TN510 - Introduction to Analytic Number Theory - 2^{nd} term

Exercises Sheet

Exercise 1. Write down all character $\bmod 6, 7, 8, 9$.

Exercise 2. Show that $\sum_{n\leq x} \varphi(x) = cx^2 + O(x)$ and compute c.

Exercise 3. Show that,

- $\cos(z)=\frac{1}{2}\left(\exp(iz) + \exp(-iz)\right)$.
- $\sin(z)=\frac{1}{2i}\left(\exp(iz) - \exp(-iz)\right)$.
- $\exp(x + iy)=\exp(x)(\cos(y) + i\sin(y))$.
- $\exp(in\alpha) = \cos(n\alpha) + i\sin(n\alpha)$.

Exercise 4. Show that $\log(z) = \log|z| + i\arg(z)$, where $|1 - z| < 1$.

Exercise 5. Show that $\displaystyle\int_0^\infty \frac{1}{1 + x^n} = \frac{\pi}{n\sin\pi/n}$.

Exercise 6. Determine the Hadamard factorisation of

- $\cos(\pi z)$,
- $e^z - 1$.

Exercise 7. Show that for $|z| < 1$ we have, $\displaystyle\prod_{n=1}^{\infty} e^{\frac{z^n}{n}} = \sum_{n=0}^{\infty} z^n$

Exercise 8. Determine a sequence $(a_n)_n \in \mathbb{C}$ such that $\prod_n (1 + a_n)$ convergent but $\sum_n a_n$ divergent.

Exercise 9. Determine a sequence $(a_n)_n \in \mathbb{C}$ such that $\sum_n a_n$ convergent but $\prod_n (1 + a_n)$ divergent.

Exercise 10. let f be an (integral) entire function of finite order. Deduce form the characterisation of nowhere vanishing functions that if f misses two distinct values then f is constant (does the conclusion of the theorem remain valid under the hypothesis that f is an entire function?).

Exercise 11. Compute $\operatorname*{Res}_{z=-n}\Gamma(z), \forall n \in \mathbb{N}$.

Exercise 12. Show that $\frac{\Gamma'}{\Gamma}(3/2) = -\gamma + 2 - \log 2$, where γ Euler constant.

Exercise 13. Let $y > 1, c > 0$ and $T > 0$. Show that

$$\left|\frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{y^s}{d}s - 1\right| < y^c \min\left(1, \frac{1}{T|\log y|}\right).$$