Family Name .... *Name* .... *Student ID(Matricola):* ....

Solve the problems adding to the replies short and essential explenations. *Please write the solutions in the designed areas. NO EXTRA SHEETS WILL BE ACCEPTED.* 1 Problem = 4 marks. Duration: 2 hours. No questions allowed in the first hour and in the last 20 minutes.

1	2	3	4	5	6	7	8	TOTAL

1. Compute $\gcd(1332, 1406)$ using the Extended Euclidean Algorithm and deduce a Bezout Identity.

2. After having stated and proved a formula to compute the $p-$adic valuation of $n!$, apply it to compute the 5–adic valuation $v_5(121!)$.

3.

a Show that a positive integer has an odd number of positive divisors if and only if it is a perfect square

b Show that $2\varphi(n) = \begin{cases} \varphi(2n) & \text{if } n \text{ is even} \\ \varphi(4n) & \text{if } n \text{ is odd} \end{cases}$

c Determine all integers n such that $\sigma(n) = 7$.

4. After having stated Gauss Theorem of existence of primitive roots modulo integers, compute all primitive roots modulo 4394.

5. Find all solutions of $x^3 - x + 1 \equiv 0 (\mathrm{mod} 140)$ by using the Chinese remainder Theorem.

6. After having stated the important properties of the Legendre–Jacobi Symbols, compute $\left(\frac{1332}{1407}\right)_J$.

7. Prove that if n is an odd positive integer, then

$$\left(\frac{13}{n}\right)_{\mathrm{J}} = \begin{cases} 1 & \text{if } n \equiv \pm 1, \pm 3, \pm 4 \bmod 13 \\ 0 & \text{if } 13 \mid n \\ -1 & \text{if } n \equiv \pm 2, \pm 5, \pm 6 \bmod 13. \end{cases}$$

8. Let p be a prime, let $a \in \{0, \cdots, p-1\}$ and let $N_3(p, a)$ be the number of integers $x \in \{0, \ldots, p-1\}$ such that $x^3 \equiv a \bmod p$. Show that
   a) If $p \equiv 2 \bmod 3$, then $N_3(p, a) = 1$
   b) If $p \equiv 1 \bmod 3$, then $N_3(p, a) \in \{0, 3\}$
   c) $N_3(3, a) = 1$