Family Name .... *Name* .... *Student ID(Matricola):* ....

Solve the problems adding to the replies short and essential explenations. *Please write the solutions in the designed areas. NO EXTRA SHEETS WILL BE ACCEPTED.* 1 Problem = 4 marks. Duration: 2 hours. No questions allowed in the first hour and in the last 20 minutes.

1	2	3a	3b	3c	3d	4	5	6	TOTAL

1. Calculate the continued fraction expansion of $\sqrt{87}$

2. An irrational number has continued fraction expansion $[\overline{2,5}]$. Compute it.

3. Solve the following problems:

a. Show that if p is a prime number such that $p = x^2 + 5y^2$ for suitable $x, y \in \mathbf{Z}$, then either $p \equiv 1 \bmod 20$ or $p \equiv 9 \bmod 20$. **hint:***study the identity modulo 5 and modulo 4. Then apply Chinese Remainder Theorem*

b. Prove that for any prime p, there exists $k \in \{1, 2, 3, 4, 5\}$ such that $kp = x^2 + 5y^2$ for some $x, y \in \mathbf{Z}$.
hint: *apply the pigeon holes principle*

c. prove that, if $x, y \in \mathbf{Z}$, then $x^2 + 5y^2 \not\equiv 2, 3, 7, 18 \bmod 20$ and deduce that if p is prime with $p \equiv 1 \bmod 20$ or $p \equiv 9 \bmod 20$ then either $p = x^2 + 5y^2$ or $4p = x^2 + 5y^2$.
hint: *first do some computation and then apply 3.b observing that if* $5 \mid x^2 + 5y^2$*, then* $5 \mid x$*.*

d. prove that if $4 \mid x^2 + 5y^2$, then $2 \mid \gcd(x, y)$. Finally deduce that if p is prime,

$$p \equiv 1, 9 \bmod 20 \qquad \Longleftrightarrow \qquad p = x^2 + 5y^2,\ \exists x, y \in \mathbf{Z}.$$

4. Show that if $\alpha \in \mathbf{R}, 0 \leq \alpha \leq 1$, then it exists a set $S \subset \mathbf{N}$ which has natural density α.
hint: *Consider the sequence* $([\beta n])_{n \in \mathbf{N}}$ *for a suitable* $\beta \in \mathbf{R}$.

5. Let $a, b \in \mathbf{N}$. Compute the number of ways to express $6^a \cdot 65^b$ as the sum of two squares.

6. State Merten's Theorems on the distribution of primes and give some ideas on their proofs.