Finite fields, Permutation Polynomials. Computational aspects with applications to public key cryptography

King Fahd University of Petroleum and Minerals

Dhahran, Saudi Arabia

Workshop on Industrial Mathematics

March 1, 2004

Private key versus Public Key

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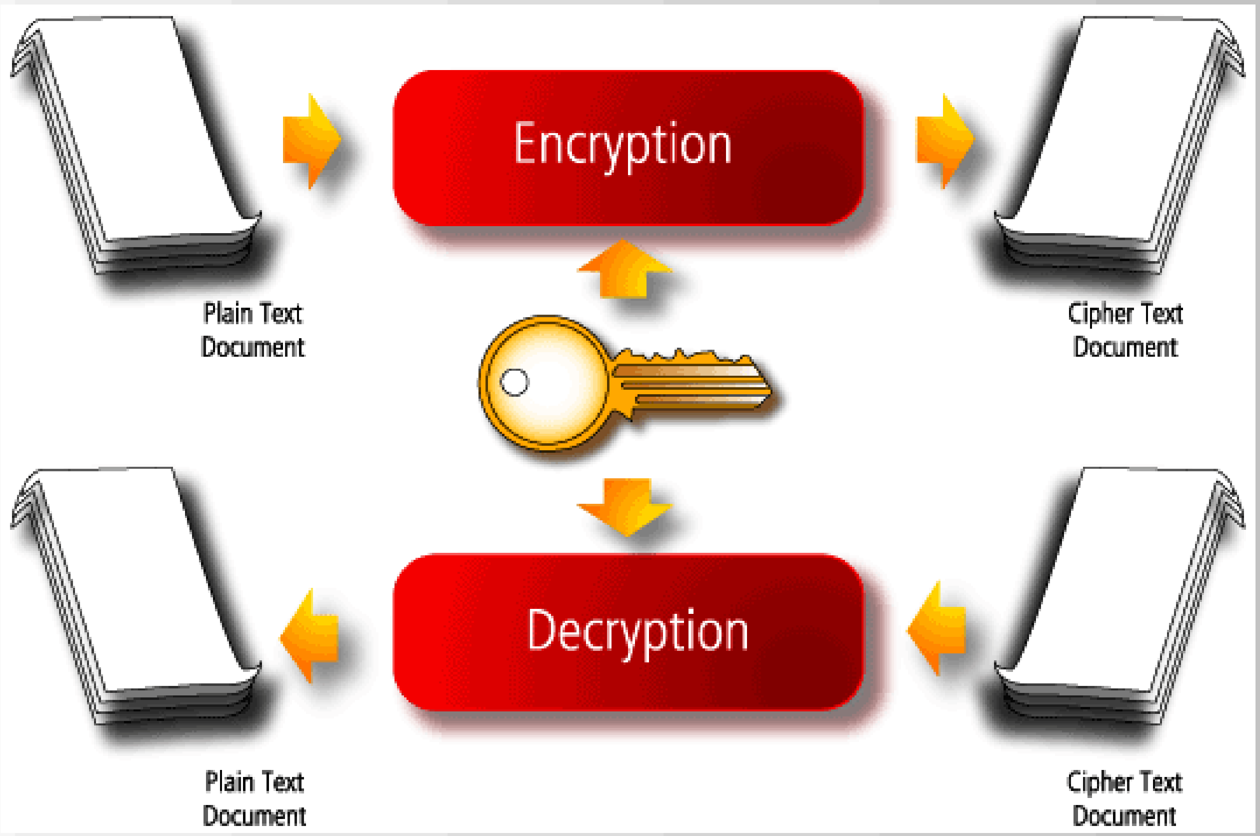

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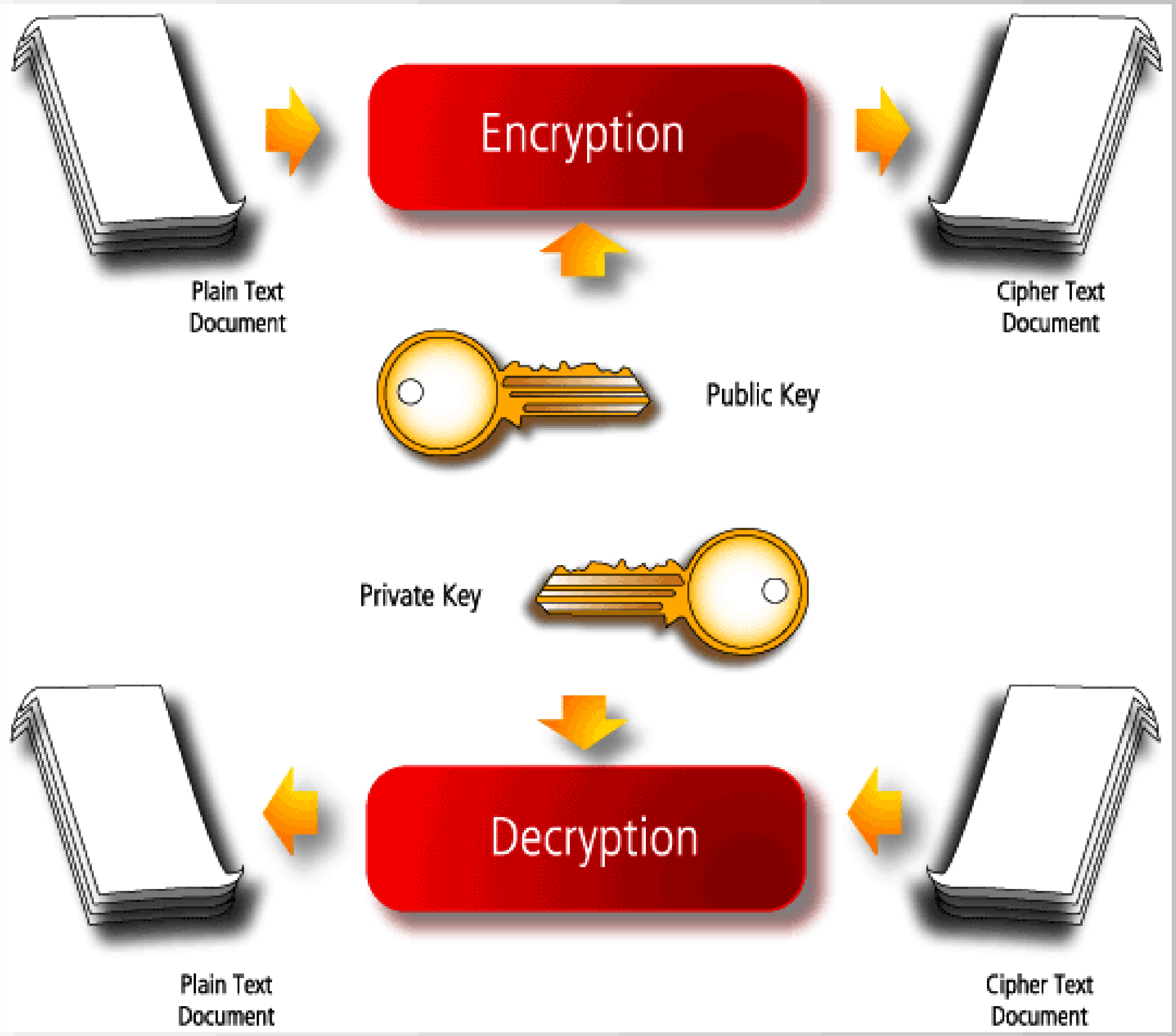

