Factoring integers, Producing primes and the RSA cryptosystem

Harish-Chandra Research Institute

Allahabad (UP), INDIA

February, 2005

$$RSA_{2048} = 25195908475657893494027183240048398571429282126204$$

032027777137836043662020707595556264018525880784406918290641249
515082189298559149176184502808489120072844992687392807287776735
971418347270261896375014971824691165077613379859095700097330459
748808428401797429100642458691817195118746121515172654632282216
869987549182422433637259085141865462043576798423387184774447920
739934236584823824281198163815010674810451660377306056201619676
256133844143603833904414952634432190114657544454178424020924616
515723350778707749817125772467962926386356373289912154831438167
899885040445364023527381951378636564391212010397122822120720357

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http://www.rsa.com/rsalabs/challenges/factoring/numbers.html/

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Theorem. If $a\in\mathbb{N}\quad\exists!\ p_1<p_2<\cdots<p_k$ *primes* s.t. $a=p_1^{\alpha_1}\cdots p_k^{\alpha_k}$

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Regrettably: RSAlabs believes that factoring in one year requires:

number	computers	memory
RSA_{1620}	1.6×10^{15}	120 Tb
RSA_{1024}	$342,000,000$	170 Gb
RSA_{760}	215,000	4Gb.

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Challenge Number	Prize ($US)
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RSA_{896}	$75,000
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RSA_{1536}	$150,000
RSA_{2048}	$200,000

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Challenge Number	Prize ($US)	Status
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RSA_{640}	$20,000	Not Factored
RSA_{704}	$30,000	Not Factored
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Carissan's ancient Factoring Machine

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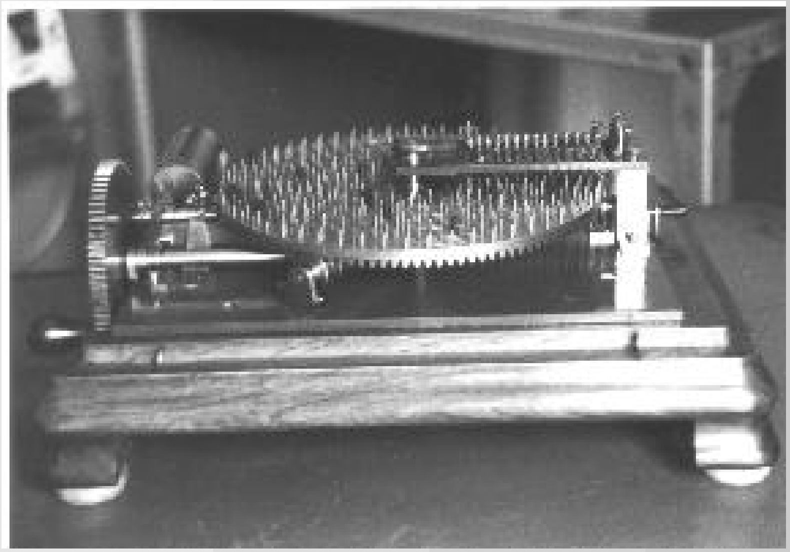

Figure 1: Conservatoire Nationale des Arts et Métiers in Paris

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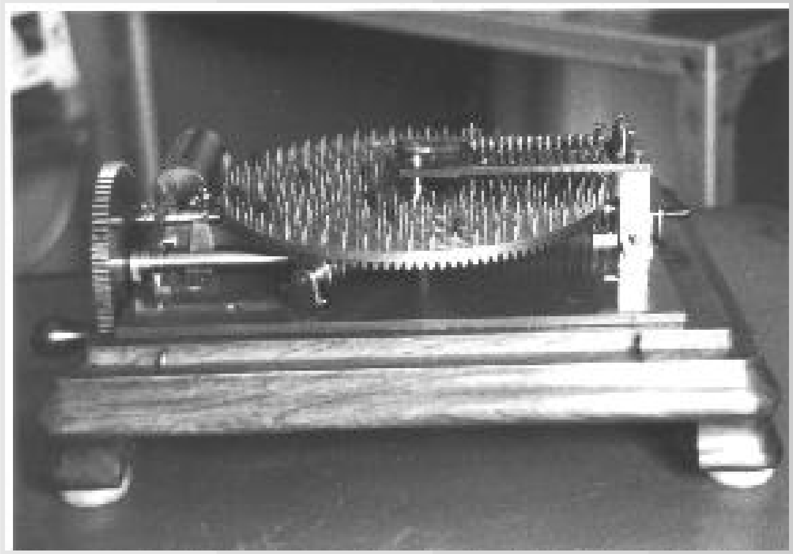

