)$, is the function $U_{f,t}(\cdot)$ defined for $x \geqslant 0$ as

$$U_{f,t}(x) = f(x)\pi_+(x - t) + m_\beta \pi_-\big(x - (t + 1)\big) \tag{8.11}$$

and for $x \leqslant 0$ as

$$U_{f,t}(x) = f(x)\pi_-(x + t) - Tm_\beta \pi_+\big(x + (t + 1)\big). \tag{8.12}$$

If f depends on t too, we consider for each t its (π, t) regularization, using the same symbol. Set

$$\psi(x, t) = m(x, t) - \mathrm{e}^{-t} m(x, 0) \tag{8.13}$$

and define $u(\cdot, t) = U_{\psi,t}(\cdot)$, i.e. the (π, t) regularization of $\psi(x, t)$. Then from (8.10)

$$\sum_{i=1}^{2} |u_i(x, t) - m_i(x, t)| \leqslant c[e^{-\alpha t} + e^{-t}] \qquad (x, t) \in \mathbb{R} \times \mathbb{R}^+. \tag{8.14}$$

Since (8.9) $\|u(t) - \bar{m}_{a(\infty)}\|_\infty \to 0$ as $t \to \infty$ and by construction $u_i(t) - \chi_i \in L^2$, for $i = 1, 2$. Therefore for all t large enough, there exists $a(t)$, see theorem 4.2, such that

$$\|u(t) - \bar{m}_{a(t)}\|_{a(t)} \leqslant c_0\, e^{-\alpha t} \tag{8.15}$$

and $a(t) \to a(\infty)$ when $t \to \infty$ exponentially fast. Since (8.7) and properties of $\bar{m}_{a(t)}$ we obtain that $\|u(t) - \bar{m}_{a(\infty)}\|_\infty \leqslant c_0\, e^{-\alpha t}$. By (8.14) we get

$$\|m(t) - \bar{m}_{a(\infty)}\|_\infty \leqslant c\, e^{-\alpha t}.$$

We denote by the same letter α the decay rate even though it changes line by line. Theorem is proved. □

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References

[1] Bates P W and Chmaj A 1999 An integral differential model for phase transitions: stationary solutions in higher space dimensions *J. Stat. Phys.* **95** 1119–39

[2] Bates P W, Fife P C, Ren X and Wang X 1997 Traveling waves in a convolution model for phase transitions *Arch. Rat. Mech. Anal.* **138** 105–36

[3] Carlen E A, Carvalho M C and Orlandi E 1998 Free energy inequalities and the rate of relaxation to instanton for interfaces profiles in Glauber dynamics *Nonlinear Diff. Eqns Appl.* **5** 205–18

[4] Carlen E A, Carvalho M C and Orlandi E 1999 Algebraic rate of decay for the excess free energy and stability of fronts for a non-local phase kinetics equation with a conservation law, I *J. Stat. Phys.* **95** 1069–117

[5] Carlen E A, Carvalho M C and Orlandi E 2000 Algebraic rate of decay for the excess free energy and stability of fronts for a non-local phase kinetics equation with a conservation law, II *Comm. P.D.E.* **25** 847–86

[6] Chen X 1997 Existence, uniqueness and asymptotic stability of travelling waves in nonlocal evolution equations *Adv. Diff. Eqns* 125–60

[7] De Masi A, Gobron T and Presutti E 1995 Travelling fronts in non local evolution equations *Arch. Rat. Mech. Anal.* **132** 143–205

[8] De Masi A, Orlandi E, Presutti E and Triolo L 1994 Glauber evolution with Kac potentials: I. Mesoscopic and macroscopic limits, interface dynamics *Nonlinearity* **7** 633–96

[9] De Masi A, Orlandi E, Presutti E and Triolo L 1994 Stability of the interface in a model of phase separation *Proc. R. Soc. Edinburgh A* **124** 1013–22

[10] De Masi A, Orlandi E, Presutti E and Triolo L 1994 Uniqueness of the instanton profile and global stability in non-local evolution equations *Rendiconti di Matematica.* Serie VII **14** 693–723

[11] Cassandro M, Orlandi E and Picco P 1999 Typical configurations for one dimensional random field Kac model *Ann. Prob.* **27** 1414–67

[12] Cassandro M, Orlandi E, Picco P and Vares M E Typical configurations for one dimensional random field Kac model: localization of phases, in preparation

[13] Fife P and McLeod J B 1977 The approach of solutions of nonlinear diffusion equations to travelling front solutions *Arch. Rat. Mech. Anal.* **75** 335–61

[14] Ermentrout G B and McLeod J B 1993 Existence and uniqueness of travelling waves for a neural network *Proc. Rat. Soc. Edinburgh* A **123** 461–78

[15] Kato T 1966 *Pertubation Theory for Linear Operators* (New York: Springer)
[16] Orlandi E and Triolo L 1997 Travelling fronts in nonlocal models for phase separation in an external field *Proc. R. Soc. Edinburgh* A **127** 823–35
[17] Penrose O 1991 A mean field equation of motion for the dynamic Ising model *J. Stat. Phys.* **63** 975–86
[18] Presutti E 1999 *From Statistical Mechanics towards Continuum Mechanics (Lecture Notes)* Max-Planck Institute, Leipzig