Classical General Examples of PKC

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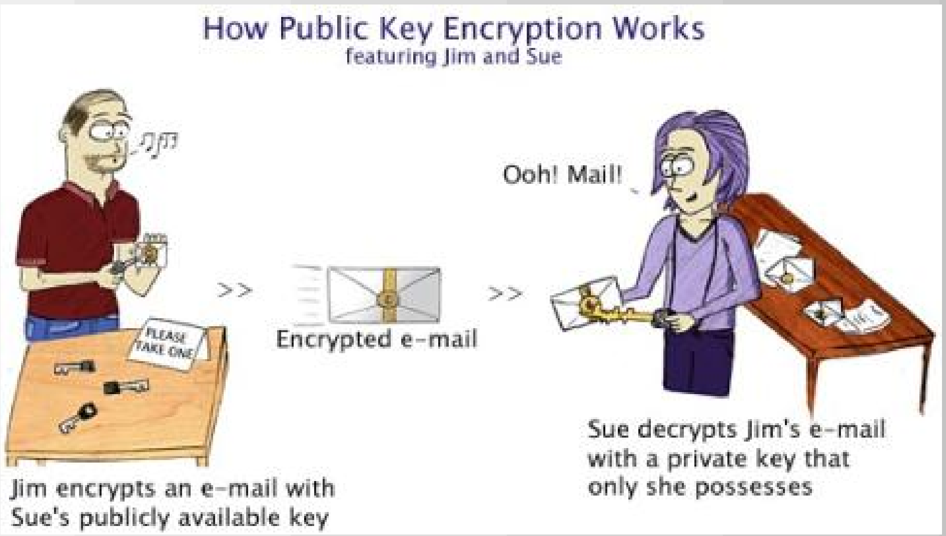
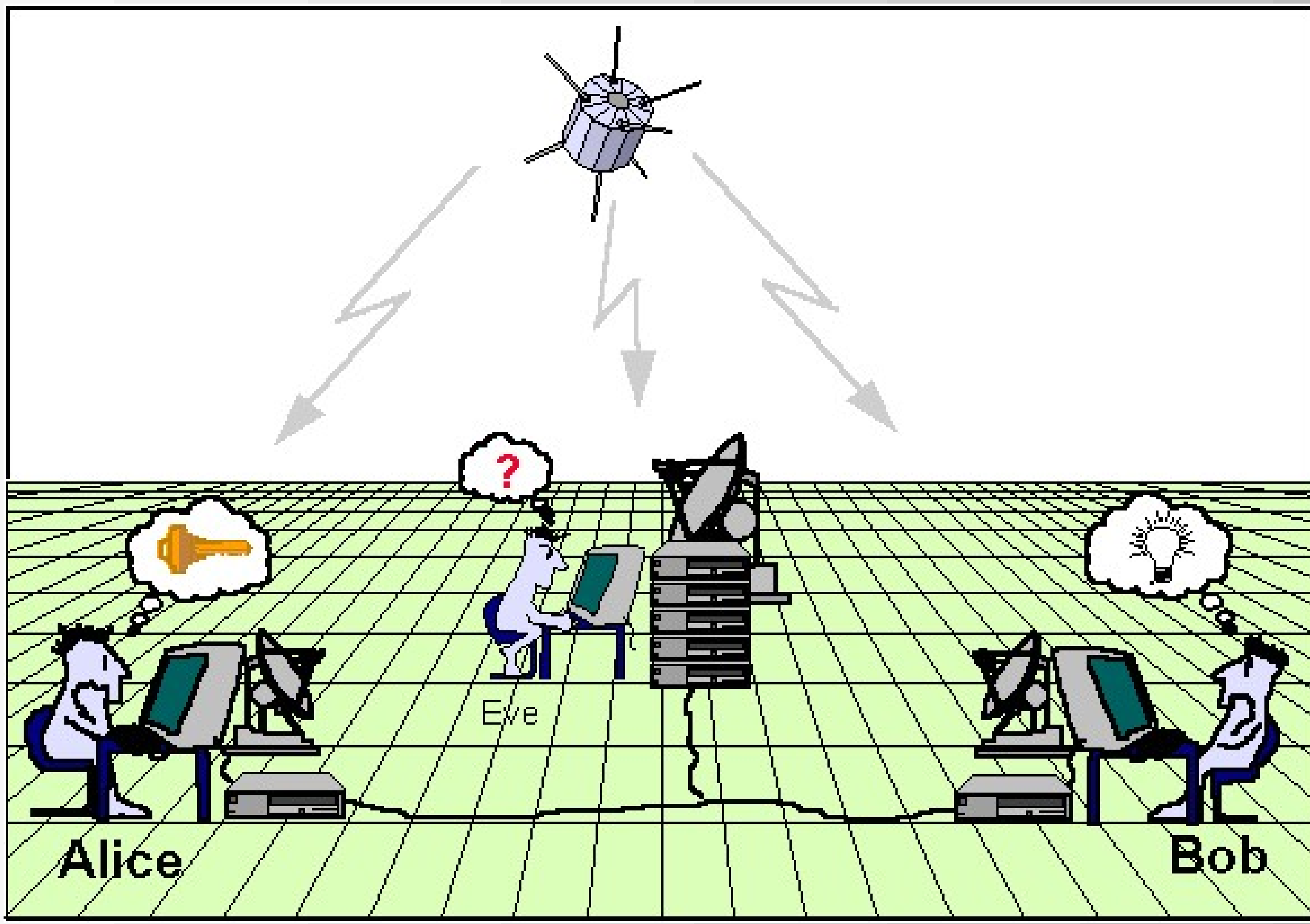


Diffie–Hellmann key exchange 1/5



Diffie–Hellmann key exchange 2/5

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1.
2.
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Diffie–Hellmann key exchange 2/5

1. **Alice** and **Bob** agree on a prime p and a *generator* g in $\mathbb{Z}/p\mathbb{Z}$
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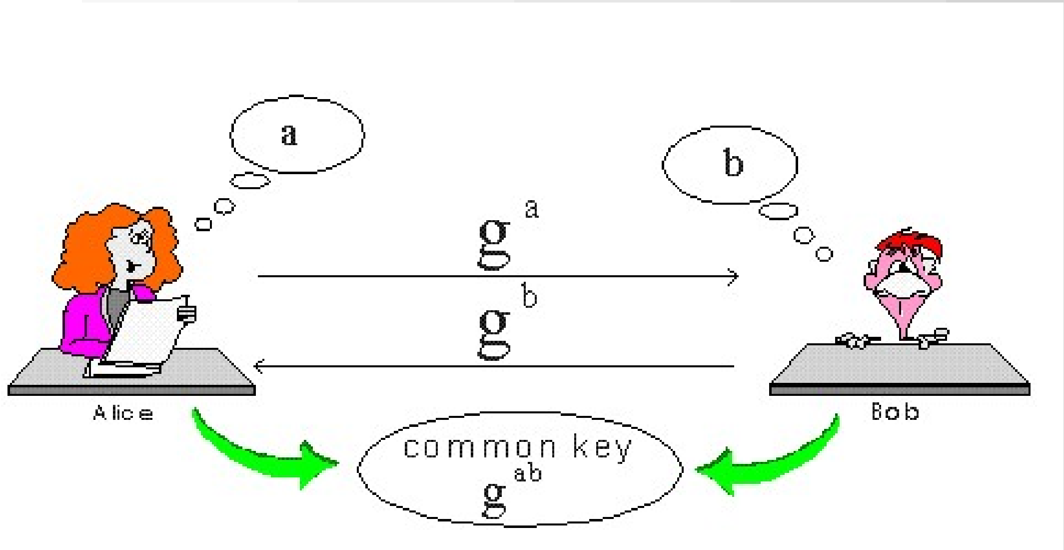

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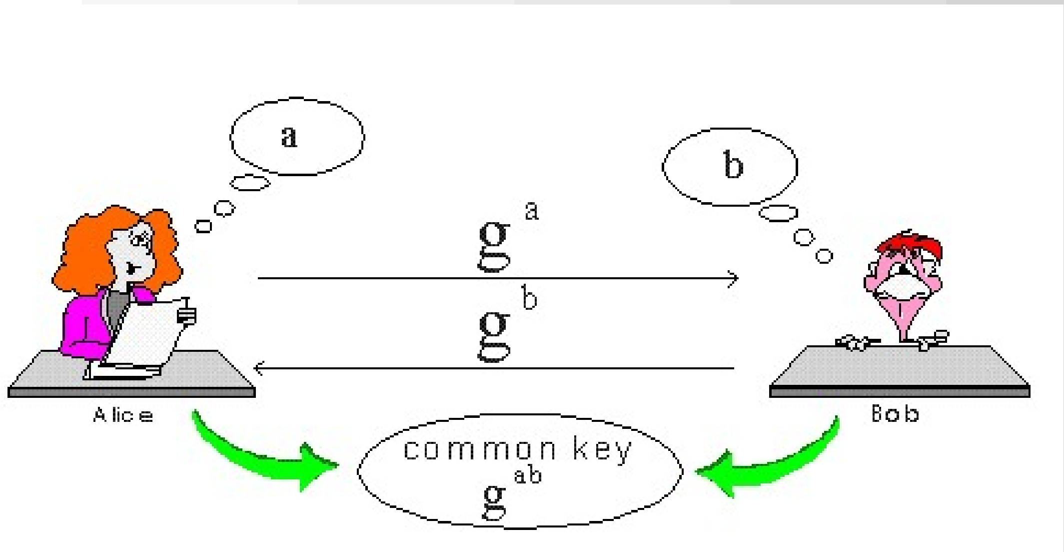

what is a generator of $\mathbb{Z}/p\mathbb{Z}$?