Figure 1: Conservatoire Nationale des Arts et Métiers in Paris

`http://www.math.uwaterloo.ca/ shallit/Papers/carissan.html`

Figure 2: Lieutenant Eugène Carissan

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$225058681 = 229 \times 982789$	2 minutes
$3450315521 = 1409 \times 2418769$	3 minutes
$3570537526921 = 841249 \times 4244329$	18 minutes

Contemporary Factoring

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D.Atkins, M. Graff, A. Lenstra, P. Leyland

$$
\begin{aligned}
RSA_{129} = {} & 114381625757888867669235779976146612010218296721242362562561842935706 \\
& 935245733897830597123563958705058989075147599290026879543541 = \\
= {} & 3490529510847650949147849619903898133417764638493387843990820577 \times \\
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&\quad 88934784717997257891267332497625752899781833797076537244027146743531593354333897 =\\
&= 102639592829741105772054196573991675900716567808038066803341933521790711307779 \times\\
&\quad 106603488380168454820927220360012878679207958575989291522270608237193062808643
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$$RSA_{576} = 188198812920607963838697239461650439807163563379417382700763356422988859715234 665485319060606504743045317388011303396716199692321205734031879550656996221305168759307650257059 =$$
$$= 398075086424064937397125500550386491199064362342526708406385189575946388957261768583317 \times 472772146107435302536223071973048224632914695302097116459852171130520711256363590397527$$

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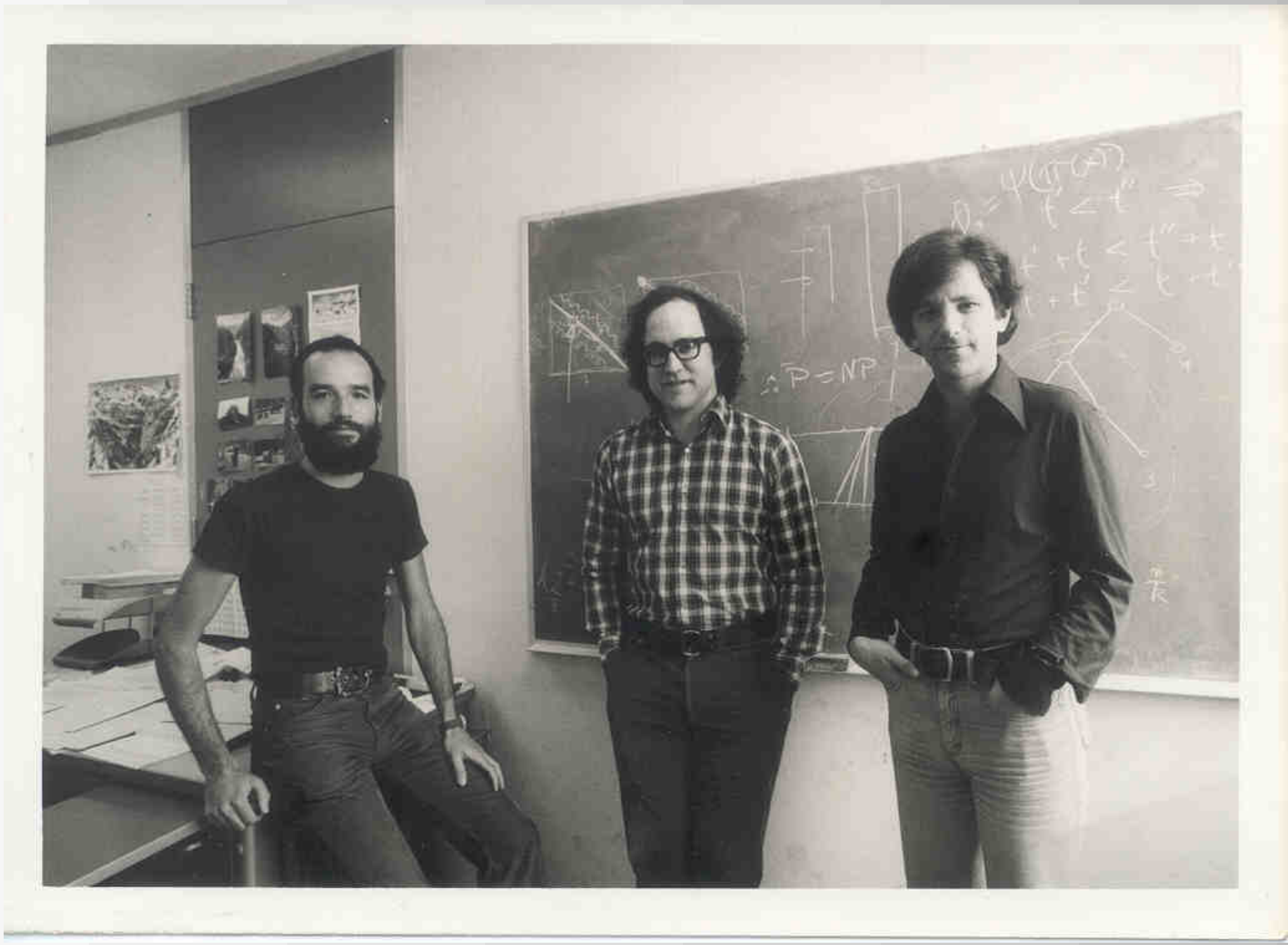
All: "sub–exponential running time"

