Tutorial 2

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1. Find the minimal polynomial of the following complex numbers:

   (a) $\sqrt[4]{2},\ \sqrt[5]{4},\ \sqrt[3]{2}+1,\ \dfrac{\sqrt[4]{2}}{\sqrt{2}+1}$ over $\mathbb{Q}$.

   (b) $\sqrt[3]{2}+\sqrt{2}$ over $\mathbb{Q}(\sqrt{2})$.

   (c) $\sqrt{2}+\sqrt{5}$ over $\mathbb{Q}(\sqrt{10})$.

   (d) ζ_{16} over $\mathbb{Q}(i)$.

2. Calculate the degree of the following extensions:

$$\mathbb{Q}\subset\mathbb{Q}(\sqrt{7},\sqrt{13}),\quad \mathbb{Q}\subset\mathbb{Q}(\sqrt{12},\sqrt{15}),\quad \mathbb{Q}\subset\mathbb{Q}(\sqrt{2},i),\quad \mathbb{Q}\subset\mathbb{Q}(\sqrt{3},\sqrt{5},\sqrt{60}),$$
$$\mathbb{Q}(\sqrt{3})\subset\mathbb{Q}(\sqrt{3},\sqrt{6}),\quad \mathbb{Q}\subset\mathbb{Q}(\frac{2}{3}\sqrt[3]{2},\sqrt[3]{2}-1),\quad \mathbb{Q}(\pi)\subset\mathbb{Q}(\pi,\sqrt[3]{7}),\quad \mathbb{Q}(\pi)\subset\mathbb{Q}(\pi^4).$$

3. Calculate the minimal polynomial of $\cos\frac{2\pi}{7}$ and $\cos\frac{2\pi}{9}$ over $\mathbb{Q}$.

4. Describe the lattice of the subfields of $\mathbb{Q}(\zeta_5),\mathbb{Q}(\zeta_7),\mathbb{Q}(\zeta_9),\mathbb{Q}(\zeta_{11})$.

5. Let $\mathbb{Q}\subseteq K$ be a field extension such that $[K:\mathbb{Q}]=10$. Explain why $\sqrt[3]{2}\notin K$.

6. Given $f(x)=x^5-x^3-6x-\frac{1}{3}x^4+\frac{1}{3}x^2+2\in\mathbb{Q}[X]$. Which of the following rings is an integral domain and/or a field:

$$\frac{\mathbb{Q}[X]}{(f(X))},\quad \frac{\mathbb{R}[X]}{(f(X))},\quad \frac{\mathbb{C}[X]}{(f(X))}\quad ?$$

   (Repeat the exercise for $f(x)=x^3-5x+3$.)

7. Check whether $\pi+\frac{1}{\pi}$ is algebraic or trancendental over $\mathbb{Q}$ and $\mathbb{Q}(\pi^2)$.

8. Let $\alpha=\sqrt[4]{3}$ and $K=\mathbb{Q}(\alpha)$.
   (a) Determine the minimal polynomial of α.
   (b) Verify that $\mathbb{Q}(\sqrt{3})\subset K$ and determine the minimal polynomial of α over $\mathbb{Q}(\sqrt{3})$.
   (c) Put $\beta=\sqrt{3}+\sqrt[4]{27}-2$, why $\beta\in K$?

9. Let $f(x)=x^3-5x-1\in\mathbb{Q}[x]$.
   (a) Check whether $\frac{\mathbb{Q}[X]}{(f(X))}$ is an integral domain and/or a field.
   (b)Let $\alpha\in\mathbb{C}:f(\alpha)=0$. Determine $[\mathbb{Q}(\alpha):\mathbb{Q}]$ and describe $\mathbb{Q}(\alpha)$.

(c) Find the inverse of $\alpha + 1, \quad \alpha^2 + \alpha + 1, \quad 2 + \alpha \quad$ and $\quad \alpha^3 - 5\alpha \quad$ in $\mathbb{Q}(\alpha)$.

10. Determine the splitting field in $\mathbb{C}$ of the following polynomials and calculate their degree:
(a)$(x^2 - 5)(x^3 - 7)$ over $\mathbb{Q}$.
(b)$x^4 + 30x^2 + 45$ over $\mathbb{Q}$.
(c)$x^4 - x^2 + 5$ over $\mathbb{Q}$.