Diffie–Hellmann key exchange 3/5

- ☞
- ☞
- ☞
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Diffie–Hellmann key exchange 3/5

- A generator (or primitive root) g of a prime number p is a number
-
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-
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Diffie–Hellmann key exchange 3/5

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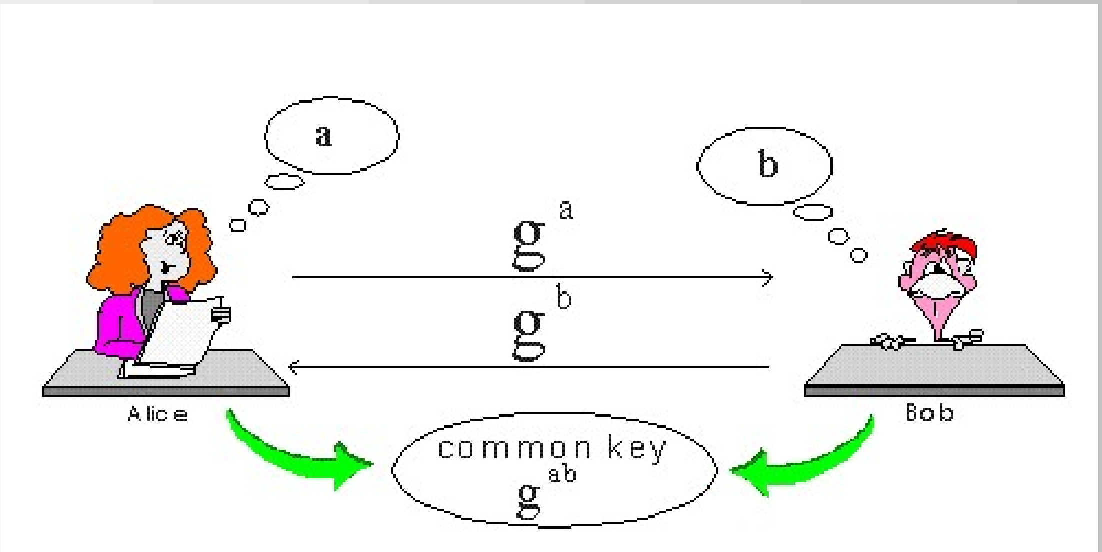
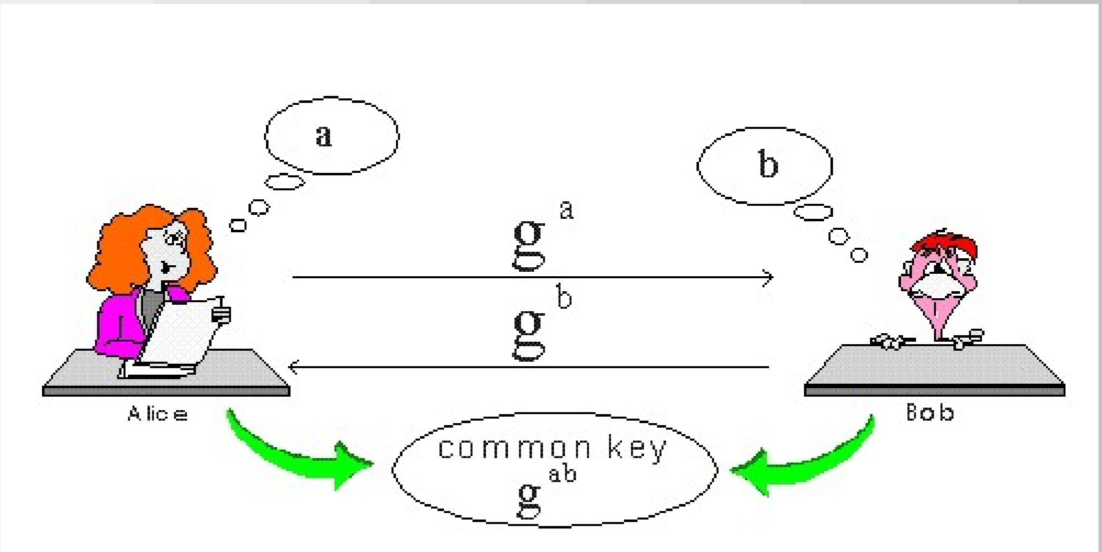
- Computing discrete logs appears infeasible in general

Diffie–Hellmann key exchange 4/5



Diffie–Hellmann key exchange 4/5



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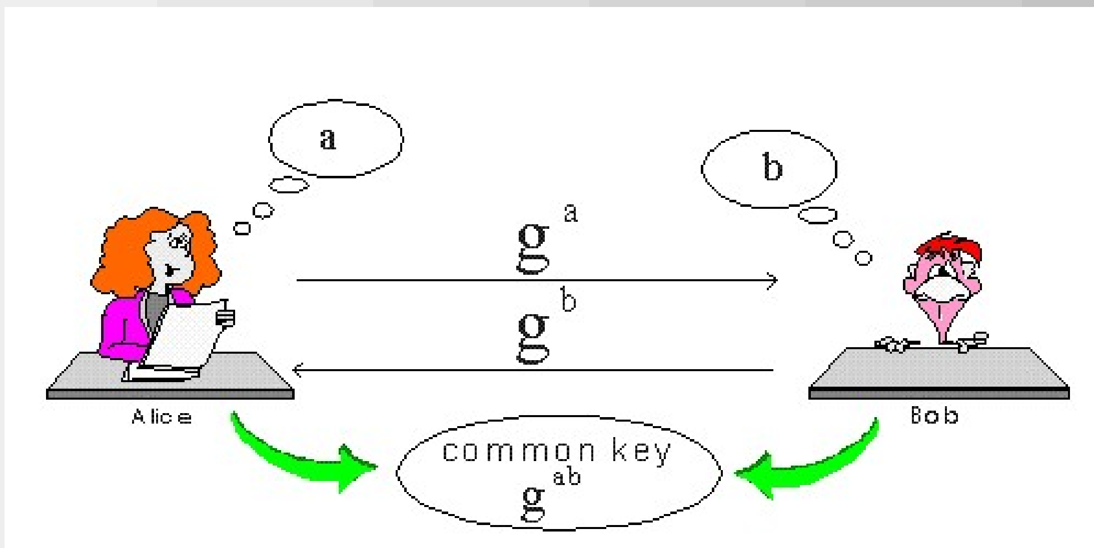
Diffie–Hellmann key exchange 4/5



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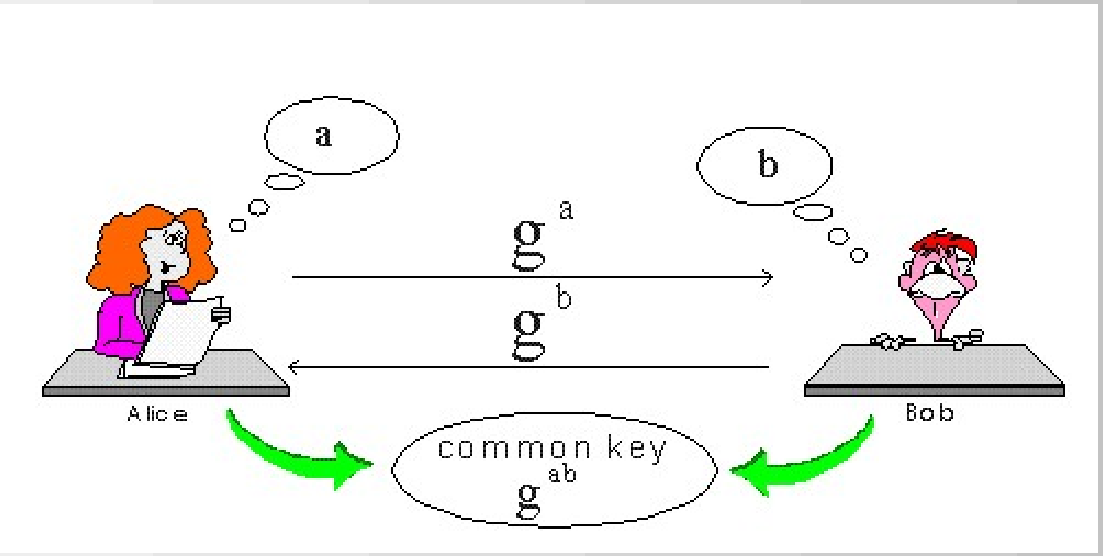
Diffie–Hellmann key exchange 4/5



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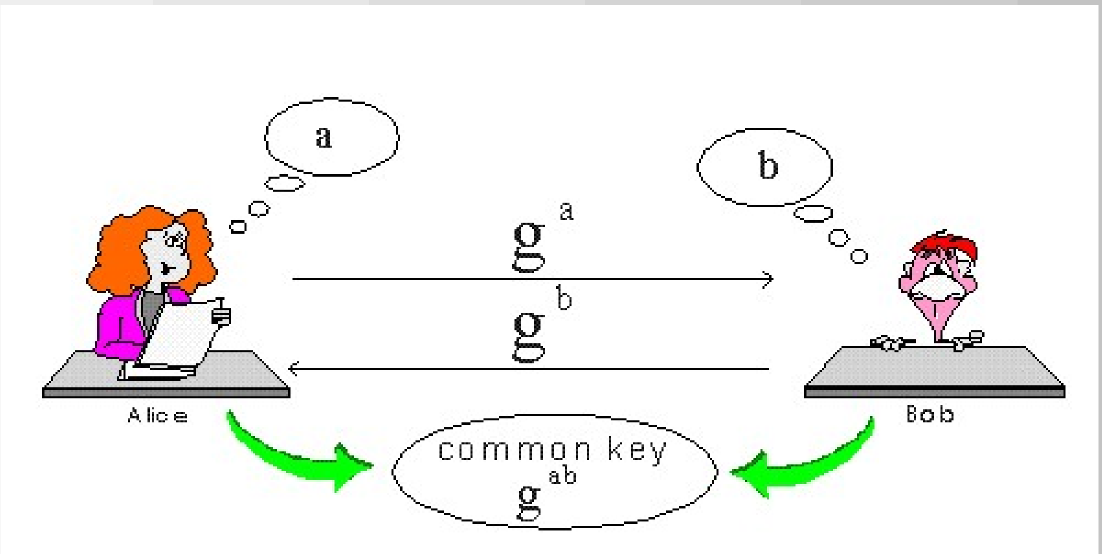
Diffie–Hellmann key exchange 4/5