RSA



Adi Shamir, Ron L. Rivest, Leonard Adleman (1978)

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$$\boxed{\begin{array}{ccc} A\ (Alice) & \longrightarrow & B\ (Bob) \\ & \uparrow & \\ & C\ (Charles) & \end{array}}$$

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1. KEY GENERATION — Bob has to do it
2. ENCRYPTION — Alice has to do it
3. DECRYPTION — Bob has to do it
4. ATTACK — Charles would like to do it

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✍

✍

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-
-
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Problem: How does Bob do all this?- We will go came back to it!

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$$\texttt{Sukumar} \leftrightarrow 19 \cdot 26^6 + 21 \cdot 26^5 + 11 \cdot 26^4 + 21 \cdot 26^3 + 12 \cdot 26^2 + 1 \cdot 26 + 18 = 6124312628$$

Note. Better if texts are not too short. Otherwise one performs some *padding*

Alice: Encryption

Represent the message $\mathcal{P}$ as an element of $\mathbb{Z}/M\mathbb{Z}$

(for example) $\boxed{A \leftrightarrow 1 \quad B \leftrightarrow 2 \quad C \leftrightarrow 3 \quad \ldots \quad Z \leftrightarrow 26 \quad AA \leftrightarrow 27 \ldots}$

`Sukumar` $\leftrightarrow 19 \cdot 26^6 + 21 \cdot 26^5 + 11 \cdot 26^4 + 21 \cdot 26^3 + 12 \cdot 26^2 + 1 \cdot 26 + 18 = 6124312628$

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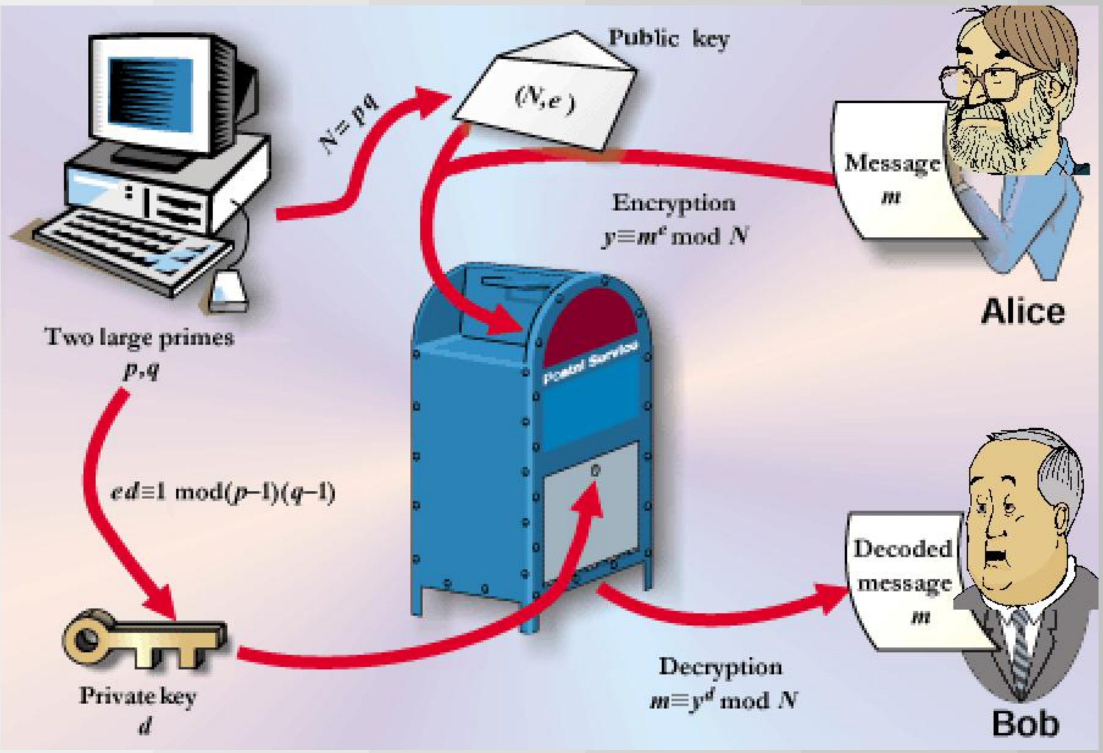
Example(cont.):$d = 65537^{-1} \bmod \varphi(9049465727 \cdot 8789181607) = 57173914060643780153$

