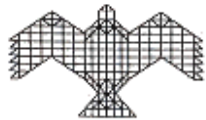


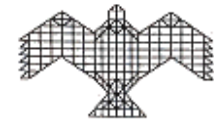
Exponential Sums and Enumeration of Permutation Polynomials

Francesco Pappalardi

Conference on Zeta Functions in honor of
Prof. K. Ramachandra on his 70th birthday



National Institute of Advanced Studies
NIAS



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$$f \equiv f_\sigma \pmod{x^q - x}$$



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$$L(x) = \sum_{s=0}^{r-1} \alpha_s x^{q^s} \text{ is a PP } \Leftrightarrow \det(\alpha_{i-j}^{q^j}) \neq 0$$



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S. Konyagin, FP (2002), P. Das (2002)

$$|\#M_q - (q - 1)!| \leq \sqrt{2e/\pi} q^{q/2}$$



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$$\times \quad m_{\mathcal{C}}(q) = \#\{\sigma \in \mathcal{C}, \partial f_\sigma = q - c_{\mathcal{C}}\} \quad (\text{minimal degree})$$



Another way to count

$$q - c_\sigma \leq \partial f_\sigma \leq q - 2 \quad \Leftrightarrow \quad \sigma \in \mathcal{S}(\mathbb{F}_q) \setminus \{id\} \quad \text{and } q > 2$$

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Theorem *C. Malvenuto, FP (2002)*

 If $\mathcal{C} \neq [2], [3], [2\ 2]$, then

$$M_{\mathcal{C}}(q) = \frac{\#\mathcal{C}}{q} + O_{\mathcal{C}}\left(\frac{1}{q^2}\right) \quad \text{if } \text{char } \mathbb{F}_q \rightarrow \infty$$

 Explicit Formulas for $M_{\mathcal{C}}(q)$ if $c_{\mathcal{C}} \leq 6$



Formulas for non-maximal degree PP



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$$\begin{aligned}
 M_{[4]}(q) &= \frac{1}{4} q(q-1)(q-5-2\eta(-1)-4\eta(-3)) \\
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 &\quad - [148 + 100\eta(-1) + 24\eta(-2) + 44\eta(-3) + 40\eta(-7)]) \\
 M_{[3\ 3]}(q) &= \frac{q(q-1)}{18} (q^3 - 13q^2 + [62 + 9\eta(-1) + 4\eta(-3)]q - [150 + 99\eta(-1) + 42\eta(-3) + 72\eta(-7)]) \\
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Theorem *C. Malvenuto, FP (to appear)*

- If $q \equiv 1 \pmod k$ then $m_{[k]}(q) \geq \frac{\varphi(k)}{k} q(q-1)$
- If $\text{char}(\mathbb{F}_q) \geq 2 \cdot 3^{\lfloor k/3 \rfloor - 1}$ then $m_{[k]}(q) \leq \frac{(k-1)!}{k} q(q-1)$



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Let $\alpha = (e - 2)/3e = 0.08808 \dots$ and $d < \alpha q$. Then

$$\left| \mathcal{N}_d - \frac{q!}{q^d} \right| \leq 2^d d q^{2+q-d} \binom{q}{d} \left(\frac{2d}{q-d} \right)^{(q-d)/2}.$$

It follows that

$$\mathcal{N}_d \sim \frac{q!}{q^d}$$

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Note: The best possible value for α in the theorem is 0.5 . In fact $\partial f_\sigma \neq (q-1)/2$ if q is odd. Therefore

$$\mathcal{N}_{(q-1)/2} = 0$$



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Need to evaluate n_S . Let $e_p(u) = e^{\frac{2\pi i u}{p}}$ and $\text{Tr}(\alpha) \in \mathbb{F}_p$ be the trace of $\alpha \in \mathbb{F}_q$.



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The proof of the Theorem is an evolution of this method.



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Lemma 2. *If $d \leq q/3$, $P \in \mathbb{F}_q[x]$, $\partial(P) = d$ and $\mu(P) \geq \mu \geq 2$, then*

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with root $\alpha \approx 0.03983478542171344979957755901$



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Planning to adopt the method of for arbitrary k_1, \dots, k_d where d grows *slowly* as $q \rightarrow \infty$