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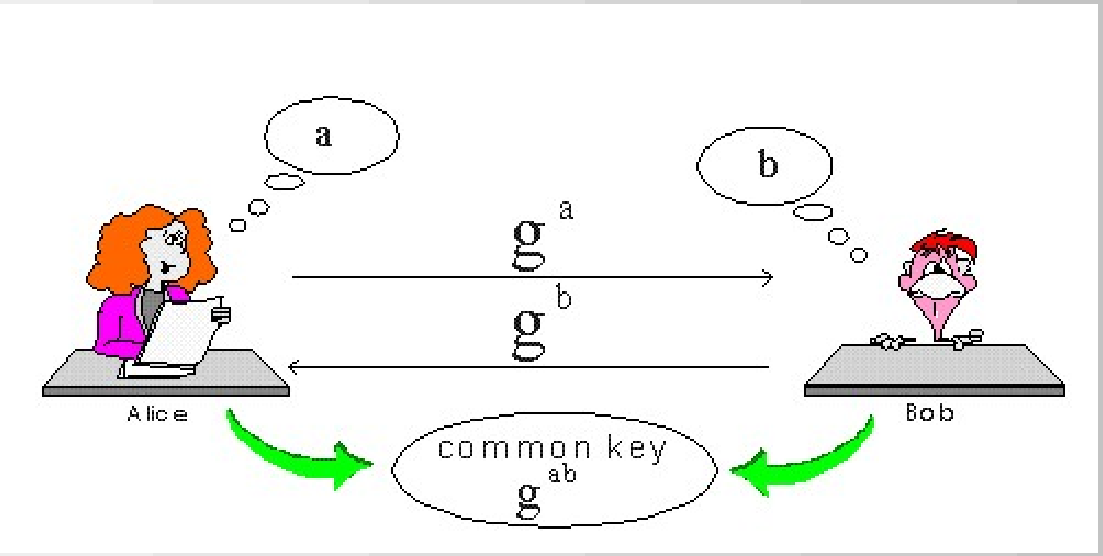
Diffie–Hellmann key exchange 4/5



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$$g^X \equiv \alpha \bmod p$$

Diffie–Hellmann key exchange 5/5

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A “criptographically meaningful size” example:

Diffie–Hellmann key exchange 5/5

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$p = 370273307460967425842481081357528298315386585184169353328410050632472746552261503118421027658$
$721711241508544733578984012456938357678209461867245573821426204444288523552318347549870943602$
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Diffie–Hellmann key exchange 5/5

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$g = 5$

$a =$ 2308840902039895388227917479653026722679565668038909847198111704018348814235352410395561538395030079070601651217032418664096044274135079002294214909329210457060330466911747378679898500024210343154844771162635809902530822

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a = 230884090203989538822791747965302672226795656680389098471981117040183488142353524103955615383\
9
503007907060165121703241866409604427413507900229421490932921045706033046691174737867989850
002421034315484477116263580990253082

b = 202628627712040976052737350793757540205242681192017941068774728007392912193775762330719406560
040933311164190467406050768556042798567906868136988403326100887782675574881508824219596637
05180574380470308541288799465419522 89

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$5^a =$ 2494514241078932628924844425756891566223499407710247477336124609623103292094965304814697324109595757601247732395287229562052325375814376804042234303084056865342398577185839357814166518479146351026737882783508710913577680

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503007907060165121703241866409604427413507900229421490932921045706033046691174737867 98985
00024210343154844771162635809902530822

$b =$ 202628627712040976052737350793757540205242681192017941068774728007392912193775762330719406560
040933311164190467406050768556042798567906868136988403326100887782675574881508824219596 63
70518057438047030854128879946541952289

$5^a =$ 249451424107893262892484442575689156622349940771024747733612460962310329209496530481469732410
959575760124773239528722956205232537581437680404223430308405686534239857718583935781416 65
18479146351026737882783508710913577680

$5^b =$ 287293760357523957032946092556813694596882586743260552838382768832192594422702357607546631218
640014853957893014446177932232015947060973983603311951612138362147414988242010983310457 62
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$b =$ 20262862771204097605273735079375754020524268119201794106877472800739291219377576233071940656004093331116419046740605076855604279856790686813698840332610088778267557488150882421959663705180574380470308541288799465419522 89

$5^a =$ 24945142410789326289248444257568915662234994077102474773361246096231032920949653048146973241095957576012477323952872295620523253758143768040422343030840568653423985771858393578141665184791463510267378827835087109135776 80

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$5^{ab} =$ 36674172125349300306071275329964633749875664216293811088694156172838197865927916343627669411439682348921744440103868565092597181273385376288526293344498755858906626836268436664512871223950829209587369115457329515844644 96

Discrete Logarithms Computation

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Some classical algorithms:

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NOTE: The last two are "very special" for $\mathbb{Z}/p\mathbb{Z}$

Discrete Logarithms computation Records 1/2

A. Joux et R. Lercier, 1998.

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$$\begin{aligned} p &= \lfloor 10^{89}\pi \rfloor + 156137 \\ &= 314159265358979323846264338327950288419716939937510582097494459230781640628620899862959619, \\ & g = 2, \end{aligned}$$

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\end{aligned}
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$$\begin{aligned} y &= \lfloor 10^{89} e \rfloor \\ &= 27182818284590452353602874713526624977572470936999595749669676277240766303535475945713821 7 \end{aligned}$$

$$2^X \equiv y \bmod p$$

$$y = g^{1767138072114216962732048234071620272302057952449914157493844716677918658538374188101093},$$

Discrete Logarithms computation Records 1/2

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$$\begin{aligned} y &= \lfloor 10^{89} e \rfloor \\ &= 2718281828459045235360287471352662497757247093699959574966967627724076630353547594571382 17 \end{aligned}$$

$$2^X \equiv y \bmod p$$

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Discrete Logarithms computation Records 1/2

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Discrete Logarithms computation Records 1/2

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$$
\begin{aligned}
p &= \lfloor 10^{89}\pi \rfloor + 156137 \\
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& g = 2, \\
y &= \lfloor 10^{89} e \rfloor \\
&= 271828182845904523536028747135266249775724709369995957496696762772407663035354759457138217
\end{aligned}
$$

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$y+2 = g^{3089883293350445253338277649145014072371680345775342279270337839998667742527392786788373 01}$,

$y+3 = g^{6580688800278838010371298688366325318718350540545118893505511320988794936425513481529784 6}$,

Discrete Logarithms computation Records 1/2

A. Joux et R. Lercier, 1998.

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$y + 2 = g^{3089883293350445253338277649145014072371680345775342279270337839998667742527392786788373 01}$,

$y + 3 = g^{6580688800278838010371298688366325318718350540545118893505511320988794936425513481529784 6}$,

$y + 4 = g^{4069601088212869919975316593460491889486849045436061788784458793535379546218510507897709 3}$

Discrete Logarithms computation Records 1/2

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It took 4.5 months... on a Pentium PRO 180 MHz

Discrete Logarithms computation Records 2/2

Discrete Logarithms computation Records 2/2

A. Joux et R. Lercier (CNRS / Ecole Polytechnique)

Discrete Logarithms computation Records 2/2

A. Joux et R. Lercier (CNRS / Ecole Polytechnique)

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Discrete Logarithms computation Records 2/2

A. Joux et R. Lercier (CNRS / Ecole Polytechnique)

① 1999 $p \cong 10^{100}$

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Discrete Logarithms computation Records 2/2

A. Joux et R. Lercier (CNRS / Ecole Polytechnique)

① 1999 $p \cong 10^{100}$

`500MHz quadri-processors Dec Alpha Server`

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Discrete Logarithms computation Records 2/2

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Discrete Logarithms computation Records 2/2

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Discrete Logarithms computation Records 2/2

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Discrete Logarithms computation Records 2/2

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Discrete Logarithms computation Records 2/2

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$p = \lfloor 10^{119}\pi \rfloor + 207819,\ g = 2$

Discrete Logarithms computation Records 2/2

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Discrete Logarithms computation Records 2/2

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$p = \lfloor 10^{119}\pi \rfloor + 207819,\ g = 2$

$y = \lfloor 10^{119} \rfloor$

$$y = g^{\substack{262112280685811387636008622038191827370390768520656974243035 \\ 380382193478767436018681449804940840373741641452864730765082}},$$

ElGamal Cryptosystem 1/2

Alice wants to sent a message $x \in \mathbb{Z}/p\mathbb{Z}$ to **Bob**

ElGamal Cryptosystem 1/2

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SETUP:

ElGamal Cryptosystem 1/2

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SETUP:

1.

2.

ElGamal Cryptosystem 1/2

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SETUP:

1. **Alice** and **Bob** agree on a prime p and a *generator* g in $\mathbb{Z}/p\mathbb{Z}$
2.