$D($`JGEBNBAUYTCOFJ`$) =$

$25439695120356558116^{57173914060643780153} (\mathrm{mod}\, 79537397720925283289) =$ `Sukumar`

RSA at work



Public key
(N,e)
N = pq
Message
m
Encryption
y≡m^e mod N
Alice
Two large primes
p,q
Postal Service
ed≡1 mod(p−1)(q−1)
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Repeated squaring algorithm

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To encrypt with $e = 2^{16}+1$, only 17 operations in $\mathbb{Z}/M\mathbb{Z}$ are enough

Key generation

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Problem. Produce a random prime $p \approx 10^{100}$

	Probabilistic algorithm (type Las Vegas)
1.	Let $p = \textsc{Random}(10^{100})$
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False Metropolitan Legend: Check primality is equivalent to factoring

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Therefore

$$0.0043523959267 < Prob\left((\textsc{Random}(10^{100}) = \texttt{prime}\right) < 0.004371422086$$

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$$2^{93960} \equiv 1 \pmod{93961} \quad \text{but} \quad 93961 = 7 \times 31 \times 433$$

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④ If m is composite $\Longrightarrow$ $Prob(m \text{ PSPF in base } a) \leq 0,25$

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MILLER RABIN ALGORITHM WITH k ITERATIONS
$N = (m-1)/2$
`for` $j = 0$ `to` k `do` $a =$`Random`(m)
`if` $a^N \not\equiv \pm 1 \bmod m$ `then OUPUT=(`m `composite): END`
`endfor OUTPUT=(`m `prime)`

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`for` $j = 0$ `to` k `do` $a =$`Random`(m)
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Miller–Rabin primality test

Let $m \equiv 3 \bmod 4$

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In the real world, software uses Miller Rabin with $k = 10$

Deterministic primality tests

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Theorem. (Miller, Bach) If m is composite, then

$$\textbf{GRH} \Rightarrow \exists a \leq 2\log^2 m \text{ s.t. } a^{(m-1)/2} \not\equiv \pm 1 \pmod{m}.$$

(i.e. *m is not SPSP in base a.*)

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Deterministic Polynomial time algorithm

It runs in $O(\log^5 m)$ operations in $\mathbb{Z}/m\mathbb{Z}$.

Certified prime records

Certified prime records

- ✎
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- ✎
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Certified prime records

- $2^{20996011} - 1$, 6320430 digits (discovered in 2003)
-
-
-
-
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- $2^{20996011} - 1$, 6320430 digits (discovered in 2003)
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- $1176694^{131072} + 1$, 795695 digits (discovered in 2003)

The AKS deterministic primality test

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Department of Computer Science & Engineering,
I.I.T. Kanpur, Agost 8, 2002.

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http://www.cse.iitk.ac.in/news/primality.html

How does the AKS work?

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Theorem. (AKS) Let $n \in \mathbb{N}$. Assume q, r primes, $S \subseteq \mathbb{N}$ finite:

- $q|r-1$;
- $n^{(r-1)/q} \bmod r \notin \{0, 1\}$;
- $\gcd(n, b-b') = 1, \quad \forall b, b' \in S$ (distinct);
- $\binom{q+\#S-1}{\#S} \geq n^{2\lfloor\sqrt{r}\rfloor}$;
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Then n is a power of a prime

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Many simplifications and improvements: Bernstein, Lenstra, Pomerance.....

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  The two problems are polynomially equivalent

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- **Private key (or symmetric)**
  - Lucifer
  - DES
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- **Public key**
  - RSA
  - Diffie–Hellmann
  - Knapsack
  - NTRU