Exercises-Analytic Number Theory Course

25/04/2013

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1. Show that

   (a) χ_ω is well defined.

   (b) χ_ω has period q.

   (c) χ_ω is totally multiplicative.

   (d) The set $\{\chi_\omega : \mathbb{Z} \to \mathbb{C}, \omega \in \mathbb{C}, \omega^{q-1} = 1\}$ does not depend on primitive root g (mod q).

   (e)
   $$\sum_{w}^{q} \chi_\omega(a)^{-1}\chi_\omega(n) = \begin{cases} 0 & \text{if } a \not\equiv n (\bmod q) \\ q-1 & \text{otherwises.} \end{cases}$$

   (f)
   $$\sum_{a \in \mathbb{Z}/q\mathbb{Z}} \chi_\omega(a) = \begin{cases} 0 & \text{if } \omega \neq 1 \\ q-1 & \text{otherwis} \end{cases}$$

2. Explore Davenport page 9 line 13-26.

3. Explore Davenport page 11 line 6 to end.

4. Prove 3 equalities of $\Gamma(s)$ function.

   (a) $\Gamma(s)\Gamma(1-s) = (\pi)/(\sin(\pi * s))$

   (b) $\Gamma(s)\Gamma(s+1/2) = (\sqrt{\pi})2^{1-2s}\Gamma(2s)$

   (c) $\Gamma(s)\Gamma(1/2(1-s)) = (1/\sqrt{\pi})2^{1-s}(\cos(\pi * s/2))\Gamma(s)$

5. Give another proof of formula of Wallis.

6. If $\alpha > 3$ and $n \in \mathbb{Z}/2^\alpha\mathbb{Z}$, there exists unique u, v where $v \in \mathbb{Z}/2\mathbb{Z}, v \in \mathbb{Z}/2^{\alpha-2}$ and $n \equiv (-1)^u * 5^v$.

7. Show that $\Gamma(1/2) = \sqrt{\pi}$.

8. Show that, for all $x > 1$, $\omega(x) < 1/2 * x^{-1/2}$.

9. Prove the 3 formula of Merten's.

   (a) $\sum_{p \le x} \log p/p = \log x + O(1)$

   (b) $\sum_{p \le x} 1/p = \log\log x + A + O(1/\log x)$

   (c) $\prod_{p \le x} \log(1 - 1/p \sim \exp^{-\delta} / \log x$

10. Compute $\forall n \in \mathbb{N}$, $\zeta(-n)$.

11. Write a report on the identity on the last line of page 67 (Davenport): Let $\chi(\mod q)$ be an *imprimitive* character induced by the *primitive* character $\chi_1(\mathrm{mod} q_1)$. Then

$$\tau(\chi) = \mu(q/q_1)\chi_1(q/q_1)\tau(\chi_1).$$

12. Write a one page report on Hurwitz Zeta Function.