ElGamal Cryptosystem 1/2

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SETUP:

1. **Alice** and **Bob** agree on a prime p and a *generator* g in $\mathbb{Z}/p\mathbb{Z}$
2. **Bob** picks a secret b, $0 < b \leq p - 1$,

ElGamal Cryptosystem 1/2

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ElGamal Cryptosystem 1/2

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ENCRYPTION: (Alice)

ElGamal Cryptosystem 1/2

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① **Alice** picks a secret k,

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ElGamal Cryptosystem 1/2

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ElGamal Cryptosystem 1/2

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$$E(x) = (\alpha, \gamma) \in \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$$

ElGamal Cryptosystem 2/2

ElGamal Cryptosystem 2/2

DECRYPTION: (Bob)

ElGamal Cryptosystem 2/2

DECRYPTION: (Bob)

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ElGamal Cryptosystem 2/2

DECRYPTION: (Bob)

① **Bob** computes

②

ElGamal Cryptosystem 2/2

DECRYPTION: (Bob)

① **Bob** computes

$$D(\alpha, \gamma) = \gamma \cdot \alpha^{p-1-b} \bmod p$$

②

ElGamal Cryptosystem 2/2

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ElGamal Cryptosystem 2/2

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ElGamal Cryptosystem 2/2

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since $g^{k(p-1)} \bmod p = 1$ by

ElGamal Cryptosystem 2/2

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ElGamal Cryptosystem 2/2

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Eve can decrypt the message if he can compute the discrete logarithm X,

ElGamal Cryptosystem 2/2

DECRYPTION: (Bob)

① **Bob** computes

$$D(\alpha, \gamma) = \gamma \cdot \alpha^{p-1-b} \bmod p$$

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$$D(E(x)) = D(\alpha, \gamma) = x \cdot g^{bk} \cdot g^{k(p-1-b)} = x$$

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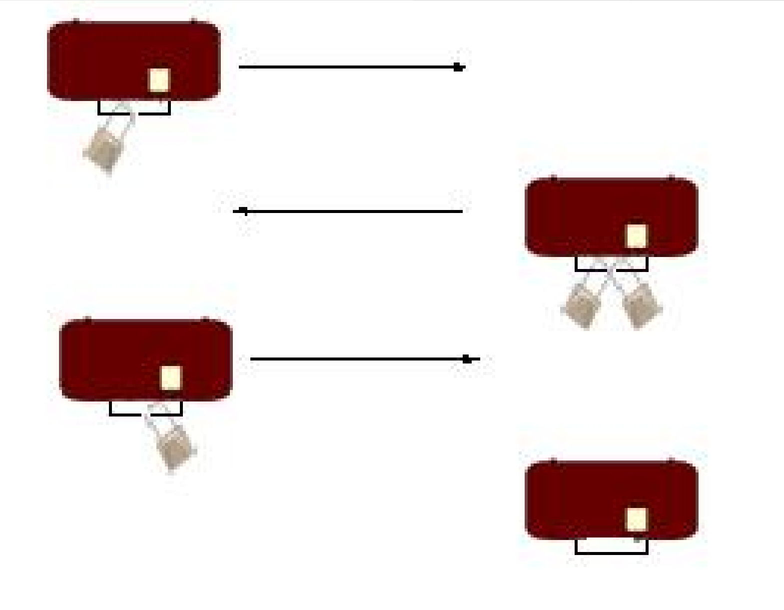
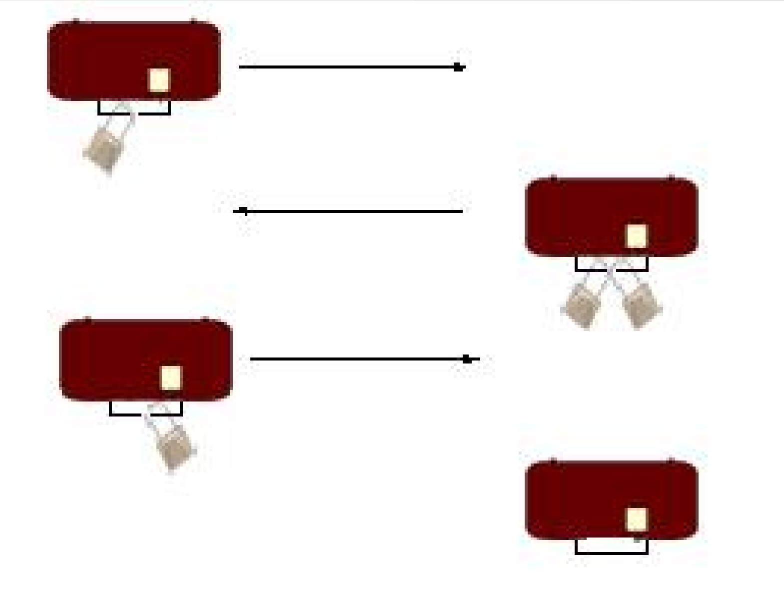
$$\beta = g^X \bmod p$$

Massey Omura 1/2

Massey Omura 1/2

Alice  Bob

Massey Omura 1/2

Alice  **Bob**

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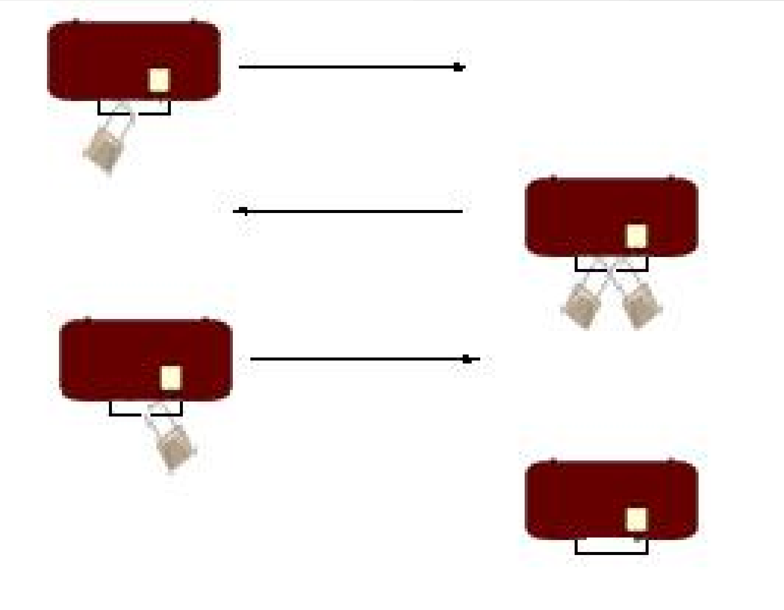
Massey Omura 1/2

Alice  **Bob**

① **Alice** and **Bob** each picks a secret key $k_A, k_B \in \{1, \ldots, p-1\}$

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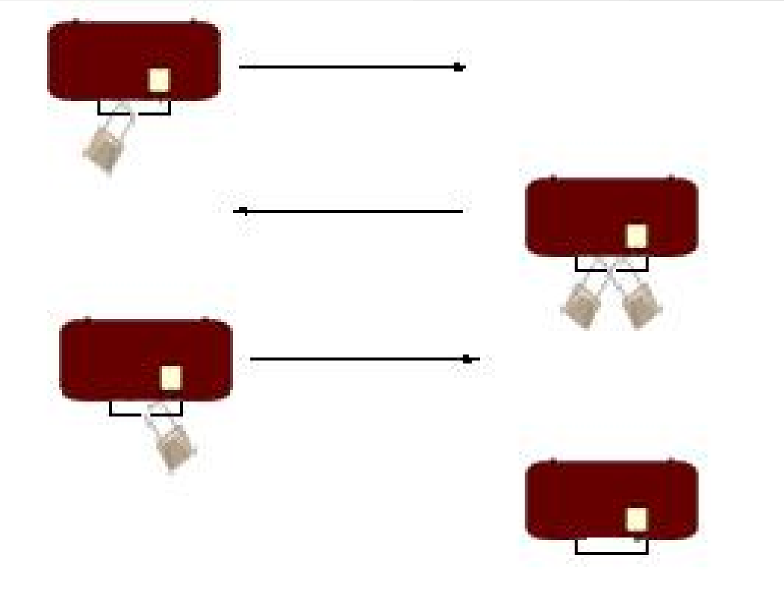
Massey Omura 1/2

Alice  **Bob**

① **Alice** and **Bob** each picks a secret key $k_A, k_B \in \{1, \ldots, p-1\}$

② They compute $l_A, l_B \in \{1, \ldots, p-1\}$ such that

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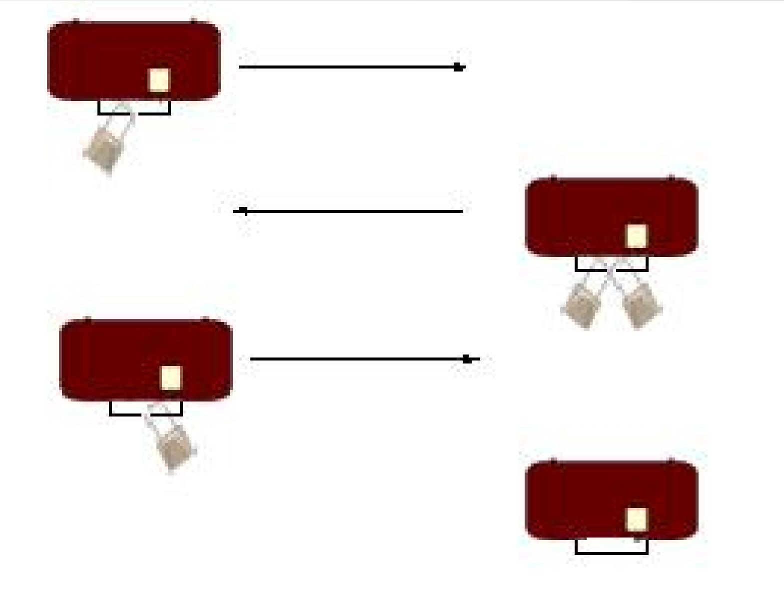
Massey Omura 1/2

Alice  **Bob**

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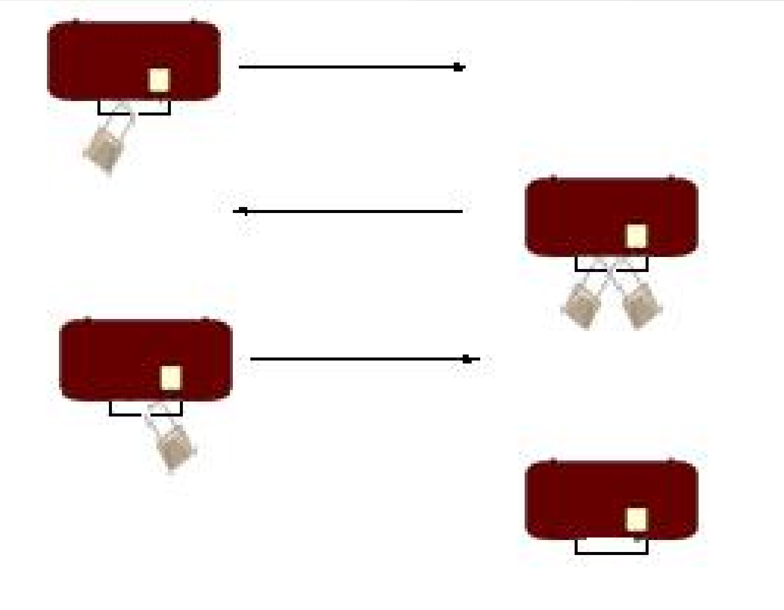
Massey Omura 1/2

Alice  **Bob**

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③ $k_A l_A = 1(\bmod p-1)$ and $k_B l_B = 1(\bmod p-1)$

④ **Alice** key is (k_A, l_A) (k_A to lock and l_A to unlock)

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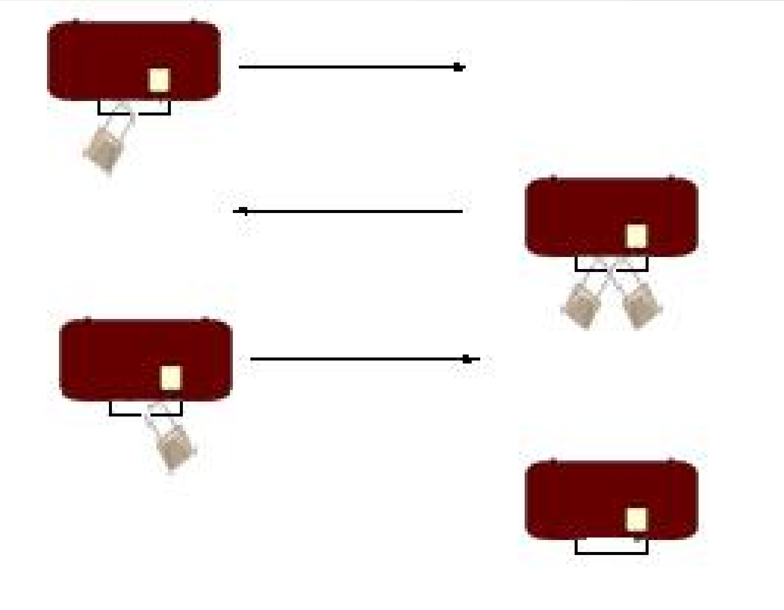
Massey Omura 1/2



Alice Bob

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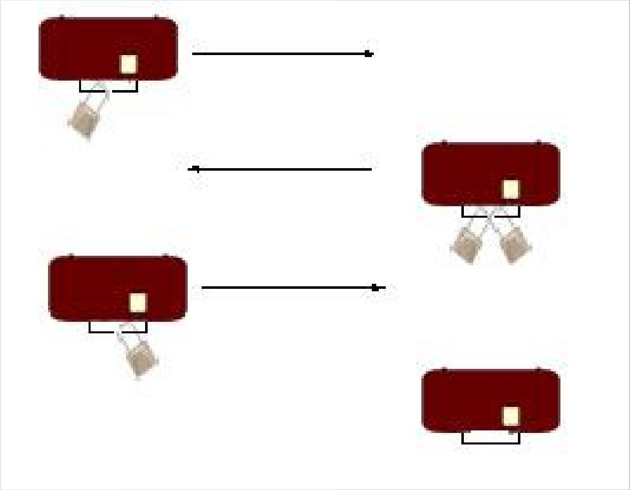
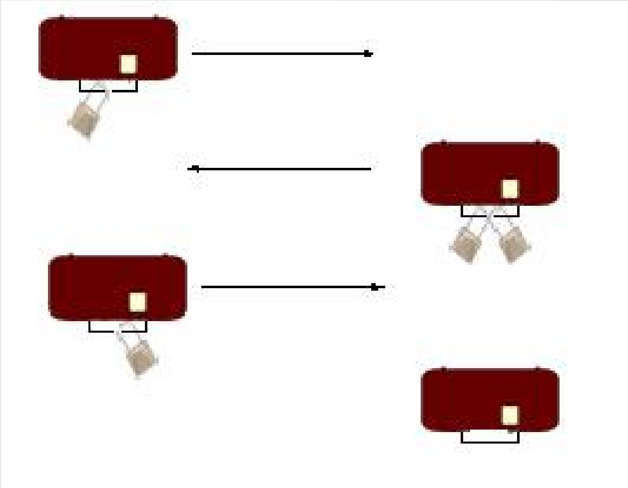
⑤ **Bob** key is (k_B, l_B) (k_B to lock and l_B to unlock)

Massey Omura 2/2

Alice (k_A, l_A)  **Bob** (k_B, l_B)

Massey Omura 2/2

Alice (k_A, l_A)  **Bob** (k_B, l_B)

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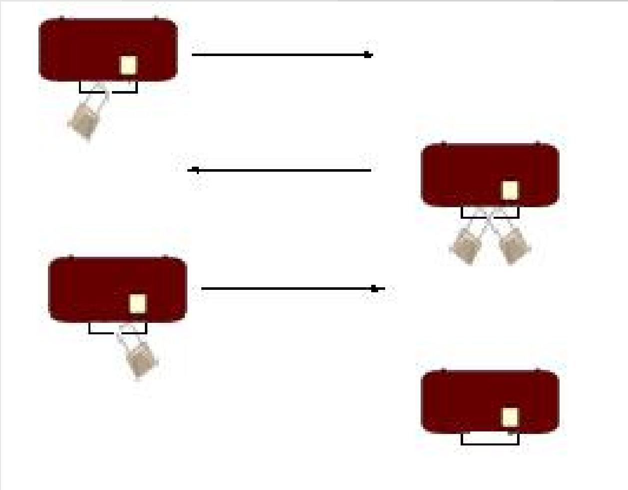
Massey Omura 2/2

Alice (k_A, l_A)  **Bob** (k_B, l_B)

① To send the message P, **Alice** computes and sends $M = P^{k_A} \bmod p$

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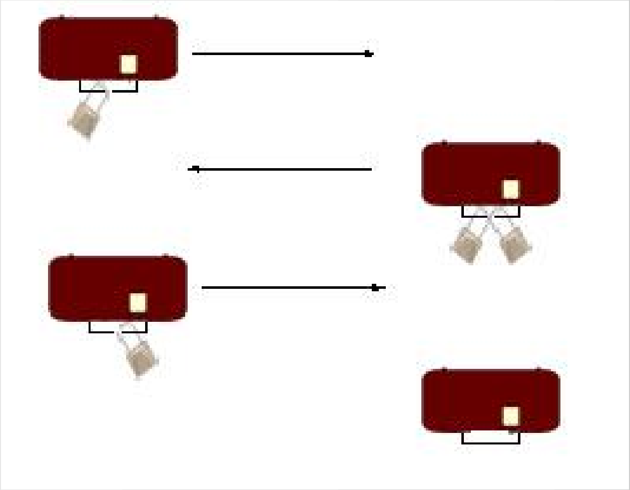
Massey Omura 2/2

Alice (k_A, l_A)  **Bob** (k_B, l_B)

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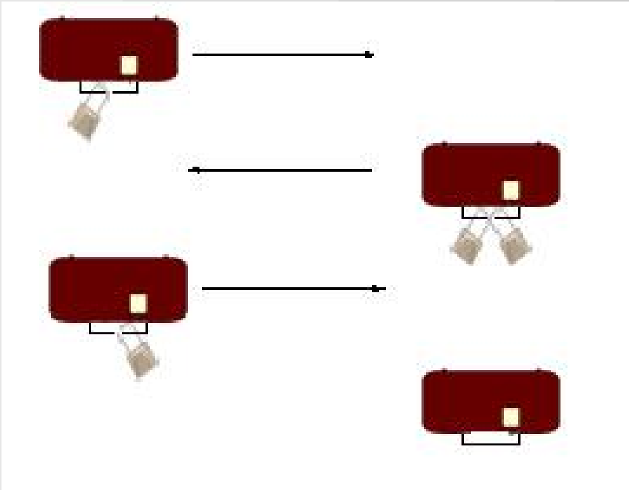
Alice (k_A, l_A)  **Bob** (k_B, l_B)

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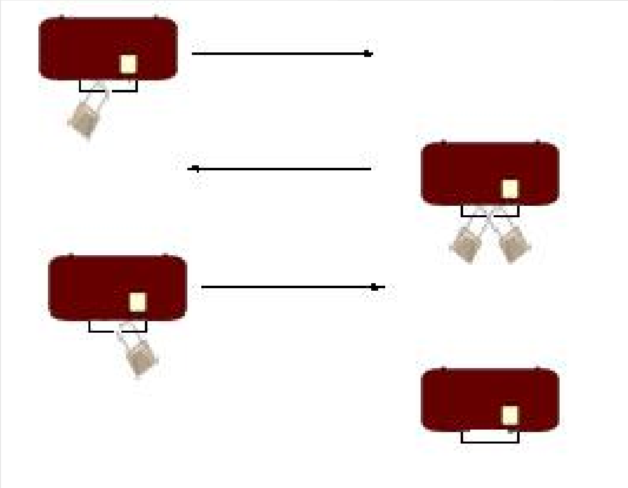
Alice (k_A, l_A)  **Bob** (k_B, l_B)

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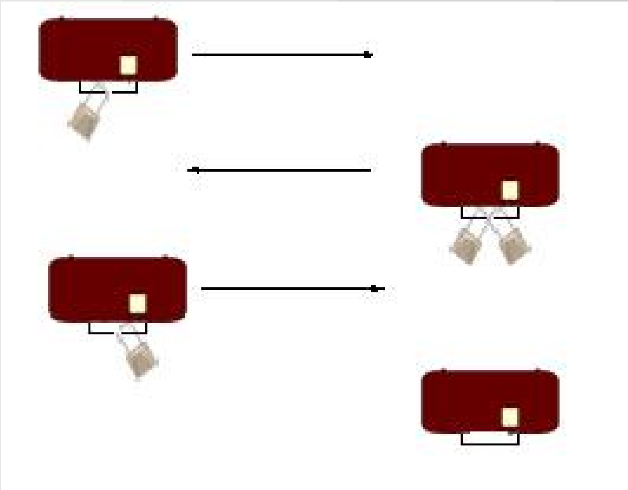
Alice (k_A, l_A)  **Bob** (k_B, l_B)

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It works: $P = L^{l_B} = N^{l_A l_B} = M^{k_B l_A l_B} = P^{k_A k_B l_A l_B}$ by Fermat Little Theorem

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- $\mathbb{F}^*_{p^m} = \mathbb{F}_{p^m} \setminus \{0\}$ is a cyclic group under multiplication

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- Good to find f sparse

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- **Problem.** Find functions with sparse interpolation polynomial

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$$\Longrightarrow \quad \text{unique with degree} \leq q - 1$$

- If $c_h = \#\{c \in \mathbb{F}_q \mid h(c) \neq c\}$,

$$q - c_h \leq \partial f_h \leq q - 2$$

- **Problem.** Find functions with sparse interpolation polynomial

Better if they are $\rightsquigarrow$Permutation polynomials

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$$D_m(D_n(x,\pm 1), \pm 1) = D_{mn}(x, \pm 1)$$

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Problem. Find new classes of PP

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(S. Konyagin, FP – 2002) $M_q = \{\sigma \in \mathcal{S}(\mathbb{F}_q) \mid \partial f_\sigma < q-2\}$

$$|\#M_q - (q-1)!| \leq \sqrt{2e/\pi}\, q^{q/2}$$

A recent result

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$$\mathcal{N}_d = \#\left\{\sigma \in \mathcal{S}(\mathbb{F}_q) \mid \partial(f_\sigma) < q - d - 1\right\}$$

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Theorem *S. Konyagin, FP – 2003*

Let $\alpha = (e-2)/3e = 0.08808\cdots$ *and* $d < \alpha q$. *Then*

$$\left|\mathcal{N}_d - \frac{q!}{q^d}\right| \leq 2^d d q^{2+q-d} \binom{q}{d} \left(\frac{2d}{q-d}\right)^{(q-d)/2}.$$

It follows that

$$\mathcal{N}_d \sim \frac{q!}{q^d}$$

if $d \leq \alpha q$ *and* $\alpha < 0.03983$

Other ways of counting

If $\sigma \in \mathcal{S}(\mathbb{F}_q)$, $\boxed{c_\sigma = \#\{a \in \mathbb{F}_q \mid \sigma(a) \neq a\}}$

$$\sigma \neq id \Rightarrow \quad q - c_\sigma \leq \partial f_\sigma \leq q - 2$$

(since $f_\sigma(x) - x$ has at least $q - c_\sigma$ roots)

Consequences.

- 2–cycles have degree $q-2$
- 3–cycles have degree $q-2$ or $q-3$
- k–cycles have degree in $[q-k, q-2]$

(Wells)
$$\boxed{\#\{\sigma \in 3\text{–cyle},\ \partial(f_\sigma) = q-3\} = \begin{cases} \frac{2}{3}q(q-1) & q \equiv 1 \bmod 3 \\ 0 & q \equiv 0 \bmod 3 \\ \frac{1}{3}q(q-1) & q \equiv 0 \bmod 3 \end{cases}}$$

More enumeration functions

- σ_1, σ_2 conjugated $\Longrightarrow$ $c_{\sigma_1} = c_{\sigma_2}$
- $\mathcal{C}$ conjugation class of permutations
- $c_{\mathcal{C}} = \#\{$ elements $\in \mathbb{F}_q$ moved by any $\sigma \in \mathcal{C}\}$
  (i.e. $c_{\mathcal{C}} = c_\sigma$ for any $\sigma \in \mathcal{C}$ $\quad q - c_{\mathcal{C}} \leq f_\sigma$)
- $\mathcal{C} = [k]$=k–cycles $\Longrightarrow$ $c_{[k]} = k$
- Natural enumeration functions:
  - ✗ $m_{\mathcal{C}}(q) = \#\{\sigma \in \mathcal{C}, \partial f_\sigma = q - c_{\mathcal{C}}\}$ (minimal degree)
  - ✗ $M_{\mathcal{C}}(q) = \#\{\sigma \in \mathcal{C}, \partial f_\sigma < q - 2\}$ (non-maximal degree)

Permutation Classes with non maximal degree

Let $\mathcal{C} = (m_1, \ldots, m_t)$ be the class of permutations with m_1 1-cycles, ..., m_t t-cycles. The number $c_{\mathcal{C}}$ of elements in $\mathbb{F}_q$ moved by any element of $\mathcal{C}$ is

$$c_{\mathcal{C}} = 2m_2 + 3m_3 + \cdots + tm_t$$

$$\boxed{M_{\mathcal{C}}(q) = \#\{\sigma \ \in \mathcal{C}, \partial f_\sigma < q-2\}}$$

Theorem 1 (C. Malvenuto, FP - 2002). $\exists N = N_{\mathcal{C}} \in \mathbb{N}$, $f_1, \cdots, f_N \in \mathbb{Z}[x]$, f_i monic, $\partial f_i = c_{\mathcal{C}} - 3$ such that if $q \equiv a \bmod N$, then

$$M_{\mathcal{C}}(q) = \frac{q(q-1)}{m_2! 2^{m_2} \cdots m_t! t^{m_t}} f_a(q)$$

k–cycles with minimal degree

$$m_{[k]}(q) = \#\{\sigma\ k\text{–cycle}, \partial f_\sigma = q - k\}$$

THEOREM 2 (C. Malvenuto, FP - 2003).

- If $q \equiv 1 \bmod k \quad \Longrightarrow$

$$m_{[k]}(q) \geq \frac{\varphi(k)}{k} q(q-1).$$

- If $q = p^f$, $p \geq 2 \cdot 3^{[k/3]-1} \quad \Longrightarrow$

$$m_{[k]}(q) \leq \frac{(k-1)!}{k} q(q-1).$$

Consequences of Theorem 1

- $\dfrac{M_{\mathcal{C}}(q)}{\#\mathcal{C}} = \dfrac{1}{q} + O\left(\dfrac{1}{q^2}\right)$
- If $\mathcal{C}$ is fixed,

$$\mathrm{Prob}(\partial f_\sigma < q - 2 \mid \sigma \in \mathcal{C}) \sim \frac{1}{q}$$

- If $q = 2^r$, $\mathcal{C}_r$ is the conjugation class of r transposition,

$$M_{\mathcal{C}_r}(q) = \frac{q!}{r!2^r(q-2r+1)!} - \frac{q-2(r-1)(2r-1)}{2r} M_{\mathcal{C}_{r-1}}(q)$$

- One can compute $M_{\mathcal{C}}(q)$ for $c_{\mathcal{C}} \leq 6$

Table 1. $\#c_{\mathcal{C}} \leq 6$, (q odd)

- $M_{[4]}(q) = \frac{1}{4}q(q-1)\left(q-5-2\eta(-1)-4\eta(-3)\right)$

- $M_{[2\ 2]}(q) = \frac{1}{8}q(q-1)(q-4)\left\{1+\eta(-1)\right\}$

- $$M_{[5]}(q) = \frac{1}{5}q(q-1)\left(q^2-(9-\eta(5)-5\eta(-1)+5\eta(-9))\,q+ +26+5\eta(-7)+15\eta(-3)+15\eta(-1)\right)$$

- $$M_{[2\ 3]}(q) = \frac{1}{6}q(q-1)\left(q^2-(9+\eta(-3)+3\eta(-1))q+ +(24+6\eta(-3)+18\eta(-1)+6\eta(-7))\right)+ \eta(-1)(1-\eta(9))q(q-5).$$

Table 2. $\#c_{\mathcal{C}} \leq 6$, (q even)

- $M_{[4]}(2^n) = \frac{1}{4}2^n(2^n-1)(2^n-4)(1+(-1)^n)$
- $M_{[2\ 2]}(2^n) = \frac{1}{8}2^n(2^n-1)(2^n-2)$
- $M_{[5]}(2^n) = \frac{1}{5}2^n(2^n-1)(2^n-3-(-1)^n)(2^n-6-3(-1)^n)$
- $M_{[2\ 3]}(2^n) = \frac{1}{6}2^n(2^n-1)(2^n-3-(-1)^n)(2^n-6)$.

Table 3. $\#c_{\mathcal{C}} = 6$, (q odd, $3 \nmid q$)

$$
\begin{aligned}
M_{[6]}(q) &= \tfrac{q(q-1)}{6}\{q^3 - 14\,q^2 + [68 - 6\,\eta(5) - 6\,\eta(50)]q - \\
&\quad [154 + 66\,\eta(-3) + 93\,\eta(-1) + 12\,\eta(-2) + 54\,\eta(-7)]\} \\
M_{[4\ 2]}(q) &= \tfrac{q(q-1)}{8}(q^3 - [14 - \eta(2)]q^2 + \\
&\quad [71 + 12\,\eta(-1) + \eta(-2) + 4\,\eta(-3) - 8\,\eta(50)]q \\
&\quad -[148 + 100\,\eta(-1) + 24\,\eta(-2) + 44\,\eta(-3) + 40\,\eta(-7)]) \\
M_{[3\ 3]}(q) &= \tfrac{q(q-1)}{18}(q^3 - 13\,q^2 + [62 + 9\,\eta(-1) + 4\,\eta(-3)]q \\
&\quad -[150 + 99\,\eta(-1) + 42\,\eta(-3) + 72\,\eta(-7)]) \\
M_{[2\ 2\ 2]}(q) &= \tfrac{q(q-1)}{48}(q^3 - [14 + 3\,\eta(-1)]q^2 + [70 + 36\,\eta(-1) + 6\,\eta(-2)]q \\
&\quad -[136 + 120\,\eta(-1) + 48\,\eta(-2) + 8\,\eta(-3)])
\end{aligned}
$$

Table 4. $\#c_{\mathcal{C}} = 6$

$$M_{[6]}(3^n) = \frac{3^n(3^n-1)}{6}\{3^{3n} - [14+2(-1)^n]3^{2n} + [71+39(-1)^n]3^n - [162+147(-1)^n]\}$$

$$M_{[4\ 2]}(3^n) = \frac{3^n(3^n-1)}{8}\{3^{3n} - [14+3\,(-1)^n]3^{2n} + [72+40\,(-1)^n]3^n - [164+140\,(-1)^n]\}$$

$$M_{[3\ 3]}(3^n) = \frac{3^n(3^n-1)}{18}\{(1+(-1)^n)\,3^{3\,n} - [14+15\;(-1)^n]3^{2\,n} + [71+81\;(-1)^n]3^n - [150+171\;(-1)^n]\}$$

$$M_{[2\ 2\ 2]}(3^n) = \frac{3^n(3^n-1)}{48}\{3^{3n} - [14+3(-1)^n]3^{2n} + [76+36(-1)^n]3^n - +[168+120(-1)^n]\}$$

Table 5. $\#c_{\mathcal{C}} = 6$

$$M_{[6]}(2^n) = \frac{2^n(2^n-1)}{6} \{(2^n - 3 - (-1)^n)(2^{2n} - (11 - (-1)^n)2^n + (41 + 7(-1)^n))\}$$

$$M_{[4\ 2]}(2^n) = \frac{2^n(2^n-1)}{8} \{(2^n - 3 - (-1)^n)(2^{2n} - 11 \cdot 2^n + 37 + (-1)^n)\}$$

$$M_{[3\ 3]}(2^n) = \frac{2^n(2^n-1)}{18} \{(2^n - 3 - (-1)^n)(2^{2n} - (10 - (-1)^n)2^n + 45 - 3(-1)^n))\}$$

$$M_{[2\ 2\ 2]}(2^n) = \frac{2^n(2^n-1)}{48} \{(2^n - 2)(2^n - 4)(2^n - 8)\}$$

Sketch of the Proof of Theorem 2. (1/3)

STEP 1. Translate the problem into one on counting points of an algebraic varieties

$$m_k(q) = \frac{q(q-1)}{k} n_k(q)$$

where $n_k(q) = \{\sigma \in [k] \mid \partial f_\sigma = q - k, \sigma(0) = 1\}$.
Need to show $|n_k(q)| \leq (k-1)!$. Now

$$f_\sigma(x) = \sum_{c \in \mathbb{F}_q} \sigma(c) \left(1 - (x-c)^{q-1}\right) = A_1 x^{q-2} + A_2 x^{q-3} + \cdots + A_{q-1}.$$

with $A_j = \sum_{c \in \mathbb{F}_q} \sigma(c) c^j = \sum_{c \in \mathbb{F}_q} \sigma(c) \left(c^j - c^{j-1}\right) = \sum_{\substack{c \in \mathbb{F}_q \\ \sigma(c) \neq c}} (\sigma(c) - c) c^j.$

Sketch of the Proof of Theorem 2. (2/3)

If $\sigma = (0,\ 1,\ x_1,\ x_2,\ \ldots,\ x_{k-2}) \in \mathcal{S}(\mathbb{F}_q)$,

$$A_j(\sigma) = (1 - x_1) + (x_1 - x_2)x_1^j + \cdots (x_{k-2} - x_{k-2})x_{k-3}^j + x_{k-2}^{j+1}.$$

Def. (Affine k–th Silvia set)

$$\mathcal{A}_k : \begin{cases} (1 - x_1) + x_1(x_1 - x_2) + \cdots + x_{k-3}(x_{k-3} - x_{k-2}) + x_{k-2}^2 & = \quad 0 \\ (1 - x_1) + x_1^2(x_1 - x_2) + \cdots + x_{k-3}^2(x_{k-3} - x_{k-2}) + x_{k-2}^3 & = \quad 0 \\ & \vdots \\ (1 - x_1) + x_1^{k-2}(x_1 - x_2) + \cdots + x_{k-3}^{k-2}(x_{k-3} - x_{k-2}) + x_{k-2}^{k-1} & = \quad 0 \end{cases}$$

$$n_k(q) = \#\{\underline{x} = (x_1, \ldots, x_{k-2}) \in \mathbb{F}_q^{k-2} \mid \underline{x} \in \mathcal{A}_k(\mathbb{F}_q), x_i \neq x_j\} \leq \#\mathcal{A}_k(\mathbb{F}_q)$$

$$\boxed{\dim_{\overline{\mathbb{F}}_q} \mathcal{A}_k = 0 \quad \overset{\text{Bezout Thm.}}{\Longrightarrow} \quad \#\mathcal{A}(\mathbb{F}_q) \leq (k-1)!}$$

Sketch of the Proof of Theorem 2. (3/3)

STEP 2.

Theorem. If $\mathbf{K}$ is an algebrically closed field,

$$\mathrm{char}(\mathbf{K}) = \begin{cases} 0 & \text{or} \\ > 2 \cdot 3^{[k/3]-1}. \end{cases}$$

Then

$$\boxed{\dim_{\mathbf{K}} \mathcal{A}_k = 0.}$$

NOTE.

- Proof is based on finding projective hyperplanes disjoint from $\mathcal{A}_k$
- There are examples of small values of q with $\dim_{\mathbf{K}} \mathcal{A}_k > 0$